

Mobility

Screws

Plücker coordinates

Freedoms and  
Constraints

Mobility Criterion

Mobility Criterion

A short treatise on robots' geometry, kinematics, and dynamics.

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# Standard Texts – Modeling and Control

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## Robot Modeling and Control

Spong, Mark W., Seth Hutchinson, and Mathukumalli Vidyasagar. Robot modeling and control. Vol. 3. New York: Wiley, 2006.

## Mathematical Modeling of Robots

Murray, R. M., Li, Z., & Sastry, S. S. (1994). A Mathematical Introduction to Robotic Manipulation. In Book (Vol. 29). <https://doi.org/10.1.1.169.3957>

# Texts – Modeling, Control, and Mechanisms

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## Robot Modeling and Control

Lynch, K. M., & Park, F. C. (2017). Modern Robotics  
Mechanics, Planning, and Control.

## Mechanisms' Kinematic Geometry

Hunt, Kenneth H., and Kenneth Henderson Hunt.  
Kinematic geometry of mechanisms. Vol. 7. Oxford  
University Press, USA, 1978.

# Texts – Screws and Kinematics

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## Screw Theory

Ball, Robert Stawell. A Treatise on the Theory of Screws. Cambridge university press, 1998.

## Mechanisms' Kinematic Geometry

Hunt, K. H. (2019). Structural Kinematics of In-Parallel-Actuated Robot-Arms. 105(December 1983), 705–712.

# Outline

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## Mechanism Components

Kinematic geometry. Mechanisms.

Joints: Joint closure; Pairs; Couplings.

Lower pairs and linkages; Higher and lower pairs.

Motions: Planar and spherical motions.

Synthesis: Type-, number-, and size-syntheses.

# Preamble.

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## Mechanics

Mechanics is an indirect study of nature via **bodies** – essentially the mathematical abstractions of common natural things; the **mass** is an *allocation* in *place* to each body; **geometry**, deals with the **theory of places**.

## Geometry

Geometry, deals with the **theory of places**; geometry is the bedrock of **robotics**, **control theory**, and many fields of **modern engineering and the physical sciences**.

# Mechanics Overview.

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## Definition (Motion)

When a **place** undergoes **body transformation** in the course of **time**, we have **motion**.

# Preamble – Mass, Body, Rigid Body Motion.

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## Definition (Body – Truesdell, 1977.)

By a **body**, we shall mean the **closure of an open set** in some **measure space**  $\Omega$  over which a **non-negative measure  $M$** , called the **mass**, is defined, and that  $M$  can be extended to a Borel measure over the  $\sigma-$  algebra of Borel sets in  $\Omega$ .

# Preamble – Mass, Body, Rigid Body Motion.

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## Bodies – Truesdell, 1977.

That in **mechanics** which deals with

- (i) **mass points**, which occupy a single point at any one time;
- (ii) **rigid bodies**, which never deform;
- (iii) **strings and rods and jets**, which are 1-dimensional; membranes and shells, that sweep out surfaces;
- (iv) **space-filling fluids and solids** e.t.c. **are termed bodies.**

# Statics, Dynamics, Rigid Body (Motion).

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## Statics and Dynamics

That which studies **putative equilibria** is referred to as **statics**. That which concerns motion of all sorts is referred to as **dynamics**. The dynamics that are specific to **particular bodies** are termed **constitutive**.

# Statics, Dynamics, Rigid Body (Motion).

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## The Rigid Body

A rigid body does not stretch, buckle, contract, bend, twist, nor deform. Well, not really!

## The Rigid Body

As engineers, we judge kinematic rigid hardware with the expectation that kinematic changes do not depart from rigid-body predictions.

# Statics, Dynamics, Rigid Body (Motion).

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## The Rigid Body

We expect that **localized stresses**, **active noise**, **vibrations** and **heat** e.t.c will not cause **reasonable departures** from expectations.

## Rigid Body Motion

That **motion** that **preserves distance** between all points in a body is termed a **rigid body motion**.

# Statics, Dynamics, Rigid Body (Motion).

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## Rigid Body Motion

At issue are components of a rigid body's **movement** w.r.t to a fixed or moving **frame of reference**. In its most basic form, this movement is parameterized by displacement (and is sometimes time-varying e.g. for a continuum body). When solving for the movements of bodies, it is often useful to include velocities (**twists**) in order to characterize the motion.

# Kinematics vs. Kinetics

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## Dynamics

$$\dot{x} = f(t; x, u), \quad x(t_0) = t_0 \quad (1)$$

$$\dot{x} = f(t; x) + g(t; x, u), \quad x(t_0) = t_0 \quad (2)$$

## Definition (Kinematics.)

**Kinematics** is the English version of the word *cinématique* coined by A.M. Ampère (1775-1836), who translated it from the Greek word *kίνημα*.

# Kinematics vs. Kinetics.

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## Definition (Truesdell)

That part of a system's **dynamics** that involves its **motion** by **displacement** – both linear and angular – and **separated from motions owing to forces and torques**, together with the successive derivatives with respect to time of all such displacements (this includes velocities, accelerations, and hyper accelerations) all form the **kinematics** of a **rigid, continuum or laminae** of bodies.

# Kinematics vs. Kinetics

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## Kinetics

The **motion** of bodies can also be conceived as resulting from the **forces' action**. **Energy, temperature, and calory** of a body are resultant effects of gains or loss of heat. Motions arising as a result of these are called **kinetics**.

# Kinematics vs. Kinetics.

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## Definition (Kinetics – Technical Definition)

That part of a system's **dynamics** that involves its **motion** by **forces, energy, torque, inertia, dynamic stability, and equilibrium** and similar properties all form the **kinetics** of a rigid, continuum or laminae of bodies.

# Kinematic Geometry.

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## Definition (Kinematic Geometry)

The solid geometry of relatively moving rigid bodies is termed the **kinematic geometry** of the rigid body. With motion, we'd have to include the successive derivatives of the displacement such as acceleration e.t.c as the 'laws of motion' stipulates in mechanics.

# Joints and Links

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## Links

Links may be rigid mechanical parts, elastic, (vulcanized) rubber components, diaphragms, conveyor belts, spring-damper systems e.t.c.

## An Elementary Joint or Kinematic Pair.

An **elementary joint** or a **kinematic pair** consists of touching two **links** together at one point – then ensuring a single contact point is **continuously maintained** throughout **relative movement**.

