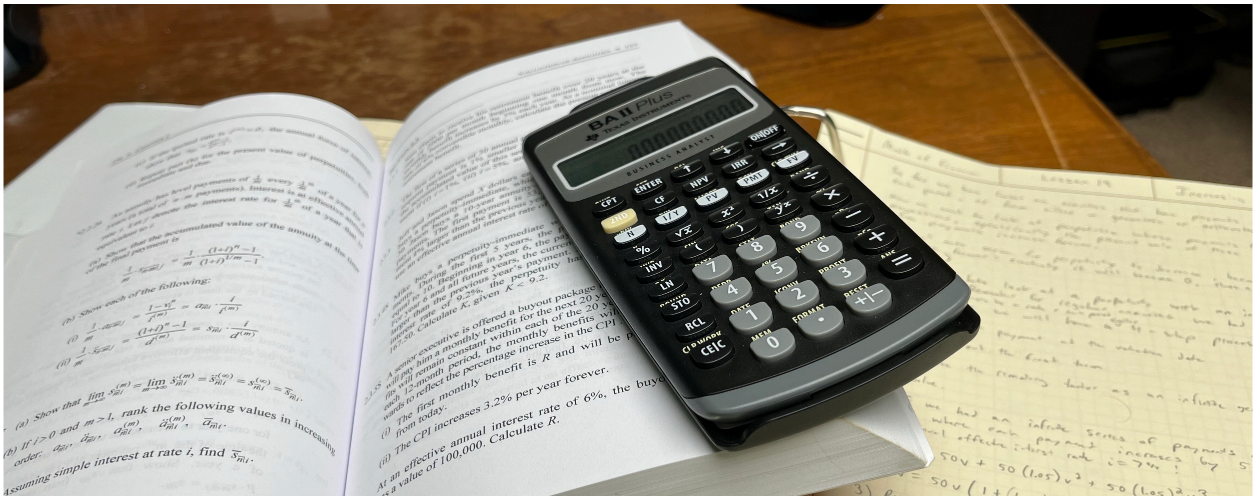
Financial Mathematics Worksheet Packet

To Accompany the Lesson & Examples Videos from JK Math



Hi there! Thanks for downloading these worksheets. The first two pages of this PDF are to inform you on what is inside this packet and how it is set up!

The worksheets in this PDF were designed such that the entire packet can be duplex printed (print on both sides of paper!). In fact I would recommend printing duplex as the majority of the worksheets for each video are 2 pages in length, meaning that they would only be 1 page once printed. There are some exceptions to this, as some are 4 pages in length resulting in 2 full pages if printed duplex, but there are only a few of those throughout this packet.

I hope you enjoy these worksheets and find them to be useful for learning financial mathematics! If you have any questions please don't hesitate to reach out.

- Josh

JK Mathematics

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Included Topics:

JK Mathematics
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Name: ____________________

Financial Mathematics Lesson 1
Compound Interest

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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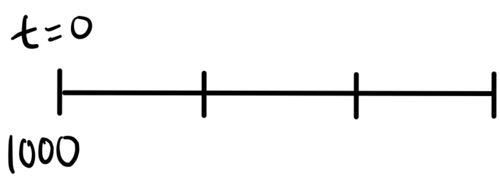
Compound interest is one of a few types of interest that accrues, or generates money (called interest) on an investment over periods of time.

In particular, the interest accrued by compounding is reinvested and then also generates interest in the next period.

Thus, compound interest can be viewed as the interest accrued on both an investment and accumulated interest from previous periods of time.

For Example:



Value at $t = 1$: Value at $t = 2$: Value at $t = 3$:

<u>Future Value Formula For Compound Interest:</u>

$\boxed{FV = C(1+i)^n}$ → **OR** → $\boxed{A(t) = A(0) \cdot a(t)}$

C = initial deposit / investment
i = compound interest rate
n = # of time periods (years in our example)

$A(t)$ = accumulation at time t
$A(0)$ = accumulation at time 0 (initial investment)
$a(t) = (1+i)^t$ = accumulation factor
t = same as n

Ex.

You deposit \$600 into an account that earns compound interest at a rate of 4% per year. How much will you accumulate in 5 years?

Name: ____________________

Financial Mathematics Lesson 1 Examples: Compound Interest

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

John deposits \$10,000 into an account that earns compound interest at a rate of $5\,\%$ per year. Find the amount of interest in his account at the end of years 1, 2, and 3, as well as the amount of interest credited in each year.

Ex. 2

Carl deposits \$100 into an account earning $10\,\%$ compound interest per year. Find how much will he accumulate in 10 years as well as his total interest earned over the period.

Ex. 3

Bob deposits \$500 into an account earning 2 % interest compounded monthly. How much is in the account after 3 years?

Name: ____________________

Financial Mathematics Lesson 2
Basic Cash Flows

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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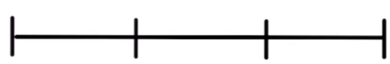
In most cases, investment accounts are busier than just depositing an amount of money and then letting it sit for a period of time.

Often times, there are multiple deposits (cash flows in) and withdrawals (cash flows out) that may occur in the future for the account.

As such, these cash flows will effect the accumulated amount in an account over time, and how it will be calculated. To demonstrate how to calculate the accumulation of an account with various cashflows, consider the following example, which can be solved in two ways:

<u>Ex. 1</u> (Long Method)

Every year, Tom's account earns interest at a compound rate of 2%. On 1/1/2017, Tom deposits \$400, on 1/1/2019, he withdrawals \$100, and on 1/1/2021 he deposits \$200. What is his balance on 1/1/2022?



Ex. 1 (Short Method)

The short method is algebraically equal to the long method, and so there is no reason to use the long method going forward. (Write the work showing that the methods are equal in the space below)

Ex. 2

Sarah deposits \$1000 today, deposits another \$1000 in 2 years, and in 6 years from today, she withdrawals \$500. How much is in her account at the end of 8 years, if her yearly compound rate is 1%?

Name: ____________________

Financial Mathematics Lesson 2 Examples: Basic Cash Flows

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Max makes a deposit into an account earning 7% compound interest per year on 1/1/2016. He makes a withdrawal of \$50 on 1/1/2018, and a deposit of \$250 on 1/1/2019. The amount in his account on 1/1/2020 is \$1193.35. What was Max's initial deposit on 1/1/2016?

Ex. 2

Connor deposits \$5000 into an account paying a compound interest rate of $3\,\%$ per year for a total of ten years. The account has a restriction that if Connor makes a withdrawal within the first four and one-half years, his account will be charged a penalty of $6\,\%$ of the withdrawal. Despite this, Connor withdrawals an amount k at the end of each years 2, 3, 5, and 8. The balance in his account at the end of the tenth year is \$5000. Find k.

Name: ________________

Financial Mathematics Lesson 3
Average Annual Rate of Return

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

There are going to be occasions where the interest rate for an investment over a certain period of time will be different for each year. In these cases, knowing the average rate for the entire period may be of interest. This rate is called the average annual rate of return.

The process for finding the average annual rate of return is best demonstrated with an example. Below are the return rates for the Apple stock from 2012 to 2016:

Year	2012	2013	2014	2015	2016
i	32.24%	8.07%	40.62%	-3.01%	12.48%

We can find the average annual rate of return for this 5 year period using the following two steps:

Step 1: Multiply the accumulations factors for each year to find the value of an investment.

Step 2: Set the value equal to $(1+i)^n$ and solve for i. (n is the number of years)

This two-step process gave us the average annual rate of return for the Apple stock in that 5 year period. The rate describes the average growth of an investment in that stock for each year.

In other words, if an investment had that average annual rate of return as the return rate for all 5 of those years, it would be worth the same amount at the end of the 5 years as it would when using the 5 different rates for each year. You can see that is true by compounding the average annual rate of return for those 5 years. (Write that work in the space below)

Ex.

Bill deposits money into an account that has a yearly compound interest rate of $1\,\%$ in year one, $0.5\,\%$ in year two, and $1.25\,\%$ in year three. What is Bill's average annual interest rate for the 3 year period?

Name: ___________________

Financial Mathematics Lesson 3 Examples: Average Annual Rate of Return

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Find the average annual rate of return for the Microsoft stock from 2009 to 2012 using the table below.

Year	2009	2010	2011	2012
i	56.79 %	-8.43 %	-6.99 %	2.89 %

Ex. 2

Arnold receives interest on an account that he made a deposit in. For the first 3 years he receives $4\,\%$ interest compounded yearly, for the next 2 years he receives $6\,\%$ interest compounded yearly, and for the last 5 years he receives $9\,\%$ interest compounded yearly. What was his average annual rate of return over the 10 year period?

Name: ____________________

Financial Mathematics Lesson 4
The Annual Effective Rate

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

In practice, interest may be credited more frequently or less frequently than one time each year. For example, some bank accounts may pay interest semiannually, quarterly, monthly, or even once every two years. We call these **non-yearly** interest rates.

To distinguish between yearly rates and non-yearly rates, we typically use different letters to denote them in our calculations:

$j \rightarrow$

$i \rightarrow$

However, sometimes when dealing with scenarios involving non-yearly rates, we may still be interested in knowing the corresponding annual rate, which we will refer to as the **annual effective rate** (This rate is assumed to be a compound interest rate).

We can find an annual effective rate i given a non-yearly rate j using the following formula:

$$\boxed{(1+j)^m - 1 = i}$$

Where m is the number of periods per year

As an example, let $j = 0.5\,\%$ be a quarterly rate and find the annual effective rate i :

We would call the quarterly rate $j = 0.5\,\%$ from that example an **equivalent rate of interest** to the annual effective rate i that was found. This is because both rates would result in the same accumulated amount over the same period of time. (In the space below, write the work showing this is true for a deposit of $500 over a one year period)

Ex.

George deposits money into an account that pays interest compounded monthly at a rate of $0.7\,\%$. What is his annual effective interest rate?

Now, there is also another way to find an unknown annual effective rate for a specific type of scenario. If you know an accumulated amount for a particular year and the accumulated amount one year later, the annual effective rate can easily be found by determining the interest credited in that year and dividing it by the original amount.

For instance, if you have $1000 at $t = 0$ and $1070 at $t = 1$, then:

$$i = \frac{1070 - 1000}{1000} =$$

This can summarized by: $$i = \frac{A(1) - A(0)}{A(0)}\ ,\ [0, 1]$$

And generalized by: $$\boxed{i = \frac{A(t) - A(t-1)}{A(t-1)}\ ,\ [t-1, t]}$$

Name: ___________________

Financial Mathematics Lesson 4 Examples: The Annual Effective Rate

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Patty places money into an account earning $15\,\%$ interest every two years. What is her annual effective interest rate?

Ex. 2

Larry deposits money into an account with an annual effective interest rate of $5\,\%$. What are the equivalent semi-annual, quarterly, and monthly compound interest rates?

Ex. 3

Gina deposits \$100 into an account for 25 months and it accumulates to \$120.54. What is the annual effective rate?

Ex. 4

Devan deposits \$3000 in an account and one year later has \$3150. What was his annual effective interest rate?

Name: ________________

$\ddot{a}_{\overline{n}|i}$ Financial Mathematics Lesson 5
Simple Interest

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

Similar to compound interest, simple interest is another type of interest that accrues, or generates interest on an investment over periods of time. The difference between compound interest and simple interest is in *how* they generate that interest.

When calculating the future value of an investment using a compound interest rate, the accumulation factor was $a(t) = (1 + i)^t$, where t is the number of time periods. However, for simple interest, the accumulation factor is different.

It looks like this: $\boxed{a(t) = (1 + i \cdot t)}$ (simple interest accumulation factor)

This accumulation factor is used to calculate the future value of an investment that has a simple interest rate.

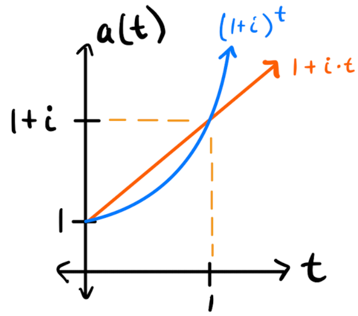
For example, if \$1000 is placed into an account with 7% simple interest for 3 years, then the future value would be:

Compare this to the future value of the investment if the 7% interest rate was a compound interest rate instead (use the same value of t):

Ex.

Kyle invests \$500 in an account with $4\,\%$ annual simple interest. How much does he accumulate in 6 months? Is this less than or greater than the amount he would accumulate if his interest was compounded yearly for the same period?

To compare compound interest and simple interest, it can be helpful to graph their accumulation factors as functions of t on the same coordinate plane.



$a(t) = (1+i)^t$
(compound)

$a(t) = (1 + i \cdot t)$
(simple)

As it turns out, the accumulation factor for compound interest is an **exponential** function, while the accumulation factor for simple interest is a **linear** function.

What we learn from this graph is that between $t = 0$ and $t = 1$, a simple interest rate i would generate more interest than a compound interest rate i, but for $t \geq 1$, a compound interest rate would generate more interest than the simple interest rate.

So, if t is measured in years, then until one year has passed ($t = 1$), a simple interest rate i will accrue more interest, but after the one year mark, a compound interest rate i will accrue more interest. (You verify that this is true by revisiting the results of the two examples in this lesson.)

Name: ____________________

Financial Mathematics Lesson 5 Examples: Simple Interest

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

Morgan puts \$100 into one account with 5 % simple interest for 100 days, \$200 into another account with 6 % simple interest for 8 months, and \$300 into a third account with 7 % simple interest for 2 years and 3 months. How much does each account accumulate over their respective time periods? Which accounts accumulate more with their simple interest rates than they would if they had compound interest rates?

Ex. 2

Jill deposits \$400 into an account with an interest rate $\frac{i}{2}$ compounded semiannually, and her friend James deposits \$700 into an account with a simple interest rate i annually. They both earn the same amount of interest during the last 6 months of the fifth year. Find i.

Ex. 3

Joanna placed money in an account earning $5\,\%$ annual simple interest. For the time interval $[2, 3]$, what is the effective annual interest rate?

Name: ____________

Financial Mathematics Lesson 6
Present Value & The Equation of Value

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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The present value of an investment can be viewed as the opposite of the future value. Instead of determining the accumulation of an investment over time, present value refers to how much should be invested **today** in order to generate a desired amount in the future.

So, in the equation for calculating future value, $A(t) = A(0) \cdot a(t)$ where $a(t) = (1+i)^t$, the present value refers to the value of $A(0)$ (the initial deposit/investment), whereas the future value refers to the value of $A(t)$ (the accumulation of the investment over time).

To find an equation for the present value, solve for $A(0)$:

We can rewrite this equation as $\boxed{P(0) = P(t) \cdot v^t}$ where $\boxed{v^t = \dfrac{1}{(1+i)^t}}$.

(Note: The notation of $P(0)$ is interchangeable with PV, and v^t is interchangeable with v_i^t)

So, if you wanted to have \$1000 in 5 years using an account with an annual effective rate of 3%, then the present value, or the amount that needs to be deposited today, would be:

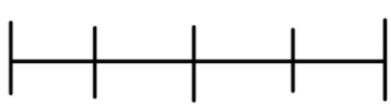
The solution to a present value problem can be checked by calculating the future value for that same scenario. So for our example, the future value in 5 years (using the same interest rate) of that deposit/present value that was found should be \$1000: (show this below)

Now that you are familiar with both concepts of future value and present value and how they are related, we can introduce the idea of the equation of value. In a way, the concept of the equation of value was first introduced in Lesson 2 (Basic Cash Flows), but it was not mentioned explicitly.

