

A short treatise on robots' kinematic geometry and kinetics.

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# Table of Contents I

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

## 1 Introduction

- Textbooks

## 2 Kinematic Geometry

- Mechanics
- Mechanisms
- Pairs and Linkages
- Serial Chains
- Hyperredundant and Parallel robots

## 3 Mobility

- Screws and a History Snippet
- Freedoms and Constraints
- Mobility Criterion

## 4 Motions

- Movement in  $\mathbb{R}^3$
- Special Orthogonal Properties



# Table of Contents II

## ● Composition of Rotations

### 5 Screw Theory

- Displacement & Twist
- Force & Wrench
- Screws in Plücker Coordinates
- Group Theory Connections

# Standard Texts – Modeling and Control

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Robot Modeling and Control

Spong, Mark W., Seth Hutchinson, and Mathukumalli Vidyasagar. Robot modeling and control. Vol. 3. New York: Wiley, 2006.

## Mathematical Modeling of Robots

Murray, R. M., Li, Z., & Sastry, S. S. (1994). A Mathematical Introduction to Robotic Manipulation. In Book (Vol. 29). <https://doi.org/10.1.1.169.3957>

# Texts – Modeling, Control, and Mechanisms

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Robot Modeling and Control

Lynch, K. M., & Park, F. C. (2017). Modern Robotics  
Mechanics, Planning, and Control.

## Mechanisms' Kinematic Geometry

Hunt, Kenneth H., and Kenneth Henderson Hunt.  
Kinematic geometry of mechanisms. Vol. 7. Oxford  
University Press, USA, 1978.

# Texts – Screws and Kinematics

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Screw Theory

Ball, Robert Stawell. A Treatise on the Theory of Screws. Cambridge university press, 1998.

## Mechanisms' Kinematic Geometry

Hunt, K. H. (2019). Structural Kinematics of In-Parallel-Actuated Robot-Arms. 105(December 1983), 705–712.

# Lecture One Outline

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Mechanism Components

Kinematic geometry. Mechanisms.

Joints: Joint closure; Pairs; Couplings.

Lower pairs and linkages; Higher and lower pairs.

Motions: Planar and spherical motions.

Synthesis: Type-, number-, and size-syntheses.

# Preamble.

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Mechanics

Mechanics is an indirect study of nature via **bodies** – essentially the mathematical abstractions of common natural things; the **mass** is an *allocation in place* to each body.

## Geometry

Geometry, deals with the **theory of places**; geometry is the bedrock of **robotics, control theory**, and many fields of **modern engineering and the physical sciences**.

# Mechanics Overview.

## Introduction

Textbooks

## Kinematic Geometry

### Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

### Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

### Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

### Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Motion)

When a **place** undergoes **body transformation** in the course of **time**, we have **motion**.

# Preamble – Mass, Body, Rigid Body Motion.

## Introduction

Textbooks

## Kinematic Geometry

### Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

### Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

### Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Body – Truesdell, 1977.)

By a **body**, we shall mean the **closure of an open set** in some **measure space**  $\Omega$  over which a **non-negative measure  $M$** , called the **mass**, is defined, and that  $M$  can be extended to a Borel measure over the  $\sigma-$  algebra of Borel sets in  $\Omega$ .

# Preamble – Mass, Body, Rigid Body Motion.

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Bodies – Truesdell, 1977.

That in mechanics which deals with

- (i) mass points, which occupy a single point at any one time;
- (ii) rigid bodies, which never deform;
- (iii) strings and rods and jets, which are 1-dimensional; membranes and shells, that sweep out surfaces;
- (iv) space-filling fluids and solids e.t.c. are termed bodies.

# Statics, Dynamics, Rigid Body (Motion).

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Statics and Dynamics

That which studies **putative equilibria** is referred to as **statics**. That which concerns motion of all sorts is referred to as **dynamics**. The dynamics that are specific to **particular bodies** are termed **constitutive**.

# Statics, Dynamics, Rigid Body (Motion).

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Rigid Body

A rigid body does not stretch, buckle, contract, bend, twist, nor deform. Well, not really!

## The Rigid Body

As engineers and roboticists, we judge kinematic rigid hardware with the expectation that kinematic changes do not depart from rigid-body predictions.

# Statics, Dynamics, Rigid Body (Motion).

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Rigid Body

We expect that localized stresses, active noise, vibrations and heat e.t.c will not cause reasonable departures from expectations.

## Rigid Body Motion

That motion that preserves distance between all points in a body is termed a rigid body motion.

# Statics, Dynamics, Rigid Body (Motion).

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Rigid Body Motion

At issue are components of a rigid body's **movement** w.r.t to a fixed or moving **frame of reference**. In its most basic form, this movement is parameterized by **displacement** (and is sometimes time-varying e.g. for a continuum body). When solving for the movements of bodies, it is often useful to include velocities (**twists**) in order to characterize the **motion**.

# Kinematics vs. Kinetics

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Dynamics

$$\dot{x} = f(t; x, u), \quad x(t_0) = t_0 \quad (1)$$

$$\dot{x} = f(t; x) + g(t; x, u), \quad x(t_0) = t_0 \quad (2)$$

## Definition (Kinematics.)

**Kinematics** is the English version of the word *cinématique* coined by A.M. Ampère (1775-1836), who translated it from the Greek word *kίνημα*.

# Kinematics vs. Kinetics.

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Definition (Truesdell)

That part of a system's **dynamics** that involves its **motion** by **displacement** – both linear and angular – and **separated from motions owing to forces and torques**, together with the successive derivatives with respect to time of all such displacements (this includes **velocities**, **accelerations**, and **hyper accelerations**) all form the **kinematics** of a **rigid**, **continuum** or **laminae** of bodies.

# Kinematics vs. Kinetics

## Introduction

Textbooks

## Kinematic Geometry

### Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

### Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

### Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

### Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Kinetics

The motion of bodies can also be conceived as resulting from the forces' action. Energy, temperature, and calory of a body are resultant effects of gains or loss of heat. Motions arising as a result of these are called kinetics.

# Kinematics vs. Kinetics.

## Introduction

Textbooks

## Kinematic Geometry

### Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

### Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

### Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

### Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Kinetics – Technical Definition)

That part of a system's **dynamics** that involves its **motion** by **forces, energy, torque, inertia, dynamic stability, and equilibrium** and similar properties all form the **kinetics** of a rigid, continuum or laminae of bodies.

# Kinematic Geometry.

## Introduction

Textbooks

## Kinematic Geometry

### Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

### Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

### Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Kinematic Geometry)

The solid geometry of relatively moving rigid bodies is termed the **kinematic geometry** of the rigid body. With motion, we'd have to include the successive derivatives of the displacement such as acceleration e.t.c as the 'laws of motion' stipulates in mechanics.

# Joints and Links

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Links

Links may be rigid mechanical parts, elastic, (vulcanized) rubber components, diaphragms, conveyor belts, spring-damper systems e.t.c.

## An Elementary Joint or Kinematic Pair.

An elementary joint or a kinematic pair consists of touching two links together at one point – then ensuring a single contact point is continuously maintained throughout relative movement.

# Joint (Contact) Kinematics

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Contact Kinematics

A body may **slide** or **slip** across a **plane** or **surface**, or **roll** over another body.

## Joints

**Joints** are the result of the **connecting points** between two or more **rigid bodies**.

# Definition of a Mechanism

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

## Definition (Author's Definition)

A connection of mechanical, magnetic, electrical, hydraulic, or pneumatic components forming an assemblage, meant for moving rigid, semi-rigid or non-rigid bodies via a controlled generation of (sometimes constrained) motion.

## Kenneth Hunt (1978)

A means of transmitting, controlling, or constraining the relative movement between parts. Whenever we have an higher pair or more, we have a mechanism.

# Mechanism Examples

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

## Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

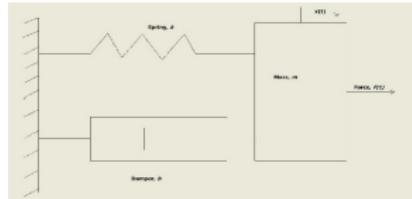
Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

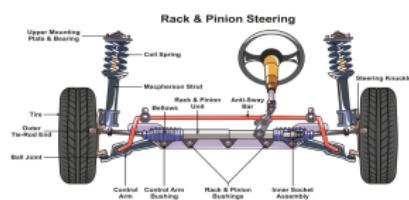
## Spring-Mass-Damper System



## Excavator



## Car suspension



## Daimler Plant



# Lower Pairs, Higher Pairs, Linkages

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Lower and Higher Pairs

When elements of pairs touch one another over a **substantial region of a surface covering a line, curve-surface, or point of contact**, we have **lower pairs**. When they touch **along a discrete line, curve-surface, or point of contact**, we have **higher pairs**.

## Linkage (Hunt, 1978)

If all joints of a **mechanism or mechanical movement** belong to lower pairs, we have a **linkage**.

# Prismatic Pairs or *P*-pairs

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

## Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

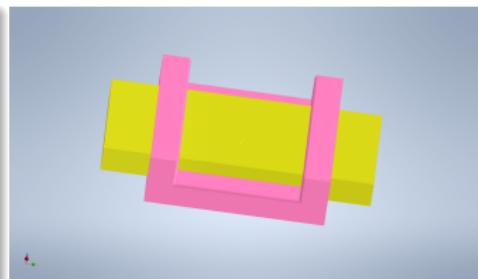
Force & Wrench

Screws in Plücker Coordinates

Group Theory

Hunt, 1978

Formed by receding the axis of the revolution surface between two pairs to  $\infty$  so that the **curve** that produces the surface moves parallel to itself, **tracing a cylinder**; or a **polygonal-tracing curve** generates a **prism**.



# Revolute Pairs or *R*-pairs

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

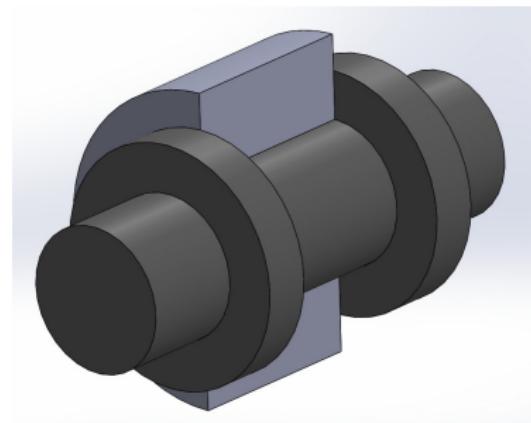
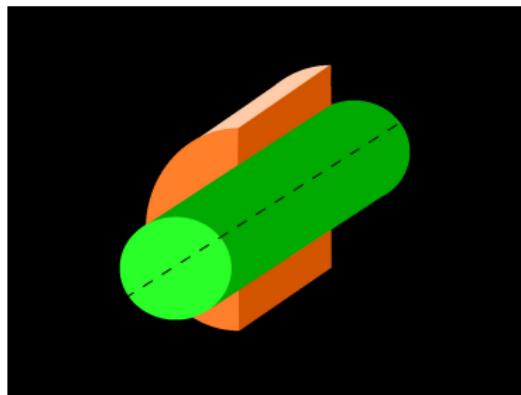
Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

One convex surface and one non-convex surface for a one degree of rotational freedom around the one joint the two surfaces make.



Revolute or Hinge or Turning or simply *R*-pairs with and without shoulder cutaway geometries. Credit: Wikimedia commons.

# Helical- & U-Joints

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

## Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory



Helical Joint

©McMaster Carr, May 2022.