# Joint (Contact) Kinematics

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## Contact Kinematics

A body may **slide** or **slip** across a plane or surface, or **roll** over another body.

## Joints

Joints are the result of the connecting points between two or more rigid bodies.

## Links

Links may be rigid mechanical parts, elastic, (vulcanized) rubber components, diaphragms, conveyor belts, spring-damper systems e.t.c.

# Definition of a Mechanism

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## Definition (Author's Definition)

A **connection** of mechanical, magnetic, electrical, hydraulic, or pneumatic components forming an **assemblage**, meant for moving rigid, semi-rigid or non-rigid bodies via a **controlled generation** of (sometimes constrained) **motion**.

## Kenneth Hunt (1978)

A means of **transmitting**, **controlling**, or **constraining** the relative movement between parts. Whenever we have an **higher pair** or more, we have a mechanism.

# Mechanism Examples

## Mobility

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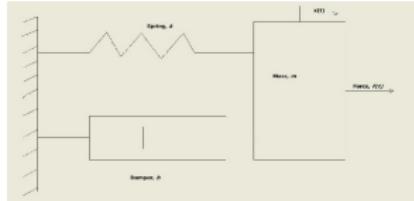
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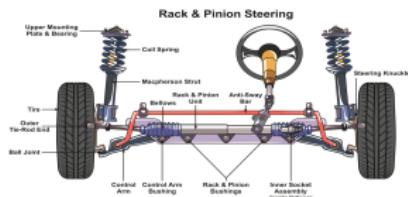
### Spring-Mass-Damper System



### Excavator



### Car suspension



### Daimler Plant



# Lower Pairs, Higher Pairs, Linkages

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## Lower and Higher Pairs

When elements of pairs touch one another over a **substantial region of a surface covering a line, curve-surface, or point of contact**, we have **lower pairs**. When they touch **along a discrete line, curve-surface, or point of contact**, we have **higher pairs**.

## Linkage (Hunt, 1978)

If all joints of a **mechanism or mechanical movement** belong to lower pairs, we have a **linkage**.

# Prismatic Pairs or *P*-pairs

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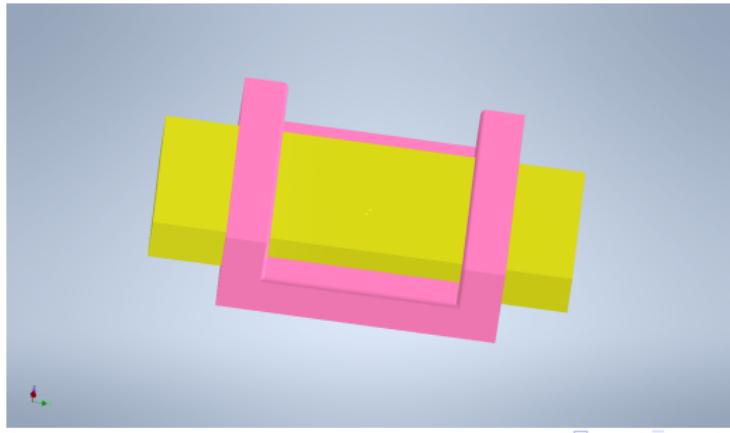
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Hunt, 1978

Formed by receding the axis of the revolution surface between two pairs to  $\infty$  so that the **curve** that produces the surface moves parallel to itself, **tracing a cylinder**; or a **polygonal-tracing curve** generates a **prism**.



# Revolute Pairs or *R*-pairs

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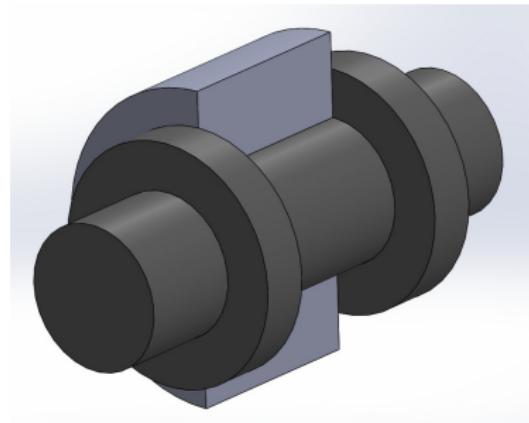
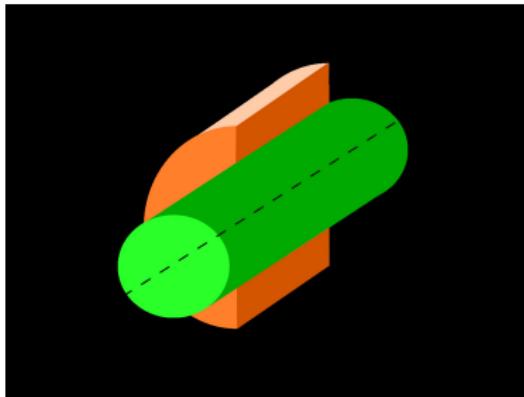
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One convex surface and one non-convex surface for a one degree of rotational freedom around the one joint the two surfaces make.



Revolute or Hinge or Turning or simply *R*-pairs with and without shoulder cutaway geometries. Credit: Wikimedia commons.

# Helical- & U-Joints

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Helical Joint

©McMaster Carr, May 2022.

