

RBOT 250: Homework Solutions

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Homework 1. SCARA Robot Mobility Analysis:

Using the mobility condition, determine and explain why the SCARA robot of Fig 2.5 has the number degrees of freedom that you find.

Solution 1. We have the following general equation from the mobility condition:

$$\mathfrak{M} = 6(n - g - 1) + \sum_{i=1}^g f_i. \quad (1)$$

where n is the number of links, g is the total number of joints and f_i are the respective relative degrees of freedoms of the individual joints. The SCARA manipulator has three joints with an *RRP* configuration *i.e.*, $g = 3$. These connect four links (including the end-effector) so that $n = 4$. Each of the joints have $f_i = 1$. Plugging these into (1), we find that

$$\mathfrak{M} = 6(4 - 3 - 1) + \sum_{i=1}^3 f_i = 3 \quad (2)$$

Hence, the SCARA manipulator is a 3DOF robot.

Homework 2. Parallel Robot Mechanism Analysis

With the *Grübler-Kutzbach's mobility condition* that we have learned, analyze the mobility criteria of the mechanism of Fig. 1. Hint: This mechanism is made up of two chains: chains $A_3 B_3 B_1 A_1$ and $A_2 B_2 B_4 A_4$, and there is a fixed distance between the *U*-joints, $A_2 A_3$ as well as A_1, A_4 .

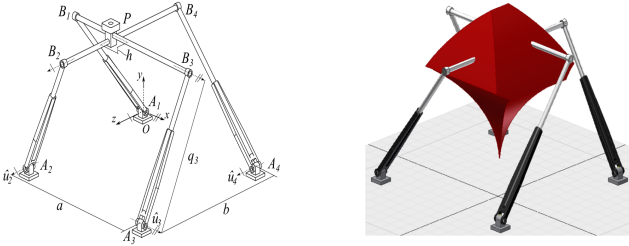


Fig. 1. *Left:* A Parallel Planar Robot Mechanism. *Right:* Workspace of the mechanism.

Solution 2. This robot has four RPRP-type kinematic chains. Three of these chains have their prismatic pair closest to the base; they serve as active joints. The fourth kinematic chain is completely passive. Points B_1 and B_2 are connected by a rod, which is perpendicular to the rod that connects points B_2 and B_4 . Both rods are connected to the moving platform by prismatic joints, which are separated from each other by a vertical offset h . Point $P = (P_x, P_y, P_z)$ is the interconnecting point for all the chains on the mobile platform and the top rods. Notice that the rotational axes of the revolute joints *i.e.*, \hat{u}_i are parallel to the x -axes of joints 1 and 3; also, they are parallel to the z -axis for joints 2 and 4.

There are many ways of solving this mobility problem. However, note that this mechanism has multiple closed loops and care must be taken when using the formulas we have introduced in our notes. Here, I will be using Gogu's method as proposed in [?]. It decomposes the mechanism into the different closed loops in order to properly analyze the mobility constraint. The mobility of the mechanism is

$$\mathfrak{M} = \sum_{i=1}^g f_i - r \quad (3)$$

with r being the number of joint parameters that lose independence after closing the loops of the mechanism. We define r as

$$r = \sum_{j=1}^k SH_j - SF r_l \quad (4)$$

where k is the number of closed-loops in the mechanism, SH_j is the connectivity of the j 'th closed loop H_j when separated from the mechanism; SF is the overall connectivity of the mechanism and r_l is the total number of parameters that lose independence in the closed loops. We may find the variables as follows

$$SH_j = \dim(RH_j), \quad SF = \dim(RF), \quad r_l = \sum_{j=1}^k r_l^{H_j} \quad (5)$$

where RH_j is the velocity vector associated with the interest point P in a closed loop H_j ; RF is the resultant velocity vector formed from the intersection of RH_j *i.e.* $RF = RH_1 \cap RH_2 \dots \cap RH_k$ and $r_l^{H_j}$ is the number of parameters that lose independence in the loop H_j . We define the variable $r_h^{H_j}$ as

$$r_l^{H_j} = SG_1^{H_j} + SG_2^{H_j} - SF_{H_j} \quad (6)$$

with $SG_i^{H_j}$, $i = 1, 2$ being the connectivity of $G_i^{H_j}$ in H_j , whereas SF_{H_j} is the loop's connectivity. We define

$$SG_i^{H_j} = \dim(RG_i^{H_j}), \quad SF_{H_j} = \dim(RF_{H_j}) \quad (7)$$

where $RG_i^{H_j}$ are the velocity vectors for $G_i^{H_j}$ and RF_{H_j} is the resultant velocity vector formed by the intersection of the $RG_i^{H_j}$ s.

