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A short treatise on robots' kinematic geometry and kinetics.

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Dissemination Venue: Microsoft Research RL Group,  
New York City, NY 10012

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## Robot Modeling and Control

Spong, Mark W., Seth Hutchinson, and Mathukumalli Vidyasagar. Robot modeling and control. Vol. 3. New York: Wiley, 2006.

## Mathematical Modeling of Robots

Murray, R. M., Li, Z., & Sastry, S. S. (1994). A Mathematical Introduction to Robotic Manipulation. In Book (Vol. 29). <https://doi.org/10.1.1.169.3957>

# Texts – Modeling, Control, and Mechanisms

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## Robot Modeling and Control

Lynch, K. M., & Park, F. C. (2017). Modern Robotics  
Mechanics, Planning, and Control.

## Mechanisms' Kinematic Geometry

Hunt, Kenneth H., and Kenneth Henderson Hunt.  
Kinematic geometry of mechanisms. Vol. 7. Oxford  
University Press, USA, 1978.

# Texts – Screws and Kinematics

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## Screw Theory

Ball, Robert Stawell. A Treatise on the Theory of Screws. Cambridge university press, 1998.

## Mechanisms' Kinematic Geometry

Hunt, K. H. (2019). Structural Kinematics of In-Parallel-Actuated Robot-Arms. 105(December 1983), 705–712.

# Lecture One Outline

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## Mechanism Components

Kinematic geometry. Mechanisms.

Joints: Joint closure; Pairs; Couplings.

Lower pairs and linkages; Higher and lower pairs.

Motions: Planar and spherical motions.

Synthesis: Type-, number-, and size-syntheses.

# Preamble.

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## Mechanics

Mechanics is an indirect study of nature via **bodies** – essentially the mathematical abstractions of common natural things; the **mass** is an *allocation in place* to each body.

## Geometry

Geometry, deals with the **theory of places**; geometry is the bedrock of **robotics, control theory**, and many fields of **modern engineering and the physical sciences**.

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## Definition (Motion)

When a **place** undergoes **body transformation** in the course of **time**, we have **motion**.

# Preamble – Mass, Body, Rigid Body Motion.

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## Definition (Body – Truesdell, 1977.)

By a **body**, we shall mean the **closure of an open set** in some **measure space**  $\Omega$  over which a **non-negative measure  $M$** , called the **mass**, is defined, and that  $M$  can be extended to a Borel measure over the  $\sigma-$  algebra of Borel sets in  $\Omega$ .

# Preamble – Mass, Body, Rigid Body Motion.

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## Bodies – Truesdell, 1977.

That in **mechanics** which deals with

- (i) **mass points**, which occupy a single point at any one time;
- (ii) **rigid bodies**, which never deform;
- (iii) **strings and rods and jets**, which are 1-dimensional; membranes and shells, that sweep out surfaces;
- (iv) **space-filling fluids and solids** e.t.c. are termed **bodies**.

# Statics, Dynamics, Rigid Body (Motion).

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## Statics and Dynamics

That which studies **putative equilibria** is referred to as **statics**. That which concerns motion of all sorts is referred to as **dynamics**. The dynamics that are specific to **particular bodies** are termed **constitutive**.

# Statics, Dynamics, Rigid Body (Motion).

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## The Rigid Body

A rigid body does not stretch, buckle, contract, bend, twist, nor deform. Well, not really!

## The Rigid Body

As engineers and roboticists, we judge kinematic rigid hardware with the expectation that kinematic changes do not depart from rigid-body predictions.

# Statics, Dynamics, Rigid Body (Motion).

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## The Rigid Body

We expect that localized stresses, active noise, vibrations and heat e.t.c will not cause reasonable departures from expectations.

## Rigid Body Motion

That motion that preserves distance between all points in a body is termed a rigid body motion.

# Statics, Dynamics, Rigid Body (Motion).

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## Rigid Body Motion

At issue are components of a rigid body's **movement** w.r.t to a fixed or moving **frame of reference**. In its most basic form, this movement is parameterized by **displacement** (and is sometimes time-varying e.g. for a continuum body). When solving for the movements of bodies, it is often useful to include velocities (**twists**) in order to characterize the **motion**.

# Kinematics vs. Kinetics

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## Dynamics

$$\dot{x} = f(t; x, u), \quad x(t_0) = t_0 \quad (1)$$

$$\dot{x} = f(t; x) + g(t; x, u), \quad x(t_0) = t_0 \quad (2)$$

## Definition (Kinematics.)

**Kinematics** is the English version of the word *cinématique* coined by A.M. Ampère (1775-1836), who translated it from the Greek word *κίνημα*.

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## Definition (Truesdell)

That part of a system's **dynamics** that involves its **motion** by **displacement** – both linear and angular – and **separated from motions owing to forces and torques**, together with the successive derivatives with respect to time of all such displacements (this includes **velocities**, **accelerations**, and **hyper accelerations**) all form the **kinematics** of a **rigid**, **continuum** or **laminae** of bodies.

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## Kinetics

The motion of bodies can also be conceived as resulting from the forces' action. Energy, temperature, and calory of a body are resultant effects of gains or loss of heat. Motions arising as a result of these are called kinetics.

# Kinematics vs. Kinetics.

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## Definition (Kinetics – Technical Definition)

That part of a system's **dynamics** that involves its **motion** by **forces, energy, torque, inertia, dynamic stability, and equilibrium** and similar properties all form the **kinetics** of a rigid, continuum or laminae of bodies.

# Kinematic Geometry.

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## Definition (Kinematic Geometry)

The solid geometry of relatively moving rigid bodies is termed the **kinematic geometry** of the rigid body. With motion, we'd have to include the successive derivatives of the displacement such as acceleration e.t.c as the 'laws of motion' stipulates in mechanics.

# Joints and Links

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## Links

Links may be rigid mechanical parts, elastic, (vulcanized) rubber components, diaphragms, conveyor belts, spring-damper systems e.t.c.

## An Elementary Joint or Kinematic Pair.

An elementary joint or a kinematic pair consists of touching two links together at one point – then ensuring a single contact point is continuously maintained throughout relative movement.

# Joint (Contact) Kinematics

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## Contact Kinematics

A body may **slide** or **slip** across a **plane** or **surface**, or **roll** over another body.

## Joints

**Joints** are the result of the **connecting points** between two or more **rigid bodies**.

# Definition of a Mechanism

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## Definition (Author's Definition)

A **connection** of **mechanical, magnetic, electrical, hydraulic, or pneumatic components** forming an **assemblage**, meant for moving **rigid, semi-rigid or non-rigid bodies** via a **controlled generation** of (sometimes constrained) **motion**.

Kenneth Hunt (1978)

A means of **transmitting**, **controlling**, or **constraining** the relative movement between parts. Whenever we have an **higher pair** or more, we have a mechanism.

# Mechanism Examples

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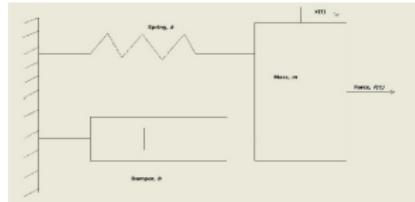
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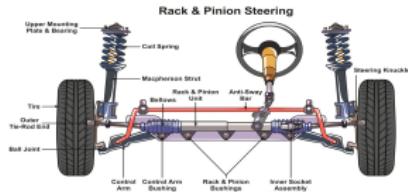
## Spring-Mass-Damper System



## Excavator



## Car suspension



## Daimler Plant



# Lower Pairs, Higher Pairs, Linkages

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## Lower and Higher Pairs

When elements of pairs touch one another over a **substantial region of a surface covering a line, curve-surface, or point of contact**, we have **lower pairs**. When they touch **along a discrete line, curve-surface, or point of contact**, we have **higher pairs**.

## Linkage (Hunt, 1978)

If all joints of a **mechanism or mechanical movement** belong to lower pairs, we have a **linkage**.

# Prismatic Pairs or *P*-pairs

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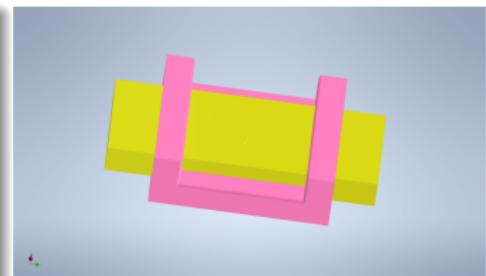
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Hunt, 1978

Formed by receding the axis of the revolution surface between two pairs to  $\infty$  so that the **curve** that produces the surface moves parallel to itself, **tracing a cylinder**; or a **polygonal-tracing curve** generates a **prism**.



# Revolute Pairs or *R*-pairs

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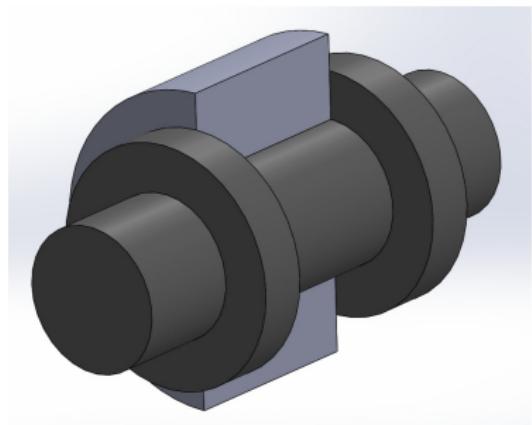
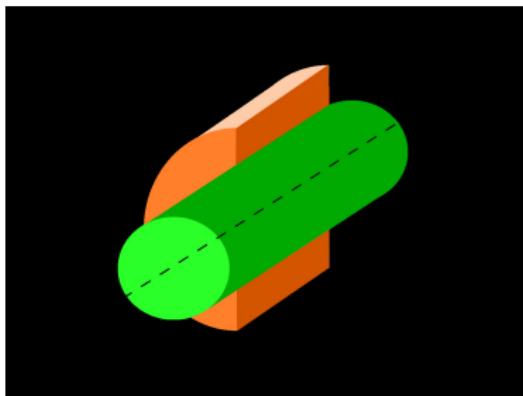
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One convex surface and one non-convex surface for a one degree of rotational freedom around the one joint the two surfaces make.