Universal Joint

# Common Lower Kinematic Pairs

## Mobility

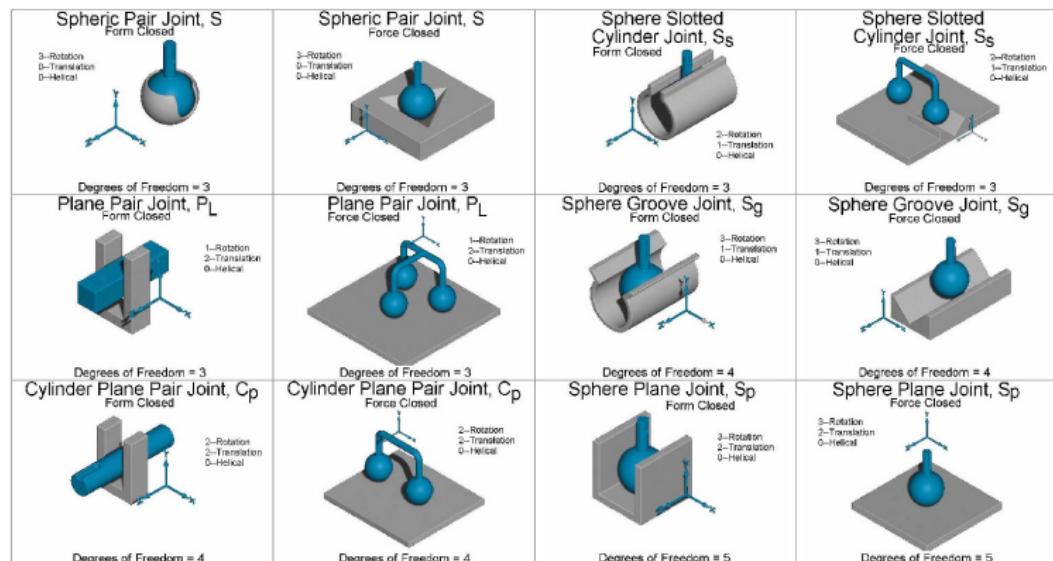
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Credit: Wharton and Singh, 2001.

# Common Lower Kinematic Pairs

## Mobility

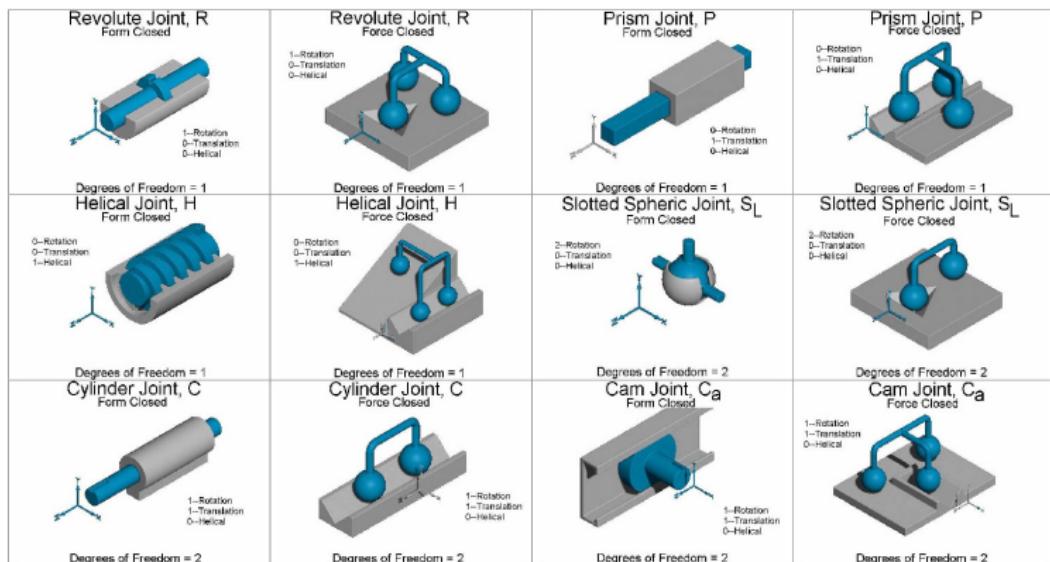
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Credit: Wharton and Singh, 2001.

# Kinematic Geometry of Common Actuations

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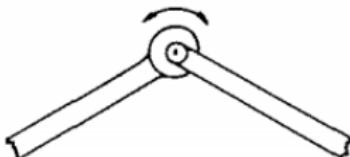
Plücker coordinates

Freedoms and Constraints

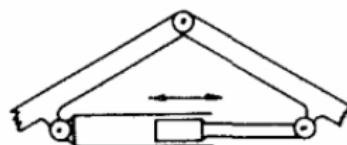
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## In-series vs. Parallel-actuated lower pairs



(a)



(b)

(a): In-series-actuated kinematic pair with a rotary joint that is actuated “about” the hinge. (b): Prismatic joint actuated “across” a hinge. Reprinted from Hunt, Kenneth. Structural Kinematics of In-Parallel-Actuated Robot Arms. Transactions of ASME. 1983.

# Kinematic Chains

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## Kinematic Chains (Reuleaux, 1975)

We can explain the structural similarity of many mechanisms by parts of **kinematic chains** connected by pairs.

### Kinematic chains

Kinematic chains are essentially the basic building structure of **mechanisms ... and robots!**

# Open Kinematic Chains

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## Chains

Open kinematic chains are based off the anthropomorphic construction of the human hand with cantilevered beam structures.

## Chain Mechanisms and Error Amplification

Amplifies errors from waist (or base frame) all the way to the tool frame. Control difficult.

## Control

Feedforward control: High power and precision hydraulic actuators for servo motors.

Sensory feedback control: Force sensing (Ernst, 1962).

# A short treatise on robots' geometry, kinematics, and dynamics.

- └ Pairs, Linkages, and Configurations
  - └ Serial Chains
    - └ Open Kinematic Chains

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The PUMA arm is the world's first serial kinematic chain. Developer: Victor Scheinman, Stanford student in the '50's. Made several iterations. Patent Rights: Joe Engelberger, (Danbury Unimation, 1961). Joe – father of robotics – created world's first robotics company in '61.

# Open Kinematic Robot Mechanisms

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## Definition (Ken Salisbury Jr., 1982)

*"[Robots are] our fascination with constructing mechanical analogues of ourselves... [this fascination] has led us to place all sorts of hopes and expectations in robot capabilities."*



The  
PUMA  
(1956).  
Stäubli  
Robot



The Stanford Arm  
(Infolab 1969).

# Open Kinematic Chains

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Open kinematic chains provide unstructured environmental interaction.

Project MAC, MIT.

Tomovic and Boni's pressure sensed grasp.

Binary robot vision system (McCarthy et al, 1963).

# Open Kinematic Chains

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Stanford Manipulator.

Boston arm.

The AMF (American Machines and Foundry) arm.

General electric's walking robot (1969).

# Long Walk Towards Direct Drive Robot Arms

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The 50's, 60's and 70's witnessed use of hydraulics for (feedforward) position control.

For feedback control, force sensors and pressure sensors were used in closed-loop scenarios.

Electrical actuation meant that robots had to be operated at high speeds. Needs for gear reduction for safe operations at low speeds.