Universal Joint

# Common Lower Kinematic Pairs

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedom and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

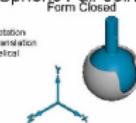
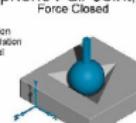
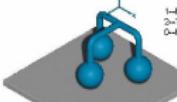
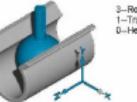
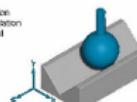
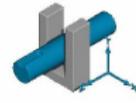
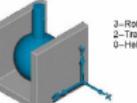
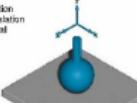
## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Groves Theory

<b>Spheric Pair Joint, S</b> Form Closed  Degrees of Freedom = 3	<b>Spheric Pair Joint, S</b> Force Closed  Degrees of Freedom = 3	<b>Sphere Slotted Cylinder Joint, S<sub>s</sub></b> Form Closed  Degrees of Freedom = 3	<b>Sphere Slotted Cylinder Joint, S<sub>s</sub></b> Force Closed  Degrees of Freedom = 3
<b>Plane Pair Joint, P<sub>L</sub></b> Form Closed  Degrees of Freedom = 3	<b>Plane Pair Joint, P<sub>L</sub></b> Force Closed  Degrees of Freedom = 3	<b>Sphere Groove Joint, S<sub>g</sub></b> Form Closed  Degrees of Freedom = 3	<b>Sphere Groove Joint, S<sub>g</sub></b> Force Closed  Degrees of Freedom = 3
<b>Cylinder Plane Pair Joint, C<sub>p</sub></b> Form Closed  Degrees of Freedom = 4	<b>Cylinder Plane Pair Joint, C<sub>p</sub></b> Force Closed  Degrees of Freedom = 4	<b>Sphere Plane Joint, S<sub>p</sub></b> Form Closed  Degrees of Freedom = 4	<b>Sphere Plane Joint, S<sub>p</sub></b> Force Closed  Degrees of Freedom = 5

Credit: Wharton and Singh, 2001.

# Common Lower Kinematic Pairs

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedom and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

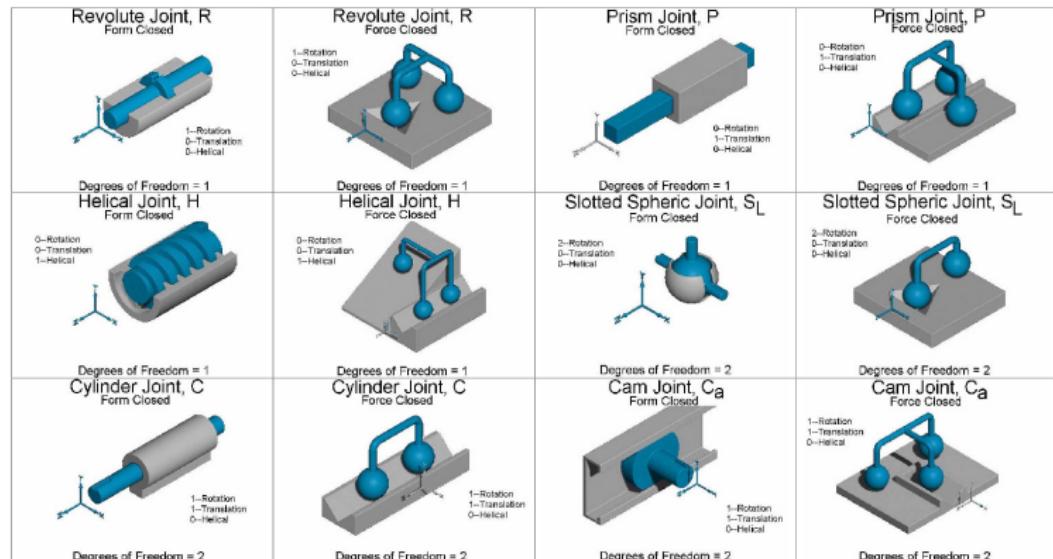
## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Groves Theory



Credit: Wharton and Singh, 2001.

# Kinematic Geometry of Common Actuations

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

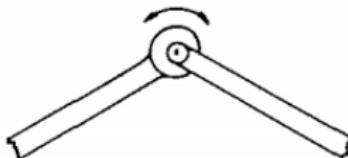
Displacement &  
Twist

Force & Wrench

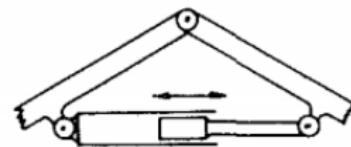
Screws in Plücker  
Coordinates

Group Theory

## In-series vs. Parallel-actuated lower pairs



(a)



(b)

(a): In-series-actuated kinematic pair with a rotary joint that is actuated “about” the hinge. (b): Prismatic joint actuated “across” a hinge. Reprinted from Hunt, Kenneth. Structural Kinematics of In-Parallel-Actuated Robot Arms. Transactions of ASME. 1983.

# Kinematic Chains

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Kinematic Chains (Reuleaux, 1975)

We can explain the structural similarity of many mechanisms  
by parts of **kinematic chains** connected by pairs.

## Kinematic chains

Kinematic chains are essentially the basic building structure  
of mechanisms ... and robots!

# Open Kinematic Chains

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Chains

Open kinematic chains are based off the anthropomorphic construction of the human hand with cantilevered beam structures.

## Chain Mechanisms and Error Amplification

Amplifies errors from waist (or base frame) all the way to the tool frame. Control difficult.

## Control

Feedforward control: High power and precision hydraulic actuators for servo motors.

Sensory feedback control: Force sensing (Ernst, 1962).

# A short treatise on robots' kinematic geometry and kinetics.

## └ Kinematic Geometry

### └ Serial Chains

#### └ Open Kinematic Chains

##### Chains

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Sensor feedback control: Force sensing (Ernst, 1962).

The PUMA arm is the world's first serial kinematic chain. Developer: Victor Scheinman, Stanford student in the '50's. Made several iterations. Patent Rights: Joe Engelberger, (Danbury Unimation, 1961). Joe – father of robotics – created world's first robotics company in '61.

# Open Kinematic Lower Pairs

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Ken Salisbury Jr., 1982)

*"[Robots are] our fascination with constructing mechanical analogues of ourselves... [this fascination] has led us to place all sorts of hopes and expectations in robot capabilities."*



The PUMA Robot  
(1956).



The Stanford Arm  
(Infolab 1969).

# Open Kinematic Chains

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

Open kinematic chains provide unstructured environmental interaction.

Project MAC, MIT.

Tomovic and Boni's pressure sensed grasp.

Binary robot vision system (McCarthy et al, 1963).

# Open Kinematic Chains

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

Stanford Manipulator.

Boston arm.

The AMF (American Machines and Foundry) arm.

General electric's walking robot (1969).

# Long Walk Towards Direct Drive Robot Arms

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

The 50's, 60's and 70's witnessed use of hydraulics for (feedforward) position control.

For feedback control, force sensors and pressure sensors were used in closed-loop scenarios.

Electrical actuation meant that robots had to be operated at high speeds. Needs for gear reduction for safe operations at low speeds.

With gear reduction came backlash, friction, and associated expenses.

# A short treatise on robots' kinematic geometry and kinetics.

## └ Kinematic Geometry

### └ Serial Chains

#### └ Long Walk Towards Direct Drive Robot Arms

The 50's, 60's and 70's witnessed use of hydraulics for (feedforward) position control.

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CMU DD I/II Arms: Workspace is donut shaped. OD: 90cm; ID: 21.7cm;  $1.8m^2$  workspace area. Built by Harry Asada. Structural design similar to aircraft gimbal arm; Uses Samarium Cobalt rare earth magnet brushless DC motors on first 3 joints, and AlNiCo magnets on tip joints. No belts, transmissions making for faster transmitting of motions, less friction, low energy, low compliance. Each joint has complex AL housing which enables: (i) Control of geometrical relationships of bearing assembly; (ii) Control of servo components to bearing assembly; (iii) Controls of rotational axes to consecutive joints.

# Direct Drive Robot Mechanism: CMU DD I Arm

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

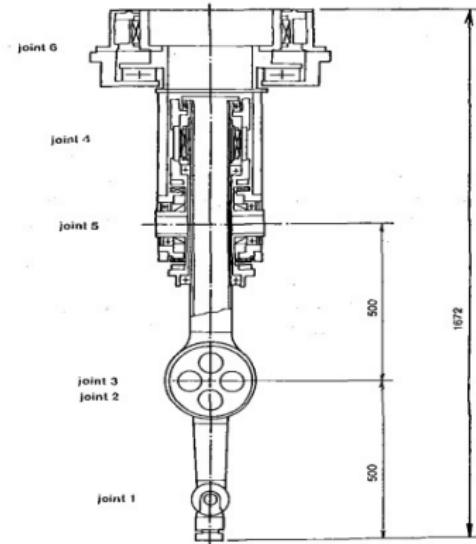
Displacement &  
Twist

Force & Wrench

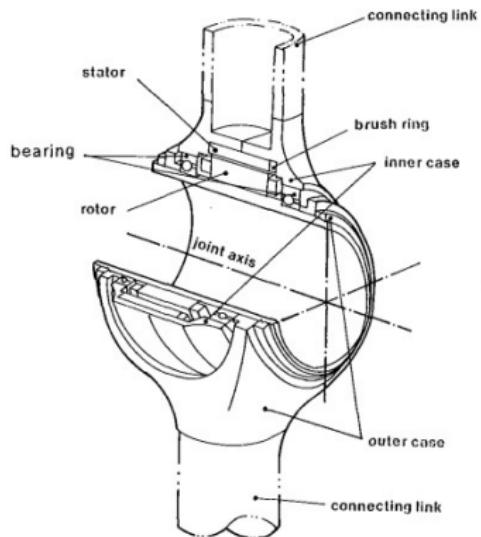
Screws in Plücker  
Coordinates

Group Theory

Along came Harry Asada.



Arm Schematics Transmission



Joint schematic

# Direct Drive Robot Mechanism: CMU DD I Arm

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

## Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

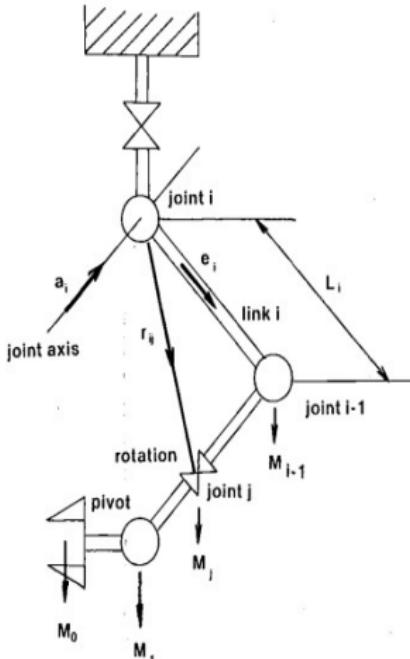
## Screw Theory

Displacement & Twist

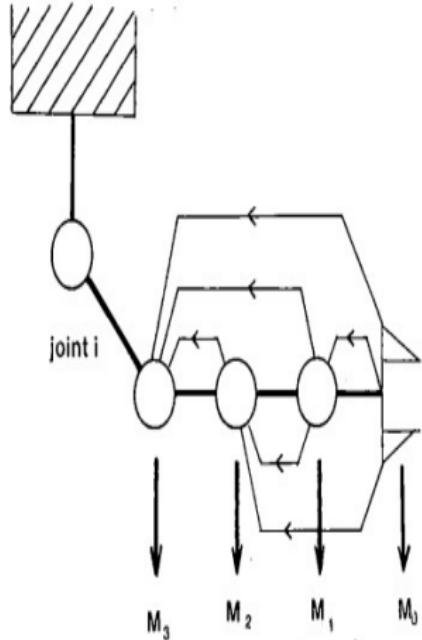
Force & Wrench

Screws in Plücker Coordinates

Group Theory



Kinematic model



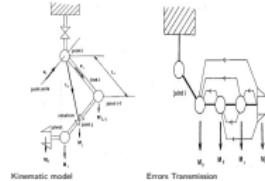
Errors Transmission

# A short treatise on robots' kinematic geometry and kinetics.

## └ Kinematic Geometry

### └ Serial Chains

#### └ Direct Drive Robot Mechanism: CMU DD I Arm



First direct-drive robot without a gearbox. Selective compliance in X-Y directions given its articulated jointed arms. One-freedom motion along Z direction given its constrained arm. New generations such as Cobra i600/i800 include power amplifiers, system and servo controls etc embedded in the robot's base. Kuka Scara arm: Lightweight, fast, powerful, low maintenance, energy consumption, investment costs etc.