The equation of value balances cash flows while considering their "time values." This is best demonstrated visually in an example problem using a timeline:

<u>Ex.</u>

How much should you deposit today in an account earning $4\,\%$ compounded annually so that you are paid \$1000 in 2 years and \$1500 in 4 years?



Name: ____________________

Financial Mathematics Lesson 6 Examples: Present Value & The Equation of Value

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Jeff wants to accumulate \$5000 in order to make a payment in 3 years. If he makes a deposit into an account today earning 10 % effective annual interest, how much should his deposit be?

Ex. 2

Lily will receive \$3000 at the end of each year for the next 5 years. With an annual effective interest rate of 8 %, find today's present value of all the payments Lily will receive.

Ex. 3

Charlie and Carly want to set up a trust fund for their three children ages 2, 4, and 5. They want the fund to pay \$20,000 to each child when they are 18, and pay \$75,000 to each child when they are 21. If the trust fund earns $9\,\%$ annual effective interest, how much should Charlie and Carly invest today?

Name: ____________________

$\ddot{a}_{\overline{n}|i}$ Financial Mathematics Lesson 7
Nominal Interest Rates

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Nominal interest rates are another type of interest rate that you will encounter, but they are actually related to the interest rates that you are already familiar with. This includes annual effective rates i, and non-yearly effective rates j.

By definition, nominal annual interest rates are just interest rates that are expressed "as stated" without any adjustment for the full effect of compounding. What this implies is that nominal rates are just another way to represent the interest rates we already know.

The notation for nominal annual interest rates is $i^{(m)}$ where m is number of times the rate is compounded, or **convertible,** per year.

To distinguish between nominal rates and other interest rates, it is important to pay attention to the wording:

Interest Rate:

Previously: "Monthly compound interest rate of 2%" ⟶

Now: "Nominal annual rate of 24% convertible monthly" ⟶

Those two rates are actually equivalent. A nominal rate $i^{(m)}$ divided by the number of times it is convertible per year m, yields j, the equivalent non-yearly rate per period m.

Conversion Formula 1: $\boxed{j = \frac{i^{(m)}}{m}}$

Now, since an effective annual rate i can be found from a non-yearly rate j using $i = (1+j)^m - 1$, and we know $j = \dfrac{i^{(m)}}{m}$, then we can also find an effective annual rate from a nominal interest rate with a different conversion formula:

Conversion Formula 2: $\boxed{i = \left(1 + \dfrac{i^{(m)}}{m}\right)^m - 1}$

Practice using the two conversion formulas by find the following equivalent rates for a nominal annual rate of 12% compounded monthly:

Effective Monthly Rate:	Effective Annual Rate:

It's important to know how to convert from a nominal interest rate to an effective annual rate or non-yearly rate because the "raw" percentage of a nominal interest rate will **never** actually be used in calculations. It will always need to be converted into equivalent rates.

<u>Ex.</u>

If the nominal annual interest rate is $18\,\%$, find the equivalent effective annual rates for $m = 2, m = 3, m = 6$.

Name: ___________________

Financial Mathematics Lesson 7 Examples: Nominal Interest Rates

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

What is the present value of $2000 due at the end of 8 years if $i^{(2)} = 0.05$?

Ex. 2

Using a nominal interest rate of i convertible semi-annually, an investment of \$1200 today and \$2300 at the end of year 1, will accumulate to \$3700 at the end of year 2. Calculate i.

Name: ________________

Financial Mathematics Lesson 8
Discount Rates

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Up to this point, all the interest rates we have been using pay/charge interest at the end of a period. This is referred to as "**interest payable in appears**." However, interest can also be paid at the beginning of a period, which is referred to as "**interest payable in advance**." Rates that pay interest in this way are called discount rates and are denoted with d.

For example, if you borrow $100 for one year at a discount rate of 10%, then that 10% interest would be charged at the beginning of the period. Now, 10% of $100 is $10, so the loan of $100 would be immediately charged $10 which means that you only receive $90 at the beginning of the period, while still needing to pay back the loan of $100 a year later.

So in this scenario: $A(1) =$, and $A(0) =$

Using these values, the effective annual interest rate for this scenario would be:

$$i = \frac{A(1) - A(0)}{A(0)} =$$

(It is different than the discount rate!)

Similarly, the effective annual **discount rate** can be represented by $\boxed{d = \frac{A(1) - A(0)}{A(1)}}$.

So, we can conclude that effective annual interest measures growth based on the initially invested amount (in this case $A(0)$), but an effective annual discount rate measures growth based on the accumulated amount at the end of the period (in this case $A(1)$).

By re-arranging $d = \dfrac{A(1) - A(0)}{A(1)}$, we can find more equations regarding discount rates:

This results in the following: $\boxed{v = 1 - d}$ $\boxed{v^t = (1 - d)^t}$ $\boxed{d = \dfrac{i}{1 + i}}$

<u>Ex.</u>

If $d = 0.03$, find v and i.

Similar to regular interest rates, discount rates can be simple or nominal:

Simple Discount Rates:

$$\boxed{(1 - d \cdot t)}$$

(Present Value Factor)

Nominal Annual Discount Rates
Convertible m times per year:

$$\boxed{1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m}$$

(Conversion formula)

To review, the differences between using effective interest rates and discount rates can be summarized in the following table: (Write the appropriate equation in each space)

	Future Value	Present Value
i		
d		

Name: ____________________

$\ddot{a}_{\overline{n}|i}$ Financial Mathematics Lesson 8 Examples: Discount Rates

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

The present value of an amount X in one quarter year is \$1994. What is X if the account has a 6 % effective annual discount rate? What is X if the account has an annual 6 % simple discount rate?

Ex. 2

If the nominal discount rate covnertible monthly is 36 %, what is the effective *monthly* discount rate, the effective *annual* discount rate, and the effective annual interest rate?

Ex. 3

Seth deposits \$250 into an account. A year and a half later, his deposit is worth \$400. What was Seth's nominal discount rate convertible semi-annually?

Name: ___________________

Financial Mathematics Lesson 9
Force of Interest

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Consider a scenario where the accumulated value of an investment is $A(t)$ and t is measured in years. The amount of interest earned by that investment in a ½ year period from time t to time $t+\frac{1}{2}$ would be:

And the semiannual interest rate i for that period would be:

Now, let's say that we want to express that rate as a nominal annual rate convertible semiannually, $i^{(2)}$. Then that nominal rate would be:

But what if we were to continuously increase the amount of times that the nominal annual rate is convertible for in a year? What happens as we increase the value of m towards infinity? To determine this, first generalize the form of that nominal annual rate by replacing each 2 with m in the equation from the previous step:

Using this general form, we can find $i^{(\infty)}$, a nominal annual rate that is convertible infinitely, or **convertible continuously**. This rate is what we refer to as the force of interest, and it can be found by taking the limit as m approaches ∞ of the equation above.

So we have: $i^{(\infty)} = \lim_{m \to \infty} \left(m \cdot \dfrac{A\left(t + \frac{1}{m}\right) - A(t)}{A(t)} \right)$

To evaluate this limit, first make it simpler by letting $\dfrac{1}{m} = h$ and then making the appropriate substitutions:

Then, rewrite the limit by pulling out any values that will not be effected by the limit:

The resulting limit should now look very familiar! It resembles the limit definition of a derivative from calculus! $\left(f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} \right)$

So, the limit can be replaced with the derivative of $A(t)$:

This results in the definition of the force of interest, which is denoted with δ_t .

$$\boxed{\delta_t = \frac{A'(t)}{A(t)}}$$

Similar to how the derivative represents the instantaneous rate of change in calculus, the force of interest is interpreted to be the **instantaneous rate of growth** of an investment per dollar invested at time t.

Using the definition of the force of interest, we can derive expressions for δ_t depending on the type of interest accumulation:

Simple Interest: $A(t) = A(0)(1 + i \cdot t)$

$$\delta_t = \frac{A'(t)}{A(t)} =$$

Compound Interest: $A(t) = A(0)(1 + i)^t$

$$\delta_t = \frac{A'(t)}{A(t)} =$$

Now, we can also use the force of interest to describe investment growth for future value and present value scenarios.

To do this, recall the derivative rule for the natural log function from calculus:

$$\frac{d}{du}\left[\ln(u)\right] = \frac{1}{u} \cdot u' = \frac{u'}{u}$$

This derivative rule is similar to the form of $\delta_t = \dfrac{A'(t)}{A(t)}$. By realizing this, we can write δ_t in a different way:

Then, by integrating this new form of δ_t from $t = 0$ to $t = n$, where n is some value of t in the future, we would have:

$$\int_0^n \delta_t \, dt =$$

By using the quotient property of the natural log, $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$, the result of the integral can be simplified further:

Now, solve for $A(n)$ by raising both sides of the equation as exponents of e :

$$\int_0^n \delta_t \, dt \ = \ \ln\left(\frac{A(n)}{A(0)}\right)$$

The resulting equation is the **future value** formula when using a force of interest:

$$\boxed{A(n) = A(0) \cdot e^{\int_0^n \delta_t \, dt}}$$

OR

$$\boxed{A(n) = A(0) \cdot \exp\left(\int_0^n \delta_t \, dt\right)}$$

If the equation is solved for $A(0)$ instead, then the result is the **present value** formula when using a force of interest:

$$\boxed{A(0) = A(n) \cdot e^{-\int_0^n \delta_t \, dt}}$$

OR

$$\boxed{A(0) = A(n) \cdot \exp\left(-\int_0^n \delta_t \, dt\right)}$$

Name: ___________________

Financial Mathematics Lesson 9 Examples: Force of Interest

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

If i is an annual effective interest rate of $5\,\%$, what is δ_t?
If i is an annual simple interest rate of $5\,\%$, what is δ_2?

Ex. 2

Given $\delta_t = 0.07 + 0.006t$, calculate the accumulated value over 5 years of an investment of \$100 made at time $t = 0$.
What is the accumulation if the \$100 is invested at $t = 2$ instead?

Ex. 3

For the period from time $t = 0$ to time $t = 2$, the force of interest is defined as follows:

$$\delta_t = \begin{cases} 0.06 & 0 \leq t \leq 1 \\ 0.06 + 0.02(t-1) & 1 < t \leq 2 \end{cases}$$

If \$1000 is invested at $t = 0$, find $A(1)$ and $A(2)$.

Name: ____________________

Financial Mathematics Lesson 10
Converting Interest Rates

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

One of the most crucial skills to develop in financial mathematics is the ability to correctly and efficiently convert between different types of interest rates.

So, now that you have seen all of the different types of interest rates you might encounter, it is time to take a comprehensive look at converting between them.

As a review, write in the type of interest rate that is represented by each notation below:

i -

j -

d -

b -

δ_t -

$i^{(m)}$ -

$d^{(m)}$ -

There are a variety of formulas that can be used to convert between these different rates. Some should be familiar, and others will be new. Do note however that not every type of rate will explicitly be equated to every other type of rate with a formula. In these instances, you may need to chain/use multiple formulas to convert from one type of rate to another.

Converting between j and i : $\boxed{i = (1+j)^m - 1}$

Ex.

What is the equivalent monthly effective interest rate if the annual effective interest rate is $6\,\%$?

Converting between b and d : $\boxed{d = 1 - (1-b)^m}$

Ex.

What is the equivalent annual effective discount rate if the quarterly effective discount rate is $8\,\%$?

Converting between $i^{(m)}$, j, and i : $\boxed{j = \dfrac{i^{(m)}}{m}}$ $\boxed{i = \left(1 + \dfrac{i^{(m)}}{m}\right)^m - 1}$

Ex.

If the nominal annual interest rate is $10\,\%$ convertible semiannually, what is the effective semiannual interest rate? The effective annual interest rate?

Converting between $d^{(m)}$, b, and d : $\boxed{b = \dfrac{d^{(m)}}{m}}$ $\boxed{d = 1 - \left(1 - \dfrac{d^{(m)}}{m}\right)^m}$

__Ex.__

If the nominal annual discount rate is 12 % convertible monthly, what is the effective monthly discount rate? The effective annual discount rate?

Converting between i and d : $\boxed{d = \dfrac{i}{1+i}}$

__Ex.__

What is the equivalent annual effective **discount** rate if the annual effective **interest** rate is 4 %?

Converting between i and δ_t (compound) : $\boxed{\delta_t = \ln(1+i)}$

__Ex.__

What is the force of interest if the effective annual **interest** rate is 9 %?

Converting between d and δ_t (compound) : $\boxed{\delta_t = -\ln(1-d)}$

Ex.
What is the force of interest if the effective annual **discount** rate is 7%?

Converting between i and δ_t (simple) : $\boxed{\delta_t = \dfrac{i}{1+i \cdot t}}$

Ex.
If the annual **simple interest** rate is 5%, what is δ_3?

Converting between d and δ_t (simple) : $\boxed{\delta_t = \dfrac{d}{1-d \cdot t}}$

Ex.
If the annual **simple discount** rate is 5%, what is δ_5?

Name: ____________________

Financial Mathematics Lesson 10 Examples: Converting Interest Rates

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

What is the equivalent annual effective discount rate if the two-year effective discount rate is 15%?

Ex. 2

What is the equivalent semiannual effective interest rate if the quarterly effective interest rate is 3%?

Ex. 3

What is the nominal annual interest rate convertible monthly if the quarterly effective interest rate is $2\,\%$?

Ex. 4

If the annual effective discount rate is $5\,\%$, what is the equivalent semiannual effective interest rate?

Ex. 5

What is the force of interest when the effective annual discount rate is $4\,\%$? When the monthly effective interest rate is $0.1\,\%$?

Name: ____________________

Financial Mathematics Lesson 11
Future Value of Annuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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An annuity is a series of periodic payments. For example, suppose you have 10 payments of \$100 and you deposit them one at a time into an account at the end of each year for a total of 10 years. This series of payments would be referred to as an annuity.

Now, if those deposits were made in an account that pays an effective annual interest rate of 5%, how would you determine the future value of the 10 deposits?

Using our current knowledge of calculating the future value for cash flows:

$$FV =$$

But this method will not be very efficient as the number of payments grows larger. It can quickly become a lot of work. To find a more efficient method, recall the concept of a geometric series and its sum from calculus:

Geometric Series: $\displaystyle\sum_{k=0}^{n} a \cdot r^k = a\left(r^0 + r^1 + r^2 + \cdots + r^{n-1} + r^n\right) = a \cdot \frac{1 - r^{n+1}}{1 - r}$

The general form of a geometric series is comparable to the method above that we would have used to calculate the future value of the annuity.

By letting $a = 100$, $r = 1.05$, and $n = 9$, we can express the future value of the annuity as a geometric series:

$$FV =$$

Then, using the sum of a geometric series, the future value of the annuity can be condensed into a simpler expression, which can then be easily evaluated to find the future value:

$$FV =$$

This sum can be generalized for any series of payments in order to create a much more efficient method of calculating the future value of an annuity.

That generalized form is shown below, but it can be simplified further into a nicer form:

$$FV = X \cdot \frac{1-(1+i)^n}{1-(1+i)} =$$

This results in the formula for the future value of an annuity, which is represented by the following notation:

Future Value of An Annuity: $\boxed{s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}}$

Note: n is the number of payments and i is the effective interest rate. Also, this notation assumes that the amount of each payment is $1, so if the payment is larger than $1, then the value of the amount X needs to be multiplied by the notation: $X \cdot s_{\overline{n}|i}$.

Practice using this notation with the example on the next page.

Ex.

Fred invests \$300 per year for a total of 20 years into an account that has an annual effective interest rate of 7%. How much does Fred have after the last \$300 is invested?

Note: The $s_{\overline{n}|i}$ notation can only be used provided that the following three conditions are satisfied:

1) The payments are of equal amount.