With gear reduction came backlash, friction, and associated expenses.

# A short treatise on robots' geometry, kinematics, and dynamics.

## └ Pairs, Linkages, and Configurations

### └ Serial Chains

#### └ Long Walk Towards Direct Drive Robot Arms

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CMU DD I/II Arms: Workspace is donut shaped. OD: 90cm; ID: 21.7cm;  $1.8m^2$  workspace area. Built by Harry Asada. Structural design similar to aircraft gimbal arm; Uses Samarium Cobalt rare earth magnet brushless DC motors on first 3 joints, and AlNiCo magnets on tip joints. No belts, transmissions making for faster transmitting of motions, less friction, low energy, low compliance. Each joint has complex AL housing which enables: (i) Control of geometrical relationships of bearing assembly; (ii) Control of servo components to bearing assembly; (iii) Controls of rotational axes to consecutive joints.

# Direct Drive Robot Mechanism: CMU DD I Arm

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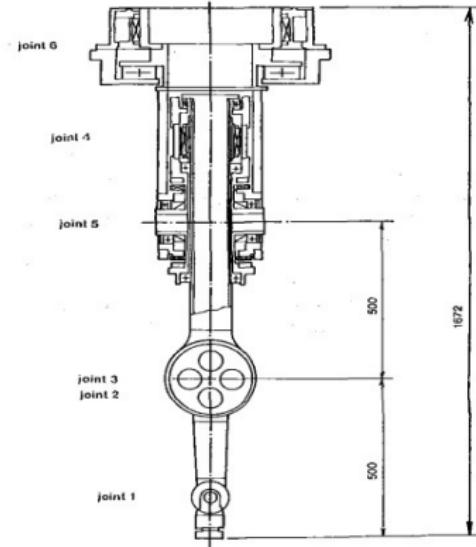
Plücker coordinates

Freedoms and Constraints

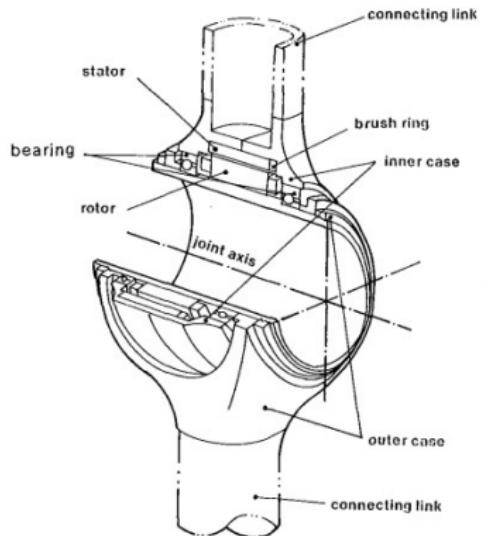
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Along came Harry Asada.



Arm Schematics Transmission



Joint schematic

# Direct Drive Robot Mechanism: CMU DD I Arm

## Mobility

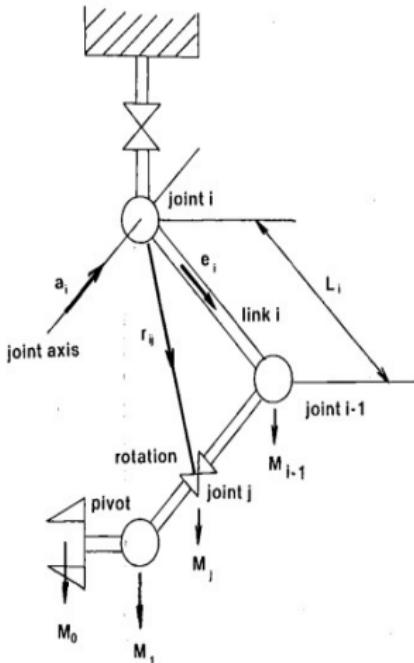
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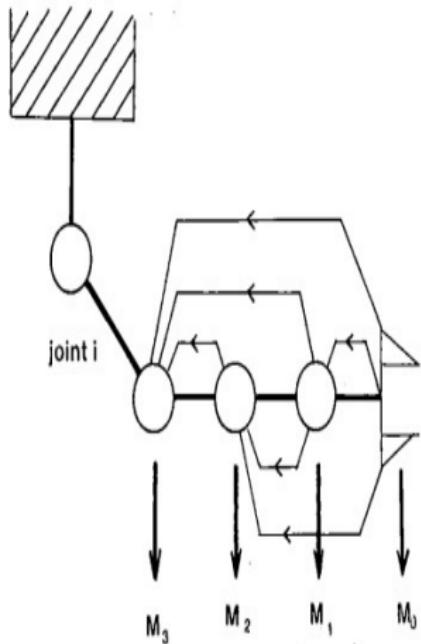
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Kinematic model



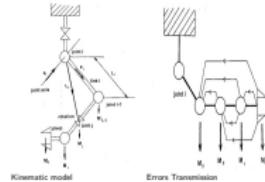
Errors Transmission

A short treatise on robots' geometry, kinematics, and dynamics.

└ Pairs, Linkages, and Configurations

  └ Serial Chains

    └ Direct Drive Robot Mechanism: CMU DD I



First direct-drive robot without a gearbox. Selective compliance in X-Y directions given its articulated jointed arms. One-freedom motion along Z direction given its constrained arm. New generations such as Cobra i600/i800 include power amplifiers, system and servo controls etc embedded in the robot's base. Kuka Scara arm: Lightweight, fast, powerful, low maintenance, energy consumption, investment costs etc.

# SCARA Robot Mechanisms

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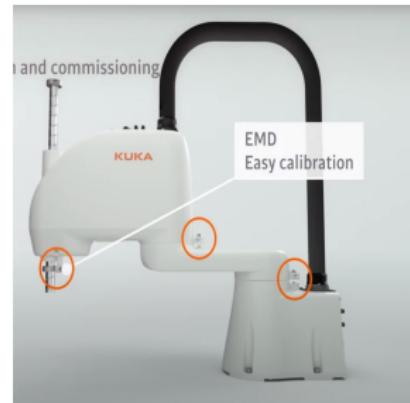
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The Adept One  
SCARA robot  
(Debut 1984).



Kuka's SCARA  
arm, 2022.  
©Kuka Robotics

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## The Stäubli anthropomorphic arm.