Revolute or Hinge or Turning or simply *R*-pairs with and without shoulder cutaway geometries. Credit: Wikimedia commons.

# Helical- & U-Joints

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Helical Joint

©McMaster Carr, May 2022.



Universal Joint

# Common Lower Kinematic Pairs

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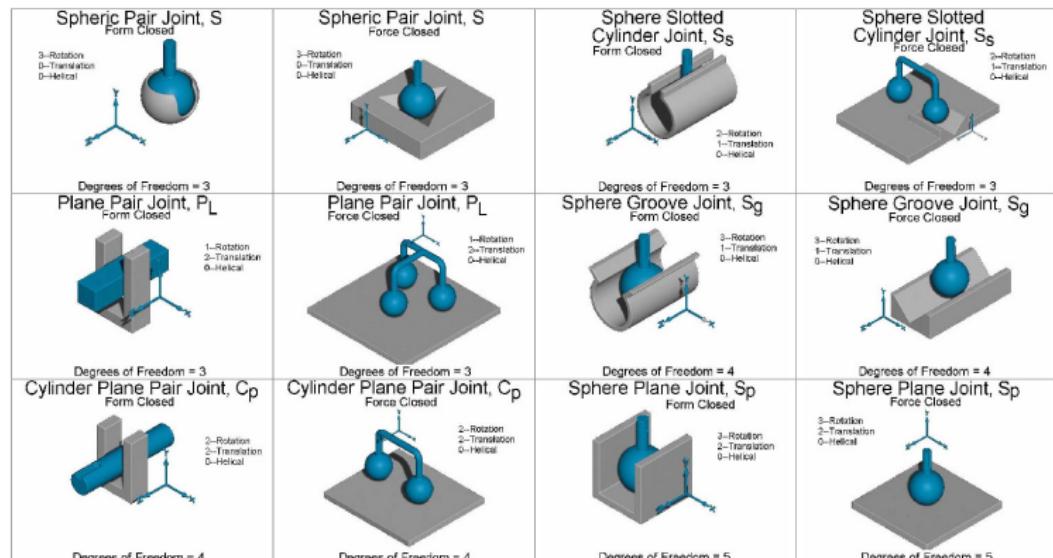
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Credit: Wharton and Singh, 2001.

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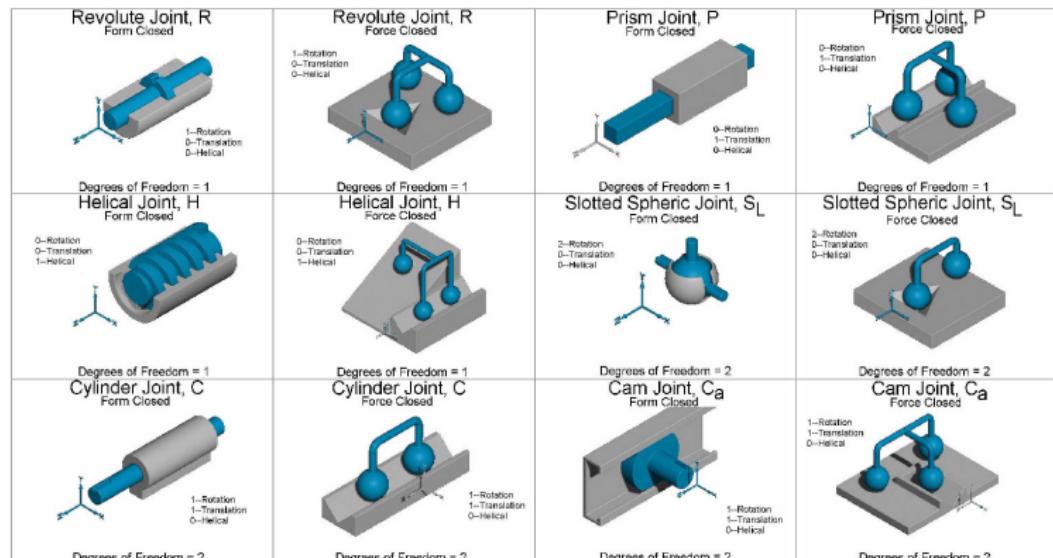
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Credit: Wharton and Singh, 2001.

# Kinematic Geometry of Common Actuations

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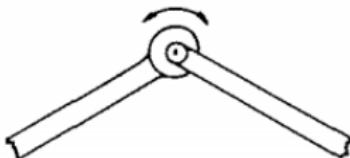
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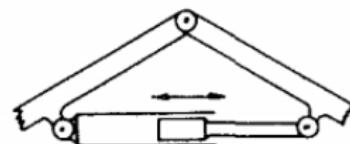
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## In-series vs. Parallel-actuated lower pairs



(a)



(b)

(a): In-series-actuated kinematic pair with a rotary joint that is actuated “about” the hinge. (b): Prismatic joint actuated “across” a hinge. Reprinted from Hunt, Kenneth. Structural Kinematics of In-Parallel-Actuated Robot Arms. Transactions of ASME. 1983.

# Kinematic Chains

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## Kinematic Chains (Reuleaux, 1975)

We can explain the structural similarity of many mechanisms  
by parts of **kinematic chains** connected by pairs.

## Kinematic chains

Kinematic chains are essentially the basic building structure  
of mechanisms ... and robots!

# Open Kinematic Chains

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## Chains

Open kinematic chains are based off the anthropomorphic construction of the human hand with cantilevered beam structures.

## Chain Mechanisms and Error Amplification

Amplifies errors from waist (or base frame) all the way to the tool frame. Control difficult.

## Control

Feedforward control: High power and precision hydraulic actuators for servo motors.

Sensory feedback control: Force sensing (Ernst, 1962).

# A short treatise on robots' kinematic geometry and kinetics.

- └ Pairs, Linkages, and Configurations
  - └ Serial Chains
    - └ Open Kinematic Chains

**Chains**  
Open kinematic chains are based off the anthropomorphic construction of the human hand with cantilevered beam structures.

**Chain Mechanisms and Error Amplification**  
Amplifies errors from wrist (or base frame) all the way to the tool frame. Control difficult.

**Control**  
Feedforward control: High power and precision hydraulic actuators for servo motors.  
Sensory feedback control: Force sensing (Ernst, 1962).

The PUMA arm is the world's first serial kinematic chain. Developer: Victor Scheinman, Stanford student in the '50's. Made several iterations. Patent Rights: Joe Engelberger, (Danbury Unimation, 1961). Joe – father of robotics – created world's first robotics company in '61.

# Open Kinematic Lower Pairs

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## Definition (Ken Salisbury Jr., 1982)

*“[Robots are] our fascination with constructing mechanical analogues of ourselves... [this fascination] has led us to place all sorts of hopes and expectations in robot capabilities.”*



The PUMA Robot  
(1956).



The Stanford Arm  
(Infolab 1969).

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Open kinematic chains provide unstructured environmental interaction.

Project MAC, MIT.

Tomovic and Boni's pressure sensed grasp.

Binary robot vision system (McCarthy et al, 1963).

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Stanford Manipulator.

Boston arm.

The AMF (American Machines and Foundry) arm.

General electric's walking robot (1969).

# Long Walk Towards Direct Drive Robot Arms

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The 50's, 60's and 70's witnessed use of hydraulics for (feedforward) position control.

For feedback control, force sensors and pressure sensors were used in closed-loop scenarios.

Electrical actuation meant that robots had to be operated at high speeds. Needs for gear reduction for safe operations at low speeds.

With gear reduction came backlash, friction, and associated expenses.

# A short treatise on robots' kinematic geometry and kinetics.

## └ Pairs, Linkages, and Configurations

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The 50's, 60's and 70's witnessed use of hydraulics for (feedforward) position control.

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CMU DD I/II Arms: Workspace is donut shaped. OD: 90cm; ID: 21.7cm;  $1.8m^2$  workspace area. Built by Harry Asada. Structural design similar to aircraft gimbal arm; Uses Samarium Cobalt rare earth magnet brushless DC motors on first 3 joints, and AlNiCo magnets on tip joints. No belts, transmissions making for faster transmitting of motions, less friction, low energy, low compliance. Each joint has complex AL housing which enables: (i) Control of geometrical relationships of bearing assembly; (ii) Control of servo components to bearing assembly; (iii) Controls of rotational axes to consecutive joints.

# Direct Drive Robot Mechanism: CMU DD I Arm

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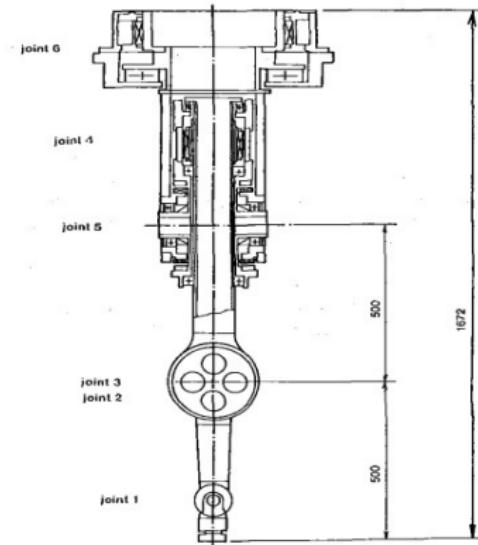
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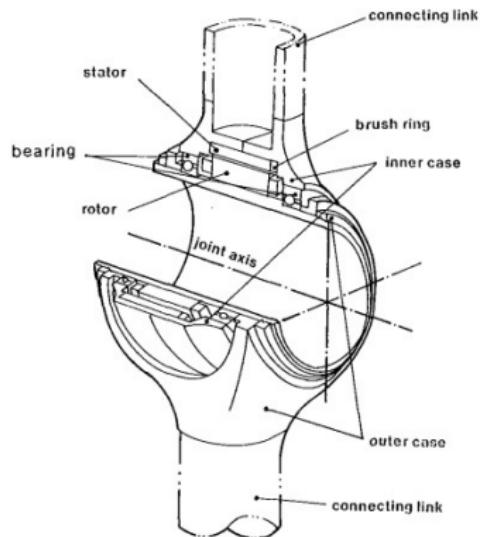
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Along came Harry Asada.