2) The payments are made at equal intervals of time, with the same frequency as the interest rate is compounded.

3) The accumulated value is found at the time of and including the final payment.

If the third requirement for using the notation is not met because the accumulated value is found at a time **after** the final payment is made, the notation can still be used as long as an adjustment is made. (Next page)

In the case of the example from the beginning of the lesson, suppose that we wanted to know the future value **3 years after** the final payment was made for the 10 payments of $100 paid every year for 10 years with an effective annual rate of 5%.

Using the notation, the future value *at year 10* would be:

Now, to find the future value 3 years later, for which no more payments are being made, we just need to multiply by the accumulation factor of $(1+i)$ to the power of 3. This will generate the interest for the value at year 10 for an additional 3 years.

So, the future value 3 years later would be:

To generalize this, the future value of an annuity at $t = n + k$, where k is the amount of extra periods, would be:

$$\boxed{s_{\overline{n|}i} \cdot (1+i)^k}$$

Name: ____________________

Financial Mathematics Lesson 11 Examples: Future Value of Annuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

Ryan invests \$200 at the end of each year for 15 years into an account that has a semiannual effective interest rate of $3\,\%$. How much does Ryan have after the last \$200 is invested?

Ex. 2

Zach deposits \$30 per month for 10 years into an account paying $0.5\,\%$ monthly interest for the first 3 years, and $0.8\,\%$ monthly interest the last 7 years. How much is in Zach's account after the final \$30 is deposited?

Ex. 3

In a series of 30 payments, the first 10 payments are \$10 each, the second 10 payments are \$20 each, and the final 10 payments are \$30 each. The payments are equally spaced and the interest rate is 5 % per payment period. Find the accumulated value at the time of the final payment.

Name: ____________________

Financial Mathematics Lesson 12
Present Value of Annuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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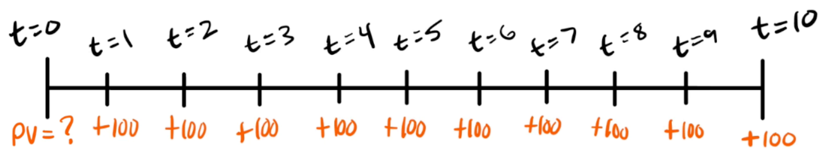
Suppose you want to open an account with a single deposit today such that you can withdrawal payments of $100 each year for 10 years from the account where the first withdrawal takes place one year from today. Given an effective annual interest rate of 6%, what should be the amount of that initial deposit?



The initial deposit in that scenario is the present value of an annuity. Using our current methods for calculating the present value of cash flows, the present value would be:

$$PV =$$

But this method will not be very efficient as the number of payments/withdrawals increases. As such, we desire a more efficient method. To find another method, similar to calculating the future value of an annuity, recall the concept of a geometric series and its sum from calculus:

Geometric Series: $$\sum_{k=0}^{n} a \cdot r^k = a\left(r^0 + r^1 + r^2 + \cdots + r^{n-1} + r^n\right) = a \cdot \frac{1 - r^{n+1}}{1 - r}$$

To match the general form of a geometric series, factor out $100v$ in the method used above:

$$PV =$$

Now, by letting $a = 100v$, $r = v$, and $n = 9$, we can express the present value of the annuity as a geometric series and then condense it into a simpler expression using the sum of a geometric series. Use that sum to calculate the present value for the scenario:

$$PV =$$

This sum can be generalized for finding the present value of any series of payments. That generalized form is shown below, but can be simplified further into a nicer form:

$$PV = X \cdot v \cdot \frac{1 - v^n}{1 - v} =$$

This results in the formula for the present value of an annuity (with payments $X = 1$), which is represented by the following notation:

Present Value of An Annuity: $\boxed{a_{\overline{n|}i} = \frac{1 - v^n}{i}}$

Ex.

Daniel wants to make a deposit into an account today so that he can withdrawal \$200 each year for 15 years, with the first withdrawal taking place one year from now. If the effective annual interest rate is 4%, how much should he deposit?

Note: The $a_{\overline{n|}i}$ notation can only be used provided that these three conditions are met:

1) The payments are of equal amount.

2) The payments are made at equal intervals of time, with the same frequency as the interest rate is compounded.

3) The valuation point is one payment period before the first payment is made.

Name: ________________

Financial Mathematics Lesson 12 Examples: Present Value of Annuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Sam wants to make a deposit today in an account paying a 2% effective quarterly interest rate so he can make withdrawals of \$500 each year for 20 years. If Sam wants to make the first withdrawal one year from today, how much should his deposit be?

Ex. 2

Joseph makes a deposit of \$6000 today into an account paying a nominal annual interest rate $i^{(2)} = 0.09$ so that he will be able to make withdrawals of equal amounts twice a year for 30 years starting one period after he makes his deposit. What is the amount of the withdrawals Joseph can make?

Ex. 3

In a series of 10 payments, the first 2 payments are \$20 each, the next 4 payments are \$30, and the last 4 payments are \$50 each. The payments are equally spaced and the interest rate is $4\,\%$ per payment period. What is the present value of the payments, given that the first payment takes place one period from today?

Name: ____________________

Financial Mathematics Lesson 13
Perpetuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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A perpetuity is a stream of payments that is infinite. In other words, it is an annuity, or a series of periodic payments, that never ends.

Now, suppose we want to make a deposit today in order to receive an infinite number of periodic payments in the future, how much should the deposit be? That deposit would be the present value of an annuity where the number of payments n is infinite.

So, we can represent this using the notation for the present value of an annuity as follows:

$$a_{\overline{n}|i} = \frac{1 - v^n}{i} \longrightarrow a_{\overline{\infty}|i} = ?$$

To determine the formula for the present value of a perpetuity, we cannot simply plug in ∞ as if it were any other number. Instead, we need to take the limit of the present value of an annuity formula as n approaches ∞:

$$a_{\overline{\infty}|i} = \lim_{n \to \infty} \frac{1 - v^n}{i} =$$

This results in the simple formula for the present value of a perpetuity: $\boxed{PV = \frac{1}{i}}$

Note that the formula assumes that the amount of each payment to be received in the future is \$1. If the amount of the payments is larger than \$1, then the amount of each payment X needs to be multiplied by the formula: $PV = X \cdot \frac{1}{i}$

Additionally, just like when using the formula for the present value of an annuity, it is important that the payment frequency and the interest rate frequency are the same in order to use the present value of a perpetuity formula (For example, *quarterly* payments **and** a *quarterly* interest rate).

Practice using this formula with the following example problems.

Ex.

Find the present value of a perpetuity that pays \$10 per year indefinitely at an annual rate of 4%.

Ex.

Find the present value of an annuity that pays \$20 per year for 15 years, starting one year from today, with an effective annual interest rate of 5%.
Compare this to the present value of a perpetuity with the same payment amount and interest rate.

Name: ____________________

Financial Mathematics Lesson 13 Examples: Perpetuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

How much should be deposited today in order to have a perpetuity that pays monthly payments of $50 indefinitely with an annual effective interest rate of 7%?

Ex. 2

If a deposit of $10,000 is made today to set up a perpetuity that makes quarterly payments indefinitely with a nominal annual interest rate of $i^{(4)} = 0.12$, what is the amount of each payment?

Ex. 3

Tyler wants to purchase a perpetuity paying \$2000 per year with the first payment to be received 11 years from today. He can purchase it by either:

1. paying \$1800 per year at the end of each year for 10 years; or
2. paying X per year at the end of each year for the first 5 years and then nothing for the next 5 years.

Find X.

Name: ____________________

Financial Mathematics Lesson 14
Annuity-Immediate vs Annuity-Due

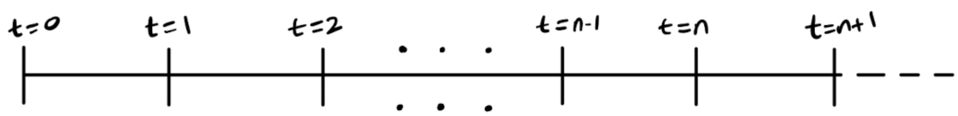
Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

Every annuity we have worked with thus far has been a type of annuity that is known as an annuity-immediate, where the periodic payments are made at the **end of each period**.

Annuity-Immediate Formulas: $\boxed{FV = s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}}$ $\boxed{PV = a_{\overline{n}|i} = \dfrac{1 - v^n}{i}}$

For future value scenarios, the accumulation of the deposits is valued *immediately* after the final deposit is made. So, given that n deposits of X are made at the end of each period, beginning with $t = 1$, the future value of the annuity is valued at $t = n$.

For present value scenarios, the annuity-immediate is valued one period *before* the first deposit is made, which in this case would be $t = 0$.



However, there is another type of annuity known as an annuity-due. For an annuity-due, the periodic payments/deposits are made at the **beginning of each period**.

In present value scenarios, an annuity-due is valued at the time the first deposit is made, which in this case is at $t = 1$. The present value of an annuity-due is denoted by $\ddot{a}_{\overline{n}|i}$.

In future value scenarios, an annuity due is valued one payment period *after* the final deposit. So, since the final deposit is made at $t = n$, the future value will be valued at $t = n + 1$. The future value of an annuity-due is denoted by $\ddot{s}_{\overline{n}|i}$.

After filling in the timeline (previous page), it is clear that the present and future values of an annuity-due are valued **one payment period after** the valuation of their annuity-immediate counterparts. So, we can multiply the values by $(1 + i)$ to bring them forward one year, and simplify the result to find distinct present and future value formulas for an annuity-due:

$$FV = \ddot{s}_{\overline{n}|i} = s_{\overline{n}|i}\,(1 + i)$$

$$PV = \ddot{a}_{\overline{n}|i} = a_{\overline{n}|i}\,(1 + i)$$

Annuity-Due Formulas: $\boxed{FV = \ddot{s}_{\overline{n}|i} = \dfrac{(1 + i)^n - 1}{1 - v}}$ $\boxed{PV = \ddot{a}_{\overline{n}|i} = \dfrac{1 - v^n}{1 - v}}$

Note: We can also calculate the present value of a perpetuity-due by multiplying the formula for a perpetuity-immediate by $(1 + i)$:

$$PV = \ddot{a}_{\overline{\infty}|i} = a_{\overline{\infty}|i}\,(1 + i) =$$

Ex.

Payments of \$60 are invested each year starting today for ten years. With an effective annual interest rate of 5%, what is the present value of these payments? What is the future value after ten years?

Name: ___________________

Financial Mathematics Lesson 14 Examples: Annuity-Immediate vs Annuity-Due

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Eric makes rental payments of $1500 per month and wants to know the present value over a 12-month period. The payments are made at the start of each month. The current nominal annual interest rate is 6 % convertible monthly. Find the present value.

Ex. 2

A company wants to invest $3700 every 6 months for 5 years to purchase a delivery truck. The investment will be compounded at an effective annual interest rate of 12.36 %. The initial investment will be made now, meaning all payments afterwards will be made at the beginning of every six months. What is the future value of the investments?

Ex. 3

Chuck will make deposits of \$500 at the end of each quarter for 10 years. At the end of 15 years, Chuck will use the fund to make annual payments of Y at the beginning of each year for 5 years, after which the fund is exhausted. The annual effective rate of interest is 7%. Determine Y.

Name: ____________

Financial Mathematics Lesson 15
Continuous Annuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

In each type of annuity that we have worked with thus far, the periodic payments were always made at specific points in time. In other words, they are practical annuities.

However, for theoretical purposes, it can be useful to consider annuities where the payments are made **continuously** over a period of time, known as continuous annuities.

For example, suppose an annuity *continually* pays a payment of \$1 per period with an effective annual rate i from $t = 0$ to some time in the future $t = n$. If we wanted to find the **future value** of this annuity at $t = n$, it will not be possible to calculate the accumulation of each continuous payment in the same way that we would for payments of other annuities.

Instead, we can determine the future value by using calculus. Specifically, by integrating the accumulation factor of $(1+i)^t$ from $t = 0$ to $t = n$ (as seen below), the result will be the accumulated value at $t = n$ for the continuous annuity:

$$\int_0^n (1+i)^t \, dt =$$

So, the future value of a continuous annuity (\$1 payments): $\boxed{FV = \bar{s}_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{\delta}}$

Similarly, to find the **present value** of a continuous annuity, integrate the present value factor v^t from $t = 0$ to $t = n$:

$$\int_0^n v^t \, dt =$$

So, the present value of a continuous annuity (\$1 payments): $\boxed{PV = \bar{a}_{\overline{n}|i} = \frac{1 - v^n}{\delta}}$

Note: The only difference between the continuous annuity formulas and their regular annuity-immediate counterparts is the denominator (δ instead of i). So the following is true:

$$\boxed{\bar{s}_{\overline{n}|i} = s_{\overline{n}|i} \cdot \frac{i}{\delta}} \qquad \boxed{\bar{a}_{\overline{n}|i} = a_{\overline{n}|i} \cdot \frac{i}{\delta}}$$

Ex.

An annuity pays \$100 continuously throughout the year for 12 years with an annual effective interest rate of 7%. Determine the present value and the future value of the annuity.

Now, if the interest for a continuous annuity is generated using δ_t , the force of interest, then we need to integrate the accumulation and present value factors for the force of interest:

$$\boxed{\bar{s}_{\overline{n}|\delta_t} = \int_0^n \exp\left(\int_t^n \delta_t \, dt\right) dt} \qquad \boxed{\bar{a}_{\overline{n}|\delta_t} = \int_0^n \exp\left(-\int_0^t \delta_t \, dt\right) dt}$$

Name: ____________________

Financial Mathematics Lesson 15 Examples: Continuous Annuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

Jason deposits \$200 into an account at the end of every month for 5 years. If the account has an effective annual interest rate of 12%, what is the future value of the account? Compare this to the future value if Jason were to theoretically make these deposits continually throughout each month instead.

Ex. 2

The force of interest at time t is $\delta_t = \dfrac{t^3}{10}$. Find the present value of a four-year continuous annuity which has a rate of payments at time t of $5t^3$.

Ex. 3

Find the future value of a two-year continuous annuity which has payments at time t of t^4 based on a force of interest at time t of $\delta_t = \dfrac{1}{t}$.

Name: ____________________

Financial Mathematics Lesson 16
Geometric Annuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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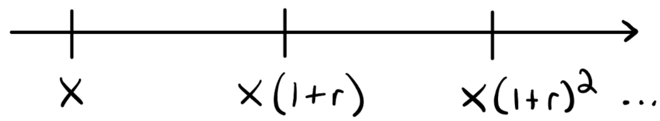
Sometimes an annuity will have periodic payments that **increase or decrease** with every payment period. One way that the payments can increase or decrease is in the form of a **geometric progression**. When the payments of an annuity form this type of progression, it is known as a geometric annuity.



In a geometric annuity, each subsequent payment X increases or decreases by a particular rate or percent r. For example, if every payment increases by 3% each period, then $r = .03$. Or if every payment decreases by 3% each period, then $r = -.03$.

To find the present or future value of a geometric annuity, use the following 4 step process:

1) Value each payment at the valuation date.

2) Factor out the first term.

3) Rewrite the remaining factor as a sum/geometric series.

4) Compute.

Practice using this process in a present and future value scenario with the examples on the next page. (Note: There is no special notation for geometric annuities)

Ex.

Find the present value of a series of payments made at the end of each year for 10 years starting one year from today that increase by 5% each year with an annual effective interest rate $i = 7\%$.

Step 1:

Step 2:

Step 3:

Step 4:

Ex.