The Staubli 6-DOF Arm is an example of a Spherical Manipulator.  
Reprinted from DirectIndustry's Webpage.

# Serial mechanisms research in the 80's

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## Mechanisms in the 80's

With the 80's came the arrival of PCs. Lots of research went into computational algorithms for the kinematics and kinetics of (mostly) anthropomorphic robot arms.

## Active control schemes

Efficient recursive Lagrangian and computational methods for the gravitational and Coriolis forces in Newton-Euler equations.

## Feedback Linearization

Dynamics feedback linearization for precise bounds on manipulator performance.

# Serial mechanisms research in the 90's

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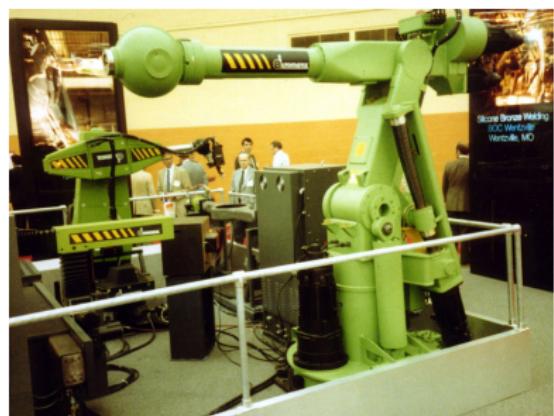
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## Automatix

Reconfigurable robots for various assembly ops.

## Robotworld

First industrial-scale reconfigurable robot and with machine vision components. RAIL scripting OS originally based on Motorola 68000, later on replaced by Apple Macintosh II.



©Wikipedia

# Hyper-redundant Continuum Robots

## Mobility

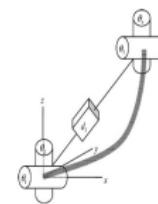
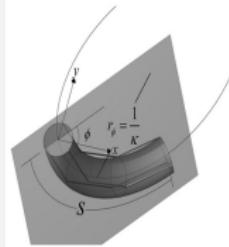
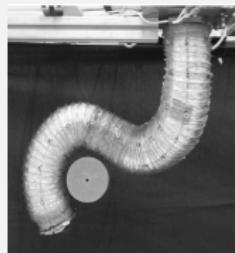
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The elephant trunk continuum robot. Jones & Walker, T-RO 2006.  
Inspiration: Muscular hydrostats in nature.

# Hyper-redundant Kinematic Chains

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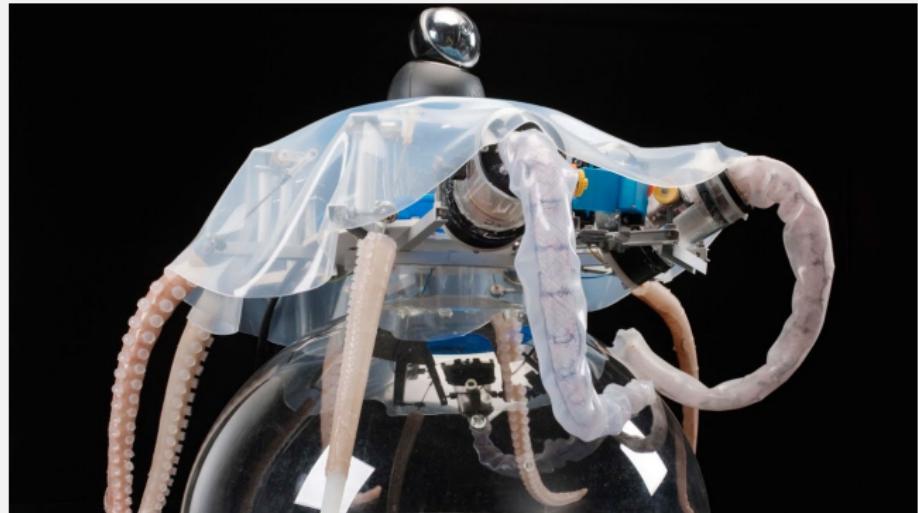
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An octopus-inspired soft robot. ©Cecilia Laschi.

# Parallel Robots

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Mehlet 2015

A **parallel robot** is made up of an end-effector with  $n$  degrees of freedom, and of a fixed base, linked together by at least two independent kinematic chains. Actuation takes place through  $n$  simple actuators.

# Parallel mechanisms: Stewart-Gough Platforms

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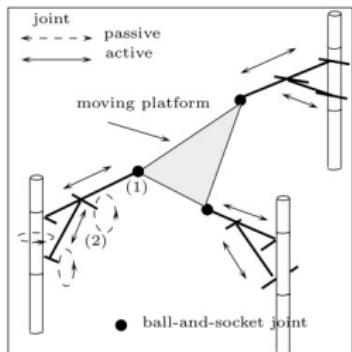
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Principles of a moving platform to test tyre wear and tear (Gough, 1947). Prototype, 1955.



Left: Stewart's 1965 mechanism. Right: The original 1954 octahedral hexapod proposed by Gough. Courtesy: [Parallemic.org](http://Parallemic.org).

# Truss Robots

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A multi-DOF Truss Robot. Courtesy of Penngineering (ICRA 2022,  
Philadelphia, PA).

# Closed kinematic chains

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Connection degree  $\geq 3$ .



A Stewart-Gough platform. SolidWorks Drawing Courtesy of Andrew Belcher. UChicago, 2018.

# A Soft Stewart Platform

## Mobility

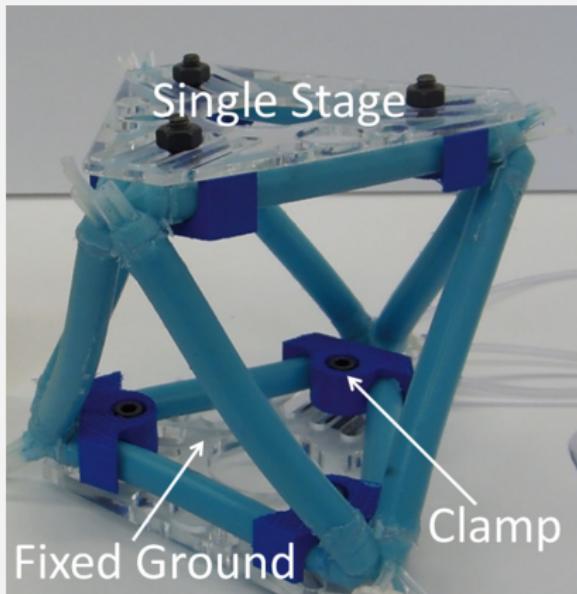
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A soft 6-6 Stewart manipulator. Jonathan Hopkins, 2015.