For this mechanism, we have $g = 14$ and $k = 2$; furthermore, we have $RH_1 = \{v_x, v_y, v_z, \omega_x\}$ and $RH_2 = \{v_x, v_y, v_z\}$ so that the loops' connectivity are $SH_1 = 4$ and $SH_2 = 4$. Hence $RF = RH_1 \cap RH_2 = \{v_x, v_y, v_z\}$ and the mechanism's connectivity is thus $SF = \dim(RF) = 3$.

If we disconnect the limbs, the velocity vector of $RG_1^{H_1} = RG_2^{H_1} = \{v_x, v_z, \omega_x\}$ and $RG_1^{H_2} = RG_2^{H_2} = \{v_x, v_y, \omega_z\}$,

so that the connectivity of each limb is $SG_{1,2}^{H_1} = SG_{1,2}^{H_2} = 3$. Thus,

$$\begin{aligned} RF_{H_1} &= RG_1^{H_1} \cap RG_2^{H_1} = \{v_x v_z \omega_x\}, \\ RF_{H_2} &= RG_1^{H_2} \cap RG_2^{H_2} = \{v_x v_z \omega_z\} \end{aligned} \quad (8)$$

so that $SG_{H_1} = SF_{H_2} = 3$. Moreover, we see that $r_l^{H_1} = r_l^{H_2} = 3$ and $r_l = 6$. Thus, we have

$$r = SH_1 + SH_2 - SF + r_l = 11. \quad (9)$$

Since $p = 14$ and $f_i = 1$, we have

$$\mathfrak{M} = 14 - 11 = 3. \quad (10)$$

Hence, the mechanism has only 3 degrees of freedom. It is a planar parallel manipulator.

Homework 3. What is the geometric meaning of equation 3.1.6 on a twist axis to you. Define *pure rotation* and a *pure translation* in terms of equation 3.1.6¹.

Solution 3. Twists and pitches of twists.

- 1) The pitch of the twist is the ratio of the magnitude of a point on the axis of the twist to the magnitude of the angular velocity about the axis of the twist. By this, we see that the pitch of the twist is the unit velocity traveled along the axis of the twist.
- 2) A pure rotation occurs when the numerator of equation 3.1.6 is zero *i.e.* when a point travels only in the angular velocity direction of the twist axis; this is called a zero pitch twist.
- 3) A pure translation occurs when the right hand side of 3.1.6 is infinite. This corresponds to an infinite pitch twist *i.e.* a point on the twist axis only travels along the linear velocity direction of the twist axis.

Homework 4. A unit screw, twist or wrench is one where the magnitude of the screw, twist or wrench is 1.

- 1) What is the geometric meaning of a unit screw to you?
- 2) Consult the identified reference materials and explain what a reciprocal screw is in no more than five sentences.

Solution 4. Here is a geometric meaning of the screw:

- 1) Imagine a nut fitted upon a mechanical screw. As we tighten the nut around the threads of the screw, there exists a rectilinear distance by which the screw travels into the nut. This rectilinear distance is called the *pitch* of the screw. Thus, the pitch is a linear magnitude. When a nut is rotated about the threads of a screw, the rectilinear distance by which the nut moves when rotated through a particular angle is the product of the pitch and the the circular measure of the angle. *A screw then may be geometrically seen as a straight line with which a definite linear magnitude, i.e. the pitch, travels in space.* Often with twist and wrenches, we wish to associate a magnitude other than 1 so that there are ∞^6 different twists and wrenches if we consider their magnitudes as

well. With this interpretation we can think of a unit screw as defining only an axis and a pitch. Then we can think of a given magnitude as defining a twist (if its units are rotation/time) or a wrench (if its units are force) acting along the screw. In both cases the pitch is expressed in length units. These six-spaces of (infinitesimal) twists or wrenches can be considered to be vector spaces in that they are closed under vector addition and scalar multiplication.