# SCARA Robot Mechanisms

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

## Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

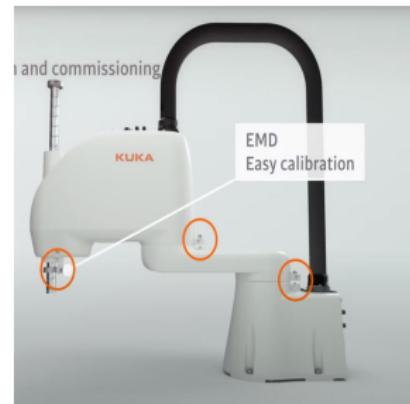
Force & Wrench

Screws in Plücker Coordinates

Group Theory



The Adept One  
SCARA robot  
(Debutted 1984).



Kuka's SCARA  
arm, 2022.  
©Kuka Robotics

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

## Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

# The Stäubli anthropomorphic arm.



# Serial mechanisms research in the 80's

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Mechanisms in the 80's

With the 80's came the arrival of PCs. Lots of research went into computational algorithms for the kinematics and kinetics of (mostly) anthropomorphic robot arms.

## Active control schemes

Efficient recursive Lagrangian and computational methods for the gravitational and Coriolis forces in Newton-Euler equations.

# Serial mechanisms research in the 80's

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Feedback Linearization

Dynamics feedback linearization for precise bounds on manipulator performance.

## Automatix

Reconfigurable robots for various assembly ops.

# Serial mechanisms research in the 90's

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

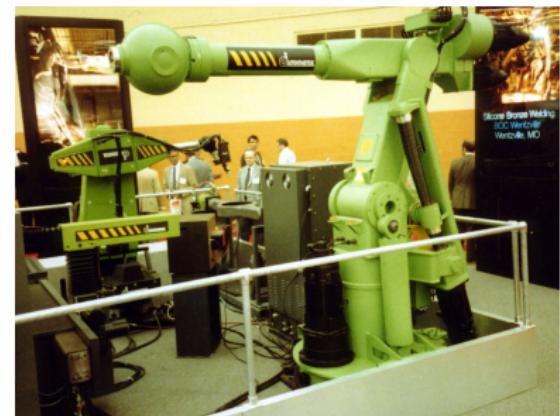
Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Robotworld

First industrial-scale reconfigurable robot and with machine vision components. RAIL scripting OS originally based on Motorola 68000, later on replaced by Apple Macintosh II.



©Wikipedia

# Hyper-redundant Continuum Robots

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

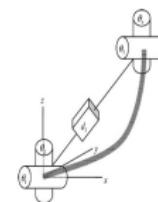
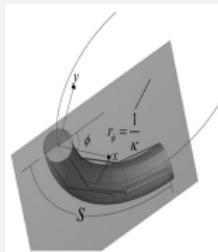
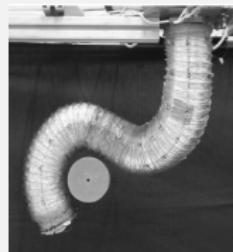
## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Grassmann



The elephant trunk continuum robot. Jones & Walker, T-RO 2006.  
Inspiration: Muscular hydrostats in nature.

# Hyper-redundant Kinematic Chains

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

## Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory



An octopus-inspired soft robot. ©Cecilia Laschi.

# Parallel Robots

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Mehlet 2015

A **parallel robot** is made up of an end-effector with  $n$  degrees of freedom, and of a fixed base, linked together by at least two independent kinematic chains. Actuation takes place through  $n$  simple actuators.

# Parallel mechanisms: Stewart-Gough Platforms

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

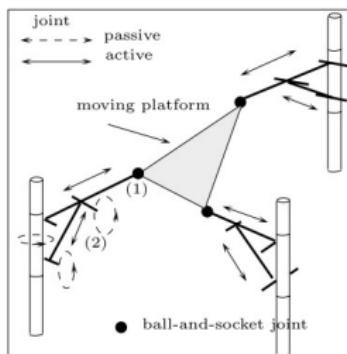
Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

Principles of a moving platform to test tyre wear and tear (Gough, 1947). Prototype, 1955.



Left: Stewart's 1965 mechanism. Right: The original 1954 octahedral hexapod proposed by Gough. Courtesy: [Parallemic.org](http://Parallemic.org).

# Truss Robots

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

## Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

## Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Groß Theory



A multi-DOF Truss Robot. Courtesy of Penngineering (ICRA 2022,  
Philadelphia, PA).

# Closed kinematic chains

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

## Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

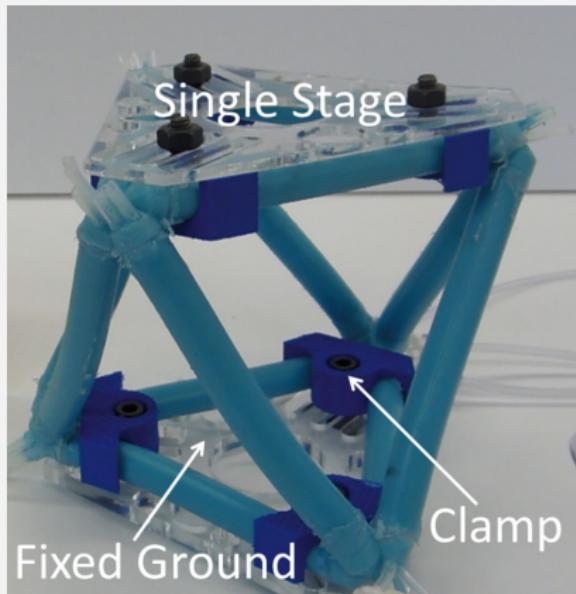
Group Theory

Connection degree  $\geq 3$ .



A Stewart-Gough platform. SolidWorks Drawing Courtesy of Andrew Belcher. UChicago, 2018.

# A Soft Stewart Platform



A soft 6-6 Stewart manipulator. Jonathan Hopkins, 2015.

# Lecture II Outline

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedom and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Freedom and Structure

### Freedoms, Constraints, and Mobility.

Motion of linkages: Screws and spatial motions.

Freedom and Mobility: Freedoms, unfreedoms, connectivity, mobility;

Grübler-Kutzbach's mobility criterion and examples;

Type-, size- and number-syntheses.

# Degrees of Freedom and Kinematic Structure

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Connection Degree)

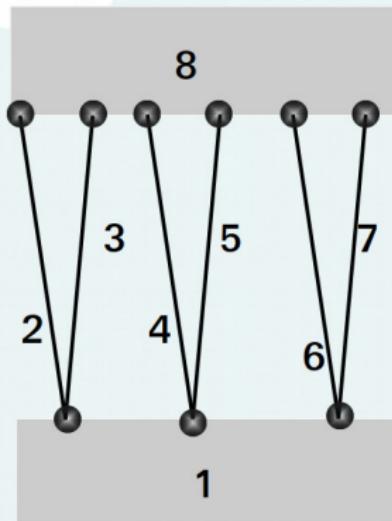
For any manipulator joint, we shall mean its connection degree to be the number of links attached it.

## Quiz

What is the connection degree of the u-joints of a Stewart-Gough platform.

# Members and Dual Graphs

## Dual graph of a Stewart platform



# Degrees of Freedom and Structure

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Members and Freedoms

Degrees of freedoms (or freedoms) concerns the relative motion of members of a pair that do not touch one another directly.

## Connectivity

By the dual graph of the Stewart platform as seen on Frame 54, the total number of freedoms that connect the two members (1 and 8) that do not connect to one another directly is six.

# Planar Linkages

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

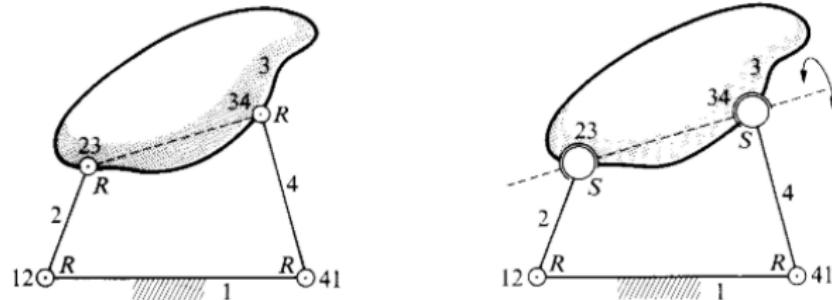
Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Four Bar Linkages

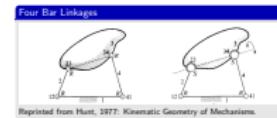


Reprinted from Hunt, 1977: Kinematic Geometry of Mechanisms.

# A short treatise on robots' kinematic geometry and kinetics.

- └ Mobility

- └ Planar Linkages



The planar *RRRR* linkage, (*left*) is modified in (*right*) to an *RSSR* linkage to allow spatial spin-movement of the coupler 3; the connectivity  $\mathcal{C}_{13} = 2$ .

# Freedom from Connectivity

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

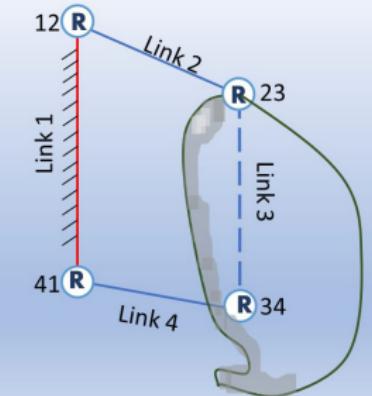
Displacement &  
Twist

Force & Wrench

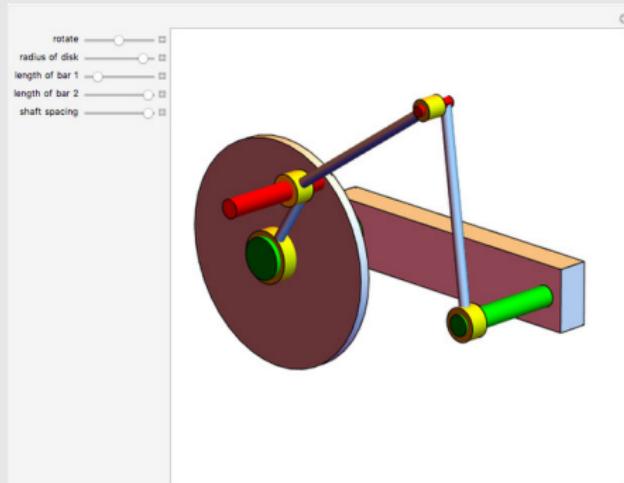
Screws in Plücker  
Coordinates

Group Theory

## A (Hacked) Four-Bar Linkage



# A Four Bar Linkage



Courtesy of Sándor Kabai, Wolfram Demonstrations Project, October 2007.

# Couplings and Freedom

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Couplings and Freedom

Links 2&4 complete a **coupling or connection** between links 1&3.

## Connectivity

The  $R$ -pairs are said to have a **connectivity** of  $C_{ij} = 1$  for all  $i, j = 1, 2, 3, 4$ . Thus, total degree of freedom is 1.

# Mobility of Mechanisms

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Mobility and Relative Mobility, $\mathfrak{M}$

Simply put, the number of a mechanism's freedoms is its **mobility**, or **relative mobility**,  $\mathfrak{M}$ .

## The Mobility, $\mathfrak{M}$

It specifies the **independent variables** needed to **determine** every relative location of a **mechanism's members** with respect to one another.

## A Note on Serial and Parallel Mobility

A little tricky to determine for parallel mechanisms but straightforward for serial mechanisms.

# Mobility of Mechanisms

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Quiz

What is the mobility  $M$  of the *RSSR* four bar linkage of Frame 56? Why?

## Quiz

What is the mobility  $M$  of the *RRRR* four bar linkage of Frame 57? Why?

## Definition (The mobility criterion (well, not yet))

Let's not get ahead of ourselves. A little introduction to screws are in order for us to grasp the **Grübler-Kutzbach** mobility criterion.

# Rigid Body Displacements and Forces

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Inhomogeneity of Displacements and Angles

Quiz: Three translations and three rotations are ill-posed for uniquely determining the freedoms of a body. Why?

They are **not homogeneous**.

For true **kinematic wholeness and generality**, displacement that is **purely translatory** and **purely rotary** is needed.