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## Freedom and Structure

Freedoms, Constraints, and Mobility.

Motion of linkages: Screws, and spatial motions.

Freedom and Mobility: Freedoms, unfreedoms, connectivity, mobility;

Grübler-Kutzbach's mobility criterion and examples.

# Degrees of Freedom and Structure

Mobility

Screws

Plücker coordinates

Freedoms and  
Constraints

Mobility Criterion

Mobility Criterion

## Definition (Connection Degree)

For any manipulator joint, we shall mean its connection degree to be the number of links attached it.

## Quiz

What is the connection degree of the u-joints of a Stewart-Gough platform.

# Members and Dual Graphs

Mobility

Screws

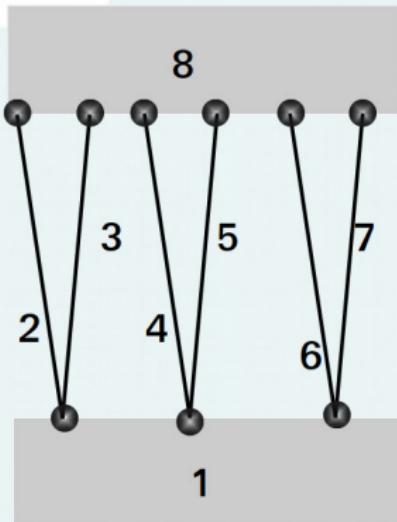
Plücker coordinates

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Mobility Criterion

Dual graph of a Stewart platform



# Degrees of Freedom and Structure

Mobility

Screws

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## Members and Freedoms

Degrees of freedoms (or freedoms) concerns the relative motion of members of a pair that do not touch one another directly.

## Connectivity

By the dual graph of the Stewart platform as seen on Frame 5, the total number of freedoms that connect the two members (1 and 8) that do not connect to one another directly is six.

# Planar Linkages

Mobility

Screws

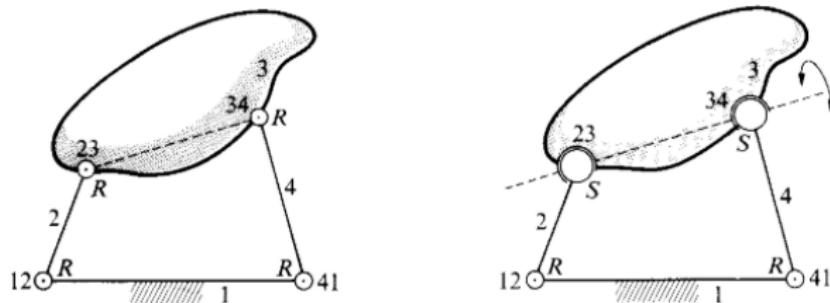
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## Four Bar Linkages

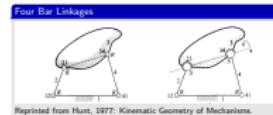


Reprinted from Hunt, 1977: Kinematic Geometry of Mechanisms.

# A short treatise on robots' geometry, kinematics, and dynamics.

- └ Mobility

- └ Planar Linkages



The planar  $RRRR$  linkage, (*left*) is modified in (*right*) to an  $RSSR$  linkage to allow spatial spin-movement of the coupler 3; the connectivity  $\mathcal{C}_{13} = 2$ .

# Freedom from Connectivity

Mobility

Screws

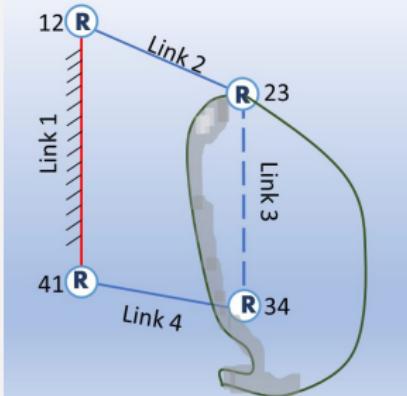
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Freedoms and Constraints

Mobility Criterion

Mobility Criterion

## A (Hacked) Four-Bar Linkage



# The Four Bar Linkage

Mobility

Screws

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Freedoms and  
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## Couplings and Freedom

Links 2&4 complete a **coupling or connection** between links 1&3.

## Connectivity

The  $R$ -pairs are said to have a **connectivity** of  $\mathcal{C}_{ij} = 1$  for all  $i, j = 1, 2, 3, 4$ . Thus, total degree of freedom is 1.

# Mobility of Mechanisms

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Mobility Criterion

## The Mobility and Relative Mobility, $\mathfrak{M}$

Simply put, the number of a mechanism's freedoms is its **mobility**, or **relative mobility**,  $\mathfrak{M}$ .

## The Mobility, $\mathfrak{M}$

It specifies the **independent variables** needed to **determine** every relative location of a **mechanism's members** with respect to one another.

## A Note on Serial and Parallel Mobility

A little tricky to determine for parallel mechanisms but straightforward for serial mechanisms.

# Mobility of Mechanisms

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Mobility Criterion

## Quiz

What is the mobility  $\mathfrak{M}$  of the *RSSR* four bar linkage of Frame 7? Why?

## Quiz

What is the mobility  $\mathfrak{M}$  of the *RRRR* four bar linkage of Frame 8? Why?

## Definition (The mobility criterion (well, not yet))

Let's not get ahead of ourselves. A little introduction to screws are in order for us to grasp the **Grübler-Kutzbach** mobility criterion.

# Unique Location of a Rigid Body in 3D Space

Mobility

Screws

Plücker coordinates

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Mobility Criterion

## Inhomogeneity of Displacements and Angles

Quiz: Three translations and three rotations are ill-posed for uniquely determining the freedoms of a body. Why?

They are **not homogeneous**.

For true **kinematic wholeness** and **generality**, displacement that is **purely translatory** and **purely rotary** is needed.

# Screws for Kinematic Generality

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## Need for Screws

From a kinematic standpoint, **six homogeneous screw coordinates** – each having an **independent screw freedom** – are needed to **uniquely determine a rigid body's location**.