Find the future value at year 10 of a series of payments made at the end of each year for 10 years that increase by 5% each year with an annual effective interest rate $i = 7\%$.

Step 1:

Step 2:

Step 3:

Step 4:

Name: ___________________

Financial Mathematics Lesson 16 Examples: Geometric Annuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

An insurance company has an obligation to pay the medical costs for a claimant. The cost of annual claims today are $3000 and medical inflation is 6 % per year. The claimant is expected to live for 10 more years. Claim payments are made at yearly intervals with the first payment to be made one year from today. Find the present value of the obligation if the annual effective interest rate is 5 %.

Ex. 2

Nick wants to make deposits into an account with a 4 % annual interest rate. His first payment is \$1000 and he decreases the payment amount by 2 % every year thereafter. If he makes these deposits for 15 years at the end of each year, what is the future value after 15 years?

Name: ____________________

Financial Mathematics Lesson 17
Arithmetic Increasing Annuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Another way that the payments of an annuity can increase or decrease is in the form of an **arithmetic progression**. When the payments of an annuity form this type of progression, it is known as an arithmetic annuity.

Unlike geometric annuities whose payments form a geometric progression by increasing or decreasing by a certain rate r each period, arithmetic annuities involve payments that increase or decrease by a **set amount** each period. In this particular lesson we are only going to focus on arithmetic annuities where the payments *increase* every period.

Compare the payment progression for a geometric annuity versus an arithmetic annuity given an initial payment of \$100 if the geometric progression is an increase by a rate r, and the arithmetic progression is an increase by an additional \$100 every period:

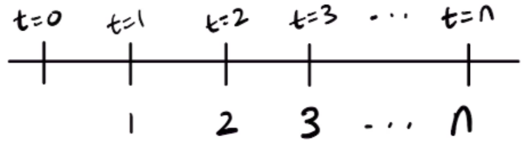
Geometric:

Arithmetic:

The simplest form of an arithmetic increasing annuity would be to start with an initial payment of \$1 that increases by \$1 every payment period thereafter. Analyzing this simplest case will allow us to find nice closed formulas for calculating the future value and present value of arithmetic increasing annuities.

The process to find these closed formulas will seem fairly abstract, but it is ultimately necessary to obtain them.

Arithmetic Increasing Annuities



Begin by expressing the present value of the arithmetic annuity by valuing each of the payments at the valuation date of $t = 0$: (assume this is an annuity-immediate)

$$PV =$$

We will denote the present value of an arithmetic increasing annuity with $(Ia)_{\overline{n}|i}$.

$$(Ia)_{\overline{n}|i} =$$

However, to make working with this equation easier, we will temporarily let $s = (Ia)_{\overline{n}|i}$.

Additionally, rewrite each present value factor using $v^t = \dfrac{1}{(1+i)^t}$:

$$s =$$

Now, create a second equation that brings s one year forward by multiplying by $(1+i)$:

$$s(1+i) =$$

Next, subtract s from $s(1+i)$:

$$s(1+i) - s =$$

Then, distribute s through $(1+i)$ and rewrite part of the equation as a geometric series:

Simplify by removing any terms that cancel, and then rewrite the geometric series by calculating its sum $\left(\text{Recall: } \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} \right)$:

We can now recognize that the sum of that geometric series is actually the formula for the present value of an annuity due, $\ddot{a}_{\overline{n}|i}$. Rewrite the equation using that notation:

By diving both sides by i we can solve for s :

$s =$

Now, substituting back for s with the notation of the present value of an arithmetic increasing annuity yields the following formula:

$$\boxed{(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}}$$

The formula above calculates the present value of an arithmetic increasing annuity-immediate that begins with a payment of \$1 and increases by \$1 each period. By multiplying that formula by some payment amount X, we can calculate the present value of the same annuity except where the first payment is X and increases by X every period.

From this formula for the present value of an arithmetic increasing annuity-immediate, we can find all other formulas regarding an annuity with payments that form an arithmetic increasing progression. This includes the future value of an arithmetic increasing annuity-immediate, and the future and present values of an arithmetic increasing **annuity-due**.

Arithmetic Increasing Annuities

Fill in this table with the appropriate formulas for each type of increasing arithmetic annuity:

	Annuity-Immediate	Annuity-Due
PV $\cdot (1+i)^n$	$(Ia)_{\overline{n}\vert i} =$	$(I\ddot{a})_{\overline{n}\vert i} =$
FV	$(Is)_{\overline{n}\vert i} =$	$(I\ddot{s})_{\overline{n}\vert i} =$

(Optional) Show the work for finding the formula for the future value of an arithmetic increasing annuity-immediate in the space below:

$$(Is)_{\overline{n}|i} = (1+i)^n \cdot (Ia)_{\overline{n}|i} =$$

Name: ___________________

Financial Mathematics Lesson 17 Examples: Arithmetic Increasing Annuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Samuel starts to make annual deposits with an initial deposit one year from today of \$7. He increases the amount of each subsequent payment by \$7 until he reaches his final payment of \$77. What is the future value of Samuel's payments if the account he makes the deposits in pays an annual effective interest rate of 5%?

Ex. 2

An annuity immediate has a first payment of \$200 and increases by \$100 each year until it reaches \$600, after which the payments stop. If the annual effective interest rate is 4%, find the present value of the annuity.

Name: ________________

Financial Mathematics Lesson 18
Arithmetic Decreasing Annuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Previously we looked at arithmetic increasing annuities where the payments of the annuity form an increasing arithmetic progression. Each payment increased by a set amount. Now we want to look at arithmetic *decreasing* annuities, where the payments form a *decreasing* arithmetic progression. Each payment will decrease by a set amount.

As a comparison, complete the arithmetic sequences below that represent the amount of the payments for an annuity where each payment changes by an amount of 100:

Increasing: 100,

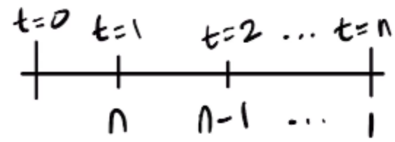
Decreasing: 900,

A simpler or more general form of the decreasing arithmetic progression would be an initial payment of n that decreases by 1 each period until the payment amount reaches 1, where then the payments stop. This creates a sequence of: $n,\ n-1,\ n-2, \ldots, 1$.

Analyzing this simplest case will allow us to find nice closed formulas for calculating the future vale and present value of arithmetic decreasing annuities.

The process is very similar to what was used to find the closed formulas for arithmetic increasing annuities. It will seem fairly abstract, but ultimately is necessary to obtain the them.

Arithmetic Decreasing Annuities



Begin by expressing the present value of the arithmetic annuity by valuing each of the payments at the valuation date of $t = 0$: (assume this is an annuity-immediate)

$$PV =$$

We will donate the present value of an arithmetic decreasing annuity with $(Da)_{\overline{n}|i}$.

$$(Da)_{\overline{n}|i} =$$

However, to make working with this equation easier, we will temporarily let $s = (Da)_{\overline{n}|i}$.

Additionally, rewrite each present value factor using $v^t = \dfrac{1}{(1+i)^t}$:

$$s =$$

Now, create a second equation that brings s one year forward by multiplying by $(1+i)$:

$$s(1+i) =$$

Next, subtract s from $s(1+i)$:

$$s(1+i) - s =$$

Then, distribute s through $(1+i)$ and pull out a negative sign from the present value factors:

We can now recognize the sum of the present value factors in that equation to be the present value of a regular annuity-immediate, $a_{\overline{n}|i}$. Rewrite the equation using that notation, and simplify the other side of the equation by removing any terms that cancel:

By dividing both sides of the equation by i we can solve for s :

$s =$

Now, substituting back for s with the notation of the present value of an arithmetic decreasing annuity yields the following formula:

$$\boxed{(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i}}$$

The formula above calculates the present value of an arithmetic decreasing annuity-immediate that begins with a payment of n dollars and decreases by $1 each period.

By multiplying the formula by some other payment amount X, you would be calculating the present value of the same annuity except where the first payment is $X \cdot n$ and decreases by X every period (Note that if $X = 1$, the first payment is $1 \cdot n = n$, so we know this is true).

From this formula for the present value of an arithmetic decreasing annuity-immediate, we can find all other formulas regarding an annuity with payments that form a decreasing arithmetic progression. This includes the future value of an arithmetic decreasing annuity-immediate, and the future and present values of an arithmetic decreasing **annuity-due**.

Fill in this table with the appropriate formulas for each type of decreasing arithmetic annuity:

	Annuity-Immediate	Annuity-Due
PV $\cdot (1+i)^n$	$(Da)_{\overline{n}\vert i} =$	$(D\ddot{a})_{\overline{n}\vert i} =$
FV	$(Ds)_{\overline{n}\vert i} =$	$(D\ddot{s})_{\overline{n}\vert i} =$

(Optional) Show the work for finding the formula for the future value of an arithmetic decreasing annuity in the space below:

$$(Ds)_{\overline{n}|i} = (1+i)^n \cdot (Da)_{\overline{n}|i} =$$

Ex.

A 10-year annuity-immediate has a first payment of \$2000 and decreases by \$200 each year thereafter. Assuming an effective annual interest rate of 6 %, calculate the present value.

Name: ____________________

Financial Mathematics Lesson 18 Examples: Arithmetic Decreasing Annuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

A 15-year annuity-immediate has a first payment of \$4500 and each subsequent year the payment decreases by \$300. At an effective annual interest rate of 7 %, determine the future value.

<u>Ex. 2</u>

An annuity-due pays \$20 per year for 10 years and then decreases by \$1 per year for 19 years. If the effective annual interest rate is $5\,\%$, calculate the present value.

Name: ______________________

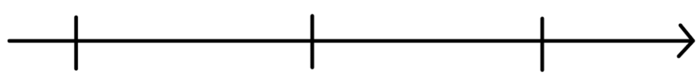
Financial Mathematics Lesson 19
Increasing Perpetuities

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

In the same way that annuities with a finite number of payments can have payments that form a geometric or arithmetic progression, annuities with an infinite number of payments, which are called perpetuities, can also have payments that form these different progressions.

In particular, we are interested in the present value of increasing perpetuities, or an annuity with an infinite number of increasing payments. We are not interested in "decreasing perpetuities," because at some point, depending on the type of progression, the payments either become 0 or infinitely small. Thus, at some point the payments will "stop"because they are either nonexistent or of a negligible amount, defeating the purpose of a perpetuity.

An increasing perpetuity can have payments that either form a geometric or arithmetic progression. Fill in the timeline below with an example of each:



Geometric:

Arithmetic:

We will begin with finding the present value of **geometric perpetuities** by revisiting the 4 step process for calculating geometric annuities, but with some minor adjustments:

1) Value each payment at the valuation date.

2) Factor out the first term.

3) Rewrite the remaining factor as an infinite geometric series.

4) Solve.

$$\left(\text{Note: The sum of an infinite geometric series is: } \sum_{k=0}^{\infty} m^k = \frac{1}{1-m}\right)$$

Ex.

Find the present value of an infinite series of payments made at the end of each year starting one year from today with an initial payment of \$50 that increases by 5 % each year with an annual effective interest rate $i = 7\%$.

Step 1:

Step 2:

Step 3:

Step 4:

Now, if you were to calculate the present value of some other geometric perpetuities, you would start to recognize a pattern.

The final expression of each calculation will be the first term from Step 1 divided by $\left(1 - \frac{1+r}{1+i}\right)$ where r is the rate that each payment increases by, and i is the effective interest rate.

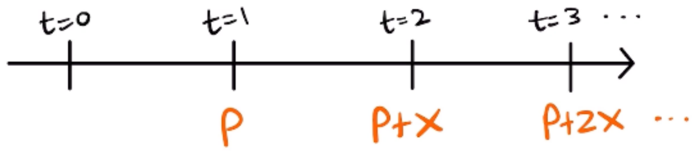
As a result of this pattern, we can shorten the 4 step process to be a simpler 2 step process:

1) Value each payment at the valuation date.

2) Solve using the formula: $PV = \dfrac{\text{[First Term]}}{\left(1 - \dfrac{1+r}{1+i}\right)}$

This 2 step process will work for finding the present value of any geometric perpetuity.

To find the present value of an increasing perpetuity with an *arithmetic* progression, or an **arithmetic perpetuity**, consider the timeline below:



We will treat the payments for an arithmetic perpetuity as **two series** of payments. One series will involve each payment of P that is made every period, and the other series will involve the increase in the payments of X every period after the intimal payment.

Separate the payments into those two series visually in the timeline below:

The present value of the first series of payments involving a consistent amount P would be calculated using the present value notation for a regular perpetuity:

$$PV =$$

The present value of the second series of payments involving the increase of an amount X each period would be calculated using the present value notation for a regular arithmetic increasing annuity-immediate, except where the number of payments n would be ∞ :

$$PV =$$

Now, by adding the two present value calculations together, we would be calculating the present value of both series of payments, which is the arithmetic perpetuity. However, do notice that the second series of payments begins one period **after** the first series, so it will need to be brought back one period by multiplying by v. So the combined equation is:

$$PV =$$

At this point we know that $a_{\overline{\infty}|i} = \dfrac{1}{i}$ and $(Ia)_{\overline{n}|i} = \dfrac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$, but we do not know what the notation $(Ia)_{\overline{\infty}|i}$ is equal to. To find this, we cannot simply plug ∞ into the formula. Instead we will need to take the limit of $\dfrac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$ as n approaches ∞.

To start, notice that as n approaches ∞ for $\ddot{a}_{\overline{n}|i}$, we are calculating the present value of a perpetuity-due. So $\ddot{a}_{\overline{\infty}|i} = \dfrac{1}{1-v} = \dfrac{1}{d}$, where d is the discount rate. This gives us:

$$\lim_{n\to\infty} \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} =$$

Now, the only part of the limit that is directly effected by n approaching ∞ is nv^n in the numerator. By taking the limit of this part separately, we can rewrite the present value factor and then use L'Hopital's rule to evaluate the limit:

$$\lim_{n\to\infty} nv^n =$$

Since the evaluation of that limit is 0, we have: $\left(\text{Recall that } d = \dfrac{i}{1+i}\right)$

$$\lim_{n\to\infty} \frac{\frac{1}{d} - nv^n}{i} =$$

Substituting this value back into the present value equation gives the equation below. Simplify it by rewriting the present value factor v and canceling any common factors:

$$PV = \frac{P}{i} + X \cdot \frac{1+i}{i^2} \cdot v =$$

This results in the present value formula for an arithmetic increasing perpetuity-immediate:

$$\boxed{PV = (Ia)_{\overline{\infty}|i} = \frac{P}{i} + \frac{X}{i^2}}$$

(Where P is the first payment and X is the amount that each payment increases by)

Name: ____________________

$\ddot{a}_{\overline{n}|i}$ Financial Mathematics Lesson 19 Examples: Increasing Perpetuities

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

A perpetuity-immediate has an initial payment of \$100. Each subsequent payment increases by \$25 for each year. If the annual effective interest rate is 2%, find the present value.

Ex. 2

The first payment of a perpetuity-immediate is \$500. Each subsequent payment increases by 3% each year. Determine the present value if the annual effective interest rate is 4%.

Ex. 3

At an annual effective interest rate of i, the present value of a perpetuity-immediate starting with a payment of \$300 the first year and increasing by \$50 for each year thereafter is \$92,000. Find i.

Name: ________________

Financial Mathematics Lesson 20
Level Payment Amortization of Loans

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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One application of annuities is the process of repaying a loan with a series of payments. When a loan is repaid in this way, the total of the payments must cover the original amount of then loan (called the **principal**) as well as the accumulated interest on the loan.

As such, each loan payment is split up into two components: (1) the amount of principal, and (2) the amount of interest.