# Need for Unique Representations

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

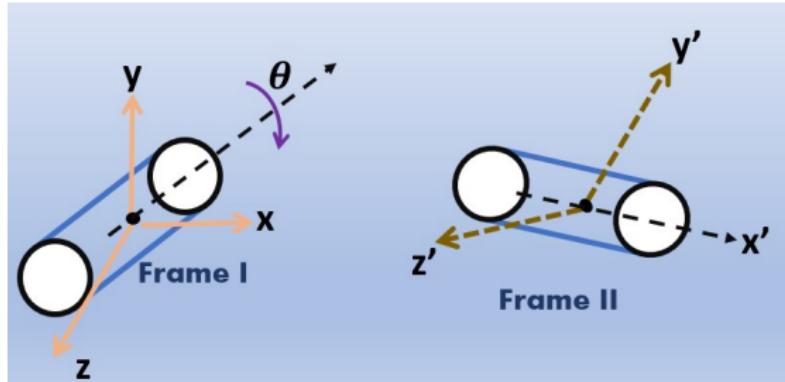
## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory



There exists infinite possible ways of movement by which the point on the rigid body in Frame I can be effected to be transferred to the location of the point on Frame II and vice versa.

# Screws for Kinematic Generality

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Need for Screws

From a kinematic standpoint, **six homogeneous screw coordinates** – each having an **independent screw freedom** – are needed to **uniquely determine a rigid body's location**.

## Definition (What is a screw anyway?)

A **screw** is a **straight line** in space, called **the axis**, with an associated direction, called **pitch**,  $p$ .



# Screws in Mechanics: A History Snippet

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Screws Through Time

Michel Chasles(1793 – 1880): Rigid Body Displacements (Kinematics).

Louis Poinsot (1777 – 1859): Geometrical Mechanics (Kinetics).

Sir Robert Stawell Ball, F.R.S, LL.D. (1840-1913): Irish Astronomer who popularized screw theory in his day.

# Screws in Mechanics: A History Snippet

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedom and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Screws Through Time

### The Death of Screw Theory

After World War I, interest in Screw theory declined. Several possible reasons for this.

- ▶ Ball died in 1913, he had no students.
- ▶ Other British/Irish mathematicians who might have carried these ideas forward died young.
  - ▶ Clifford, a contemporary and friend of Ball's died 1879 aged 33.
  - ▶ Charles Jasper Joly (1864–1906) studied under Helmholtz and Königs in Berlin, was a successor of Ball's as Royal Astronomer of Ireland.
  - ▶ Arthur Buchheim (1859–1888), studied under Klein at Leipzig. Taught at Manchester grammar school, died aged 29.
- ▶ Relativity became popular, and Euclidean geometry less so. Ball joked, "The Theory of Screws is now all done; it is quite obsolete; it is all going over into non-Euclidean space."

Kept alive in Soviet Union, Kotelnikov and others.

### Rediscovered by Mechanical Engineers

In the 1960s two mechanical engineers in Australia rediscovered Ball's work. Ball's theory of screws was just what they needed to study mechanisms.

- ▶ Kenneth Henderson Hunt (1920–2002) was born in the UK, worked at Monash University. "Kinematic Geometry of Mechanisms", first published in 1978. Applied screw theory to the problem of designing constant velocity joints.
- ▶ Jack Raymond Phillips (1923–2009) University of Sydney. Studied agricultural machinery, (trailed disc ploughs) and the mechanics of the lobster's claw. Two volume work "Freedom in Machinery: Introducing Screw Theory".

Courtesy of J.M. Selig, IROS 2018 Screw Theory Tutorial.

# Screws for Kinematic Generality

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Simplest (Unique) Representation of Displacements

Chasles(1793 – 1880) showed that any given displacement of a rigid body can be uniquely represented as the rotation of the body about an axis (the screw axis) followed by a translation parallel to that axis (the pitch).

## Michel Chasles and Screws

Chasles called this unique transformation screws. Chasles is responsible for the Euclidean description of the motion of a rigid body in space and he made lasting contributions to theories of rigid body dynamics.

# Screws for Kinetic Generality

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Simplest (Unique) Representation of Forces

[Poinsot](#)(1777 – 1859) showed that **any system of forces** acting on a rigid body can be represented by a **single force**, together with **a couple** acting along the normal to the force **in a plane**.

## Louis Poinsot and Geometrical Mechanics

“Everyone makes for himself a clear idea of the motion of a point, that is to say, of the motion of a corpuscle which one supposes to be infinitely small, and which one reduces by thought in some way to a mathematical point.” ~ Louis Poinsot, 1834.

# Freedoms, Unfreedoms, and Mobility

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

## Freedom and Constraints

Suppose a screw  $f = (f_1, \dots, f_6)$  “fixes” a body in 3D space.

Each **constraint**  $u_i \neq f_j$  for  $(i, j) \in \{1, \dots, 6\}$ .

Rather each  $u_i$  has influence on every  $\{f_i\}_{i=1}^6$ .

Each  $u_i$  from the six independent equations,  
 $g(s_1, s_2, s_3, s_4, s_5, s_6) = 0$ , suppresses a  
**freedom**,  $f_i$ .

Progressively relaxing each  $u_i$ , or **unfreedom**,  
adds an extra body  $f_i$ .

# Freedoms, Unfreedoms, and Mobility

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Freedom and Unfreedoms

Suppose the total **freedoms** is  $f$  and the total **unfreedoms** is  $u$ , then

$$u + f = 6.$$

Note: A rigid body's freedoms is also referred to the dimension of its **configuration space**.

## Relative Freedoms

Suppose there are a total of  $n$  **unconstrained** bodies.

Suppose further that we choose one out of the bodies as a reference body. Then the total number of **relative freedoms** is  $6(n - 1)$ .

# Freedoms, Unfreedoms, and Mobility

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

## Constraints and Joints

Now, consider  $k$  independent constraints<sup>a</sup> such as joints along points, lines, curves or surfaces.

<sup>a</sup>NB: The total allowable constraints is 5 for a body in relative motion. 6 for a fully rigid body.

## The Mobility Criterion

Let the constraint of joint,  $i$  (e.g. a joint along points, lines, curves or surfaces) be  $u_i$ . Then the mobility criterion  $\mathfrak{M}$  is

$$\mathfrak{M} = 6(n - 1) - \sum_{i=1}^k u_i. \quad (3)$$

# General Grübler-Kutzbach Mobility Criterion

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## General Grübler-Kutzbach Mobility Criterion

Recall that  $\sum_i u_i + f_i = 6$  from Frame (70) so that

$$\mathfrak{M} = 6(n - k - 1) - \sum_{i=1}^f f_i. \quad (4)$$

## Exceptions: Relative Planar and Spherical Motions

For bodies restricted to relative planar or spherical motions, the total freedoms + constraints is 3 (not 6)!

$$\mathfrak{M} = 3(n - k - 1) - \sum_{i=1}^f f_i. \quad (5)$$

# General Grübler-Kutzbach Criterion References

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Grübler-Kutzbach Mobility Criterion References

Attributed to Grübler:

Schoenflies, Arthur, and M. Grübler. "Kinematik." In Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, pp. 190-278.

Vieweg+Teubner Verlag, Wiesbaden, 1908;

Grübler, Martin Fürchtegott. Getriebelehre: eine Theorie des Zwanglaufes und der ebenen Mechanismen. Springer, 1917.

and Kutzbach:

Kutzbach, Karl. "Mechanische leitungsverzweigung, ihre gesetze und anwendungen." Maschinenbau 8, no. 21 (1929): 710-716.

# Loops

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Loops

A kinematic chain often comprises members called loops.

## Binary Link

Members in a binary link constitute a single loop. Example:  
The four-bar linkage.

## Single loops

For single loops,  $k = n$  so that  $\mathfrak{M} = \sum_{i=1}^f f_i - 6$ .

## Mobility of Mechanisms

$\mathfrak{M} \leq 1$  for at least one actuator-pair to produce mobility at a successor joint which depends on that actuator-pair's input.

# Mobility of Common Robot Configurations

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

## Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

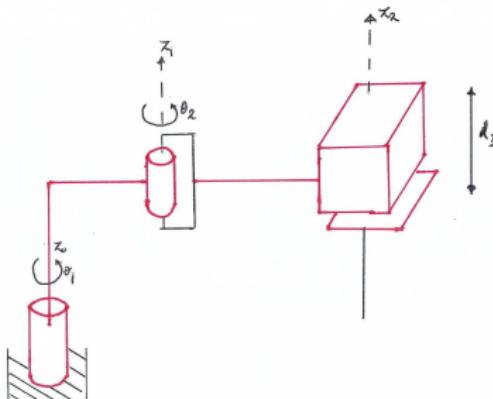
## Screw Theory

Displacement & Twist

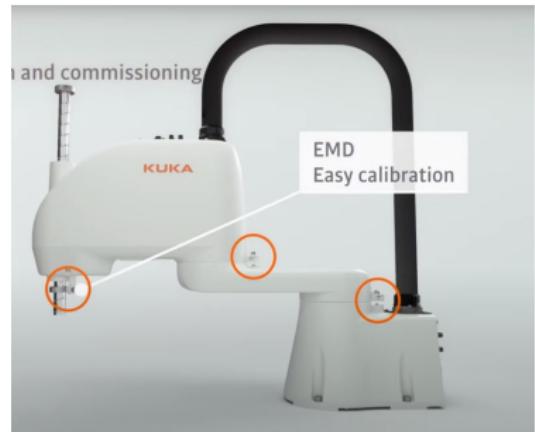
Force & Wrench

Screws in Plücker Coordinates

Group Theory



Configuration of the SCARA Arm.



Courtesy of Fanuc America Inc.

# Mobility of The SCARA Robot

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Mobility Analysis

Two rotary joints. One prismatic joint acting along the  $z$  axis, and constrained along the  $xy$  plane.

## Mobility Parameters

Four rigid bodies (links). Three constraints. Four freedoms. Therefore,  
 $\mathfrak{M} = 6(4 - 3 - 1) + 4 = 4$

# Mobility Analysis of The Universal Robot

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

## Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

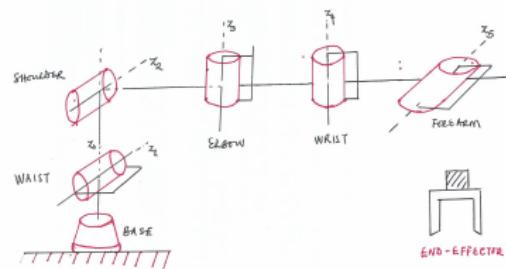
Group Theory



©Universal Robots A/S, DK.

## The Revolute Arm

Falls under so-called *RRR* kinematic arrangements. Also called a **revolute**, **elbow**, or **anthropomorphic manipulator**.



$$n = 6; k = 5; f = 6$$

$$\therefore \mathfrak{M} = 6(n - k - 1) + \sum f_i$$

$$\Rightarrow 6(6 - 5 - 1) + 6 \text{ or } \mathfrak{M} = 6.$$

# Mobility of The Stewart-Gough Platform

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

## Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Gauss Theory



# Mobility of The Stewart-Gough Platform

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Unconstrained bodies, $n$

There are six universal joints that connect the base platform to the prismatic linear actuators.

There are six spherical joints that connect the top platform to the top of the prismatic actuators.

Altogether, there are  $n = 6 + 6 + 2$  or 14 unconstrained rigid bodies.

## Constraints, $k$

Six u-joints. Six spherical joints. Six prismatic joints. Altogether, there are  $f = 6 + 6 + 6 := 18$  constraints.

## Freedoms, $f$

Each u-joints has two freedoms.

Each spherical joint has three (rotary) freedoms. Each prismatic joint has one freedom.

Altogether, there are  $f = 6 \times 2 + 6 \times 3 + 6 \times 1 := 36$  freedoms.

# Mechanism Synthesis

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Definition (Type-Synthesis)

What **type** of mechanism is appropriate for a task: A **linkage or profile mechanism?**

Type-synthesis

## Definition (Size-Synthesis)

What **major dimensions** of the mechanism is to be synthesized?

Size-synthesis

# Number Synthesis

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Number-Synthesis)

That which deals with the **freedoms and constraints** after the type- and size-synthesis of a mechanism, as well as a **kinematic chain's structural analysis** is termed the **number-synthesis** of the mechanism.?