Arm Schematics Transmission



Joint schematic

# Direct Drive Robot Mechanism: CMU DD I Arm

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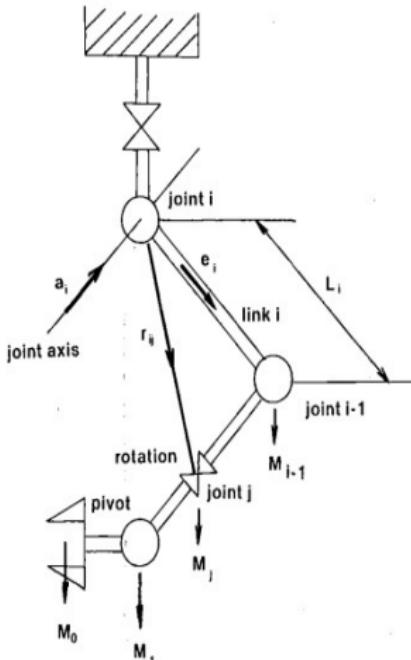
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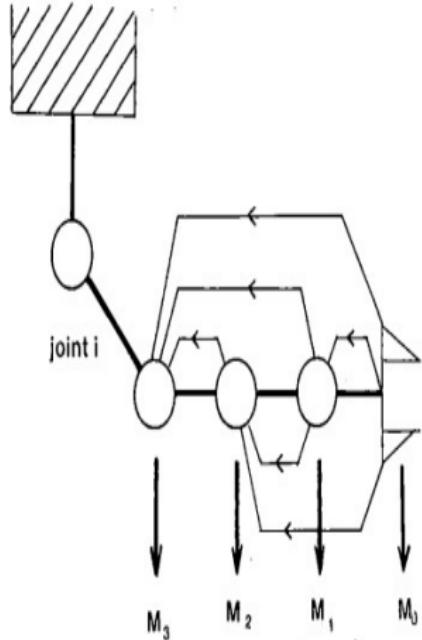
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Kinematic model



Errors Transmission

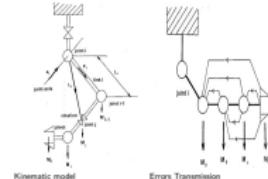
# A short treatise on robots' kinematic geometry and kinetics.

## └ Pairs, Linkages, and Configurations

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#### └ Direct Drive Robot Mechanism: CMU DD I

Arm



First direct-drive robot without a gearbox. Selective compliance in X-Y directions given its articulated jointed arms. One-freedom motion along Z direction given its constrained arm. New generations such as Cobra i600/i800 include power amplifiers, system and servo controls etc embedded in the robot's base. Kuka Scara arm: Lightweight, fast, powerful, low maintenance, energy consumption, investment costs etc.

# SCARA Robot Mechanisms

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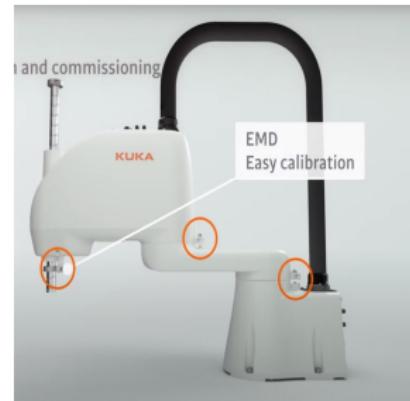
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The Adept One  
SCARA robot  
(Debutted 1984).



Kuka's SCARA  
arm, 2022.  
©Kuka Robotics

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## The Stäubli anthropomorphic arm.



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## Mechanisms in the 80's

With the 80's came the arrival of PCs. Lots of research went into computational algorithms for the kinematics and kinetics of (mostly) anthropomorphic robot arms.

### Active control schemes

Efficient recursive Lagrangian and computational methods for the gravitational and Coriolis forces in Newton-Euler equations.

# Serial mechanisms research in the 80's

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## Feedback Linearization

Dynamics feedback linearization for precise bounds on ma-  
nipulator performance.

## Automatix

Reconfigurable robots for various assembly ops.

# Serial mechanisms research in the 90's

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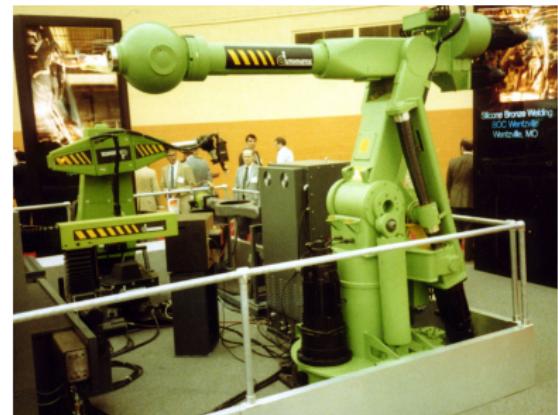
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## Robotworld

First industrial-scale reconfigurable robot and with machine vision components. RAIL scripting OS originally based on Motorola 68000, later on replaced by Apple Macintosh II.



©Wikipedia

# Hyper-redundant Continuum Robots

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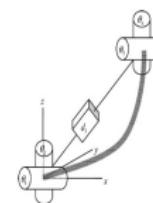
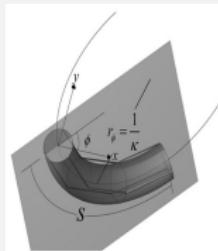
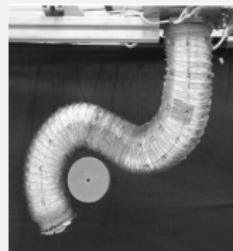
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The elephant trunk continuum robot. Jones & Walker, T-RO 2006.  
Inspiration: Muscular hydrostats in nature.

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An octopus-inspired soft robot. ©Cecilia Laschi.

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Mehlet 2015

A **parallel robot** is made up of an end-effector with  $n$  degrees of freedom, and of a fixed base, linked together by at least two independent kinematic chains. Actuation takes place through  $n$  simple actuators.

# Parallel mechanisms: Stewart-Gough Platforms

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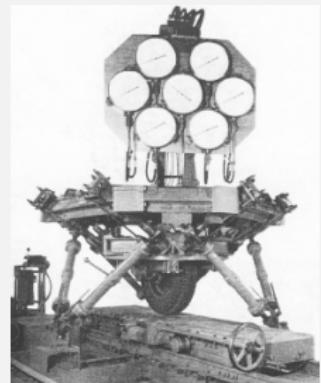
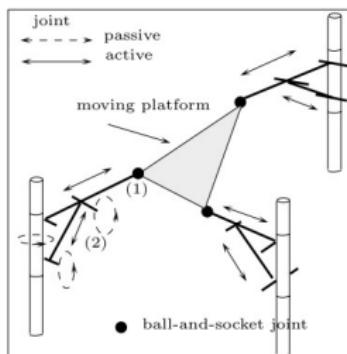
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Principles of a moving platform to test tyre wear and tear (Gough, 1947). Prototype, 1955.



Left: Stewart's 1965 mechanism. Right: The original 1954 octahedral hexapod proposed by Gough. Courtesy: [Parallemic.org](http://Parallemic.org).

# Truss Robots

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A multi-DOF Truss Robot. Courtesy of Penngineering (ICRA 2022,  
Philadelphia, PA).

# Closed kinematic chains

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Connection degree  $\geq 3$ .



A Stewart-Gough platform. SolidWorks Drawing Courtesy of Andrew Belcher. UChicago, 2018.

# A Soft Stewart Platform

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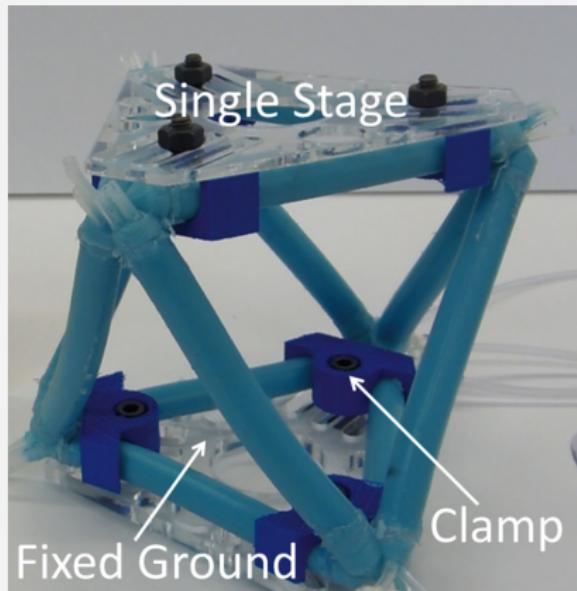
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A soft 6-6 Stewart manipulator. Jonathan Hopkins, 2015.

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## Freedom and Structure

### Freedoms, Constraints, and Mobility.

Motion of linkages: Screws and spatial motions.

Freedom and Mobility: Freedoms, unfreedoms, connectivity, mobility;

Grübler-Kutzbach's mobility criterion and examples;

Type-, size- and number-syntheses.

# Degrees of Freedom and Kinematic Structure

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## Definition (Connection Degree)

For any manipulator joint, we shall mean its connection degree to be the number of links attached it.

## Quiz

What is the connection degree of the u-joints of a Stewart-Gough platform.

# Members and Dual Graphs

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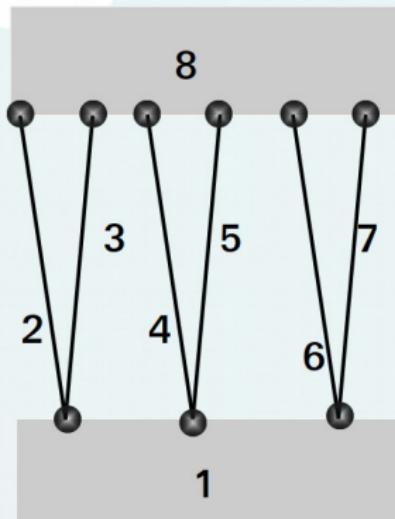
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## Dual graph of a Stewart platform



# Degrees of Freedom and Structure

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## Members and Freedoms

Degrees of freedoms (or freedoms) concerns the relative motion of members of a pair that do not touch one another directly.