There are many different ways that loan repayment can be set up, but we are only going to focus on one, which is the most common method of loan repayment known as the amortization method. Before looking at how this method works, it is important to become familiar with the various components of the method.

Amortization Method Components:

L -

K -

n -

i -

OB_t -

I_t -

PR_t -

Each of these components are related to one another in some way and play an important part in calculations for the amortization method. This relationship between the components is best demonstrated with an example problem.

Ex.

You have a loan of \$30,000 (total principal), and in order to repay the loan, you make 5 level payments at the end of each year for 5 years, at an effective annual interest rate of 8%.

Start by identifying the value of the known components:

$L =$ $n =$ $i =$ $K = ?$

To determine the value of the level payments K, set up a present value equation for an annuity-immediate with $a_{\overline{n}|i}$ using the given values of n and i, where L is the present value:

$$PV = K \cdot a_{\overline{n}|i} \implies$$

Now that we know the amount of each level payment, we can begin to find the interest paid, the principal paid, and the outstanding balance for the loan in each year.

Before the first payment at $t = 1$, it is important to make note of the original outstanding balance at $t = 0$. This will be the same amount as the loan L.

$\underline{t = 0:}$ $OB_0 =$

Then, to find the interest paid in each year, multiply the current outstanding balance by the interest rate i. The principal paid in each year will be the amount left over after subtracting the interest paid from the payment K. Subtracting the principal from the previous outstanding balance will produce the new outstanding balance for that year.

$\underline{t = 1:}$ $I_1 =$

$PR_1 =$

$OB_1 =$

$\underline{t = 2:}$ $I_2 =$

$PR_2 =$

$OB_2 =$

Now, this process could be repeated for all 5 years. We could continue to calculate the interest paid, principal paid, and the outstanding balance in each year for all 5 level payments, using the same method from the first two years. If we did, we would find that at the end of $t = 5$ the outstanding balance OB_5 would be 0 as the loan would be completely repaid. Once all 5 payments are made, the loan repayment and amortization is finished.

From the process demonstrated in the example, we can generalize calculating the interest paid, principal paid, and outstanding balance with the following formulas:

$$\boxed{I_{t+1} = OB_t \cdot i}$$

$$\boxed{\begin{aligned} PR_{t+1} &= K_{t+1} - I_{t+1} \\ &= K_{t+1} - OB_t \cdot i \end{aligned}}$$

$$OB_{t+1} = OB_t - PR_{t+1}$$

OR

$$OB_{t+1} = OB_t(1+i) - K_{t+1}$$

Additionally, since we are working with **level** payments (they do not change), there are some additional formulas that can be helpful to know:

$$\boxed{I_t = K\left(1 - v^{n-t+1}\right)}$$

$$\boxed{\begin{aligned} &PR_{t+1} = PR_t(1+i) \\ &PR_t = K \cdot v^{n-t+1} \end{aligned}}$$

If for some reason you do not have level payments, then the formulas above **cannot** be used.

Now, notice that in the second set of formulas for level payments that we have formulas for calculating the interest I_t or principal PR_t at any time t without the need for any other values of interest, principal, or the outstanding balance.

However, what is not included is a formula that calculates the outstanding balance at any time t without the need for any other values. This is because there is a bit more work involved with calculating the outstanding balance at any time, which will be demonstrated in the next example.

Ex.

A loan of \$2000 with an effective monthly interest rate of 1% is to be amortized by equal payments at the end of each month over a period of 18 months. Find the outstanding balance at the end of the first 8 months.

First, identify all known and unknown values:

$L =$ $n =$ $i =$ $K = ?$ $OB_8 = ?$

In order to calculate OB_8, we will need to know the amount of each level payment K. Similar to the example from earlier in this lesson, find K by setting up a present value equation:

$$PV = K \cdot a_{\overline{n}|i} \implies$$

Now, there are two methods to calculate the outstanding balance in this example: the prospective method and the retrospective method. Each method will give the same answer, but the thinking process behind them is different.

1) Prospective Method: (How many payments/periods are left?)

$$OB_8 =$$

2) Retrospective Method: (How much of the loan has already been paid?)

$$OB_8 =$$

We can generalize the prospective and retrospective methods with the following formulas:

$$\boxed{\text{Prospective: } OB_t = K \cdot a_{\overline{n-t}|i}}$$

$$\boxed{\text{Retrospective: } OB_t = L(1+i)^t - K \cdot s_{\overline{t}|i}}$$

Name: ________________

Financial Mathematics Lesson 20 Examples: Level Payment Amortization of Loans

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

Fill in the blanks of the following amortization schedule for a loan with 4 level payments.

t	OB_t	I_t	PR_t
0		x	x
1	732.84	57.00	217.16
2			
3			
4	0.00		

Ex. 2

A 30 year monthly payment mortgage loan for \$300,000 is offered at an effective monthly interest rate of 0.5 %. Find (1) the monthly payment, (2) the total principal and interest paid, (3) the outstanding balance in 5 years, and (4) the principal and interest paid in those 5 years.

Name: ____________________

Financial Mathematics Lesson 21
Bond Valuation

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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A bond is a debt that requires periodic interest payments called "**coupons**," for a stated term, as well as the repayment of the principal (original debt amount) at the end of the term.

Those who sell bonds are referred to as sellers, issuers, or borrowers, and those who purchase bonds are referred to as buyers, lenders, or bond holders. These terms are important when reading problems dealing with bonds.

To value the price of a bond, we need to be familiar with the various components of a bond:

Components of a Bond:

F -

C -

r -

$F \cdot r$ -

n -

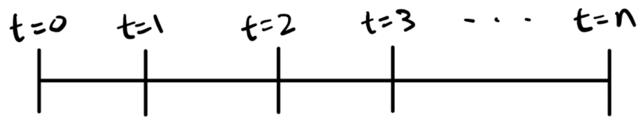
P -

j -

A couple things to remember about bonds:

- The face value F and redemption value C are the same unless stated otherwise
- Coupons are assumed to be paid semiannually unless stated otherwise
- The coupon rate and yield rate will always be given as **nominal interest rates** convertible semiannually
- The price of the bond is equal to the present value of the series of coupons plus the redemption value

Now, to calculate the price of a bond, consider the timeline below for a bond with n coupon periods:



In the timeline, label where the price of the bond P is valued, where each coupon $F \cdot r$ would be paid, and where the redemption value C would be paid.

Since the price of the bond is equal to the present value of all the payments in the timeline, we can create the following formula for the price of a bond:

$$\boxed{P = F \cdot r \cdot a_{\overline{n}|j} + C \cdot v_j^n}$$

Ex.

A 10 year bond issued for \$100 pays 5% coupons semiannually and yields a nominal interest rate of 6% convertible semiannually. Calculate the price of the bond.

What if the yield rate was 4% ? $j =$

$P =$

What if the yield rate is the same as the coupon rate at 5% ? $j =$

$P =$

From the results of this example we can conclude the following:

If $j = r$, then $P = F$ (buying at "par")

If $j > r$, then $P < F$ (buying at a discount)

If $j < r$, then $P > F$ (buying at a premium)

(The first statement is why the term "par value" is sometimes used to refer to the face value)

Name: ___________________

Financial Mathematics Lesson 21 Examples: Bond Valuation

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Find the price of a 15 year \$500 par value bond that pays 8 % coupons semiannually, and yields an annual nominal interest rate of 7 % convertible semiannually.
If this bond was purchased at the calculated price, would it be bought at a premium or at a discount?

Ex. 2

A 20 year \$1000 par value bond pays 2 % coupons semiannually. The bond is priced at \$877.98 to yield an annual nominal interest rate of 3 % compounded semiannually. What is the redemption value of the bond?

Ex. 3

A bond with a par value of \$2000 paying 8.5 % coupons semiannually, and redeemable at \$2050 is bought to yield a nominal interest rate of 10 % convertible semiannually. If the present value of the redemption amount is \$410, what is the price of the bond?

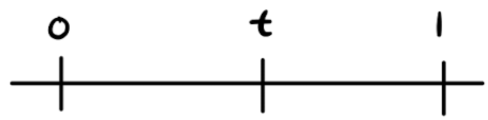
Name: ____________________

Financial Mathematics Lesson 22
Market Price of Bonds

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

Sometimes a bond will need to be priced at a particular moment in time that takes place **between two coupon periods** rather than being priced at the end of a coupon period. When this occurs, the price of the bond is referred to as the market price, or "clean price."

The market price of a bond is denoted with P_t, where t is a value between 0 and 1, and is measured from the last coupon payment/period. Label this on the timeline below:



At $t = 0$ on the timeline, we would also have another value P_0, which represents the price of the bond after the last coupon period (where the last coupon was paid).

To calculate the market price P_t, we first take the value of P_0 and accumulate it to time t by multiplying by the appropriate accumulation factor using the yield rate of the bond j. We call this the "price-plus-accrued" or sometimes refer to it as the "dirty price." This is the price of the bond between coupon periods that **includes** the accrued interest/fractional coupon.

Price-Plus-Accrued: $P_t =$

However, this value is **not** the value of the market price. By definition, the market price is the price of the bond between coupon periods that **excludes** the accrued interest/fractional coupon. Therefore we need to subtract out the fractional coupon because the entire coupon would not be paid until the next coupon period (P_t is between coupon periods).

So, if we subtract the fractional coupon amount, which would be calculated by multiplying t by the amount of each coupon for the bond, which is the face value F times the coupon rate r , then we can find the market price:

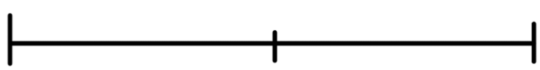
$$\boxed{\text{Market Price: } P_t = P_0(1+j)^t - t \cdot Fr}$$

Note: The numerical value of t can be found by dividing the number of days since the last coupon was paid by the number of total number of days in the coupon period:

$$\boxed{t = \frac{\text{\# of days since last coupon paid}}{\text{\# of days in coupon period}}}$$

<u>Ex.</u>

A bond with a par value of \$1000 has payment dates of January 21 and July 21 for each year of its term. The bond pays coupons at a rate of 8 % semiannually, and matures on January 21, 2009. Find the market price of the bond if it is sold on March 10, 2007, given that the last coupon paid was on January 21, 2007, and the yield rate is 9 % convertible semiannually.



Name: ________________

Financial Mathematics Lesson 22 Examples: Market Price of Bonds

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

A \$100 par value bond pays 7 % coupons semiannually. The coupons are paid on January 11 and July 11 for each year of its term, and it will mature on January 11, 2050. Determine the market price of the bond if it is sold on March 5, 2022, given that the last coupon paid was on January 11, 2022, and the yield rate is 6 % convertible semiannually.

Ex. 2

A financial newspaper lists a price on February 19, 2020 of a 5 % bond with a face amount of $1000 maturing on April 1, 2040. The yield rate was listed to be 5.5 %. Find the price that was listed in the newspaper.

Name: ____________________

Financial Mathematics Lesson 23
Amortization for Bonds

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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When working with bonds it is sometimes necessary to determine the amount of interest received or principal returned in a bond coupon or redemption payment. This can be done by **viewing a bond as an amortized loan**.

In viewing the bond as an amortized loan, we can use the same amortization method used for loan repayment in the past. However, a few of the components will change to more accurately reflect the terminology/components of a bond:

<u>Components of Amortization for Bonds:</u>

$L \longrightarrow$

$OB_t \longrightarrow$

n - # of payments (same)

I_t - interest paid (same)

$K \longrightarrow$

$\longrightarrow$

PR_t - principal paid (same)

$i \longrightarrow$

Using these new equivalent components, we can rewrite the formulas for calculating interest, principal, and the outstanding balance (book value) for the amortization method as follows:

$$\boxed{I_{t+1} = BV_t \cdot j}$$

$$\boxed{\begin{aligned} PR_{t+1} &= F \cdot r - I_{t+1} \\ &= F \cdot r - BV_t \cdot j \end{aligned}}$$

$$\boxed{\begin{gathered} BV_{t+1} = BV_t - PR_{t+1} \\ \text{OR} \\ BV_{t+1} = BV_t(1+j) - F \cdot r \end{gathered}}$$

However, note that the calculation for the final principal will be slightly different, as the final payment of a bond includes the face/redemption value F with the last coupon payment $F \cdot r$.

Ex.

A 2 year 1000 par value bond has a coupon rate of 6 % convertible semiannually. It is sold at a yield rate of 5 % convertible semiannually. Fill in the amortization table for this bond.

t	BV_t	K	I_t	PR_t
0		——	——	——
1				
2				
3				
4				

Note: If this example instead had a yield rate that was *larger* than the coupon rate $(j > r)$, the price of the bond will be less than the face value (bought at a discount), and as a result the principal amounts will be negative. This is not an issue, it simply occurs because the interest paid will be greater than the actual coupon payments. Everything will even out since the final payment includes the face/redemption value in addition to the coupon.

Some extra formulas to know regarding amortization for bonds:

$$\boxed{PR_t = F(r-j)v_j^{n-t+1}} \qquad \boxed{BV_t = F \cdot r \cdot a_{\overline{n-t}|j} + C \cdot v_j^{n-t}}$$

These formulas can calculate the principal paid or book value at any particular moment in time without the need for any other interest, principal, or book value calculations.

Name: ____________________

Financial Mathematics Lesson 23 Examples: Amortization for Bonds

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

A 2 year 100 par value bond pays 4% coupons semiannually. The bond is sold at a yield rate of 6% convertible semiannually. Fill in the amortization table for this bond.

t	BV_t	K	I_t	PR_t
0		——	——	——
1				
2				
3				
4				

Ex. 2

A 10 year bond with a par value of 10,000 and semiannual coupons at a rate of 5% convertible semiannually is bought to yield 7% convertible semiannually. Calculate (1) the book value immediately after the 5th coupon and (2) the principal immediately after the 10th coupon.

Name: ________________

Financial Mathematics Lesson 24
Callable Bonds

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Sometimes when a bond is issued, the issuer of the bond will give a range of dates where the bond can be redeemed. This means that when this type of bond is purchased, the issuer and purchaser do not know when the bond will be redeemed, but agree upon a set of dates where it *could* be redeemed.

This type of bond is referred to as a callable bond, because it can be "called," or redeemed at any time within a predetermined range of possible redemption dates. The actual date of redemption is chosen at a time in the future.

When calculating the price of a callable bond, it is best to choose the price that is most beneficial for the purchaser (the lowest price). This depends on whether the bond is purchased at a premium or a discount, which is based on the yield rate and coupon rate.

Fill in and complete the table below that outlines the general process for determining when to redeem a callable bond:

Bond Type	Coupon Rate / Yield Rate	Calculate price at...
	$r > j$	
	$r < j$	

We will demonstrate that this table is true with an example problem on the next page.

Ex.

A 10 year 1000 par value bond with 10 % semiannual coupons is callable on any coupon date after the first 6 years. If the yield rate is 8 % convertible semiannually, calculate the price of the bond.

What if the coupon rate remains the same, but the yield rate is 12% ? $j =$

Note: If the coupon rate and yield rate are the same $(r = j)$, then it does not matter when the callable bond is redeemed, the price will be the same regardless.

Name: ____________________

$\ddot{a}_{\overline{n}|i}$ Financial Mathematics Lesson 24 Examples: Callable Bonds

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

A bond with a face amount of 100 that pays 8 % semiannual coupons is callable on any coupon date from 15 and one half years after issue up to the maturity date which is 20 years from the issue. Find the price of the bond to yield a minimum nominal rate of (a) 10 %, (b) 8 %, and (c) 6 %.