- 2) **Reciprocal Screws:** Suppose a body is free to twist about a screw x and that body is in equilibrium, while being acted upon by a wrench on another screw χ ; we can extend this logic and infer that a body that is free to rotate about the screw χ will be in equilibrium, while being acted upon by a wrench on a screw x . By the principle of virtual velocities, if the body is in equilibrium the work done in a small displacement against external forces must be zero with the virtual coefficient vanishing. *A pair of screws are reciprocal when their virtual coefficient is zero.*

Homework 5. Given a matrix $\hat{m} \in so(3)$, suppose that the following relation holds,

$$\hat{m}^2 = m m^T - \|m\|^2 I \quad (11)$$

$$\hat{m}^3 = -\|m\|^2 \hat{m} \quad (12)$$

with the fact that higher powers of \hat{m} can be recursively found. Utilizing this lemma along with $m = \omega\theta$, and $\|\omega\| = 1$, show that

$$e^{\hat{\omega}\theta} = I + \hat{\omega} \sin \theta + \hat{\omega}^2 (1 - \cos \theta). \quad (13)$$

Solution 5. First recall that we can expand $e^{\hat{\omega}\theta}$ using Taylor series so that

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \theta \hat{\omega} + \frac{\theta^2}{2!} \hat{\omega}^2 + \frac{\theta^3}{3!} \hat{\omega}^3 + \frac{\theta^4}{4!} \hat{\omega}^4 + \frac{\theta^5}{5!} \hat{\omega}^5 + \\ &\quad \frac{\theta^6}{6!} \hat{\omega}^6 + \frac{\theta^7}{7!} \hat{\omega}^7 + \dots \end{aligned} \quad (14)$$

Rewriting, we find that

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \hat{\omega} \left(\theta + \frac{\theta^3}{3!} \hat{\omega}^2 + \frac{\theta^5}{5!} \hat{\omega}^4 + \frac{\theta^7}{7!} \hat{\omega}^6 + \dots \right) + \\ &\quad \hat{\omega}^2 \left(\frac{\theta^2}{2!} + \frac{\theta^4}{4!} \hat{\omega}^2 + \frac{\theta^6}{6!} \hat{\omega}^4 + \dots \right). \end{aligned} \quad (15)$$

Since we have from the lemma that

$$\hat{\omega}^2 = \omega \omega^T - \|\omega\|^2 I \quad (16a)$$

$$\hat{\omega}^3 = -\|\omega\|^2 \hat{\omega} \quad (16b)$$

we may write (15) as

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \hat{\omega} \left(\theta + \frac{\theta^3}{3!} (\omega \omega^T - \|\omega\|^2 I) + \frac{\theta^5}{5!} (\omega \omega^T - \|\omega\|^2 I)^2 + \dots \right) \\ &\quad + \hat{\omega}^2 \left(\frac{\theta^2}{2!} + \frac{\theta^4}{4!} (\omega \omega^T - \|\omega\|^2 I) + \frac{\theta^6}{6!} (\omega \omega^T - \|\omega\|^2 I)^2 + \dots \right) \\ &= I + \hat{\omega} \left(\theta - \frac{\theta^3}{3!} \hat{\omega}^2 + \frac{\theta^5}{5!} \hat{\omega}^4 - \frac{\theta^7}{7!} \hat{\omega}^6 + \dots \right) \\ &\quad + \hat{\omega}^2 \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} \hat{\omega}^2 + \frac{\theta^6}{6!} \hat{\omega}^4 + \dots \right) \end{aligned} \quad (17)$$

¹Figure out how they correspond to zero pitch and infinite pitch twists.

Recall from trigonometric identities that

$$\begin{aligned}\sin \theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\ \cos \theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\end{aligned}\quad (18a)$$

Therefore, (17) becomes

$$e^{\hat{\omega}\theta} = I + \hat{\omega} \sin \theta + \hat{\omega}^2 (1 - \cos \theta) \quad (19)$$

where we have used the identities $\hat{\omega}^2 = -1$ and $\hat{\omega}^4 = 1$ etc.