## Connectivity

By the dual graph of the Stewart platform as seen on Frame 54, the total number of freedoms that connect the two members (1 and 8) that do not connect to one another directly is six.

# Planar Linkages

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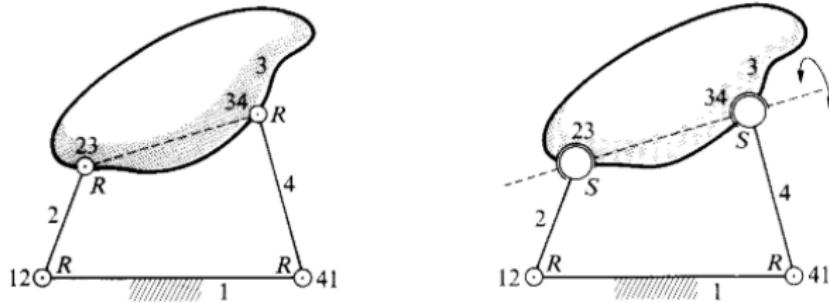
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## Four Bar Linkages

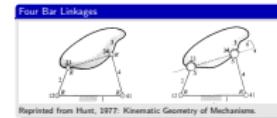


Reprinted from Hunt, 1977: Kinematic Geometry of Mechanisms.

# A short treatise on robots' kinematic geometry and kinetics.

- └ Mobility

- └ Planar Linkages



The planar  $RRRR$  linkage, (*left*) is modified in (*right*) to an  $RSSR$  linkage to allow spatial spin-movement of the coupler 3; the connectivity  $\mathcal{C}_{13} = 2$ .

# Freedom from Connectivity

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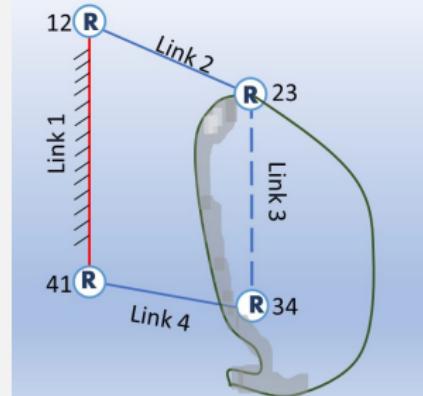
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## A (Hacked) Four-Bar Linkage



# A Four Bar Linkage

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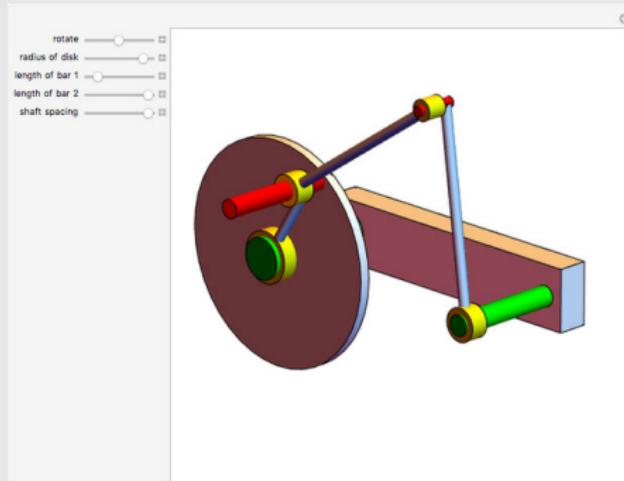
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## A Four Bar Linkage



Courtesy of Sándor Kabai, Wolfram Demonstrations Project, October 2007.

# Couplings and Freedom

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## Couplings and Freedom

Links 2&4 complete a **coupling or connection** between links 1&3.

## Connectivity

The  $R$ -pairs are said to have a **connectivity** of  $C_{ij} = 1$  for all  $i, j = 1, 2, 3, 4$ . Thus, total degree of freedom is 1.

# Mobility of Mechanisms

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## The Mobility and Relative Mobility, $\mathfrak{M}$

Simply put, the number of a mechanism's freedoms is its **mobility**, or **relative mobility**,  $\mathfrak{M}$ .

## The Mobility, $\mathfrak{M}$

It specifies the **independent variables** needed to **determine** every relative location of a **mechanism's members** with respect to one another.

## A Note on Serial and Parallel Mobility

A little tricky to determine for parallel mechanisms but straightforward for serial mechanisms.

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## Quiz

What is the mobility  $M$  of the *RSSR* four bar linkage of Frame 56? Why?

## Quiz

What is the mobility  $M$  of the *RRRR* four bar linkage of Frame 57? Why?

## Definition (The mobility criterion (well, not yet))

Let's not get ahead of ourselves. A little introduction to screws are in order for us to grasp the **Grübler-Kutzbach** mobility criterion.

# Rigid Body Displacements and Forces

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## Inhomogeneity of Displacements and Angles

Quiz: Three translations and three rotations are ill-posed for uniquely determining the freedoms of a body.  
Why?

They are **not homogeneous**.

For true **kinematic wholeness** and  
**generality**, displacement that is **purely**  
**translatory** and **purely rotary** is needed.

# Need for Unique Representations

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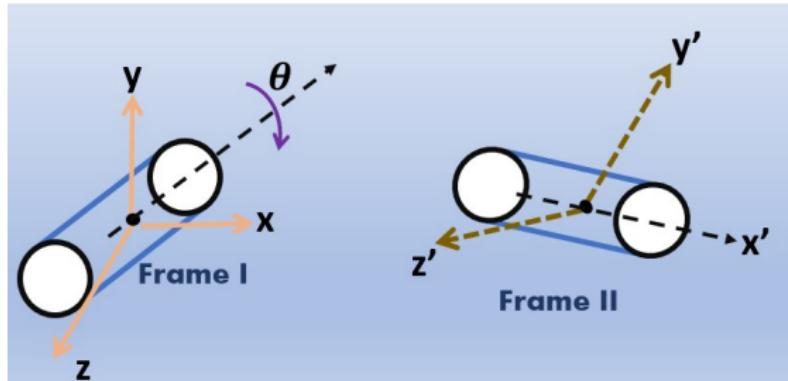
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There exists infinite possible ways of movement by which the point on the rigid body in Frame I can be effected to be transferred to the location of the point on Frame II and vice versa.

# Screws for Kinematic Generality

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## Need for Screws

From a kinematic standpoint, **six homogeneous screw coordinates** – each having an **independent screw freedom** – are needed to **uniquely determine a rigid body's location**.

## Definition (What is a screw anyway?)

A **screw** is a **straight line** in space, called **the axis**, with an associated direction, called **pitch**,  $p$ .



# Screws in Mechanics: A History Snippet

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## Screws Through Time

Michel Chasles(1793 – 1880): Rigid Body Displacements (Kinematics).

Louis Poinsot (1777 – 1859): Geometrical Mechanics (Kinetics).

Sir Robert Stawell Ball, F.R.S, LL.D.  
(1840-1913): Irish Astronomer who popularized screw theory in his day.

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## Screws Through Time

### The Death of Screw Theory

After World War I, interest in Screw theory declined. Several possible reasons for this.

- ▶ Ball died in 1913, he had no students.
- ▶ Other British/Irish mathematicians who might have carried these ideas forward died young.
  - ▶ Clifford, a contemporary and friend of Ball's died 1879 aged 33.
  - ▶ Charles Jasper Joly (1864–1906) studied under Helmholtz and Königs in Berlin, was a successor of Ball's as Royal Astronomer of Ireland.
  - ▶ Arthur Buchheim (1859–1888), studied under Klein at Leipzig. Taught at Manchester grammar school, died aged 29.
- ▶ Relativity became popular, and Euclidean geometry less so. Ball joked, "The Theory of Screws is now all done; it is quite obsolete; it is all going over into non-Euclidean space."

Kept alive in Soviet Union, Kotelnikov and others.

### Rediscovered by Mechanical Engineers

In the 1960s two mechanical engineers in Australia rediscovered Ball's work. Ball's theory of screws was just what they needed to study mechanisms.

- ▶ Kenneth Henderson Hunt (1920–2002) was born in the UK, worked at Monash University. "Kinematic Geometry of Mechanisms", first published in 1978. Applied screw theory to the problem of designing constant velocity joints.
- ▶ Jack Raymond Phillips (1923–2009) University of Sydney. Studied agricultural machinery, (trailed disc ploughs) and the mechanics of the lobster's claw. Two volume work "Freedom in Machinery: Introducing Screw Theory".

Courtesy of J.M. Selig, IROS 2018 Screw Theory Tutorial.

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## Simplest (Unique) Representation of Displacements

Chasles(1793 – 1880) showed that any given displacement of a rigid body can be uniquely represented as the rotation of the body about an axis (the screw axis) followed by a translation parallel to that axis (the pitch).

## Michel Chasles and Screws

Chasles called this unique transformation screws. Chasles is responsible for the Euclidean description of the motion of a rigid body in space and he made lasting contributions to theories of rigid body dynamics.

# Screws for Kinetic Generality

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## Simplest (Unique) Representation of Forces

[Poinsot](#)(1777 – 1859) showed that **any system of forces** acting on a rigid body can be represented by a **single force**, together with **a couple** acting along the normal to the force **in a plane**.

## Louis Poinsot and Geometrical Mechanics

“Everyone makes for himself a clear idea of the motion of a point, that is to say, of the motion of a corpuscle which one supposes to be infinitely small, and which one reduces by thought in some way to a mathematical point.” ~ Louis Poinsot, 1834.

# Freedoms, Unfreedoms, and Mobility

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## Freedom and Constraints

Suppose a screw  $f = (f_1, \dots, f_6)$  “fixes” a body in 3D space.

Each **constraint**  $u_i \neq f_j$  for  $(i, j) \in \{1, \dots, 6\}$ .

Rather each  $u_i$  has influence on every  $\{f_i\}_{i=1}^6$ .