Ex. 2

A 1000 par value bond has 7% semiannual coupons and is callable at the end of the 11th through the 16th years at par. Determine the prices to yield a minimum nominal rate of 7.5% and 6.5%.

Name: ____________________

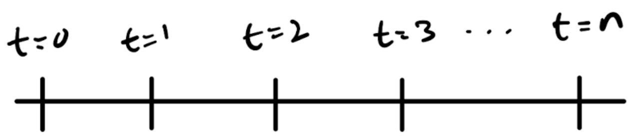
Financial Mathematics Lesson 25
Internal Rate of Return

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

A typical financial transaction involves a number of payments made, called **cashflows out**, as well as a number of payments received, called **cashflows in**, that are made at various moments in time. The interest rate for the transaction at which the value of all the cashflows out are equal to the cashflows in is called the internal rate of return (IRR).

For example, in the price of a bond, $P = F \cdot r \cdot a_{\overline{n}|j} + C \cdot v_j^n$, the yield rate j would be the IRR because it is the rate at which the price paid for the bond (cash flow out) is equal to the present value of the coupons and redemption amount to be received (cash flows in).

In a more general scenario, consider a transaction consisting of an amount C_0 invested at $t = 0$, that allows for several future payments of C_1, C_2, C_3, ... , C_n to be received at times $t = 1, 2, 3, \ldots, n$ respectively. (Write these values in the timeline below)



Set up an equation of value where C_0 is the present value of the other payments:

$C_0 =$

Now, by rewriting each present value factor using $v^t = \dfrac{1}{(1+i)^t}$ in this equation, we can identify that i is the IRR for this transaction:

$C_0 =$

So, if we know the values of C_0, C_1, C_2, and so on, we could solve for i and find the value of the IRR. To generalize this process of solving for the IRR, subtract C_0 from both sides of the equation: (also revert the present value factors back to their previous form)

$0 =$

Then, by expressing the cashflows as a finite geometric series, we find the equation below:

$$\boxed{\sum_{k=0}^{n} C_k \cdot v^{t_k} = 0}$$

The IRR is the rate that satisfies the equation above.

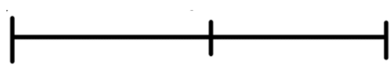
(This is true assuming that the transaction is using a compound/effective interest rate)

Ex.

A transaction has net cashflows of:

$$C_0 = -1000,\ C_1 = 450,\ C_2 = 630$$

at time 0, 1, and 2 respectively. Determine the internal rate of return (IRR).



Note: When a transaction has 3 cashflows or less, it is fairly simple to solve for the IRR by hand. However, once a transaction involves **more** than 3 cashflows, solving for the IRR by hand becomes incredibly difficult. In those scenarios it is advised to use a calculator to help with finding the IRR for the transaction.

Name: ____________

Financial Mathematics Lesson 25 Examples: Internal Rate of Return

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

A transaction has net cashflows of:

$$C_0 = -5,\ C_1 = 3.5,\ C_2 = 4$$

at time 0, 1, and 2 respectively. What is the internal rate of return (IRR) for this transaction?

Ex. 2

An investor is asked to invest \$12,000 and is promised in return a payment of \$6000 in one year, and \$6500 in the second year. Find the investor's internal rate of return.

Name: ____________________

Financial Mathematics Lesson 26
Dollar-Weighted Rate of Return

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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The dollar-weighted rate of return (DRR) is similar to the internal rate of return (IRR), in that it is a rate at which the value of all cashflows out are equal to cashflows in, but the DRR is more commonly used for reporting the return of an investment fund on a yearly basis.

The main difference between the DRR and IRR is that unlike the IRR, the DRR is based on **simple interest** rather than compound interest.

Recall the accumulation factors for compound interest and simple interest:

Compound:

Simple:

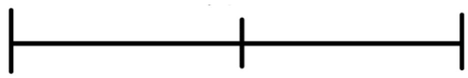
Given the initial balance, any cashflows, and the final balance of an investment fund for a particular year, we can find the dollar-weighted rate of return, which is a simple interest rate.

The general process to find the DRR is to set up an equation of value such that the final balance of the investment fund is equal to the accumulation via simple interest of the initial balance and any cashflows (including cashflows in and out).

Practice using this process with the example on the next page.

<u>Ex.</u>

An investment manager has a fund of \$100,000 at the beginning of 2019. On July 1st of that same year, a new deposit of \$110,000 was made. At the end of the year the balance of the fund was \$220,000. Find the dollar-weighted rate of return.



Compare the DRR to the IRR for the same scenario:

Name: ___________________

Financial Mathematics Lesson 26 Examples: Dollar-Weighted Rate of Return

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

An investment manager had a fund of \$120,000 at the start of the year. On February 1st, a withdrawal of \$20,000 was made. On September 1st, a deposit of \$20,000 was made. At the end of the year, the fund balance was \$125,000. Determine the dollar-weighted rate of return.

Ex. 2

Given the following information about the activity in an investment account, and the dollar-weighted rate of retun is 10 %, determine the value of X.

Date	Fund Value Before D/W	Deposit	Withdrawal
January 1	100.00		
July 1	130.00		X
October 1	115.00	2X	
December 31	120.00		

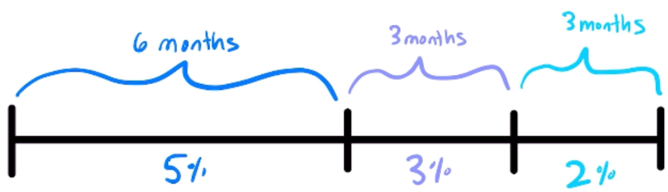
Name: ____________________

Financial Mathematics Lesson 27
Time-Weighted Rate of Return

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

The time-weighted rate of return (TRR) is used to measure growth in an account or fund for a one year period by compounding returns over successive parts of the year.

For example, consider a scenario of a one year period where for 6 months, the interest rate was 5%, and then for the next 3 months it was 3%, and then for the last 3 months it was 2%.



To find the TRR for this scenario, the growth of an investment of $1 would be calculated as such:

$$1 \cdot (1.05)(1.03)(1.02) =$$

The precent at which that investment of $1 grew would be the TRR. So, by subtracting 1, we find could find the TRR:

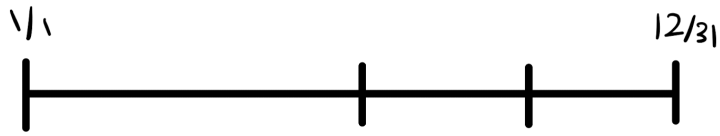
$$\text{TRR} =$$

Notice that the amount of time for which each interest rate was active during the year was not taken into account for this calculation. We compounded the rates over successive parts of the year and *ignored* the number of months for which each interest rate was active.

So, given the balance of a fund over the course of a year as well as any cashflows for that year, how can we calculate the TRR? The formula for calculating the TRR can be found by considering the general scenario on the next page.

Let A be the balance of an investment fund at the beginning of a year, B be the balance at the end of the year, $\pm C_K$ be a deposit (+) or withdrawal (−) made at time t_K, and F_K be the reported balance of the fund before a deposit or withdrawal is made at time t_K.

(Label where A, B, $\pm C_1$, $\pm C_2$, F_1, and F_2 would be in the timeline below)



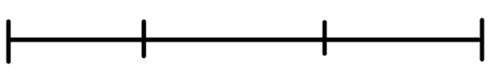
The time-weighted rate of return can be calculated using the following formula:

$$\boxed{\text{TRR } = \frac{F_1}{A} \times \frac{F_2}{F_1 + C_1} \times \frac{F_3}{F_2 + C_2} \times \cdots \times \frac{F_K}{F_{K-1} + C_{K-1}} \times \frac{B}{F_K + C_K} - 1}$$

This formula measures the growth of the investment account over a one year period by compounding the returns over successive parts of the year, ignoring the amount of time between each reported balance and cashflows. That is the idea of the TRR.

<u>Ex.</u>

An investment fund has an initial balance of \$100,000 to start the year. On April 1st, the fund increased to \$112,000 and a deposit of \$30,000 was made. On October 1st, the balance was \$125,000 and a withdrawal of \$42,000 was made. At the end of the year, the balance of the fund was \$100,000. Determine the time-weighted rate of return.



Name: ____________________

Financial Mathematics Lesson 27 Examples: Time-Weighted Rate of Return

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

<u>Ex. 1</u>

An investment fund has an initial balance of \$5,000 to begin the year. On March 1st, the fund dropped to \$4,800 and a deposit of \$800 was made. On June 1st, the fund balance was \$5,200 and a withdrawal of \$500 was made. The balance of the fund at the end of the year was \$5,500. What is the time-weighted rate of return?

<u>Ex. 2</u>

Given the following information about the activity in an investment account, determine the time-weighted rate of return.

Date	Fund Value Before D/W	Deposit	Withdrawal
January 1	100.00		
July 1	130.00		10.00
October 1	115.00	20.00	
December 31	120.00		

Ex. 3

An investment manager had a fund of $200 to start the year. On May 1st, a withdrawal of $10 was made. At the end of the year, the fund balance was $225. Given that the time-weighted rate of return is 18%, what was the balance of the fund on May 1st? (Round to the nearest whole number)

Name: ____________________

Financial Mathematics Lesson 28
Inflation & Real Rates of Interest

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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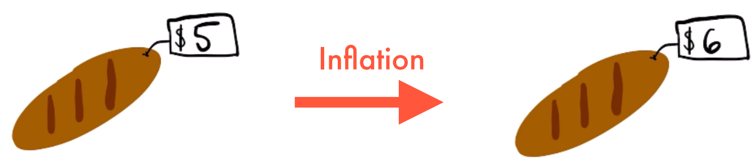
Inflation is known as the decrease in the **purchasing power of money**, or what you are able to buy with your money.

For example, if you had $10 and wanted to buy a loaf of bread that costs $5 per loaf, you could by 2 loaves. But, if the price increased to $6 due to inflation, now you can only buy 1 loaf of bread with your $10. (See below)



With **$10**, you can buy: _____ loaves of bread _____ loaf of bread

---

Due to inflation, our money today may not be worth the same amount 1 year, 2 years, or 3 years in the future and so on. Because of this, an effective interest rate for money in an account may not entirely reflect the actual amount of value added to the account.

This is where the idea of real rates of interest come into play. The real rate of interest is an interest rate that is adjusted for inflation. It will describe the growth or decay of the purchasing power of an investment *after* the effects of inflation.

To better understand real rates of interest, consider the example on the next page.

Ex.

Suppose that the forecasted inflation rate for the coming year is an effective annual rate of 15 %. What would be the real, or inflation-adjusted, rate of interest for the coming year if (1) the effective annual interest rate is 20 % and (2) the effective annual interest rate is 10 %.

To solve this problem, first assume that we have an investment of \$100 and we want to purchase an item that costs the same amount of \$100.

(1) $i =$

$FV =$

(2) $i =$

$FV =$

The end of year cost for the item due to inflation:

In (1), we can purchase the item with \$5 left over, so our purchasing power *increased* by \$5. However in (2), we are \$5 short of being able to purchase the item, so our purchasing power *decreased* by \$5. The real rate of interest would be represented by the change in purchasing power divided by the year end cost (This will measure the percent increase in purchasing power).

So, we can calculate the real rate of interest in each case:

(1) $i_{real} =$

(2) $i_{real} =$

Now, we can generalize how to find i_{real} with a formula by analyzing the different parts of the calculations above. Specifically, use the work from (1):

$i_{real} =$

This results in the general formula for real rates of interest: $\boxed{i_{real} = \dfrac{i - r}{1 + r}}$

(where i is the effective annual interest rate, and r is the effective annual inflation rate)

Name: ____________________

$\ddot{a}_{\overline{n}|i}$ Financial Mathematics Lesson 28 Examples: Inflation & Real Rates of Interest

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

If the inflation rate for the coming year is an effective annual rate of $8\,\%$, what would be the real rate of interest given that the effective annual interest rate is $9\,\%$?

Ex. 2

Riley has \$200 in an account with an effective annual interest rate of $10\,\%$. At the end of the year, Riley wants to use his account to purchase a new camera that currently costs \$200. If the predicted inflation rate for the upcoming year is $5\,\%$, will Riley be able to afford the camera with the money in his account at that time? What is his real rate of interest?

Ex. 3

Joan has an investment fund with an effective annual interest rate of 7 %. After a year of inflation, Joan is informed that her real rate of interest was -3.6 %. What was the inflation rate?

Name: ____________________

Financial Mathematics Lesson 29
The Portfolio Method

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

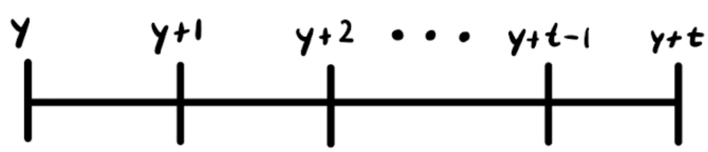
Sometimes investment funds pool money from several different individuals or corporations and are responsible for making investments for those individuals or corporations.

In a situation like this, a natural question would be: "How should the investment fund distribute the returns (the interest) between the different individuals or corporations?"

One way to distribute the interest to the various accounts is known as the portfolio method. The portfolio method applies one specific rate of interest to **all** accounts regardless of when the money entered the fund. Typically this rate is an effective (compound) annual rate and changes from year to year for all accounts.

The rates described above are referred to as **portfolio rates** and are denoted with i^y where y is the year that the rate is active. (Note: y is not an exponent, it is a superscript)

So, if an amount X is invested in year y, then the balance in the account for some year in the future, $y + t$ would be:



$A(y + t) =$

Ex.

Suppose that an investment fund credits investors using the portfolio method with the annual rates in the table below. If \$100 is invested on 1/1/17, find the balance on 1/1 of 2018, 2019, and 2020.

Year	Portfolio Rates
y	i^y
2017	3.50 %
2018	4.00 %
2019	3.25 %
2020	5.50 %

Ex.

On Jan. 1, 2006, an amount of \$1000 is invested into a fund that uses the portfolio method. If the table of annual rates below is used, what is the balance on Jan. 1, 2009?

Year	Portfolio Rates
y	i^{y+4}
2002	8.70 %
2003	9.31 %
2004	9.30 %

Name: ____________________

Financial Mathematics Lesson 29 Examples: The Portfolio Method

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

Brad invests \$500 on Jan. 1, 2021 into an investment fund that uses the portfolio method. Given that the table of annual rates below is used, calculate Brad's account accumulation on Jan. 1 of 2022, 2023, 2024, and 2025.

Year	Portfolio Rates
y	i^y
2020	2.15 %
2021	1.70 %
2022	3.90 %
2023	2.50 %
2024	4.20 %
2025	5.35 %
2026	2.95 %

Ex. 2

Janice makes an investment of $3000 on Jan. 1, 2012 into a fund that uses the portfolio method with the annual rates in the table below. One year later on Jan. 1, 2013, Janice's balance is $3271.20. If her friend Rosie makes an investment of $2000 on Jan. 1, 2010 into the same fund, what is Rosie's balance on Jan. 1, 2013?

Year	Portfolio Rates
y	i^y
2010	9.69 %
2011	8.97 %
2012	X
2013	7.66 %

Ex. 3 (Full solution on JK Math +)

On Jan. 1, 1989, Mario makes an investment of $10,000 into a fund that uses the portfolio method. If the table of annual rates below is used, what is the balance of Mario's account 4 years later?

Year	Portfolio Rates
y	i^{y+5}
1984	6.20 %
1985	7.48 %
1986	5.92 %
1987	6.39 %

Name: ____________________

Financial Mathematics Lesson 30
Spot Rates & Forward Rates

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

The concept of spot rates and forward rates are related to a larger concept known as the **term structure of interest rates**. A simple way to demonstrate what the term structure of interest rates means is to consider a scenario where you want to take out a loan.