## Definition (What is a screw anyway?)

A **screw** is a **straight line** in space, called **the axis**, with an associated direction, called **pitch**,  $p$ .



# Unique Location of a Rigid Body in 3D Space

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Plücker coordinates

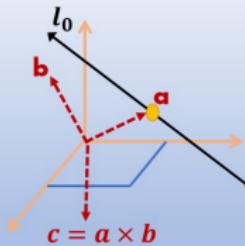
Freedoms and Constraints

Mobility Criterion

Mobility Criterion

## Definition (Screw Coordinates)

Six-vector,  $s$ , related to the Plücker coordinates (see right inset) parameterize a screw i.e.  
 $s = (s_1, s_2, s_3, s_4, s_5, s_6)$ .



## Plücker Coordinates

Let  $\mathbf{a}$  be a point on line  $\ell_0$ . Let  $\mathbf{a}$ 's direction cosine vector (to be introduced shortly) be  $\mathbf{b}$ . Then, its binormal (moment) vector is  $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ . We say the pair  $(\mathbf{b}, \mathbf{c})$  is the Plücker Coordinates of the point  $\mathbf{a}$  on axis  $\ell_0$ .

# Screws and Plücker Coordinates Relationship

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## Screw axis and Plücker Coordinates Relationship

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3 \quad (3)$$

$$c_1 = s_4 - ps_1, \quad c_2 = s_5 - ps_2, \quad c_3 = s_6 - ps_3. \quad (4)$$

$p$ : pitch! How to find it?

## Screw and Plücker Coordinates Relationship

Suppose that

$$h = \sqrt{b_1^2 + b_2^2 + b_3^2} \quad (5)$$

Then  $(b/h, c/h)$  are respectively the direction cosines of the line,  $l_0$  and its moment.

# Pitch and Magnitude of the screw

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## Pitch of a screw

$$p = \frac{s_1 s_4 + s_2 s_5 + s_3 s_6}{\sqrt{s_1^2 + s_2^2 + s_3^2}}, \quad (6)$$

$$|s| = \sqrt{s_1^2 + s_2^2 + s_3^2} \quad \text{if } p \neq \infty \quad (7)$$

$$|s| = \sqrt{s_4^2 + s_5^2 + s_6^2} \quad \text{if } p = \infty \quad (8)$$

# Plücker Coordinates Example

Mobility

Screws

Plücker coordinates

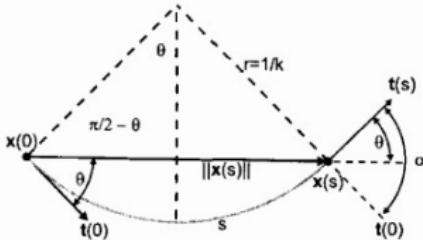
Freedoms and Constraints

Mobility Criterion

Mobility Criterion

## Chasles' Theorem Applied to The Serret-Frenet Frame

Consider a spatial curve  $C$  on the elephant continuum trunk shown earlier. Suppose  $C$  is parameterized by its arc length  $s \in [0, 1]$ . For a point  $x = [x, y, z]^T$  on  $C$ , the unit tangent vector to  $C$  is  $t = dx/ds$ . Denote by  $n$  the principal normal to  $C$  at  $n$ ; then we must have  $b = t \times n$  as the binormal. We say  $(b, n)$  together form the Plücker coordinates of the tangent  $t$ .



# Plücker Coordinates Example

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## Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force  $F$  acts at the point  $a$  in the image of Frame 14. What are the Plücker coordinates of the **line of force**?

## Homogeneous Coordinates!

Plücker Coordinates give six unit parameters of a point on a line. Plücker Coordinates are in **homogeneous coordinates!**

A short treatise on robots' geometry, kinematics, and dynamics.

└ Mobility

  └ Plücker coordinates

    └ Plücker Coordinates Example

**Poinsot's Theorem Quiz on a Force and its Moment**  
 Suppose that a force  $F$  acts at the point  $a$  in the image of Frame 14. What are the Plücker coordinates of the **line of force**?

**Homogeneous Coordinates!**  
 Plücker Coordinates give six unit parameters of a point on a line. Plücker Coordinates are in **homogeneous coordinates**.

## Poinsot's Theorem Quiz on a Force and its Moment

Imagine that a force  $F$  is acting at the point  $a$  in the image of Frame 14. Suppose that  $\tau$  is torque acting along the normal to point  $a$ . Then  $(f, \tau)$  are the Plücker coordinates of the **line of force**.

## Arithmetics on Screws

Scalar and vector arithmetic operations are valid on infinitesimal screws e.g.

$$c_1 s_1 + c_2 s_2 = 0 \text{ for } c_1, c_2 \neq 0 \text{ on screws } s_1, s_2. \quad (9)$$

# Freedoms, Unfreedoms, and Mobility

Mobility

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Freedoms and  
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Mobility Criterion

Mobility Criterion

## Freedom and Constraints

Suppose a screw  $\mathbf{f} = (f_1, \dots, f_6)$  “fixes” a body in 3D space.

Each **constraint**  $u_i \neq f_j$  for  $(i, j) \in \{1, \dots, 6\}$ .

Rather each  $u_i$  has influence on every  $\{f_i\}_{i=1}^6$ .

Each  $u_i$  from the six independent equations,  
 $g(s_1, s_2, s_3, s_4, s_5, s_6) = 0$ , suppresses a  
**freedom**,  $f_i$ .

Progressively relaxing each  $u_i$ , or **unfreedom**,  
adds an extra body  $f_i$ .

# Freedoms, Unfreedoms, and Mobility

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## Freedom and Unfreedoms

Suppose the total **freedoms** is  $f$  and the total **unfreedoms** is  $u$ , then

$$u + f = 6.$$

Note: A rigid body's freedoms is also referred to the dimension of its **configuration space**.

## Relative Freedoms

Suppose there are a total of  $n$  **unconstrained** bodies.

Suppose further that we choose one out of the bodies as a reference body. Then the total number of **relative freedoms** is  $6(n - 1)$ .

# Freedoms, Unfreedoms, and Mobility

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## Constraints and Joints

Now, consider  $k$  independent constraints<sup>a</sup> such as joints along points, lines, curves or surfaces.