Each  $u_i$  from the six independent equations,  
 $g(s_1, s_2, s_3, s_4, s_5, s_6) = 0$ , suppresses a  
**freedom**,  $f_i$ .

Progressively relaxing each  $u_i$ , or **unfreedom**,  
adds an extra body  $f_i$ .

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## Freedom and Unfreedoms

Suppose the total **freedoms** is  $f$  and the total **unfreedoms** is  $u$ , then

$$u + f = 6.$$

Note: A rigid body's freedoms is also referred to the dimension of its **configuration space**.

## Relative Freedoms

Suppose there are a total of  $n$  **unconstrained** bodies.

Suppose further that we choose one out of the bodies as a reference body. Then the total number of **relative freedoms** is  $6(n - 1)$ .

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## Constraints and Joints

Now, consider  $k$  independent constraints<sup>a</sup> such as joints along points, lines, curves or surfaces.

<sup>a</sup>NB: The total allowable constraints is 5 for a body in relative motion. 6 for a fully rigid body.

## The Mobility Criterion

Let the constraint of joint,  $i$  (e.g. a joint along points, lines, curves or surfaces) be  $u_i$ . Then the mobility criterion  $\mathfrak{M}$  is

$$\mathfrak{M} = 6(n - 1) - \sum_{i=1}^k u_i. \quad (3)$$

# General Grübler-Kutzbach Mobility Criterion

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## General Grübler-Kutzbach Mobility Criterion

Recall that  $\sum_i u_i + f_i = 6$  from Frame (70) so that

$$\mathfrak{M} = 6(n - k - 1) - \sum_{i=1}^f f_i. \quad (4)$$

## Exceptions: Relative Planar and Spherical Motions

For bodies restricted to relative planar or spherical motions, the total freedoms + constraints is 3 (not 6)!

$$\mathfrak{M} = 3(n - k - 1) - \sum_{i=1}^f f_i. \quad (5)$$

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## The Grübler-Kutzbach Mobility Criterion References

Attributed to Grübler:

Schoenflies, Arthur, and M. Grübler. "Kinematik." In Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, pp. 190-278.

Vieweg+Teubner Verlag, Wiesbaden, 1908;

Grübler, Martin Fürchtegott. Getriebelehre: eine Theorie des Zwanglaufes und der ebenen Mechanismen. Springer, 1917.

and Kutzbach:

Kutzbach, Karl. "Mechanische leitungsverzweigung, ihre gesetze und anwendungen." Maschinenbau 8, no. 21 (1929): 710-716.

# Loops

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## Loops

A kinematic chain often comprises members called loops.

## Binary Link

Members in a binary link constitute a single loop. Example:  
The four-bar linkage.

## Single loops

For single loops,  $k = n$  so that  $\mathfrak{M} = \sum_{i=1}^f f_i - 6$ .

## Mobility of Mechanisms

$\mathfrak{M} \leq 1$  for at least one actuator-pair to produce mobility at a successor joint which depends on that actuator-pair's input.

# Mobility of Common Robot Configurations

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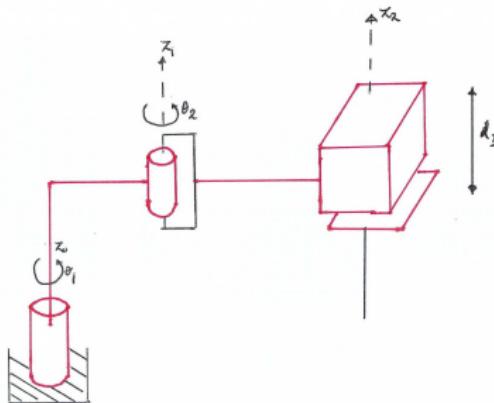
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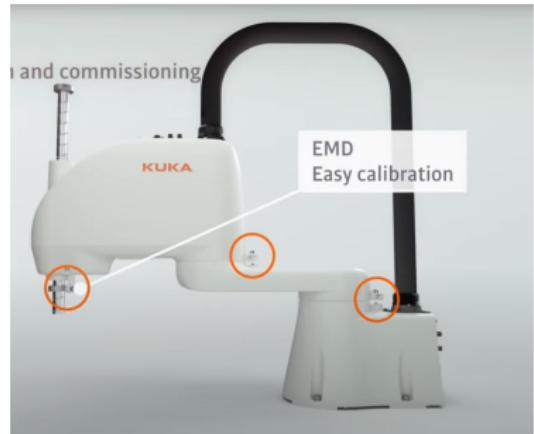
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Configuration of the SCARA Arm.



Courtesy of Fanuc America Inc.

# Mobility of The SCARA Robot

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## Mobility Analysis

Two rotary joints. One prismatic joint acting along the  $z$  axis, and constrained along the  $xy$  plane.

## Mobility Parameters

Four rigid bodies (links). Three constraints. Four freedoms. Therefore,  $\mathfrak{M} = 6(4 - 3 - 1) + 4 = 4$

# Mobility Analysis of The Universal Robot

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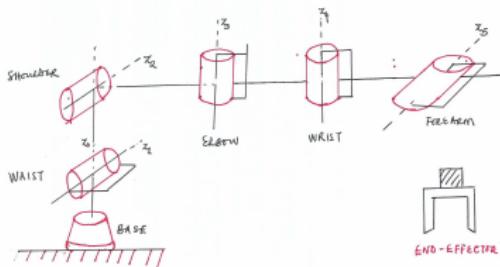
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©Universal Robots A/S, DK.

## The Revolute Arm

Falls under so-called *RRR* kinematic arrangements. Also called a **revolute**, **elbow**, or **anthropomorphic manipulator**.



$$n = 6; k = 5; f = 6$$

$$\therefore \mathfrak{M} = 6(n - k - 1) + \sum f_i$$

$$\Rightarrow 6(6 - 5 - 1) + 6 \text{ or } \mathfrak{M} = 6.$$

# Mobility of The Stewart-Gough Platform

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## Unconstrained bodies, $n$

There are six universal joints that connect the base platform to the prismatic linear actuators.

There are six spherical joints that connect the top platform to the top of the prismatic actuators.

Altogether, there are  $n = 6 + 6 + 2$  or 14 unconstrained rigid bodies.

## Constraints, $k$

Six u-joints. Six spherical joints. Six prismatic joints. Altogether, there are  $f = 6 + 6 + 6 := 18$  constraints.

## Freedoms, $f$

Each u-joints has two freedoms.

Each spherical joint has three (rotary) freedoms. Each prismatic joint has one freedom.

Altogether, there are  $f = 6 \times 2 + 6 \times 3 + 6 \times 1 := 36$  freedoms.

# Mechanism Synthesis

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## Definition (Type-Synthesis)

What **type** of mechanism is appropriate for a task: A **linkage or profile mechanism?**

Type-  
synthesis

## Definition (Size-Synthesis)

What **major dimensions** of the mechanism is to be synthesized?

Size-  
synthesis

# Number Synthesis

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## Definition (Number-Synthesis)

That which deals with the **freedoms and constraints** after the type- and size-synthesis of a mechanism, as well as a **kinematic chain's structural analysis** is termed the **number-synthesis** of the mechanism.?

# Quiz: Mobility of a Planar Parallel Mechanism

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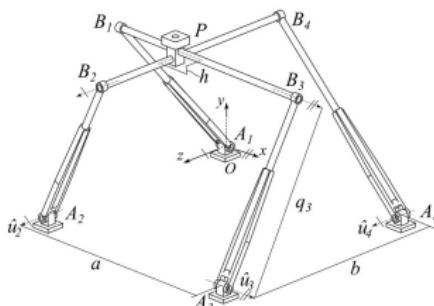
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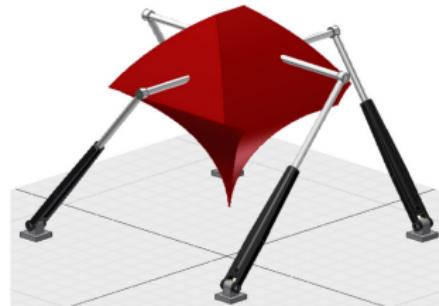
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## Quiz

Analyze the mobility of the mechanism below.



A planar parallel mechanism.



Workspace of the mechanism.

Reprint

# Quiz: Mechanism Hints

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## Hint – Mechanism Description

Point  $P = (P_x, P_y, P_z)$  is the interconnecting point for all the chains on the mobile platform and the top rods.

## Hint – Mechanism Description

The rods that connects points  $B_1$  and  $B_3$ , and points  $B_2$  and  $B_4$  are perpendicular. Both rods are connected to the moving platform by prismatic joints, which are separated from each other by a vertical offset  $h$ .  $\hat{u}_i$  signifies universal joints.

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## Rigid Body Motions

Screws and Rigid Body Transformations.

Rigid body motions: Properties; Direction cosines; Rotation compositions.

Screws (properly revisited): Chasles' and Poinsot's theorem.

Displacement and Force screws; Plücker coordinates; Homogeneous coordinates;

Rodrigues' formula; the matrix exponential.

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## Rigid Body Motions

### Screws and Rigid Body Transformations.

Group theory: The Lie algebra, motions in  $\mathfrak{se}(3)$ ;, and the Lie Group.

Transformations: Translations and rotations in  $\mathbb{R}^3$ , planar rotations,  $SO(3)$ ,  $SE(3)$  motions; homogeneous transformations; Euler and Fick angles; Brockett's exponential map formula. Paden-Kahan subproblems.

# Rigid Body Motions

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## Rigid Body Motion – Intro

A mapping  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a **rigid body motion** if

$$\|g(x) - g(y)\| = \|x - y\| \text{ for all } x, y \in \mathbb{R}^3; \quad (6)$$

$$g(x \times y) = g(x) \times g(y) \text{ for all } x, y \in \mathbb{R}^3; \quad (7)$$

## Rigid Body Motion Preserves Inner Products

For two vectors  $a$  and  $b$ ,  $\langle a, b \rangle = g(a) \times g(b)$ .