# Quiz: Mobility of a Planar Parallel Mechanism

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

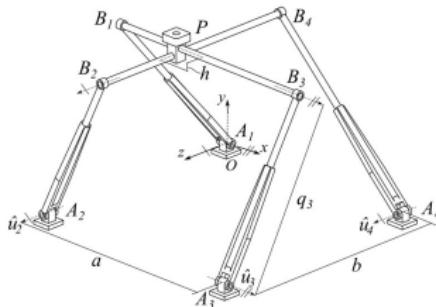
Screws in Plücker

Coordinates

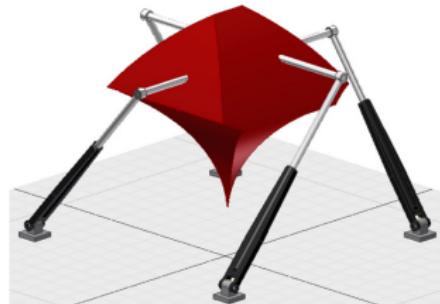
Group Theory

## Quiz

Analyze the mobility of the mechanism below.



A planar parallel mechanism.  
Reprinted from Garcia-Murillo et al.



Workspace of the mechanism.

# Quiz: Mechanism Hints

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

## Hint – Mechanism Description

Point  $P = (P_x, P_y, P_z)$  is the interconnecting point for all the chains on the mobile platform and the top rods.

## Hint – Mechanism Description

The rods that connects points  $B_1$  and  $B_3$ , and points  $B_2$  and  $B_4$  are perpendicular. Both rods are connected to the moving platform by prismatic joints, which are separated from each other by a vertical offset  $h$ .  $\hat{u}_i$  signifies universal joints.

# Lecture III Outline

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Lecture III Outline

### Rigid Body Transformations and Screws Theory.

Rigid body motions: Properties; Direction cosines; Rotation compositions; Rotation Parameterizations.

Rodrigues' formula; the matrix exponential,  $SO(3)$ ,  $SO(n)$ ,  $\mathbb{SE}(3)$  group properties.

Transformations: Translations and rotations in  $\mathbb{R}^3$ , planar rotations,  $SO(3)$ ,  $SE(3)$  motions; homogeneous transformations; Euler and Fick angles.

# Rigid Body Motions

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Rigid Body Motion – Intro

A mapping  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a **rigid body motion** if

$$\|g(x) - g(y)\| = \|x - y\| \text{ for all } x, y \in \mathbb{R}^3; \quad (6)$$

$$g(x \times y) = g(x) \times g(y) \text{ for all } x, y \in \mathbb{R}^3; \quad (7)$$

## Rigid Body Motion Preserves Inner Products

For two vectors  $a$  and  $b$ ,  $\langle a, b \rangle = g(a) \times g(b)$ .

# Rigid Body Transformations

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

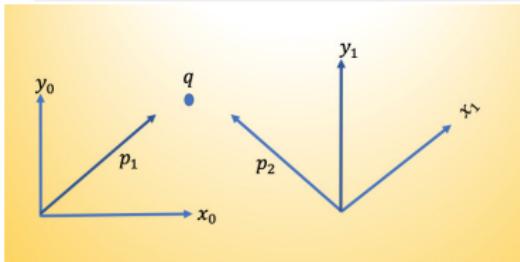
Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Translation of Point $q$ between Two Frames

For a reference frame,  $o_0x_0y_0$  and a moving coordinate frame,  $o_1x_1y_1$ , the translation of  $q$  is given as below:



$$q^0 = \begin{pmatrix} q_x^0 \\ q_y^0 \end{pmatrix}, \quad q^1 = \begin{pmatrix} q_x^1 \\ q_y^1 \end{pmatrix}$$

## Translation of Origin between Two Frames

$$o_1^0 = \begin{pmatrix} o_x^0 \\ o_y^0 \end{pmatrix}, \quad o_0^1 = \begin{pmatrix} o_x^1 \\ o_y^1 \end{pmatrix}. \quad (8)$$

# Rigid Body Transformations

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Applications to Screws

Applies to Chasles' displacement theorem and Poinsot's force and couple transformations too.

## Screw Transformations

$$\mathbf{t}_1^0 = \begin{pmatrix} t_x^0 \\ t_y^0 \end{pmatrix}, \quad \mathbf{t}_1^1 = R(-\theta)q^0 \quad (9)$$

$$\mathbf{t}_2^0 = R(\theta)q^0, \quad \mathbf{t}_2^1 = \begin{pmatrix} t_x^1 \\ t_y^1 \end{pmatrix} \quad (10)$$

where  $\theta$  is the angle coordinate frame  $o_1x_1y_1$  makes w.r.t  $o_0x_0y_0$ .

# Rotations in $\mathbb{R}^3$

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Rotations in $\mathbb{R}^3$

Conventions: Bodies' orientations are measured along a corkscrew direction, specified by a local coordinate frame. Thus, relative orientation is measured from the local coordinate frame to an inertial coordinate frame.

# Direction Cosines

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

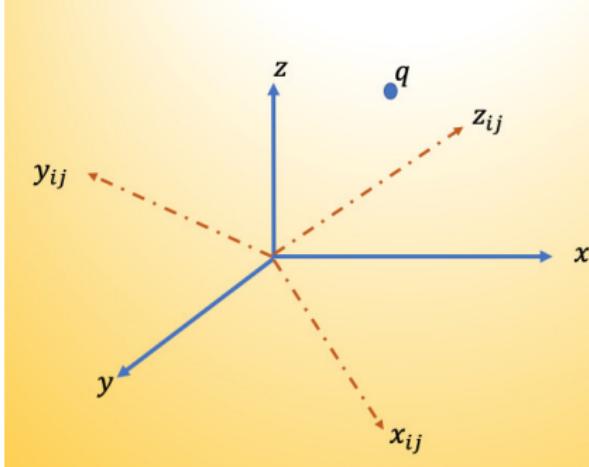
## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory



## Conventions

I: Inertial frame; J: Body frame.

$q : (x_{ij}, y_{ij}, z_{ij}) \in \mathbb{R}^3$ : coordinates of the principal axes of J relative to I.

# Rotation Matrix from Direction Cosines

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

## Rotation as Composition of Projections Between Frames

$$R_{ij} = [\mathbf{x}_{ij} \quad \mathbf{y}_{ij} \quad \mathbf{z}_{ij}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \quad (11)$$

## Rotation Matrix as Unit Axes' Dot Products

$$R_{ij} = \begin{bmatrix} \mathbf{x}_j \cdot \mathbf{x}_i & \mathbf{y}_j \cdot \mathbf{x}_i & \mathbf{z}_j \cdot \mathbf{x}_i \\ \mathbf{x}_j \cdot \mathbf{y}_i & \mathbf{y}_j \cdot \mathbf{y}_i & \mathbf{z}_j \cdot \mathbf{y}_i \\ \mathbf{x}_j \cdot \mathbf{z}_i & \mathbf{y}_j \cdot \mathbf{z}_i & \mathbf{z}_j \cdot \mathbf{z}_i \end{bmatrix}. \quad (12)$$

# Rotation Matrix from Direction Cosines

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Rotation Matrices are Direction Cosines!

$$\begin{aligned}\mathbf{x}_j \cdot \mathbf{x}_i &= \cos(\angle(\mathbf{x}_j, \mathbf{x}_i)), & \mathbf{y}_j \cdot \mathbf{x}_i &= \cos(\angle(\mathbf{y}_j, \mathbf{x}_i)), \dots \\ \dots, \mathbf{y}_j \cdot \mathbf{z}_i &= \cos(\angle(\mathbf{y}_j, \mathbf{z}_i)), & \mathbf{z}_j \cdot \mathbf{z}_i &= \cos(\angle(\mathbf{z}_j, \mathbf{z}_i)).\end{aligned}$$

## Properties of Rotation Matrices

Rows of  $R_{ij}$  are the **unit vector** coordinates of  $I$  in the frame  $J$  so that

$$R_{ij} = R_{ji}^{-1} = R_{ji}^T. \quad (13)$$

That is, the **inverse of the rotation matrix is equal to its transpose.**

# Special Orthogonal 3, SO(3)

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Orthogonal properties!

Observe:  $\det \mathbf{R} = \mathbf{r}_1^T \cdot (\mathbf{r}_2 \times \mathbf{r}_3)$ . In **corkscrew notation**,  $\det \mathbf{R} = +1$  i.e.  $\mathbf{r}_2 \times \mathbf{r}_3 = \mathbf{r}_1$  so that  $\det \mathbf{R} = \mathbf{r}_1^T \cdot \mathbf{r}_1 = +1$ . A matrix that satisfies the above property is said to possess a **special orthogonal 3, denoted SO(3), property**.

## SO(n) Property

Special orthogonal means  $\det \mathbf{R} = +1$ . The set of all SO matrices in  $\mathbb{R}^{n \times n}$  is

$$SO(n) = \{\mathbf{R} \in \mathbb{R}^{n \times n} : \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = +1\}. \quad (14)$$

# Rotations on Vectors

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

## Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Rotating a Vector

Suppose that a point  $p_j$  is on a frame  $J$ , then the vector that connects a point  $q_j$  in the frame  $J$  to  $p_j$  is  $v_j = q_j - p_j$ . Now, the rotation matrix's action on  $v_j$  is

$$\mathbf{R}_{ij}(v_j) := \mathbf{R}_{ij}q_j - \mathbf{R}_{ij}p_j = q_i - p_i = v_i. \quad (15)$$

# Planar Rotations

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

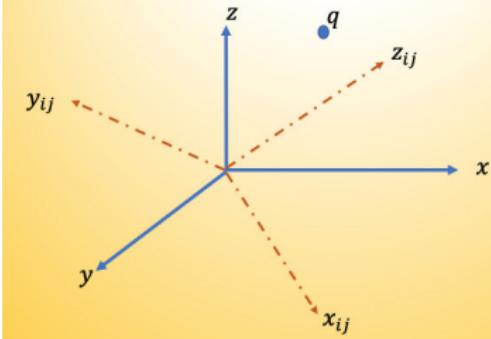
Screws in Plücker Coordinates

Group Theory

## Planar Rotations

Let the angle of rotation between the two coordinate frames be  $\theta$ . Then,

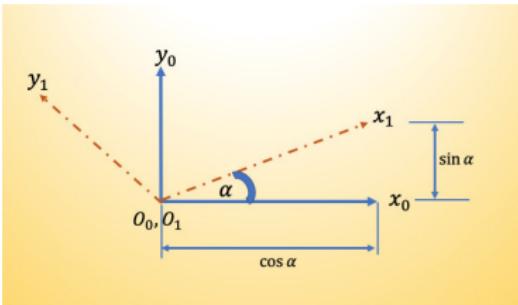
$$R_1^0 = \begin{pmatrix} x_1^0 & y_1^0 \end{pmatrix} \quad (16)$$



## Planar Rotations

It follows that

$$R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad (17)$$



# Planar Rotations

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

## Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Planar Rotations via Direction Cosines

$$\begin{aligned} R_1^0 &= \begin{bmatrix} \mathbf{x}_0 \cdot \mathbf{x}_1 & \mathbf{y}_1 \cdot \mathbf{x}_0 \\ \mathbf{x}_0 \cdot \mathbf{y}_1 & \mathbf{y}_1 \cdot \mathbf{y}_0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\cos(\pi/2 - \alpha) \\ \cos(\pi/2 - \alpha) & \cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \end{aligned} \quad (18)$$

Projection of  $y_1$  on  $x_0$  is negative because of our adopted right-handed frame.

# Axis-Angle Parameterization

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

## Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Exponential Rotation Coordinates

It can be verified that

$$e^{\tilde{\omega}\theta} = \mathbf{I} + \tilde{\omega} \sin \theta + \tilde{\omega}^2 (1 - \cos \theta), \quad (19)$$

$\tilde{\omega} \in so(3)$  and  $so(n) = \{\tilde{\omega} : \tilde{\omega} \in \mathbb{R}^n \times \mathbb{R}^n \mid \tilde{\omega}^T = -\tilde{\omega}^T\}$ .