<sup>a</sup>NB: The total allowable constraints is 5 for a body in relative motion. 6 for a fully rigid body.

## The Mobility Criterion

Let the constraint of joint,  $i$  (e.g. a joint along points, lines, curves or surfaces) be  $u_i$ . Then the mobility criterion  $\mathfrak{M}$  is

$$\mathfrak{M} = 6(n - 1) - \sum_{i=1}^k u_i. \quad (10)$$

# General Grübler-Kutzbach Mobility Criterion

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Mobility Criterion

## General Grübler-Kutzbach Mobility Criterion

Recall that  $\sum_i u_i + f_i = 6$  from Frame (21) so that

$$\mathfrak{M} = 6(n - k - 1) - \sum_{i=1}^f f_i. \quad (11)$$

## Exceptions: Relative Planar and Spherical Motions

For bodies restricted to relative planar or spherical motions, the total freedoms + constraints is 3 (not 6)!

$$\mathfrak{M} = 3(n - k - 1) - \sum_{i=1}^f f_i. \quad (12)$$

# General Grübler-Kutzbach Criterion References

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## The Grübler-Kutzbach Mobility Criterion References

Attributed to Grübler:

Schoenflies, Arthur, and M. Grübler. "Kinematik." In Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, pp. 190-278.

Vieweg+Teubner Verlag, Wiesbaden, 1908;

Grübler, Martin Fürchtegott. Getriebelehre: eine Theorie des Zwangslaufes und der ebenen Mechanismen. Springer, 1917.

and Kutzbach:

Kutzbach, Karl. "Mechanische leitungsverzweigung, ihre gesetze und anwendungen." Maschinenbau 8, no. 21 (1929): 710-716.

# Loops

Mobility

Screws

Plücker coordinates

Freedoms and  
Constraints

Mobility Criterion

Mobility Criterion

## Loops

A kinematic chain often comprises members called loops.

## Binary Link

Members in a binary link constitute a single loop. Example:  
The four-bar linkage.

## Single loops

For single loops,  $k = n$  so that  $\mathfrak{M} = \sum_{i=1}^f f_i - 6$ .

## Mobility of Mechanisms

$\mathfrak{M} \leq 1$  for at least one actuator-pair to produce mobility at a successor joint which depends on that actuator-pair's input.

# Mobility of Common Robot Configurations

## Mobility

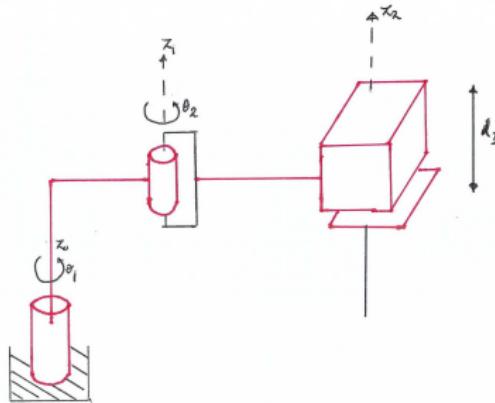
Screws

Plücker coordinates

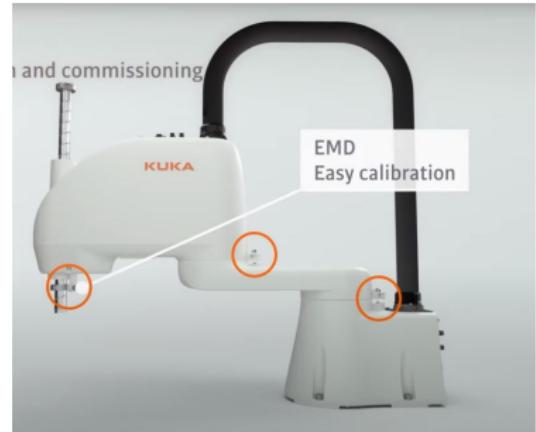
Freedoms and Constraints

Mobility Criterion

Mobility Criterion



Configuration of the SCARA Arm.



Courtesy of Fanuc America Inc.

# Mobility of The SCARA Robot

## Mobility

Screws

Plücker coordinates

Freedoms and  
Constraints

Mobility Criterion

Mobility Criterion

## Mobility Analysis

Two rotary joints. One prismatic joint acting along the  $z$  axis, and constrained along the  $xy$  plane.

## Mobility Parameters

Four rigid bodies (links).  
Three constraints. Four freedoms. Therefore,  
 $\mathfrak{M} = 6(4 - 3 - 1) + 4 = 4$

# Mobility Analysis of The Universal Robot

## Mobility

Screws

Plücker coordinates

Freedoms and Constraints

Mobility Criterion

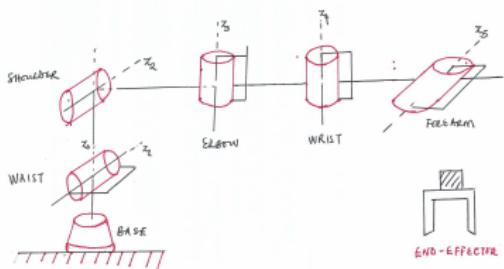
Mobility Criterion



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## The Revolute Arm

Falls under so-called *RRR* kinematic arrangements. Also called a **revolute**, **elbow**, or **anthropomorphic manipulator**.



$$n = 6; k = 6; f = 5 \times 3 := 18$$

$$\therefore \mathfrak{M} = 6(n - k - 1) + \sum f_i \\ \Rightarrow 6(6 - 1 - 5) + 18 \text{ or } \mathfrak{M} = 6.$$

# Mobility of The Stewart-Gough Platform

## Mobility

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# Mobility of The Stewart-Gough Platform

## Mobility

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### Unconstrained bodies, $n$

There are six universal joints that connect the base platform to the prismatic linear actuators.

There are six spherical joints that connect the top platform to the top of the prismatic actuators.

Altogether, there are  $n = 6 + 6 + 2$  or 14 unconstrained rigid links.

### Constraints, $k$

Six u-joints. Six spherical joints. Six prismatic joints. Altogether, there are  $f = 6 + 6 + 6 := 18$  constraints.

### Freedoms, $f$

Each u-joints has two freedoms. Each spherical joint has three (rotary) freedoms. Each prismatic joint has one freedom.

Altogether, there are  $f = 6 \times 2 + 6 \times 3 + 6 \times 1 := 36$  freedoms.