# Rigid Body Motions as Screws

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## Rigid Body Motion as a Screw Motion

The motion of a **rigid body** is precisely the same as if it were attached to the **nut of a literal mechanical screw**. Associated with the screw is its pitch.

## Definition (Screw)

That straight line with which a **definite linear magnitude** termed the pitch is associated is called the **screw**.

# Screw as a Geometric Quantity

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## Pitch of a Screw

The rectilinear distance through which (a literal nut) **nut** is translated parallel to the axis of a screw, while the nut is rotated through the angular unit of circular measure is termed the **pitch**.

## Plücker Coordinates

Let  $a$  be a point on line  $\ell_0$ . Let  $a$ 's direction cosine vector (to be introduced shortly) be  $b$ . Then, its binormal (moment) vector is  $c = a \times b$ . We say the pair  $(b, c)$  is the Plücker Coordinates of point  $a$  on axis  $\ell_0$ .

# Screw in Plücker Coordinates

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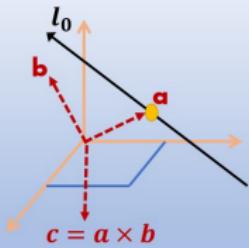
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## Definition (Screw Coordinates)

Six-vector,  $s$ , related to the Plücker coordinates, parameterize a screw i.e.

$$s = (s_1, s_2, s_3, s_4, s_5, s_6).$$



# Screws and Plücker Coordinates

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## Screw axis and Plücker Coordinates

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3; \quad (8)$$

$$c_1 = s_4 - p \cdot s_1, \quad c_2 = s_5 - p \cdot s_2, \quad c_3 = s_6 - p \cdot s_3. \quad (9)$$

## Pitch in Plücker Coordinates

$$p = \frac{s_1 s_4 + s_2 s_5 + s_3 s_6}{s_1^2 + s_2^2 + s_3^2}, \quad (10)$$

$$|s| = \sqrt{s_1^2 + s_2^2 + s_3^2} \quad \text{if } p \neq \infty, \quad (11)$$

$$|s| = \sqrt{s_4^2 + s_5^2 + s_6^2} \quad \text{if } p = \infty \quad (12)$$

# Pitch and Magnitude of the screw

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## Plücker Coordinates' Direction Cosines

Suppose that  $h = \sqrt{b_1^2 + b_2^2 + b_3^2}$ . Then  $(b/h, c/h)$  are respectively the direction cosines of the line,  $l_0$  and its moment.

## Homogeneous Coordinates!

Plücker Coordinates give six unit parameters of a point on a line. Plücker Coordinates are in homogeneous coordinates!

# Twist About a Screw (Axis)

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## Twist

A body's **twist** about a screw is a **uniform (infinitesimal) rotation** about the screw **followed by a uniform (infinitesimal) translation** about an axis parallel to the screw, through a distance that is the product of the pitch and the **circular measure of rotation**.

## Twist

A **twist** requires six algebraic quantities for its complete specification: five ( $\{t_i\}_{i=1}^5$ ) specify the screw, the sixth (or its amplitude) specifies the screw's rotaty angle,  $t_6$ .

# Twist in Plücker Coordinates

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## Definition (Twist Coordinates)

A six-vector,  $t$ , related to the Plücker coordinates parameterize a twist vector i.e.  $t = [(t_1, t_2, t_3), (t_4, t_5, t_6)]$  or  $t = (\omega, v)$ , where  $\omega = (t_1, t_2, t_3)$  and  $v = (t_4, t_5, t_6)$ .

## Plücker Coordinates of a Twist

$$b_1 = t_1, \quad b_2 = t_2, \quad b_3 = t_3 \quad (13)$$

$$c_1 = t_4 - p \cdot s_1, \quad c_2 = t_5 - p \cdot s_2, \quad c_3 = t_6 - p \cdot s_3. \quad (14)$$

# Twists in Plücker Coordinates

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## Pitch of the Twist

$$p_t = \frac{t_1 t_4 + t_2 t_5 + t_3 t_6}{t_1^2 + t_2^2 + t_3^2} = \frac{\omega \cdot v}{\omega \cdot \omega}.$$

## Pitch of the Twist

Expressed as a ratio of the magnitude of the velocity of a point on the twist axis to the magnitude of the angular velocity about the twist axis.

## Translation Distance

$d_t = t_6 \times p_t$ . The sign expresses the rotation's direction.

# Twists and Fixed Movements

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## Pure Rotation

Let pitch be zero. That which results is but **pure rotation**.

## Pure Translation

Let pitch be infinite. That which results **cannot be a finite twist**, except the amplitude be zero, whereupon the twist becomes a pure translation parallel to the screw.

# Curvilinear Displacement: Serret-Frenet Frame

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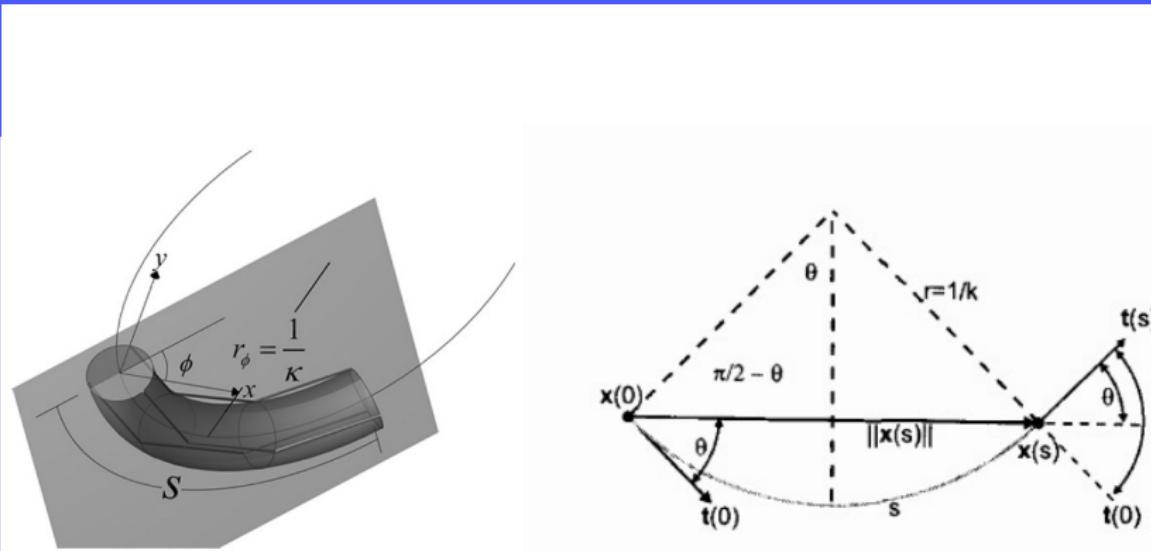
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Elephant Trunk Multi-sectional Continuum Model (left), and its Representation in the Serret-Frenet Frame.

# Plücker Coordinates Example

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## Chasles' Theorem Applied to The Serret-Frenet Frame

Consider a spatial curve  $S$  on the elephant continuum trunk shown earlier. Suppose  $S$  is parameterized by its arc length  $s \in [0, 1]$ . For a point  $x = [x, y, z]^T$  on  $S$ , the unit tangent vector at  $s$  is  $t(s) = dx/ds$ .

## Differential Kinematics and The Serret-Frenet Frame

Denote by  $n$  the principal normal to  $S$  at  $n$ ; then we must have  $b = t \times n$  as the binormal. We say  $(b, n)$  is the Plücker coordinate of the tangent  $t$ .

# Force

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## Force

Net force exerted on a body,  $\mathbf{F} = (f_x, f_y, f_z)$ .

## Couple of Force

Suppose that  $\mathbf{F}$  acts along a corkscrew axis. The resulting motion when  $\mathbf{F}$  makes an infinitesimal rotation about its screw axis is called its couple,  $\mathfrak{C} = (c_x, c_y, c_z)$ .

# Complete Wrench on a Screw

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## Wrench

A wrench requires six algebraic quantities for its complete specification: five ( $\{w_i\}_{i=1}^5$ ) specify the screw, the sixth (or its intensity),  $w_6$ , specifies the force's magnitude.

## Couple's Moment

The moment of the couple is the product of the intensity of the wrench and the screw's pitch i.e.

$$\alpha(\mathcal{C}) = w_6 \times p_w.$$

# Wrench on a Screw

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## Wrench

Simple Definition: A **force** and a **couple** both acting in a plane perpendicular to the force.

## Definition (Complete Definition)

The **resultant canonical system of forces** acting on a rigid body, **reduced to a resultant force on a point**, and acting along the **resultant couple** that is **perpendicular to the plane** in which the force acts is called **the wrench**.

# Wrench in Plücker Coordinates

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## Definition (Wrench Coordinates)

A six-vector,  $w$ , related to the Plücker coordinates parameterize a wrench vector i.e.

$w = [(w_1, w_2, w_3), (w_4, w_5, w_6)]$  or  $w = (f, m)$ , where  $f = (w_1, w_2, w_3)$  and  $m = (w_4, w_5, w_6)$ .

## Plücker Coordinates of a Wrench

$$b_1 = w_1, \quad b_2 = w_2, \quad b_3 = w_3 \quad (15)$$

$$c_1 = w_4 - p \cdot s_1, \quad c_2 = w_5 - p \cdot s_2, \quad c_3 = w_6 - p \cdot w_3. \quad (16)$$

# Wrench in Plücker Coordinates

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## Pitch of the Wrench

$$p_t = \frac{w_1 w_4 + w_2 w_5 + w_3 w_6}{w_1^2 + w_2^2 + w_3^2} = \frac{\mathbf{f} \cdot \mathbf{m}}{\mathbf{f} \cdot \mathbf{f}}.$$

## Pitch of the Wrench

Expressed as a ratio of the moment applied about a point on the axis to the magnitude of the force applied along the wrench axis.

## Wrench's Magnitude

$$\|\mathbf{f}\| = \sqrt{w_1^2 + w_2^2 + w_3^2} \text{ if } p_w = 0 \text{ else}$$
$$\|\mathbf{m}\| = \sqrt{w_4^2 + w_5^2 + w_6^2} \text{ if } p_w = \infty.$$

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## Pure Force

Let pitch be zero. That which results is **pure force** along its screw axis.