Length of Loan	Interest Rate
10 years	7.25 %
20 years	8.00 %
30 years	8.75 %

Often times, you as the borrower of a loan are given several options on how you can repay the loan. For example, in the table above you have the option of repaying the loan over a 10 year period, 20 year period, or 30 year period (starting today). Depending on the chosen length, the interest rate associated with the loan will be different.

Typically those interest rates will **increase** as the length of the loan increases (this incentivizes borrowers to pay back the loan quicker, since the shorter loan will have a smaller interest rate). The different rates in the table are known as spot rates, and they are all annual effective interest rates (compound interest rates). They are denoted with s_t , where t is the term length associated with the spot rate.

Now, consider a more basic scenario with an investment of $1 using spot rates:

Length of Investment	Spot Rate
1 year	5.00 %
2 years	6.00 %
3 years	7.00 %

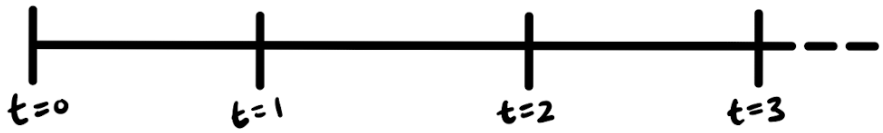
$s_1 =$

$s_2 =$

$s_3 =$

Label where these spot rates accumulate interest in the timeline on the next page.

Spot Rates & Forward Rates



<u>Accumulation Factors:</u>

s_1 :

s_2 :

s_3 :

So by using spot rates, we can accumulate an investment starting at $t = 0$ to any time in the future, $t = 1, 2, 3$ and so on, but what if we wanted to accumulate an investment from $t = 1$ to $t = 2$ or from $t = 2$ to $t = 3$? The rates that allow us to accumulate an investment for these periods of time are known as forward rates.

For example, the interest rate that accumulates interest from $t = 1$ to $t = 2$ would be a forward rate. This rate would be denoted with $f_{1,2}$. Label $f_{1,2}$ in the timeline above.

To calculate the value of a forward rate, we need to set up an equation of value using the known spot rates and the unknown forward rate we want to find.

So, to find $f_{1,2}$, set up a equation of value using s_1 and s_2 :

Solving for $f_{1,2}$ we have:

In the same way, we can set up an equation of value to find another forward rate, $f_{2,3}$:

Solving for $f_{2,3}$ gives:

By analyzing the calculations for $f_{1,2}$ and $f_{2,3}$, we can generalize the process into a formula that can be used to find a forward rate between two years that are 1 year apart:

$$\boxed{f_{n-1,n} = \frac{\left(1+s_n\right)^n}{\left(1+s_{n-1}\right)^{n-1}} - 1}$$

These forward rates are called "$n-1$ year forward rates"

(Note: $f_{0,1} = s_1$)

Now, we can also find forward rates that span more than 1 year. For example, we could also find the forward rate $f_{1,3}$ in the timeline from earlier, which would accumulate the investment from $t=1$ to $t=3$.

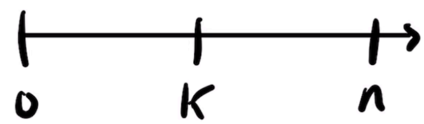
To find $f_{1,3}$, set up an equation of value using s_1 and s_3 :

Solving for $f_{1,3}$ gives:

Just like with $n-1$ year forward rates, we can determine a generalized formula for forward rates that cover a span of years bigger than 1 year by analyzing the calculation for $f_{1,3}$. In general, we can find this type of forward rate using the following equation:

$$\boxed{\left(1+s_n\right)^n=\left(1+s_k\right)^k\left(1+f_{k,n}\right)^{n-k}}$$

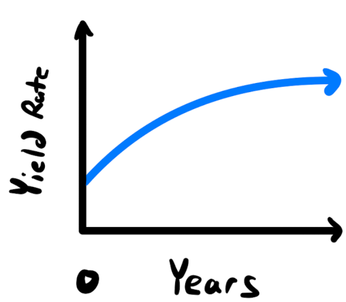
Label the spot rates from this formula in the timeline below:



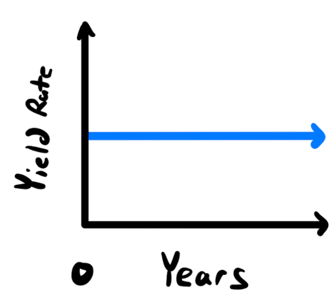
When working with spot rates and forward rates, you might encounter the term **yield curve**.

A yield curve is the graph that corresponds to a table of spot rates. It includes the spot rates given, and extends those values to be continuous. In other words, a yield curve describes spot rates for non-discrete term lengths such as 1.5 years, 1.75 years, 2.1 years, or any other term length that is not countable like 1 year, 2 years, 3 years, and so on.

Generally, a yield curve is shaped like the curve on the left below. It is increasing and concave down. However, if all spot rates are equal, then the yield curve will be flat, or a horizontal line like the curve on the right.



(General Yield Curve)



(Flat Yield Curve)

Name: ____________________

Financial Mathematics Lesson 30 Examples: Spot Rates & Forward Rates

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

Given the following yield curve, find the three-year forward rate $f_{3,4}$ and the four-year forward rate $f_{4,5}$.

Year	Spot Rate
1	4.0 %
2	4.5 %
3	5.0 %
4	5.5 %
5	6.0 %

Ex. 2

For the yield curve below, find the spot rate s_2 if it is known that $f_{2,3} = 4.2\,\%$.

Year	Spot Rate
1	3.2 %
2	?
3	3.8 %
4	3.9 %

<u>Ex. 3</u>

The spot rate for year k is given by the equation:

$$s_k = 0.07 + 0.002k - 0.001k^2$$

Find the three-year forward rate $f_{3,4}$ implied by this yield curve.

<u>Ex. 4</u> (Full Solution on JK Math +)

Given the following table of n-year forward rates, find s_4.

Year	Forward Rate
0	2.8 %
1	3.6 %
2	4.3 %
3	5.1 %

Name: ____________________

Financial Mathematics Lesson 31
Macaulay & Modified Durations

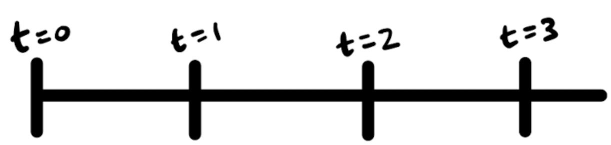
Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

The value of a fixed series of payments, such as an annuity or bond, is sensitive to changes in the interest rate, or yield rate. If the interest rate for a set of cashflows changes, the present value of the cashflows will also change, for better or for worse.

The measure of this sensitivity for the value of a series of payments to change is known as **duration**. There are two different types of duration that can be calculated: Macaulay duration ($MacD$) and modified duration ($ModD$).

The Macaulay duration is a weighted average amount of time until all payments in a series are made. The "weight" applied to each payment at time t will be the present value of the payment divided by the total present value for all the payments.

Consider the timeline below for a 3-year bond with $10 annual coupons and a $1000 par value. Fill in the timeline with the payments that would be received at each time.



Now, write out the individual present value for each of the payments as well as the total present value of all the payments:

$PV_1 =$ ________ $PV_2 =$ ________ $PV_3 =$ ________

Total PV: ____________________

To calculate the Macaulay duration for these payments, take each time t and multiply by the present value of the corresponding present value divided by the total present value:

$$MacD =$$

Simplify the expression and use an interest rate of $i = .05$ to find the Macaulay duration:

This value for the Macaulay duration is measured in **years**. It tells you in how many years you should expect to receive the present value of the cashflows.

It is important to note that the $MacD$ is a slightly smaller value than the amount of years to maturity for the bond. It will never be larger than the years to maturity, as that would imply that you expect to receive the present value of the payments *after* the bond reaches maturity, which does not make much sense and defeats the purpose of a bond.

Also, note that if the bond was a zero-coupon bond, meaning that there are no paid coupons (just a return of the redemption amount at maturity), that the value of the $MacD$ will be the same as the time to maturity. Show that this is true in the space below:

$$MacD =$$

In general, for a set of cashflows $C_1, C_2, C_3, \ldots, C_n$ the calculation for the $MacD$ can be represented with the following formula:

$$MacD = \frac{1 \cdot C_1 v^1 + 2 \cdot C_2 v^2 + 3 \cdot C_3 v^3 + \cdots + n \cdot C_n v^n}{C_1 v^1 + C_2 v^2 + C_3 v^3 + \cdots + C_n v^n} = \boxed{\frac{\sum_{t=1}^{n} t \cdot C_t \cdot v^t}{\sum_{t=1}^{n} C_t \cdot v^t}}$$

Now, to calculate the *modified* duration, we need to first revisit the present value of the general payments C_1, C_2, C_3, and so on. Their present value is as follows:

$$PV = C_1 v + C_2 v^2 + C_3 v^3 + \cdots$$

Represent this present value as a function of the interest rate i by writing out what each present value factor v is equal to in terms of i :

$$P(i) =$$

Since the derivative of a function represents the slope, or the rate at which one variable changes with respect to the other, if we were to take the derivative of this present value function $P(i)$, we will get a function that describes the change in the present value with respect to changes in the interest rate i.

The derivative of $P(i)$ is of interest to us because duration is a measurement of the sensitivity for the present value of a series of payments to change as interest rates change. As such, the rate of change for the present value and interest rate can help with measuring duration.

Find the derivative $P'(i)$ using the power rule and chain rule for derivatives on $P(i)$:

$$P'(i) =$$

Notice that the derivative is negative. This makes sense because as the interest rate for a set of cashflows increases, the present value of the cashflows decreases. For example, consider a bond with a yield rate j. Recall that as the yield rate increases compared to the coupon rate ($j > r$), that the price of the bond decreases (it would be bought at a discount). The price of the bond will only continue to decrease below par value as the yield rate grows larger.

Therefore, the price, or present value function would be a decreasing function, which is highlighted by the fact that its derivative is negative.

(Recall: in calculus, $f'(x) < 0 \rightarrow f(x)$ is dec. , $f'(x) > 0 \rightarrow f(x)$ is inc.)

Now, if we multiply the derivative of the present value function by -1, notice the similarities to the numerator of the calculation for the $MacD$:

$$-P'(i) =$$

In particular, notice that each term is only off by a factor of $(1+i)$ or v^{-1} in comparison to the terms in the numerator of the $MacD$. Additionally, the present value function $P(i)$ is the same as the denominator of the $MacD$. Therefore, we can make the following conclusion:

$$MacD = \frac{-P'(i)(1+i)}{P(i)}$$

The modified duration is this same calculation, except without the extra factor of $(1+i)$ in the numerator. Therefore, modified duration is defined as follows:

$$\boxed{ModD = \frac{-P'(i)}{P(i)}}$$

OR

$$\boxed{ModD = MacD \cdot v}$$

Use the second formula to calculate the $ModD$ for the 3-year bond from earlier in this lesson:

$$ModD =$$

Unlike the $MacD$, the $ModD$ is **not** measured in years. Instead, it tells us a percentage of how much the present value decreases per a 1% increase in the interest rate. So, in our example, for a 1% increase in the interest rate, the present value will decrease by ______ %.

Shortcut Formulas for $MacD$ of a Bond:

$$\boxed{\text{Annual Coupons:} \quad MacD = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r\left((1+i)^n-1\right)+i}}$$

Semiannual Coupons:

$$\boxed{MacD = \frac{1+j}{2j} - \frac{1+j+n(r-j)}{2r\left((1+j)^n-1\right)+2j} \quad \text{where} \quad r = \frac{1}{2}r^{(2)} \text{ and } j = \frac{1}{2}j^{(2)}}$$

Name: ____________________

Financial Mathematics Lesson 31 Examples: Macaulay & Modified Durations

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

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Ex. 1

An investment pays \$3000 at the end of year one and \$5000 at the end of year three. The investment is purchased to yield an effective annual rate of $8\,\%$. Find the Macaulay duration and Modified duration for this investment.

Ex. 2

Determine the Macaulay duration of a 10-year 1000 par value bond with $7\,\%$ annual coupons and an effective annual interest rate of $5.5\,\%$.

Ex. 3

Annuity A pays \$1 at the beginning of each year for three years. Annuity B pays \$1 at the beginning of each year for four years. The Macaulay duration of Annuity A at the time of purchase is 0.9. Note that both annuities A and B offer the same yield rate. Calculate the Macaulay duration of Annuity B at the time of purchase.

Ex. 4 (Full solution on JK Math +)

Find the Macaulay duration of a 10-year 1000 par value bond with 9% semiannual coupons and a semiannual yield rate of 8%.

Ex. 5 (Full solution on JK Math +)

An investor has $4000 worth of 5-year bonds with a modified duration of 4.625, $6000 worth of 10-year bonds with a modified duration of 9.313, and $10,000 worth of 20-year bonds with a modified duration of 19.175. What is the modified duration of this entire portfolio?

Ex. 6 (Full solution on JK Math +)

The present value of a set of cash flows is represented by the following function:

$$P(i) = 500(1+i)^{-1} + 750(1+i)^{-2} + 1000(1+i)^{-3}$$

Find the modified duration if $i = .02$.

Name: ________________

Financial Mathematics Lesson 32
Macaulay & Modified Convexity

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Recall that duration is a way to measure the sensitivity of the value of a series of payments to change based on how interest rates change. In other words, duration reflects the rate at which the present value of a set of cashflows changes as interest rates change.

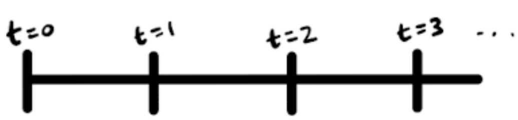
Now, **convexity** measures the curvature of the relationship between the present value of a set of cashflows and interest rates. It reflects the rate at which the *duration* of a set of cashflows changes as interest rates change.

Similar to duration, there are two different types of convexity that can be calculated for a set of cashflows: Macaulay convexity ($MacC$) and modified convexity ($ModC$).

The Macaulay convexity is a weighted average of the **squares** of the times of the payments in a series, where the weights are the same as in the Macaulay duration ($MacD$).

Recall that the definition of the $MacD$ is very similar. It is the weighted average amount of time until all payments in a series are made where the weight applied to each time t is the present value of the payment at that time divided by the present value of all the payments.

So, for a general set of cashflows C_1 , C_2 , C_3 , ... (label them in the timeline below), the $MacD$ is calculated with the following formula:



$$MacD = \frac{\sum_{t=1}^{n} t \cdot C_t \cdot v^t}{\sum_{t=1}^{n} C_t \cdot v^t}$$

Now, calculating the Macaulay convexity is actually very similar to calculating the Macaulay duration. The only difference is that each time t in the numerator of the calculation will be squared. Therefore we have:

$MacC =$

This calculation for the Macaulay convexity can be condensed and expressed as follows:

$$\boxed{MacC = \frac{\sum_{t=1}^{n} t^2 \cdot C_t \cdot v^t}{\sum_{t=1}^{n} C_t \cdot v^t}}$$

To calculate the *modified* convexity, first recall the definition of modified duration:

$$ModD = \frac{-P'(i)}{P(i)}$$

Write out the terms of the present value function $P(i)$ from this definition using a set of general cashflows C_1 , C_2 , C_3 , ... :

$P(i) =$

Take the first derivative using the power rule and chain rule for derivatives:

$P'(i) =$

It is important to notice that duration, such as the modified duration, measures a relationship between the present value for a set of cashflows and the interest rate. It is essentially the rate of change between present value and interest rates.