# Axis-Angle Parameterization

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Parameterization from Exponentiation of Axial Rotation

It can be verified that

$$e^{\tilde{\omega}\theta} = \begin{bmatrix} \omega_x^2 v_\theta + c_\theta & \omega_x \omega_y v_\theta - \omega_z s_\theta & \omega_x \omega_z v_\theta + \omega_y s_\theta \\ \omega_x \omega_y v_\theta + \omega_z s_\theta & \omega_y^2 v_\theta + c_\theta & \omega_y \omega_z v_\theta - \omega_x s_\theta \\ \omega_x \omega_z v_\theta - \omega_y s_\theta & \omega_y \omega_z v_\theta + \omega_x s_\theta & \omega_z^2 v_\theta + c_\theta \end{bmatrix}$$

where  $v_\theta = 1 - \cos \theta$ ,  $-2\pi n \leq \theta \leq 2\pi n$ .

## Angle from Exponentiation of Axial Rotation

$$\theta = \cos^{-1} \frac{\text{Trace}(\mathbf{R}) - 1}{2} \quad (20)$$

# Axis-Angle Parameterization

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Axis from Exponentiation of Axial Rotation

Equating the off-diagonal terms, we find that

$$\omega = \frac{1}{2 \sin \theta} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \quad (21)$$

Suppose  $\theta \neq 0$ , equations (20) together with (21) are the **axis-angle representation**.

# The Skew Symmetrix Matrix

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Skew Symmetric Matrix

For a rigid body whose rotation is locally parameterized by  $R$  (with  $R : RR^T = I$ ) followed by a translation,  $d$ .

Compose this transformation as  $g = \begin{pmatrix} R & I \\ 0 & 1 \end{pmatrix} \in SE(3)$ .

## Skew Symmetric Matrix

There exists a skew symmetric matrix,  $\tilde{\omega}$ , for an axis  $\omega = [\omega_x, \omega_y, \omega_z]^T$  about which the rotary and translatory movement occurs such that

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = -\tilde{\omega}^T \quad (22)$$

# Composition of Rotations

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Rotations Composition

Let the **relative orientation** of a frame  $K$  to a frame  $J$  be  $\mathbf{R}_{jk}$ , and let frame  $J$ 's **relative orientation** to frame  $I$  be  $\mathbf{R}_{ij}$ , then the **relative orientation** of frame  $K$  w.r.t  $I$  is

$$\mathbf{R}_{ik} = \mathbf{R}_{ij} \cdot \mathbf{R}_{jk}. \quad (23)$$

## Rotations Composition

Equivalent to **rotating  $J$  relative to frame  $I$  according to  $\mathbf{R}_{ij}$** ; then **aligning frame  $J$  to  $K$** , we **rotate  $K$  relative to  $I$  according to  $\mathbf{R}_{jk}$** . This frame relative to which rotation occurs is termed the **current frame**.

# Composition of Rotations About A Current Axis

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

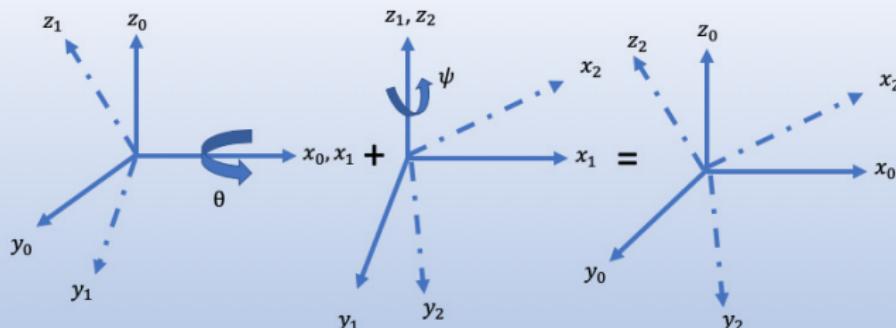
Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Composition of Rotations About A Current Axis



# Composition of Rotations About A Current Axis

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Compositions

$$\mathbf{R} = \mathbf{R}_{x,\theta} \mathbf{R}_{z,\psi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix} \cdot \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (24)$$

$$\mathbf{R} = \begin{pmatrix} c_\psi & -s_\psi & 0 \\ 0 & c_\theta c_\psi & -s_\theta \\ s_\theta s_\psi & s_\theta c_\psi & c_\theta \end{pmatrix} \quad (25)$$

Notice how the order of multiplication is carried out, owing to the axis about which we are making the transformation.

# Composition of Rotations

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Skew matrices and $SO(3)$

### Lemma (Murray, Li, and Sastry, 1994)

*The exponential of a skew-matrix  $\tilde{\omega}$  that parameterizes a rotation  $\theta$  about an axis,  $\omega$ , is in  $SO(3)$  i.e.,  $e^{\tilde{\omega}\theta} \in SO(3)$ .*

## Pre-multiplication of Rotations

A **rotation about a fixed axis** requires a **pre-multiplication**.

## Post-multiplication of Rotations

A **rotation about a current axis** necessitates a **post-multiplication**.

# Rotations' Composition

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

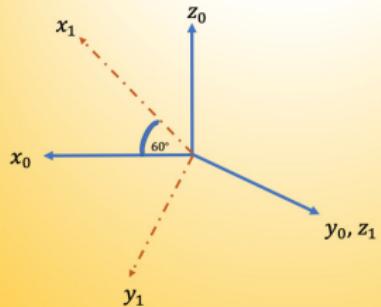
Group Theory

## Rotations Composition

Suppose all axes of the inertial frame are successively rotated by  $\beta$  around  $x_0, y_0, z_0$  respectively. What is the transformation? Verify that (1)

$R_{e,\beta} = I$  where  $e$  is the axes about which we are rotating and  $\beta$  is the angle of rotation; (2) The composition of rotations about  $\beta$  and  $\alpha$  in a successive manner implies that

$R_{z,\beta}, R_{z,\alpha} = R_{z,\beta+\alpha}$ , and (3)  
 $(R_{z,\beta})^{-1} = R_{z,-\beta}$ .



Relative orientation between  
two frames.

# Local Parameterization of Rotations in SO(3)

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

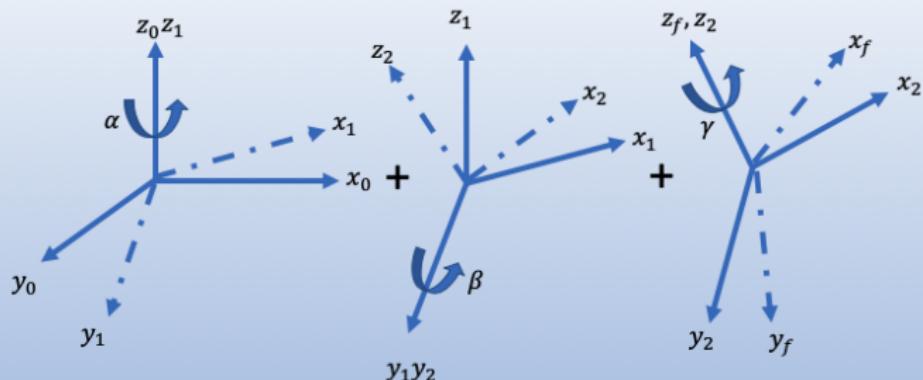
Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Theorem (Euler)

*Any rotation in  $SO(3)$  can be represented as a rotation about a fixed axis  $\omega \in \mathbb{R}^3$  through an angle  $\theta \in [0, 2\pi]$ .*



Relative orientation between two frames.

# Local Parameterizations of SO(3)

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Rotation About Principal $x$ Axis

$$\mathbf{R}_x(\theta) = e^{\tilde{x}\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad (26)$$

## Rotation About Principal $y$ Axis

$$\mathbf{R}_y(\phi) = e^{\tilde{y}\phi} = \begin{bmatrix} -\cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix} \quad (27)$$

# Local Parameterizations of SO(3)

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Rotation About Principal $z$ Axis

$$\mathbf{R}_z(\psi) = e^{\tilde{z}\psi} = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (28)$$

# Euler Angles as Parameterization of Rotations

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

*“Read Euler, read Euler, he is the master of us all.” – Pierre-Simon Laplace.*

## Euler ( $ZYZ$ ) Angles

$$\mathbf{R}_{ij}(\alpha, \beta, \gamma) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma) \quad (29)$$

$$= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix} \quad (30)$$

# Euler Angles as Parameterization of Rotations

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

*“The study of Euler’s works will remain the best school for the different fields of mathematics, and nothing else can replace it.” – Carl Friedrich Gauss.*

Euler ( $ZYZ$ ) Angles. Case  $\sin(\beta) > 0$

Given an  $R$  and  $\sin(\beta) > 0$ , the Euler angles are:

$$\beta = \arctan 2(r_{33}, \sqrt{1 - r_{33}^2}) \quad (31a)$$

$$\alpha = \arctan 2(r_{23}/\sin \beta, r_{13}/\sin \beta) \quad (31b)$$

$$\gamma = \arctan 2(r_{32}/\sin \beta, -r_{31}/\sin \beta) \quad (31c)$$

# Euler Angles as Parameterization of Rotations

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Euler ( $ZYZ$ ) Angles. Case $\sin(\beta) < 0$

Given an  $R$  and  $\sin(\beta) < 0$ , the Euler angles are:

$$\beta = \arctan 2(r_{33}, -\sqrt{1 - r_{33}^2}) \quad (32a)$$

$$\alpha = \arctan 2(-r_{23}/\sin \beta, -r_{13}/\sin \beta) \quad (32b)$$

$$\gamma = \arctan 2(-r_{32}/\sin \beta, r_{31}/\sin \beta) \quad (32c)$$

Euler angles are not unique owing to the sign of the angle about which the  $y$  axis rotates!

# Euler Angles Drawbacks

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Drawbacks

Lack of **smooth, global solutions** to Euler angles from rotation;

**Singularities** at  $\mathbf{R} = \mathbf{I}$ ;

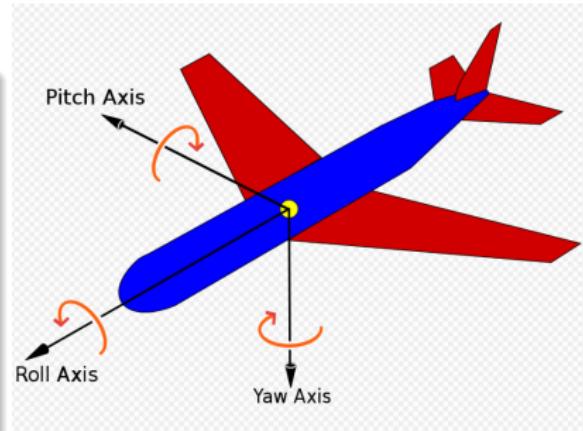
Possible infinite representation of Euler angles at specific configurations e.g.  $\mathbf{R}(\theta, 0, -\theta) = \mathbf{I}$ .

# Yaw, Pitch, and Roll Axes

<b>Introduction</b>
Textbooks
<b>Kinematic Geometry</b>
Mechanics
Mechanisms
Pairs and Linkages
Serial Chains
Hyperredundant and Parallel robots
<b>Mobility</b>
Screws and a History Snippet
Freedoms and Constraints
Mobility Criterion
<b>Motions</b>
Movement in $\mathbb{R}^3$
Special Orthogonal Properties
Composition of Rotations
<b>Screw Theory</b>
Displacement & Twist
Force & Wrench
Screws in Plücker Coordinates
Group Theory

## Fick angles

Otherwise called the **yaw, pitch, and roll angles**.  $R_{ij}$  found by rotating about the  $x$ -axis (roll), then a rotation about the  $y$ -axis (pitch), and finally a rotation about  $z$ -axis – all in the body frame.



Aircraft Principal Axes in the right-hand frame. Courtesy of Wikimedia commons.

# Quaternions

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Global Parameterization of Rotations

Instead of locally parameterizing the  $SO(3)$  group, quaternions, unlike rotation matrices, globally parameterize the  $SO(3)$  Lie Group.

## Quaternions

Formally, we represent a quaternion as follows:

$$Q = q_0 + q_x \mathbf{i} + q_y \mathbf{j} + q_z \mathbf{k}, \quad (33)$$

where  $q_0 \in \mathbb{R}$  is the scalar component of  $Q$  and  $\mathbf{q} = (q_x, q_y, q_z) \in \mathbb{R}^3$  is the vector component.

