Proof of Erdös-Faber-Lovász Conjecture

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Abstract

Proof of the 1972 conjecture by P. Erdös, V. Faber, and L. Lovász: The union of k pairwise edge-disjoint complete graphs with k vertices is k-colorable.

1 Introduction

In this article, I provide a proof of the 1972 Erdös-Faber-Lovász conjecture[1]:

> If k complete graphs, each having exactly k vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with k colors.

The proof is based on the construction. Before presenting the proof, I describe concepts and present definitions necessary for the proof itself.

1.1 Graphs, Layers, Vertices, and Columns

One can envision k complete graphs, each with k vertices, and each lying in a 2-dimensional plane, stacked, one on top of the next to form a set of k parallel layers. One can further envision that each layer is identical to the remaining $k-1$ layers. In this case, one can align the vertices of all the graphs such that we can place a column through each vertex that runs perpendicular to the k planes. And further, each column can be labeled with one of k colors, the colors which are conveyed to the vertices that the column intersects. Figure 1 shows this for $k = 4$.

Definition 1 *Λ_k is a set of k colors, and λ_i is the i^{th} color from the set Λ_k.*

Definition 2 *$v_p(i, j)$ is vertex i in graph j and has color λ_p.*

In Figure 1 the graphs are arranged such that all the vertex colors, λ_p, of a given column, c_i, are the same. If we chose the color λ_i for c_i, then for all the vertices, $v_p(i, j)$, we have $p = i$. Therefore, the vertices in c_i are denoted by $v_i(i, j)$. Because it is redundant, we drop the subscript on v.

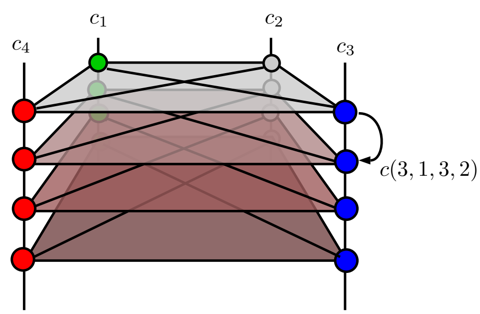

Figure 1: Example of $k = 4$ graphs stacked up in layers. Each graph has vertices of the same color aligned in columns. In this sense, we can view a graph as a layer, and the columns are colors of the vertices. The arrow labeled $c(3, 1, 3, 2)$ represents a connection between the *blue* vertices from graph $1 \to$ graph 2. A connection represents a shared vertex between two layers; and the vertices must have the same color.

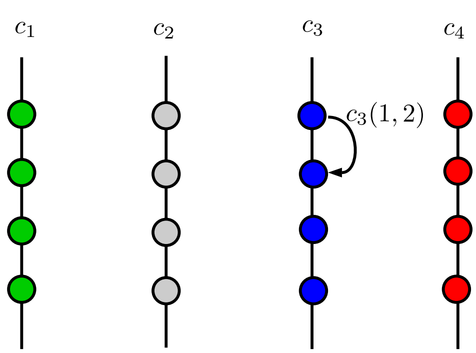

Figure 2: Example of the $k = 4$ graphs represented as columns. This representation–equivalent to the one shown in figure 1–turns out to be more convenient. Although not explicitly shown, because each layer represents a *complete* graph, it is implied that all the vertices in a given layer are connected.

Definition 3 *c_i is the i^{th} column, and has color λ_i.*

Definition 4 *$v(i, j)$ is vertex i in graph j and has color λ_i.*

Figure 1 shows that vertices $v(3, 1)$ and $v(3, 2)$ are connected–as denoted by $c(3, 1, 3, 2)$. In general, connections of the form $c(i, j, l, m)$ for which $i = l$ are connections between vertices of the same color. Therefore, connections of the form $c(i, j, i, m)$ are equivalent to a shared vertex between graphs j and m.

Definition 5 *$c(i, j, l, m)$ is a connection between $v(i, j)$ and $v(l, m)$.*

$$c(i,j,l,m) = \begin{cases} 1 & \textit{if } v(i,j) \textit{ and } v(l,m) \textit{ are connected.} \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

Figure 2 shows an additional abstraction. Here the vertices are arranged as a matrix where the rows represent at complete graph, or layer, and the columns represent the colors of the vertices. Although figure 2 does not show connections between the vertices of a given graph (i.e. no lines are drawn between the vertices within a row), because each row represents a complete graph, it is implied that all the vertices are connected.

2 Construction

The construction of the connections between graphs is based on a series of constraints that defines whether a given connection, $c(i, j, l, m)$, equals 0 or 1.

Because we care about shared vertices between graphs, we issue the first constraint that only connections between vertices of the same color are allowed.

$$c(i,j,l,m) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Constraint 2 allows us to streamline our notation from $c(i, j, i, m)$ to $c_i(j, m)$ because we allow only connections with the same color.

Definition 6 *Connections of the form $c(i, j, i, m)$ are equivalent the relation $v(i, j) = v(i, m)$, and we denote this connection as $c_i(j, m)$. The subscript on c identifies the color of the vertices, or equivalently, the column to which the vertices belong.*

Based on constraint 2 and the new notation, equation 3 provides the rules for constructing the connections between vertices that satisfy the conjecture to be proven. For k complete graphs, we have the connections (shared vertices) given by

$$c_i(j,m) = \begin{cases} 1 & \text{if } j \leq i < k \text{ and } m = i + 1. \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

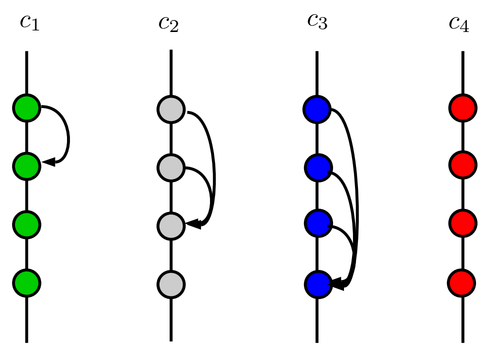

Figure 3: Example of the connections between vertices of $k = 4$ graphs as defined in the constraints of equation 2 and equation 3.

3 Proof

By construction we have k complete graphs. The k nodes of each of the k graphs are colored with colors from the set Λ_k. All graphs are identical. Each graph is connected to other graphs only by nodes of the same color–which is equivalent to sharing that node.

I must show that none of the k graphs are connected to any of the other $k-1$ graphs more than once, and that each of the k graphs is connected to all the remaining $k-1$ graphs at least once. In other words, each of the k graphs is connected to the other $k-1$ graphs exectly once.

First I show that for any j and m that meet the conditions laid out by equations 2 and 3), there is only one connection. In other words, if a graph is connected to another graph, it is only connected once.

The number of connections made between graphs j and m under our constraints is given by

$$N_c = \sum_{i=1}^{k} c_i(j,m) = c_{m-1}(j,m) = 1 \tag{4}$$

The second and third equalities are a consequnce of equation 3, where only a connection is formed if $m = i+1$ and $j < m$.

Now I have shown that if two graphs are connected under our constraints, they have one connection. Next I must show that every graph is connected to every other graph. Together, that any two graphs are only connected once, and that each graph is connected to every other graph, shows that every graph is connected to every other graph exactly once.

The total number of connections, $\tilde{N}_c$, expected when each graph is connected to every other graph exactly once, is given by

$$\tilde{N}_c = \sum_{i=i}^{k}(k-i) = \sum_{i=1}^{k-1} i \tag{5}$$

The total number of actual connections between the graphs, based on the construction is given by summing over all possible connections, as given by

$$\sigma_c = \sum_{i,j,m=1}^{k} v_i(j,m) = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{m=1}^{k} v_i(j,m) \tag{6}$$

The constraint that $m = i + 1$ from equation 3, implies that all connections $c_i(j,m) = 0$ when $m \neq i + 1$. Equation 6, the total number of connections between the graphs reduces to

$$\sigma_c = \sum_{i,j=1}^{k} \gamma_{i,j} \tag{7}$$

with the definition

$$\gamma_{i,j} \equiv c_i(j, i+1) \tag{8}$$

The additional constraint in equation 3 that $j \leq i < k$ implies that

$$\sigma_c = \sum_{i=1}^{k-1}\sum_{j=1}^{i} \gamma_{i,j} \tag{9}$$

Over this range, $0 < i < k$ and $j \leq i$, where $(i,j) \in \mathbb{N}$, we have $\gamma_{i,j} = 1$, so that

$$\sigma_c = \sum_{i=1}^{k-1}\sum_{j=1}^{i} 1 = \sum_{i=1}^{k-1} i \tag{10}$$

which matches $\tilde{N}_c$ from equation 5.

References

[1] Erdos, Paul, *"On the combinatorial problems which I would most like to see solved"* Combinatorica, Vol. 1, pp 25-42, 1981