As such, it makes sense that the definition of the $ModD$ includes the derivative of the present value function since the derivative is also related to the idea of a rate of change.

In a similar manner, convexity is a rate of change describing how duration changes with respect to interest rates. It is essentially the rate of change of the rate of change! In calculus, this is referred to as the second derivative of a function.

As such, it should be no surprise that the definition of modified convexity includes the second derivative of the present value function and is defined as follows:

$$\boxed{ModC = \frac{P''(i)}{P(i)}}$$

Find the second derivative of the present value function $P(i)$ from earlier by differentiating $P'(i)$ using the power and chain rule for derivatives:

$$P''(i) =$$

The terms of this function would be found in the numerator for the $ModC$ calculation.

Now, there is another way to calculate the modified convexity that involves using the Macaulay duration and Macaulay convexity.

Recall that the $MacD$ and $MacC$ would be defined as follows for a general set of cashflows:

$$MacD = \frac{1 \cdot C_1 v^1 + 2 \cdot C_2 v^2 + 3 \cdot C_3 v^3 + \cdots}{C_1 v^1 + C_2 v^2 + C_3 v^3 + \cdots} \qquad MacC = \frac{1^2 \cdot C_1 v^1 + 2^2 \cdot C_2 v^2 + 3^2 \cdot C_3 v^3 + \cdots}{C_1 v^1 + C_2 v^2 + C_3 v^3 + \cdots}$$

To discover the other way to calculate the modified convexity, begin by finding the sum of the $MacD$ and $MacC$ by combining like terms in the numerator (note that both expressions have a common denominator):

$$MacD + MacC =$$

Simplify the expression by reducing each term by a factor of v :

$$MacD + MacC =$$

Now, compare this expression to the modified convexity for the general set of cashflows. Use the present value function $P(i)$ and its second derivative $P''(i)$ from earlier in this lesson:

$$ModC = \frac{P''(i)}{P(i)} =$$

If not done already, rewrite the modified convexity using present value factors and then simply by reducing each term by a factor of v:

$$ModC =$$

Notice when comparing this expression for the $ModC$ to the expression for $MacD + MacC$ that the denominators are alike and the terms in the numerator only differ by a factor of v^2. As a result, we can make the following conclusions:

$$\boxed{ModC = (MacD + MacC)\,v^2} \quad \text{OR} \quad \boxed{ModC = \frac{MacD + MacC}{(1+i)^2}}$$

This makes for a nice way to calculate the modified convexity without needing to find the second derivative of the present value function. Instead, you can simply calculate the Macaulay duration and convexity (which are very similar calculations), add them together, and multiply by the present value factor squared.

Name: ____________________

Financial Mathematics Lesson 32 Examples: Macaulay & Modified Convexity

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

Chris has the following cash flows: \$40,000 in 2 years, \$25,000 in 3 years, and \$100,000 in 4 years. Given an effective annual interest rate of 7 %, find the Macaulay convexity and modified convexity of Chris's cash flows.

Ex. 2

A company has incoming assets of \$7000 in one year, \$5000 in three years, and \$9000 in five years. The company also has liabilities due of \$6000 in two years and \$13,000 in four years. Given that both the assets and liabilities for the company are valued using an effective annual interest rate of 5 %, determine whether the assets or liabilities have a larger Macaulay convexity.

Ex. 3 (Full solution on JK Math +)

The present value of a set of cash flows is represented by the following function:

$$P(i) = 3000(1+i)^{-1} + 1000(1+i)^{-2} + 2000(1+i)^{-3}$$

Find the modified convexity if $i = .03$.

Ex. 4 (Full solution on JK Math +)

The Macaulay convexity for a set of cash flows is 8.365. The same set of cash flows has a modified convexity of 10.07. Find the modified duration for this set of cash flows if the effective annual interest rate used to calculate each convexity is 4%.

Name: ________________

Financial Mathematics Lesson 33
Asset-Liability Exact Matching

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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When conducting a business, an individual or corporation will commit to making payments and receiving payments in the future at various points in time. To keep the business running, or to even be profitable, it is in their best interest to make investments such that they have funds available to use for outgoing payments as they become due.

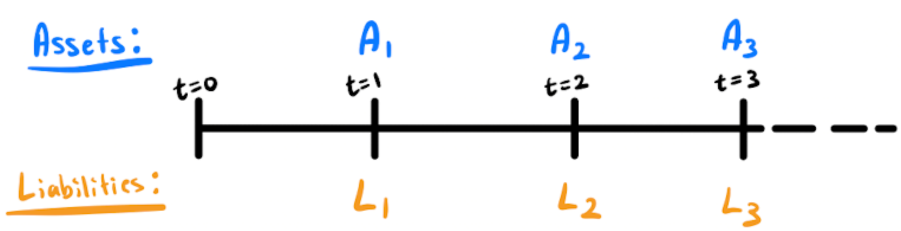
The outgoing payments are referred to as ______________ and the incoming payments, or investment funds available to pay for the outgoing payments, are referred to as _________.

Assets – ( )

Liabilities – ( )

Methods for managing assets and liabilities will be the focus of this lesson and the next.

One method to manage assets and liabilities is known as **dedication**, or more commonly, exact matching. Exact matching works like this: If the investments for a company can be arranged such that the assets exactly cover the liabilities due at each point in time, $A_t = L_t$, then the assets and liabilities are said to be "exactly matched." (See timeline below)



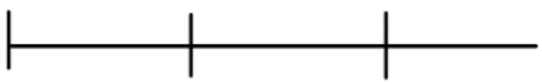
In the given timeline, if the assets and liabilities are exactly matched, we would say (Fill in):

A_1 ____ L_1 , A_2 ____ L_2 , A_3 ____ L_3 ,

<u>Ex.</u>

A company has a \$75,000 liability due in one year and a second liability of \$203,300 due in two years. The company has two types of bonds, A and B, that they can use to match these liabilities exactly. Bond A is a 2-year \$1000 par value bond with 7% annual coupons, and Bond B is a 1-year zero coupon bond redeemable at \$1000.

How many of each type of bond should the company purchase in order to match the liabilities exactly? Additionally, determine the cost for matching the liabilities exactly if the effective yield rate for both bonds is 5% annually.



Name: ____________________

Financial Mathematics Lesson 33 Examples: Asset-Liability Exact Matching

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

An insurance company must pay liabilities of \$99 at the end of one year, \$103 at the end of two years, and \$100 at the end of three years. The only investments available to the company are the three bonds detailed below. Bonds A and C are annual coupon bonds. Bond B is a zero-coupon bond.

Bond	Maturity (years)	Effective Annual Yield	Par	Coupon Rate
A	1	5 %	100	6 %
B	2	7 %	100	N/A
C	3	8 %	100	4 %

If all three bonds are redeemable at par, calculate the number of each bond that needs to be purchased to match the liabilities exactly.

Ex. 2

A company has liabilities of \$2000, \$4000, \$6000, payable at the end of years 1, 2, and 3 respectively. The investments available to the company are the following zero-coupon bonds detailed in the table below:

Maturity (years)	Effective Annual Yield	Par
1	7%	1000
2	8%	1000
3	9%	1000

Determine the cost for matching the liabilities exactly.

Ex. 3 (Full solution on JK Math +)

A company must pay liabilities of \$2396 in one year, \$1436 in two years, \$1770 in three years, and \$3210 in four years. The company can use the bonds detailed below to match their liabilities. If the company wants to match their liabilities exactly, how many of each bond type should the company purchase?

Bond	Maturity (years)	Effective Annual Yield	Par	Coupon Rate
A	1	4%	1000	3%
B	2	5%	1000	6%
C	3	6%	1000	4%
D	4	5%	1000	7%

Additional space for Ex. 3:

Name: ________________

Financial Mathematics Lesson 34
Redington & Full Immunization

Fill in this note sheet by watching the lesson video found at: https://www.youtube.com/c/JKMath

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Similar to exact matching, immunization is a method of managing assets and liabilities. Specifically, immunization focuses on protecting the value of assets and liabilities from changes in interest rates.

In order to discuss immunization, we first need to consider the present value of the assets and liabilities for a business. In each case, the present value will be represented as a function in terms of the interest rate i.

Present Value of Assets: _____

Present Value of Liabilities: _____

Given these two present values for a particular business we would be interested in knowing the **net present value**. The net present value (denoted with $P(i)$) is defined as the difference between the present value of the assets and liabilities.

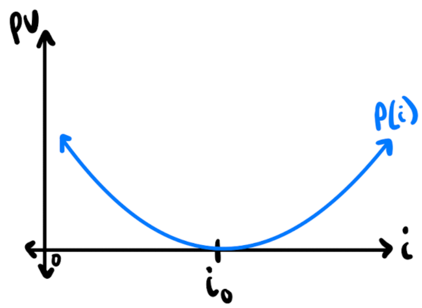
Net Present Value: $P(i) =$

In general, a business wants their net present value to be at least 0, or preferably greater than 0. They would not want it to be less than 0 because that would mean they owe more money than they are able to pay (more liabilities than assets). Essentially, the company would be in debt and have a financial loss. On the other hand, if the net present value is greater than 0, then the business has more assets than liabilities which would be a financial gain.

Now that we have established what the net present value function represents and how it it defined, we can begin to dive into the concept of immunization.

The idea of immunization is this: For a specific interest rate $i = i_0$, if possible, arrange assets and liabilities such that the graph of the net present value function $P(i)$ has a minimum value of 0 at $i = i_0$.

Visually, the graph of the net present value function should look something like the graph below when immunization is achieved. In order for the graph to look like this and have a minimum value of 0 at $i = i_0$, there are three conditions that must be met.



Conditions for Immunization at $i = i_0$:

1)

2)

3)

To summarize, in order for the graph to look like it does above, the net present value function evaluated at $i = i_0$ needs to be 0, the derivative of the net present value function evaluated at $i = i_0$ needs to be 0 (critical value/relative extremum at $i = i_0$), and the second derivative of the net present value function evaluated at $i = i_0$ needs to be greater than 0 (concave up).

Now, by analyzing these three conditions, we can make some conclusions about what needs to be true about the assets and liabilities in a portfolio in order to be immunized.

Recall that the net preset value function is defined as: $P(i) = P_A(i) - P_L(i)$. In order for $P(i_0) = 0$, then the PV of the assets and liabilities must be the same. So, from the first condition we can conclude that $P_A(i_0)$ ____ $P_L(i_0)$ in order to achieve immunization at $i = i_0$.

Now, take the derivative of the net present value function: $P'(i) =$

Using the first derivative, we can make the conclusion from the second condition for immunization at $i = i_0$ that $P_A'(i_0)$ ____ $P_L'(i_0)$.

Next, find the second derivative of the net present value function: $P''(i) =$

Using this second derivative, we can make the conclusion from the third condition for immunization at $i = i_0$ that $P''_A(i_0)$ ____ $P''_L(i_0)$.

Now, taking these conditions one step further, recall the definition of the modified duration and modified convexity:

$$ModD = \frac{-P'(i)}{P(i)} \qquad\qquad ModC = \frac{P''(i)}{P(i)}$$

Notice that the calculation for the $ModD$ includes the PV function and its first derivative. Since both the PV and its derivative for the assets and liabilities must be equal, we can make the conclusion that $ModD_A(i_0)$ _____ $ModD_L(i_0)$ in order to have immunization at $i = i_0$.

In the same way, notice that the calculation for the $ModC$ includes the PV function and its second derivative. Since the second derivative of the PV for the assets must be greater than the second derivative of the PV for the liabilities, we can make the conclusion that $ModC_A(i_0)$ _____ $ModC_L(i_0)$ in order to have immunization at $i = i_0$.

To summarize, the conditions for (Redington) immunization at $i = i_0$ are as follows:

1) The PV of the assets equals the PV of the liabilities

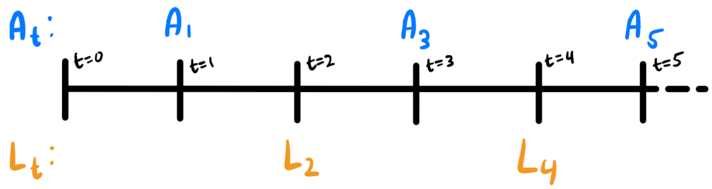
2) The duration of the assets equals the duration of the liabilities

3) The convexity of the assets is greater than the convexity of the liabilities

Note: Since $MacD$ and $ModD$ are related and $MacC$ and $ModC$ are related, it does not matter which duration is used for condition 2 or which convexity is used for condition 3.

The three conditions above are the conditions for a specific type of immunization known as Redington immunization. This is the most basic type of immunization. However, there is a second and stronger type known as full immunization.

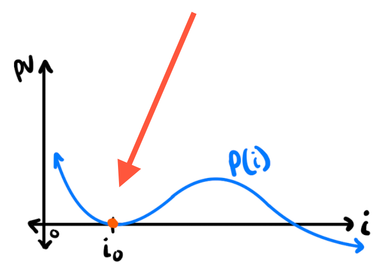
Full immunization has the same first two conditions as Redington immunization, but its third condition is different: 3) Each liability is preceded and followed by an asset. This condition can be met by arranging assets and liabilities similar to what is seen in the timeline below:



Full immunization is a stronger immunization strategy compared to Redington immunization because of what each type of immunization implies about the minimum at $i = i_0$.

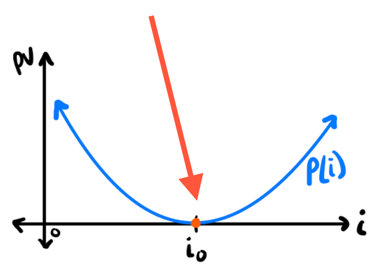
Redington immunization implies that the minimum at $i = i_0$ is a **relative/local minimum**.

So, Redington immunization structures assets and liabilities such that the risk of adverse effects to their value is eliminated for ______ changes in interest rates from $i = i_0$.



Full immunization implies that the minimum at $i = i_0$ is an **absolute/global minimum**.

So, full immunization structures assets and liabilities such that the risk of adverse effects to their value is eliminated for ______ changes in interest rates from $i = i_0$.



Since full immunization is associated with an absolute minimum and Redington immunization is associated with a relative minimum, we can conclude that full immunization implies Redington immunization but Redington immunization does **not** imply full immunization.

Note: The method of exactly matching assets and liabilities would **not** be an immunization strategy. You could test this and find that the third conditions for both types would not be met.

Name: ___________________

Financial Mathematics Lesson 34 Examples: Redington & Full Immunization

Step-by-step walkthrough for each example can be found at https://www.youtube.com/c/JKMath

Ex. 1

A company has a liability of \$53,000 due in two years. The company wants to immunize this liability at an effective annual interest rate of 5 % by using a one-year zero-coupon bond and a three-year zero-coupon bond. Determine how much of each type of bond should be bought, given that both bonds have a par value of \$1000.

Ex. 2

A company has liabilities of \$500 in one year and \$1000 in four years. The company sets up an investment program to match the duration and present value of their liabilities using an effective annual interest rate of $10\,\%$. This investment program produces asset cash flows of X today and Y in three years. Find X and Y, and determine if the investment program satisfies the conditions for Redington immunization.

Ex. 3 (Full solution on JK Math +)

Mark has a liability of \$14,000 due in eight years. To match this liability, Mark has incoming assets of \$6000 in five years and X in $8+z$ years. It is known that Mark is using a full immunization strategy with an effective annual interest rate of $3\,\%$.

Calculate $\dfrac{X}{z}$.

More space for Ex. 3: