"Constraints-Preserving" Controllers in Large-Scale Systems.

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***Abstract*—**

I. INTRODUCTION

In this paper, we shall concentrate on finding optimal policies in a large scale system that satisfy specified constraints in the inputs and states of the dynamical system. We term such constraints as "safety-preserving". These constraints usually involve states and input (or actuators) constraints that must be satisfied in a least restrictive sense.

Safety-preserving problems usually arise in the interaction among multiple agents that share a physical space such as cyber-physical systems (CPS). CPS examples may include modern manufacturing assembly lines where humans and machines jointly work together for products delivery to a supply chain being controlled by computer software resources, personalized interoperable medical devices, autonomous cars on a highway, (almost unmanned) long-hauled passenger flights, or multidimensional allocation processes with multiple constraints such as cargo loading problems (with weight and size restrictions) [1, Ch. II].

The "physical" and "cyber" couplings of such systems is critical in modern CPS infrastructure: finding feasible control laws in the presence of complex dynamics; solving for collision avoidance schemes in real-time for complex multi-agent systems, or navigating in uneven terrains while maintaining stability – all require a systematic orchestration of local control laws that must be carefully planned and executed. Therefore, the safety analysis of combined CPS systems in the presence of sensing, control, and learning becomes timely and crucial.

Differential optimal control theory and games offer a systematic paradigm for resolving the safety of multiple agents interacting over a shared space. Both problems resolve a (robustly) optimal controller by analyzing the Hamilton-Jacobi Bellman (HJB) or its Isaacs (HJI) counterpart. As HJ-type equations have no classical solution for almost all *practical* problems, stable numerical and computational methods need to be brought to bear in order to produce solutions with (approximately) optimal guarantees.

With essentially non-oscillating (ENO) [2] Lax-Friedrichs [3] schemes, consistent and monotone solutions to HJ Hamiltonians can be resolved via explicit marching schemes that solve for unique (viscosity) solutions [4], [5] to HJ-type equations with high accuracy and precision *on a grid* [6]–[8], unique (viscosity) solutions to HJ-type equations with high accuracy and precision *on a mesh*. Employing meshes for resolving inviscid Euler equations whose solutions are the derivatives of HJ equations, these methods scale exponentially with state dimensions, making them ineffective for complex systems – a direct consequence of *curse of dimensionality* [9].

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II. NOTATIONS AND DEFINITIONS.

Let us now introduce the notations that are commonly used in this article. Time variables e.g. t, t_0, τ, T will always be real numbers. We let $t_0 \le t \le t_f$ denote fixed, ordered values of t. Vectors shall be column-wise stacked and be denoted by small bold-face letters i.e. $\mathbf{e}, \mathbf{u}, \mathbf{v}$ e.t.c. Matrices will be denoted by bold-math Latin upper case fonts e.g. $\mathbf{T}, \mathbf{S}$. Exceptions: the unit matrix is $\boldsymbol{I}$; and i, j, k, p are indices. Positive, negative, increasing, decreasing e.t.c. shall refer to strict corresponding property.

The set S of all $\boldsymbol{x}$ such that $\boldsymbol{x}$ belongs to the real numbers $\mathbb{R}$, and that $\boldsymbol{x}$ is positive shall be written as $S = \{\boldsymbol{x} \,|\, \boldsymbol{x} \in \mathbb{R}, \boldsymbol{x} > 0\}$. The cardinality of S shall be written as $[S]$. We define Ω as the open set in $\mathbb{R}^n$. To avoid the cumbersome phrase "the state $\boldsymbol{x}$ at time t", we will associate the pair $(\boldsymbol{x}, t)$ with the *phase* of the system for a state $\boldsymbol{x}$ at time t. Furthermore, we associate the Cartesian product of Ω and the space $T = \mathbb{R}^1$ of all time values as the *phase space* of $\Omega \times T$. The interior of Ω is denoted by $\text{int } \Omega$; whilst the closure of Ω is denoted $\bar{\Omega}$. We denote by $\delta\Omega \, (:= \bar{\Omega} \backslash \text{int } \Omega)$ the boundary of the set Ω.

Unless otherwise stated, vectors $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are reserved for admissible control (resp. disturbance) at time t. We say $\boldsymbol{u}(t)$ (resp. $\boldsymbol{v}(t)$) is piecewise continuous in t, if for each t, $\boldsymbol{u} \in \mathscr{U}$ (resp. $\boldsymbol{v} \in \mathscr{V}$), $\mathscr{U}$ (resp. $\mathscr{V}$) is a Lebesgue measurable and compact set. At all times, any of $\boldsymbol{u}$ or $\boldsymbol{v}$ will be under the influence of a *player* such that the motion of a state $\boldsymbol{x}$ will be influenced by the will of that player. Our operational domain involves conflicting objectives between various agents e.g. with a heading convergence goal under an external disturbance' influence. For agents that are members of a local coordination group, collision avoidance shall apply so that agents within a local neighborhood cooperate to avoid entropy and predatory pursuer(s). Thus, the problem at hand assumes that of a pursuit *game*. And by a game, we do not necessarily refer to a single game, but rather a *collection of games*. Such a game will terminate when *capture* occurs, that is the distance between players falls below a predetermined threshold.

Each player in a game shall constitute either a pursuer (***P***) or an evader (***E***). The cursory reader should not interpret ***P*** or ***E*** as controlling a single agent. In complex settings, we may have several pursuers (enemies) or evaders (peaceful citizens). However, when ***P*** or ***E*** governs the behavior of but one agent, these symbols will denote the agents themselves. The first component of an agent labeled say 5 within a flock labeled 2 is written $\boldsymbol{x}_1^{(5)_2}$. Each evading agent, identified by its label i as a state superscript, is parameterized by three state components: its linear velocities $(\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)})$, and its heading $w^{(i)}$. When we need to distinguish an agent within a flock say F_j from another flock F_k, we shall use the index of the flock e.g. j as a further subscript for a particular agent's state i.e. $\boldsymbol{x}_1^{(i)_j} \neq \boldsymbol{x}_1^{(i)_k}$.

Given the various possibilities of outcomes, the question of what is "best" will be resolved by a *payoff*, Φ, whose extremal over a time interval will constitute a *value*, $\boldsymbol{V}$[1]. We adopt Isaac's [10] language so that if the payoff for a game is finite, we shall have a *game of kind*; and for a game with a continuum of payoffs, we shall have *a game of degree*. The *strategy* executed by ***P*** or ***E*** during a game shall be denoted by $\alpha \in \mathscr{A}$ (resp. $\beta \in \mathscr{B}$). With this definition, a control law e.g. $\boldsymbol{u}^{(i)}$ played by a player e.g. ***P*** will affect *agent i*; and a collection of agents under ***P***'s *willpower* be referred to as a *flock*. We shall refer to an aggregation of flocks on a state space as a *murmuration* [2].

III. Flight Model and Control

We consider the classic dart-shaped paper airplane for our numerical analysis and experiment. The paper plane is characterized by a point mass whose longitudinal motions are described by four differential equations. The integral of these differential equations yields three longitudinal paths i.e. the (1) constant-angle descent; (2) vertical oscillation; and (3) loop. The paper plane is assumed made out of plain paper, with a weight of $0.003kg$, wing span of $12cm$ and a length of $28cm$. We fix the angle of attack at $9.3°$, producing a configuration lift-drag ratio of 5.2 [11].

TO-DO:

- Describe rationale for using drones;
- Describe drone dynamics;
- Describe MPC controller;
- Add result about MPC and indicate why a basic MPC scheme fails to obey state convergence constraints;
- Add result about why MPC fails to obey input constraints.

IV. Flight and Safety

We consider a flight level and global heading orientation maintenance problem. We are inspired by air traffic control (ATC) problems whereupon airworthiness and aviation operational guidance requires Reduced Vertical Separation Minimum (RVSM) operations. RVSM airspace is any airspace or route where flight levels 290 and 410 inclusive must be vertically separated by 1000ft. RVSM allows aircraft to *safely* travel along better optimal profiles, save fuel usage, and increase airspace capacity [12].

Altitude-keeping performance of airplanes is a key element in ensuring safe operations in "RVSM-Compliant" airspace. The safety question involves strict altitude-keeping aircraft performance by tracking the altimetry system error (ASE)[3] to be within the error budget throughout a flight envelope. In our work, we consider the ASE tracking bounds a state constraint parameter. According to FAA separation standards [13],

In addition, we want the orientation of every aircraft to be within a certain bound, $\omega_{min} < \omega \leq \omega_{max}$.

Appendix

Acknowledgment

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[1] The functional $\boldsymbol{\Phi}$ may be considered a functional mapping from an infinite-dimensional space to the space of real numbers.

[2] The definition of murmurations we use here has a semblance to the murmurations of possibly thousands of starlings observed in nature.

[3] The ASE is equivalent to the pressure altitude shown to the flightcrew and free stream pressure altitude.