# Quaternions

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Quaternions

The **unit quaternions** are the **subset of all  $Q \in \mathbb{Q}$  such that  $\|Q\| = 1$** . For a rotation matrix  $R = \exp(\hat{\omega}\theta)$ , we have the **unit quaternion** as

$$Q = (\cos(\theta/2), \omega \sin(\theta/2)), \quad (34)$$

where  $\omega \in \mathbb{R}^3$  is the **axis of orientation** and  $\theta \in \mathbb{R}$  is the **angle of rotation**.

# Other Parameterization of Rotations

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Fick ( $ZYX$ ), Helmholtz ( $YZX$ ) Angles.

We could **permute the order of rotation** such as rotating successively about **different axes**. Examples include **successive rotations about  $ZYX$  axes for the Fick angles** and **successive rotations about  $YZX$  axes for Helmholtz angles**.

## Fick ( $ZYX$ ) and Helmholtz ( $YZX$ ) Angles.

These avoid **Euler angle singularities** at  $\mathbf{R} = \mathbf{I}$ . This does not preclude **singularities at other configurations**.

# Summary of Parameterization of Rotations

## Summary of Parameterizations

Rotation matrices can be parameterized in one of many ways depending on our use case. The common examples of parameterizations are

- (1) Axis-Angle representation;
- (2) Euler angles ( $ZYZ$ ) representation;
- (3) Fick angles (*i.e.*,  $ZYX$  or yaw, pitch and roll) representation;
- (4) Helmholtz angles (or  $YZX$ ) angles representation; and

# Lecture IV Outline

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Lecture IV Outline

### Screws Theory and Rigid Body Transformations.

Screws (properly revisited): Chasles' and Poinsot's theorem; Displacement and Force screws; Plücker coordinates.

Wrench; Instantaneous screw axis; Couple; Adjoint maps; Velocity transformations – in Body and Spatial Homogeneous Coordinates.

Group theory: The Lie algebra, motions in  $\mathfrak{se}(3)$ ;, and the Lie Group.

Manipulator kinematics: Brockett's exponential map formula. Paden-Kahan subproblems. Denavit-Hartenberg Conventions. short treatise on robots' kinematics

# Rigid Body Motions as Screws

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Rigid Body Motion as a Screw Motion

The motion of a **rigid body** is precisely the same as if it were attached to the **nut of a literal mechanical screw**. Associated with the screw is its pitch.

## Definition (Screw)

That straight line with which a **definite linear magnitude** termed the pitch is associated is called the **screw**.

# Screw as a Geometric Quantity

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Pitch of a Screw

The rectilinear distance through which (a literal nut) **nut** is translated parallel to the axis of a **screw**, while the nut is rotated through the **angular unit of circular measure** is termed the **pitch**.

## Plücker Coordinates

Let  $a$  be a point on line  $\ell_0$ . Let  $a$ 's direction cosine vector (to be introduced shortly) be  $b$ . Then, its binormal (moment) vector is  $c = a \times b$ . We say the pair  $(b, c)$  is the **Plücker Coordinates** of point  $a$  on axis  $\ell_0$ .

# Screw in Plücker Coordinates

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

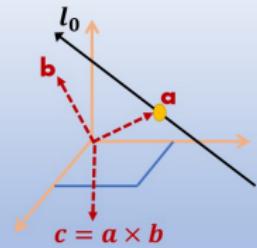
Screws in Plücker Coordinates

Group Theory

### Definition (Screw Coordinates)

Six-vector,  $s$ , related to the Plücker coordinates, parameterize a screw i.e.

$$s = (s_1, s_2, s_3, s_4, s_5, s_6).$$



# Screws and Plücker Coordinates

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Screw axis and Plücker Coordinates

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3; \quad (35)$$

$$c_1 = s_4 - p \cdot s_1, \quad c_2 = s_5 - p \cdot s_2, \quad c_3 = s_6 - p \cdot s_3. \quad (36)$$

## Pitch in Plücker Coordinates

$$p = \frac{s_1 s_4 + s_2 s_5 + s_3 s_6}{s_1^2 + s_2^2 + s_3^2}, \quad (37)$$

$$|s| = \sqrt{s_1^2 + s_2^2 + s_3^2} \quad \text{if } p \neq \infty, \quad (38)$$

$$|s| = \sqrt{s_4^2 + s_5^2 + s_6^2} \quad \text{if } p = \infty \quad (39)$$

# Pitch and Magnitude of the screw

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Plücker Coordinates' Direction Cosines

Suppose that  $h = \sqrt{b_1^2 + b_2^2 + b_3^2}$ . Then  $(b/h, c/h)$  are respectively the direction cosines of the line,  $l_0$  and its moment.

## Homogeneous Coordinates!

Plücker Coordinates give six unit parameters of a point on a line. Plücker Coordinates are in homogeneous coordinates!

# Twist About a Screw (Axis)

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Twist

A body's **twist** about a screw is a **uniform (infinitesimal) rotation** about the screw followed by a **uniform (infinitesimal) translation** about an axis parallel to the screw, through a distance that is the product of the pitch and the **circular measure of rotation**.

## Twist

A **twist** requires six algebraic quantities for its complete specification: five ( $\{t_i\}_{i=1}^5$ ) specify the screw, the sixth (or its amplitude) specifies the screw's rotaty angle,  $t_6$ .

# Twist in Plücker Coordinates

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Definition (Twist Coordinates)

A six-vector,  $t$ , related to the Plücker coordinates parameterize a twist vector i.e.  $t = [(t_1, t_2, t_3), (t_4, t_5, t_6)]$  or  $t = (\omega, v)$ , where  $\omega = (t_1, t_2, t_3)$  and  $v = (t_4, t_5, t_6)$ .

## Plücker Coordinates of a Twist

$$b_1 = t_1, \quad b_2 = t_2, \quad b_3 = t_3 \quad (40)$$

$$c_1 = t_4 - p \cdot s_1, \quad c_2 = t_5 - p \cdot s_2, \quad c_3 = t_6 - p \cdot s_3. \quad (41)$$

# Twists in Plücker Coordinates

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Pitch of the Twist

$$p_t = \frac{t_1 t_4 + t_2 t_5 + t_3 t_6}{t_1^2 + t_2^2 + t_3^2} = \frac{\omega \cdot v}{\omega \cdot \omega}.$$

## Pitch of the Twist

Expressed as a ratio of the magnitude of the velocity of a point on the twist axis to the magnitude of the angular velocity about the twist axis.

## Translation Distance

$d_t = t_6 \times p_t$ . The sign expresses the rotation's direction.

# Twists and Fixed Movements

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Pure Rotation

Let pitch be zero. That which results is but **pure rotation**.

## Pure Translation

Let pitch be infinite. That which results **cannot be a finite twist**, except the amplitude be zero, whereupon the twist becomes a pure translation parallel to the screw.

# Curvilinear Displacement: Serret-Frenet Frame

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

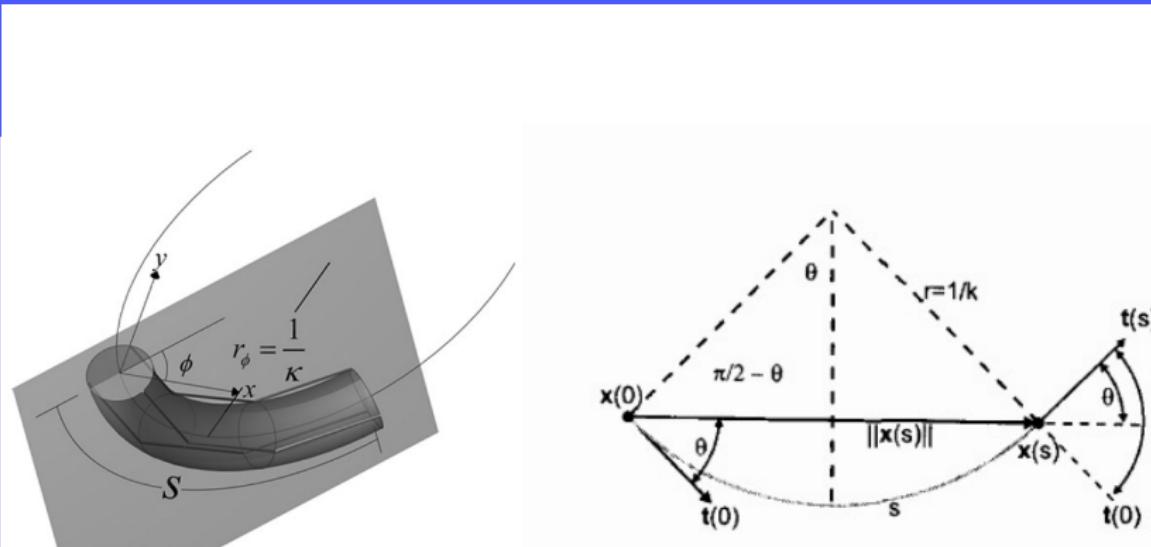
## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory



Elephant Trunk Multi-sectional Continuum Model (left), and its Representation in the Serret-Frenet Frame.

# Plücker Coordinates Example

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal

Properties

Composition of

Rotations

Screw Theory

Displacement &

Twist

Force & Wrench

Screws in Plücker

Coordinates

Group Theory

## Chasles' Theorem Applied to The Serret-Frenet Frame

Consider a spatial curve  $S$  on the elephant continuum trunk shown earlier. Suppose  $S$  is parameterized by its arc length  $s \in [0, 1]$ . For a point  $x = [x, y, z]^T$  on  $S$ , the unit tangent vector at  $s$  is  $t(s) = dx/ds$ .

## Differential Kinematics and The Serret-Frenet Frame

Denote by  $n$  the principal normal to  $S$  at  $n$ ; then we must have  $b = t \times n$  as the binormal. We say  $(b, n)$  is the Plücker coordinate of the tangent  $t$ .

# Force

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

## Screw Theory

Displacement &  
Twist

## Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Force

Net force exerted on a body,  $\mathbf{F} = (f_x, f_y, f_z)$ .

## Couple of Force

Suppose that  $\mathbf{F}$  acts along a corkscrew axis. The resulting motion when  $\mathbf{F}$  makes an infinitesimal rotation about its screw axis is called its **couple**,  $\mathfrak{C} = (c_x, c_y, c_z)$ .

# Complete Wrench on a Screw

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Wrench

A wrench requires six algebraic quantities for its complete specification: five ( $\{w_i\}_{i=1}^5$ ) specify the screw, the sixth (or its intensity),  $w_6$ , specifies the force's magnitude.

## Couple's Moment

The moment of the couple is the product of the intensity of the wrench and the screw's pitch i.e.

$$\alpha(\mathcal{C}) = w_6 \times p_w.$$

# Wrench on a Screw

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Wrench

Simple Definition: A **force** and a **couple** both acting in a plane perpendicular to the force.

## Definition (Complete Definition)

The **resultant canonical system of forces** acting on a rigid body, **reduced to a resultant force on a point**, and acting along the **resultant couple** that is **perpendicular to the plane** in which the force acts is called **the wrench**.

# Wrench in Plücker Coordinates

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Definition (Wrench Coordinates)

A six-vector,  $w$ , related to the Plücker coordinates parameterize a wrench vector i.e.

$w = [(w_1, w_2, w_3), (w_4, w_5, w_6)]$  or  $w = (f, m)$ , where  $f = (w_1, w_2, w_3)$  and  $m = (w_4, w_5, w_6)$ .

## Plücker Coordinates of a Wrench

$$b_1 = w_1, \quad b_2 = w_2, \quad b_3 = w_3 \quad (42)$$

$$c_1 = w_4 - p \cdot s_1, \quad c_2 = w_5 - p \cdot s_2, \quad c_3 = w_6 - p \cdot w_3. \quad (43)$$

# Wrench in Plücker Coordinates

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Pitch of the Wrench

$$p_t = \frac{w_1 w_4 + w_2 w_5 + w_3 w_6}{w_1^2 + w_2^2 + w_3^2} = \frac{\mathbf{f} \cdot \mathbf{m}}{\mathbf{f} \cdot \mathbf{f}}.$$

## Pitch of the Wrench

Expressed as a ratio of the moment applied about a point on the axis to the magnitude of the force applied along the wrench axis.

## Wrench's Magnitude

$$\|\mathbf{f}\| = \sqrt{w_1^2 + w_2^2 + w_3^2} \text{ if } p_w = 0 \text{ else}$$
$$\|\mathbf{m}\| = \sqrt{w_4^2 + w_5^2 + w_6^2} \text{ if } p_w = \infty.$$

# Wrenches and Fixed Movements

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Pure Force

Let pitch be zero. That which results is **pure force** along its screw axis.

## Pure Couple

Let pitch be infinite. That which results **cannot be a finite wrench**, except the intensity be zero, whereupon the wrench becomes a pure couple in a plane that is perpendicular to the screw.

# Statics and Instantaneous Kinematics

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Statics and kinematics

Statics	Instantaneous Kinematics
Force, $\mathbf{F}$ about $n$ .	Infinitesimal rotation, $\omega$
Couple, $\mathfrak{C}$ : $[\mathbf{F}] \times [\ell]$	Infinitesimal translation, $t$
$p_w = \pm \mathfrak{C}/\mathbf{F}$	Pitch of a Wrench, $w$
$ \mathbf{F} $	Intensity of Wrench

Dyname:  $(\mathbf{F}, \mathfrak{C})$ . Credits: Plücker (1866), Routh (1892).

# Plücker Coordinates Kinetics Quiz

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force  $\mathbf{F}$  acts at the point  $a$  in the image of Frame 120. What are the Plücker coordinates of the **line of force**?

# A short treatise on robots' kinematic geometry and kinetics.

## Screw Theory

### Screws in Plücker Coordinates

#### Plücker Coordinates Kinetics Quiz

Poinsot's Theorem Quiz on a Force and its Moment  
Suppose that a force  $\mathbf{F}$  acts at the point  $a$  in the image of Frame 120. What are the Plücker coordinates of the line of force?

## Poinsot's Theorem Quiz on a Force and its Moment

Imagine that a force  $\mathbf{F}$  is acting at the point  $a$  in the image of Frame 120. Suppose that  $\tau$  is torque acting along the normal to point  $a$ . Then  $(f, \tau)$  are the Plücker coordinates of the line of force.

## Arithmetics on Screws

Scalar and vector arithmetic operations are valid on infinitesimal screws e.g.

$$c_1 s_1 + c_2 s_2 = 0 \text{ for } c_1, c_2 \neq 0 \text{ on screws } s_1, s_2. \quad (44)$$

# Plücker Coordinates Kinetics Quiz

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force  $\mathbf{F}$  acts at the point  $a$  in the image of Frame 120. What are the Plücker coordinates of the **line of force**?

# Group Theory Review

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## The Euclidean Motion

Let  $\mathbb{E}^3$  denote the ordinary Cartesian 3-space that admits the standard inner product

$$\langle x, y \rangle = \sum_i x_i y_i. \quad (45)$$

## Transformations

The set of all length-preserving transformations in  $\mathbb{E}^3$  shall be denoted by  $\mathbb{E}(3) \in \mathbb{R}^6$  i.e., the family of translations and rotations<sup>a</sup>.

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<sup>a</sup>Rotations in  $\mathbb{E}^3$  are not necessarily proper.

# Group Transformation Isomorphism

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

Brockett, 1990

Euclidean transformation under group composition and  
Euclidean transformation under group multiplication preserve  
the isomorphic property.

Example: Affine Euclidean Transformations

$q$  defines a Euclidean affine transformation  $q = Rx + d$  if  $\langle R, R^T \rangle = I$  for  $(q, d) \in \mathbb{R}^3$ . Now, suppose  $q = R_1x + d_1$  and  $p = R_2q + d_2$ , then  $p = R_2R_1x + d_2$ .

# Group Transformation Isomorphism

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Example: Euclidean Transformation Identity

$$\begin{pmatrix} R_2 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2 R_1 & R_2 d_1 + d_2 \\ 0 & 1 \end{pmatrix} \quad (46)$$

## The isomorphic property (Brockett, 1990)

That matrices of the form (SE(3) matrices):  $\begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix}$  are isomorphic.

# The General Linear Group

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## $SO(3)$ as a General Linear Group

The special orthogonal group,  $SO(3)$ , is a subgroup of the general linear group

$$SO(3) = \{\mathbf{R} \in GL(n, \mathbb{R}) : \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = 1\}. \quad (47)$$

# The Lie Group

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## The Lie Group

A group with a topology operation on its set of elements such that the group can be given the structure of a differential manifold with the property that group multiplication and inversion is continuous is called a Lie group.

## The Special Euclidean Matrix Group, $SE(3)$

$SE(3)$  is a differentiable manifold, comprised of all the translations and proper rotations that moves a body from one point to another in the ordinary cartesian 3-space  $E^3$ .

# The Lie Group

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## The Special Euclidean Matrix Group, $SE(3)$

$$g = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}; g \in SE(3). \quad (48)$$

# The Lie Group

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Special Euclidean Matrix Group, $SE(3)$

$$SE(3) = \{(\mathbf{R}, d) : \mathbf{R} \in SO(3), d \in \mathbb{R}^3\} := SO(3) \times \mathbb{R}^3. \quad (49)$$

I have followed Chasles' notation, who posited that any rigid motion can be formed via a rotation, followed by a translation, and that the rotation and the translation commute i.e.  $\mathbf{R}d = d$ .

## The Special Euclidean Matrix Group, $SE(3)$

Note: Most authors' notation follow Euclid's theorem i.e. any rigid motion is a translation followed by a rotation about an axis that passes through a pre-specified (fixed) point.

$$SE(3) = \{(d, \mathbf{R}) : d \in \mathbb{R}^3, \mathbf{R} \in SO(3)\} := \mathbb{R}^3 \times SO(3). \quad (50)$$

# The Lie Group

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Special Euclidean Matrix Group, $SE(3)$

Chasles' notation allows for motion representation in form of screw motions.

## Commutativity of group operations on $SE(3)$

[Brockett, 1990]: Equation (49) imply that the Lie group is a semidirect product of simple Lie subgroup of orthogonal transformations and the abelian Lie subgroup of all translations.

# The Lie Algebra

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Lie Algebra, $\mathfrak{se}(3)$

The Lie algebra is a vector space  $\hat{\xi}$  with the antisymmetric bilinear operation  $[,] : \hat{\xi} \times \hat{\xi} \rightarrow \hat{\xi}$  which satisfies the Jacobi identity,

$$[\hat{\xi}_1, [\hat{\xi}_2, \hat{\xi}_3]] + [\hat{\xi}_2, [\hat{\xi}_3, \hat{\xi}_1]] + [\hat{\xi}_3, [\hat{\xi}_1, \hat{\xi}_2]] = 0. \quad (51)$$

NB:  $[,]$  is alternatively the Lie bracket notation with antisymmetry operation  $[\hat{\xi}_2, \hat{\xi}_3] = -[\hat{\xi}_3, \hat{\xi}_2]$ .

# The Lie Algebra Representation

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Lie Algebra Representation, $\mathfrak{se}(3)$

The Lie algebra admits the following homogeneous coordinates representation for a point  $q \in \mathbb{R}^3$  on a link that rotates with unit velocity  $\omega$ ,

$$\hat{\xi} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3), \quad \xi = (\omega^T, v^T)^T \in \mathbb{R}^6 \quad (52)$$

where  $v = -\omega \times q$ .

# The Lie Algebra Representation

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## The Lie Algebra Representation, $\mathfrak{se}(3)$

Observe:

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \equiv -\tilde{\omega}^T \in \mathfrak{so}(3) \quad (53)$$

is the **skew-symmetric form of the velocity of the tip point**,  
 $\omega \in \mathbb{R}^3$ .

# The Lie Algebra Diffeomorphisms

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## Lie Representation Snippet

Observe:

$$(\tilde{\cdot})_{SO(3)} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3) \quad (54)$$

$$(\tilde{\cdot})_{SE(3)} : \mathbb{R}^6 \rightarrow \mathfrak{se}(3) \quad (55)$$

$\tilde{\omega}(S) \in \mathfrak{se}(3)$ : e.g. **Twist parameterization** of a curve, deformation, **screw**.

$\omega(S) \in \mathbb{R}^6$ : e.g. **Motion vector** e.g. linear + angular velocities, axial, shear, bending, and torsion motion.

# The exponential map belongs to the Lie Group

Introduction

Textbooks

Kinematic

Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and

Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and

Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

The exponential map,  $\exp(\mathfrak{se}(3))$ , is an element of  $SE(3)$

Given  $g : \begin{pmatrix} \mathbf{R}(\theta) & \mathbf{d} \\ 0 & 1 \end{pmatrix} \in SE(3)$  there exists a

$\tilde{\xi} = (\tilde{\omega}, v) \in \mathfrak{se}(3)$ , such that  $\exp(\tilde{\xi}\theta) \in SE(3)$ <sup>a</sup>.

---

<sup>a</sup>Proof in Murray and Sastry, Prop 2.8.

The exponential map,  $\exp(\mathfrak{se}(3))$ , is surjective onto  $SE(3)$

Given  $g : \begin{pmatrix} \mathbf{R}(\theta) & \mathbf{d} \\ 0 & 1 \end{pmatrix} \in SE(3)$  there exists a

$\begin{pmatrix} \tilde{\omega} & \mathbf{d} \\ 0 & 0 \end{pmatrix}$ ;  $\tilde{\omega} = -\tilde{\omega}^T$ , such that  $\exp(\tilde{\omega}) = g$ <sup>a</sup>.

---

<sup>a</sup>Proof in Murray and Sastry, Prop 2.9.

# Chasles and Affine Transformations

## Introduction

Textbooks

## Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

## Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

## Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

## Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

## Group Theory

## Chasles Theorem and Affine Transformations

$$\begin{pmatrix} \mathbf{R} & \mathbf{d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{d} \\ 0 & 1 \end{pmatrix} \quad (56)$$

with  $\mathbf{R}\mathbf{d} = \mathbf{d}$ . Note  $\langle \mathbf{c}, \mathbf{d} \rangle = 0$  for  $\mathbf{c}$  and  $\mathbf{d}$  to be unique.

# Screw Motion and Exponential Map

Introduction

Textbooks

Kinematic  
Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and  
Parallel robots

Mobility

Screws and a  
History Snippet

Freedoms and  
Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal  
Properties

Composition of  
Rotations

Screw Theory

Displacement &  
Twist

Force & Wrench

Screws in Plücker  
Coordinates

Group Theory

## Screw Motion and Exponential Map (Brockett, 1990)

Range and null space of a  $\tilde{\omega}$  are orthogonal. Thus,

$$\begin{pmatrix} I & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{\omega} & d - \tilde{\omega}c \\ 0 & 0 \end{pmatrix} \quad (57)$$

establishes that every motion of the form  $\begin{pmatrix} \tilde{\omega} & d \\ 0 & 0 \end{pmatrix} \theta$  is a screw motion w.r.t some origin.

# Group Composition and Screws Connection

Introduction

Textbooks

Kinematic Geometry

Mechanics

Mechanisms

Pairs and Linkages

Serial Chains

Hyperredundant and Parallel robots

Mobility

Screws and a

History Snippet

Freedoms and Constraints

Mobility Criterion

Motions

Movement in  $\mathbb{R}^3$

Special Orthogonal Properties

Composition of Rotations

Screw Theory

Displacement & Twist

Force & Wrench

Screws in Plücker Coordinates

Group Theory

## The Lie Algebra Representation, $\mathfrak{se}(3)$

Observe:

$$(\tilde{\cdot})_{SO(3)} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3) \quad (58)$$

$$(\tilde{\cdot})_{SE(3)} : \mathbb{R}^6 \rightarrow \mathfrak{se}(3) \quad (59)$$

$\tilde{\omega}(S) \in \mathfrak{se}(3)$ : **Twist parameterization** of a curve, deformation, **screw**.

$\omega(S) \in \mathbb{R}^6$ : **Motion vector** e.g. linear + angular velocities, axial, shear, bending, and torsion motion.