## Pure Couple

Let pitch be infinite. That which results **cannot be a finite wrench**, except the intensity be zero, whereupon the wrench becomes a pure couple in a plane that is perpendicular to the screw.

# Statics and Instantaneous Kinematics

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## Statics and kinematics

| Statics   | Instantaneous Kinematics         |
|---|----------------------------------|
| Force, $\mathbf{F}$ about $n$ .                       | Infinitesimal rotation, $\omega$ |
| Couple, $\mathfrak{C}$ : $[\mathbf{F}] \times [\ell]$ | Infinitesimal translation, $t$   |
| $p_w = \pm \mathfrak{C}/\mathbf{F}$                   | Pitch of a Wrench, $w$           |
| $ \mathbf{F} $  | Intensity of Wrench              |

Dyname:  $(\mathbf{F}, \mathfrak{C})$ . Credits: Plücker (1866), Routh (1892).

# Plücker Coordinates Kinetics Quiz

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## Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force  $\mathbf{F}$  acts at the point  $a$  in the image of Frame 89. What are the Plücker coordinates of the **line of force**?

# A short treatise on robots' kinematic geometry and kinetics.

## Mobility

### Screws in Plücker Coordinates

#### Plücker Coordinates Kinetics Quiz

Poinsot's Theorem Quiz on a Force and its Moment  
Suppose that a force  $F$  acts at the point  $a$  in the image of Frame 89. What are the Plücker coordinates of the **line of force**?

## Poinsot's Theorem Quiz on a Force and its Moment

Imagine that a force  $F$  is acting at the point  $a$  in the image of Frame 89. Suppose that  $\tau$  is torque acting along the normal to point  $a$ . Then  $(f, \tau)$  are the Plücker coordinates of the **line of force**.

## Arithmetics on Screws

Scalar and vector arithmetic operations are valid on infinitesimal screws e.g.

$$c_1 s_1 + c_2 s_2 = 0 \text{ for } c_1, c_2 \neq 0 \text{ on screws } s_1, s_2. \quad (17)$$

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## Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force  $\mathbf{F}$  acts at the point  $a$  in the image of Frame 89. What are the Plücker coordinates of the **line of force**?

# Rigid Body Transformations

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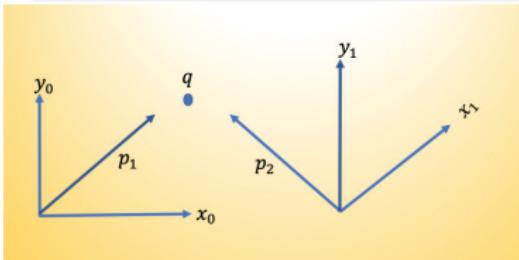
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## Translation of Point $q$ between Two Frames

For a reference frame,  $o_0x_0y_0$  and a moving coordinate frame,  $o_1x_1y_1$ , the translation of  $q$  is given as below:



$$q^0 = \begin{pmatrix} q_x^0 \\ q_y^0 \end{pmatrix}, \quad q^1 = \begin{pmatrix} q_x^1 \\ q_y^1 \end{pmatrix}$$

## Translation of Origin between Two Frames

$$o_1^0 = \begin{pmatrix} o_x^0 \\ o_y^0 \end{pmatrix}, \quad o_0^1 = \begin{pmatrix} o_x^1 \\ o_y^1 \end{pmatrix}. \quad (18)$$

# Rigid Body Transformations

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## Applications to Screws

Applies to Chasles' displacement theorem and Poinsot's force and couple transformations too.

## Screw Transformations

$$\mathbf{t}_1^0 = \begin{pmatrix} t_x^0 \\ t_y^0 \end{pmatrix}, \quad \mathbf{t}_1^1 = R(-\theta)q^0 \quad (19)$$

$$\mathbf{t}_2^0 = R(\theta)q^0, \quad \mathbf{t}_2^1 = \begin{pmatrix} t_x^1 \\ t_y^1 \end{pmatrix} \quad (20)$$

where  $\theta$  is the angle coordinate frame  $o_1x_1y_1$  makes w.r.t  $o_0x_0y_0$ .

# Rotations in $\mathbb{R}^3$

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## Rotations in $\mathbb{R}^3$

Conventions: Bodies' orientations are measured along a corkscrew direction, specified by a local coordinate frame. Thus, relative orientation is measured from the local coordinate frame to an inertial coordinate frame.

# Direction Cosines

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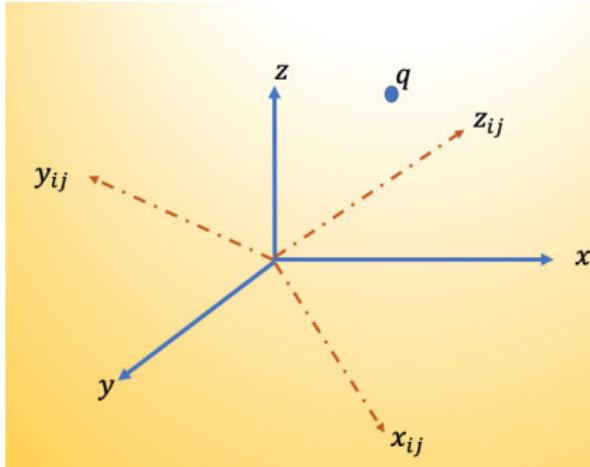
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## Conventions

$I$ : Inertial frame;  $J$ :  
Body frame.

$q : (x_{ij}, y_{ij}, z_{ij}) \in \mathbb{R}^3$ :  
coordinates of the  
principal axes of  $J$   
relative to  $I$ .

# Rotation Matrix from Direction Cosines

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## Rotation as Composition of Projections Between Frames

$$R_{ij} = [\mathbf{x}_{ij} \quad \mathbf{y}_{ij} \quad \mathbf{z}_{ij}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \quad (21)$$

## Rotation Matrix as Unit Axes' Dot Products

$$R_{ij} = \begin{bmatrix} \mathbf{x}_j \cdot \mathbf{x}_i & \mathbf{y}_j \cdot \mathbf{x}_i & \mathbf{z}_j \cdot \mathbf{x}_i \\ \mathbf{x}_j \cdot \mathbf{y}_i & \mathbf{y}_j \cdot \mathbf{y}_i & \mathbf{z}_j \cdot \mathbf{y}_i \\ \mathbf{x}_j \cdot \mathbf{z}_i & \mathbf{y}_j \cdot \mathbf{z}_i & \mathbf{z}_j \cdot \mathbf{z}_i \end{bmatrix}. \quad (22)$$

# Rotation Matrix from Direction Cosines

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## Rotation Matrices are Direction Cosines!

$$\mathbf{x}_j \cdot \mathbf{x}_i = \cos(\angle(\mathbf{x}_j, \mathbf{x}_i)), \quad \mathbf{y}_j \cdot \mathbf{x}_i = \cos(\angle(\mathbf{y}_j, \mathbf{x}_i)), \dots \\ \dots, \mathbf{y}_j \cdot \mathbf{z}_i = \cos(\angle(\mathbf{y}_j, \mathbf{z}_i)), \quad \mathbf{z}_j \cdot \mathbf{z}_i = \cos(\angle(\mathbf{z}_j, \mathbf{z}_i)).$$

## Properties of Rotation Matrices

Rows of  $R_{ij}$  are the **unit vector** coordinates of  $I$  in the frame  $J$  so that

$$R_{ij} = R_{ji}^{-1} = R_{ji}^T. \quad (23)$$

That is, the **inverse of the rotation matrix is equal to its transpose.**

# Special Orthogonal 3, SO(3)

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## Orthogonal properties!

Observe:  $\det \mathbf{R} = \mathbf{r}_1^T \cdot (\mathbf{r}_2 \times \mathbf{r}_3)$ . In **corkscrew notation**,  $\det \mathbf{R} = +1$  i.e.  $\mathbf{r}_2 \times \mathbf{r}_3 = \mathbf{r}_1$  so that  $\det \mathbf{R} = \mathbf{r}_1^T \cdot \mathbf{r}_1 = +1$ . A matrix that satisfies the above property is said to possess a **special orthogonal 3, denoted SO(3), property**.

## SO(n) Property

Special orthogonal means  $\det \mathbf{R} = +1$ . The set of all SO matrices in  $\mathbb{R}^{n \times n}$  is

$$SO(n) = \{\mathbf{R} \in \mathbb{R}^{n \times n} : \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = +1\}. \quad (24)$$

# Rotations on Vectors

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## Rotating a Vector

Suppose that a point  $p_j$  is on a frame  $J$ , then the vector that connects a point  $q_j$  in the frame  $J$  to  $p_j$  is  $v_j = q_j - p_j$ . Now, the rotation matrix's action on  $v_j$  is

$$\mathbf{R}_{ij}(v_j) := \mathbf{R}_{ij}q_j - \mathbf{R}_{ij}p_j = q_i - p_i = v_i. \quad (25)$$

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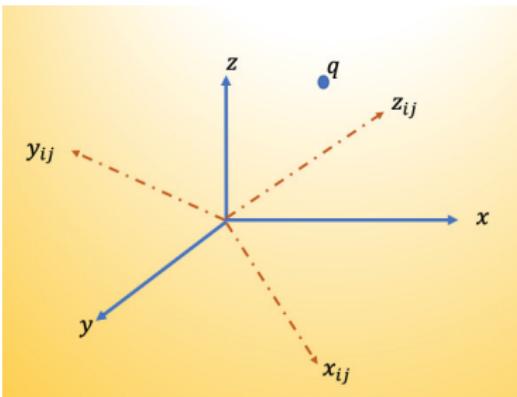
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## Planar Rotations

Let the angle of rotation  
between the two coordinate  
frames be  $\theta$ . Then,

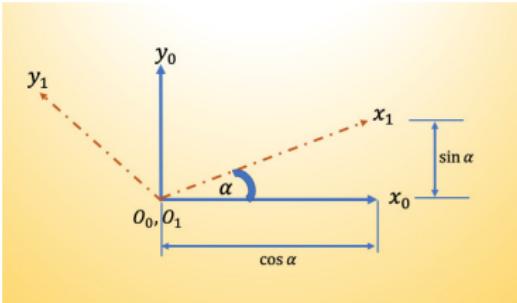
$$R_1^0 = \begin{pmatrix} x_1^0 & y_1^0 \end{pmatrix} \quad (26)$$



## Planar Rotations

It follows that

$$R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad (27)$$



# Planar Rotations

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## Planar Rotations via Direction Cosines

$$\begin{aligned} R_1^0 &= \begin{bmatrix} \mathbf{x}_0 \cdot \mathbf{x}_1 & \mathbf{y}_1 \cdot \mathbf{x}_0 \\ \mathbf{x}_0 \cdot \mathbf{y}_1 & \mathbf{y}_1 \cdot \mathbf{y}_0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\cos(\pi/2 - \alpha) \\ \cos(\pi/2 - \alpha) & \cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \end{aligned} \quad (28)$$

Projection of  $y_1$  on  $x_0$  is negative because of our adopted right-handed frame.

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## Rotations Composition

Let the **relative orientation** of a frame  $K$  to a frame  $J$  be  $\mathbf{R}_{jk}$ , and let frame  $J$ 's **relative orientation** to frame  $I$  be  $\mathbf{R}_{ij}$ , then the **relative orientation** of frame  $K$  w.r.t  $I$  is

$$\mathbf{R}_{ik} = \mathbf{R}_{ij} \cdot \mathbf{R}_{jk}. \quad (29)$$

## Rotations Composition

Equivalent to **rotating  $J$  relative to frame  $I$  according to  $\mathbf{R}_{ij}$** ; then **aligning frame  $J$  to  $K$** , we **rotate  $K$  relative to  $I$  according to  $\mathbf{R}_{jk}$** . This frame relative to which rotation occurs is termed the **current frame**.

# Composition of Rotations About A Current Axis

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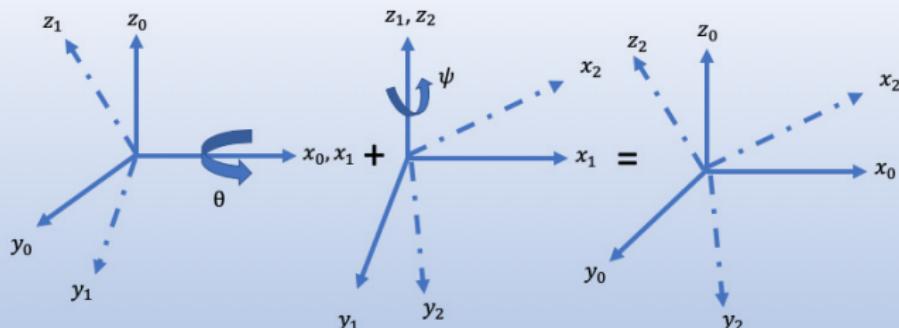
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## Composition of Rotations About A Current Axis



# Composition of Rotations About A Current Axis

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## Compositions

$$\mathbf{R} = \mathbf{R}_{x,\theta} \mathbf{R}_{z,\psi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix} \cdot \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (30)$$

$$\mathbf{R} = \begin{pmatrix} c_\psi & -s_\psi & 0 \\ 0 & c_\theta c_\psi & -s_\theta \\ s_\theta s_\psi & s_\theta c_\psi & c_\theta \end{pmatrix} \quad (31)$$

Notice how the order of multiplication is carried out, owing to the axis about which we are making the transformation.

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## Skew Symmetry Operations

What happens when the order of multiplication is reversed?

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## Skew Symmetric Matrix

$$(S)^\wedge = \begin{pmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{pmatrix} \quad (32)$$

## Skew Symmetric Matrix

Observe  $s_{ij} = -s_{ji}$  for  $i \neq j$  and  $s_{ii} = 0$

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## Pre-multiplication of Rotations

A rotation about a fixed axis requires a pre-multiplication.

## Post-multiplication of Rotations

A rotation about a current axis necessitates a post-multiplication.

# Rotations' Composition

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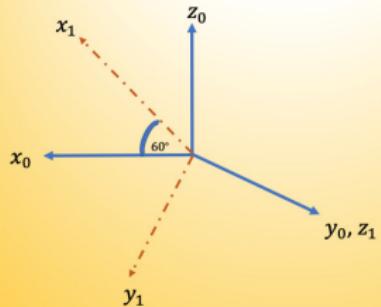
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## Rotations Composition

Suppose all axes of the inertial frame are successively rotated by  $\beta$  around  $x_0, y_0, z_0$  respectively. What is the transformation? Verify that (1)

$R_{e,\beta} = I$  where  $e$  is the axes about which we are rotating and  $\beta$  is the angle of rotation; (2) The composition of rotations about  $\beta$  and  $\alpha$  in a successive manner implies that

$R_{z,\beta}, R_{z,\alpha} = R_{z,\beta+\alpha}$ , and (3)  
 $(R_{z,\beta})^{-1} = R_{z,-\beta}$ .



Relative orientation between  
two frames.

# Euler Angles as Parameterization of Rotations

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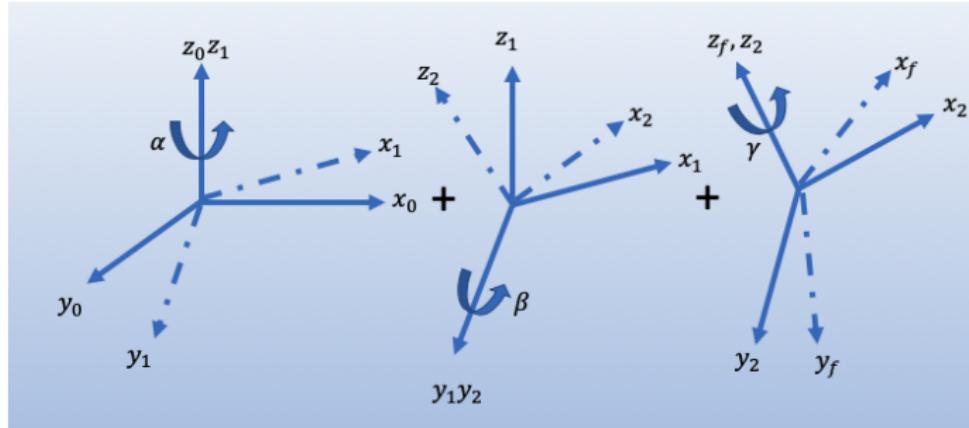
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Relative orientation between two frames.

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## Euler ( $ZYZ$ ) Angles

$$\mathbf{R}_{ij}(\alpha, \beta, \gamma) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma) \quad (33)$$

$$\begin{aligned} &= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \\ &= \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix} \end{aligned} \quad (34)$$

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## Euler ( $ZYZ$ ) Angles. Case $\sin(\beta) > 0$

$$\beta = \arctan 2(r_{33}, \sqrt{1 - r_{33}^2}) \quad (35a)$$

$$\alpha = \arctan 2(r_{23}/\sin \beta, r_{13}/\sin \beta) \quad (35b)$$

$$\gamma = \arctan 2(r_{32}/\sin \beta, -r_{31}/\sin \beta) \quad (35c)$$

where  $\arctan 2(y, x)$  determines the quadrant of the angle based on the sign of  $x$  and  $y$ .

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## Euler ( $ZYZ$ ) Angles. Case $\sin(\beta) < 0$

$$\beta = \arctan 2(r_{33}, -\sqrt{1 - r_{33}^2}) \quad (36a)$$

$$\alpha = \arctan 2(-r_{23}/\sin \beta, -r_{13}/\sin \beta) \quad (36b)$$

$$\gamma = \arctan 2(-r_{32}/\sin \beta, r_{31}/\sin \beta) \quad (36c)$$

Euler angles are not unique owing to the sign of the angle about which the  $y$  axis rotates!

# Other Axes Parameterization of Rotations

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## Fick ( $ZYX$ ), Helmholtz ( $YZX$ ) Angles.

We could **permute the order of rotation** such as rotating successively about **different axes**. Examples include **successive rotations about  $ZYX$  axes for the Fick angles** and **successive rotations about  $YZX$  axes for Helmholtz angles**.

## Fick ( $ZYX$ ) and Helmholtz ( $YZX$ ) Angles.

These avoid **Euler angle singularities** at  $\mathbf{R} = \mathbf{I}$ . This does not preclude **singularities at other configurations**.

# Fick angles and Yaw, Pitch, and Roll Axes

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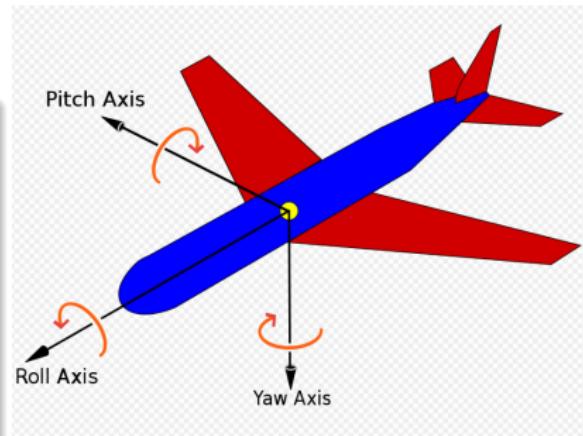
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## Fick angles

Otherwise called the yaw, pitch, and roll angles.  $R_{ij}$  found by rotating about the  $x$ -axis (roll), then the  $y$ -axis (pitch), and finally the  $z$ -axis – all in the body frame.



Aircraft Principal Axes in the right-hand frame. Courtesy of Wikimedia commons.

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## Summary of Parameterizations

Rotation matrices can be parameterized in one of many ways depending on our use case. The common examples of parameterizations are

- (1) Axis-Angle representation;
- (2) Euler angles ( $ZYZ$ ) representation;
- (3) Fick angles (*i.e.*,  $ZYX$  or yaw, pitch and roll) representation;
- (4) Helmholtz angles (or  $YZX$ ) angles representation; and