RECURSIVE COMPUTATION OF COMPOUND RANDOM VARIABLES WITH A FINITE-MIXTURE COUNT DISTRIBUTION

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1. Introduction

Compound random variables have been studied extensively in probability theory and applied fields, as they provide a natural framework for modeling aggregate quantities subject to two layers of randomness: the distribution of the summands and the distribution of the count variable. Classical treatments can be found in [**?**], where compound distributions are developed in the context of actuarial mathematics and discrete distributions.

While the concept is straightforward, calculating the exact probability distribution of a compound random variable can be a significant computational challenge. Direct methods involving multiple convolutions are often cumbersome and inefficient. A more elegant and powerful approach is the use of recursive methods, which allow for the efficient calculation of the probability $P(S_N = k)$ based on the probabilities of preceding values. This thesis explores one such recursive technique, which is particularly effective when the compounding distribution belongs to a specific class of distributions.

The primary contribution of this work is the extension and simplification of this recursive method to the case where the compounding distribution is a finite mixture of Poisson, Binomial and Negative Binomial distributions. A direct application of the recursive formula to this case leads to a computationally intensive, nested problem, where the mixing weights of the distribution must be recalculated at each step. Additionally, we derive a closed-form expression that computes these recursive weights directly, thereby eliminating some of the nested recursion and simplifying the calculation.

2. Background

Definition 1. The expected value of a discrete random variable, X, is denoted by $E(X)$, and given by

$$E[X] = \sum_{x=1}^{\infty} x \cdot f(x),$$

where $f(x)$ is the probability mass function of X.

Definition 2. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables. Let N be a nonnegative integer valued random variable that is independent of the sequence $X_1, X_2, \ldots, X_n$. Then,

$$S_N = \sum_{i=1}^{N} X_i$$

is a *compound random variable.*

Definition 3. In a compound random variable, the distribution of N is called the *compounding distribution.*

Theorem 1 (Compound Random Variable Identity)**.** *Let M be a random variable that is independent of the sequence of random variables $X_1, X_2, \ldots, X_n$ that forms a compound random variable and M satisfies*

$$P(M = n) = \frac{n\, P(N = n)}{E[N]}, \qquad n = 1, 2, 3, \ldots$$

Then, for any function h,

$$E(S_N \cdot h(S_N)) = E[N] \cdot E(X_1 \cdot h(S_M))\,.$$

Proof.

$$\begin{aligned} E(S_N \cdot h(S_N)) &= E\left[\sum_{i=1}^{N} X_i\, h(S_N)\right] \\ &= E\left[E\left(\sum_{i=1}^{N} X_i\, h(S_N) \,\middle|\, N = n\right)\right] \\ &= \sum_{n=0}^{\infty} E\left(\sum_{i=1}^{n} X_i\, h(S_n) \,\middle|\, N = n\right) P(N = n) \\ &= \sum_{n=0}^{\infty} E\left(\sum_{i=1}^{n} X_i\, h(S_n)\right) P(N = n) \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{n} E(X_i\, h(S_n))\, P(N = n). \end{aligned}$$

Since $X_1, X_2, \ldots$ are independent and identically distributed, we have

$$E(X_i \cdot h(S_N)) = E(X_j \cdot h(S_N))\,, \qquad \forall\, i, j \in \{1, 2, \ldots, n\}.$$

So,

$$\begin{aligned} E(S_N \cdot h(S_N)) &= \sum_{n=0}^{\infty} \sum_{i=1}^{n} E(X_1\, h(S_n))\, P(N = n) \\ &= \sum_{n=1}^{\infty} n \cdot E(X_1\, h(S_n))\, P(N = n). \end{aligned}$$

Then, by substitution of $P(M = n) = \dfrac{n\, P(N = n)}{E[N]}$, we have

$$\begin{aligned} &= \sum_{n=0}^{\infty} E[N] \cdot E(X_1\, h(S_n))\, P(M = n) \\ &= E[N] \sum_{n=0}^{\infty} E(X_1\, h(S_n))\, P(M = n) \\ &= E[N] \sum_{n=0}^{\infty} E(X_1\, h(S_n) \mid M = n)\, P(M = n) \\ &= E[N] \sum_{n=0}^{\infty} E(X_1\, h(S_M) \mid M = n)\, P(M = n) \\ &= E[N]\, E(E(X_1\, h(S_M) \mid M = n)) \\ &= E[N]\, E(X_1\, h(S_M))\,. \end{aligned}$$

□

Corollary 1. *Let S_N be a compound random variable and suppose $X_1, X_2, \ldots, X_n$ are positive integer valued random variables. Suppose $P(X_1 = i) = \alpha_i$, $i > 0$, and M is a random variable that is independent of the sequence $X_1, X_2, \ldots, X_n$ and satisfies*

$$P(M = n) = \frac{n\, P(N = n)}{E[N]}.$$

Then,

$$P(S_N = 0) = P(N = 0) \quad \textit{and} \quad P(S_N = k) = \frac{1}{k}\, E[N] \sum_{i=1}^{k} i\, \alpha_i\, P(S_{M-1} = k - i).$$

Proof. For a fixed k, let

$$h(x) = \begin{cases} 1 & \text{if } x = k, \\ 0 & \text{otherwise,} \end{cases}$$

and note that

$$S_N \cdot h(S_N) = \begin{cases} k & \text{if } S_N = k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(S_N \cdot h(S_N)) = k \cdot P(S_N = k).$$

Now, by the Compound Random Variable Identity, $E(S_N \cdot h(S_N)) = E[N] \cdot E(X_1 \cdot h(S_M))$. So,

$$\begin{aligned} k \cdot P(S_N = k) &= E[N] \cdot E(X_1 \cdot h(S_M)) \\ &= E[N] \cdot E(E(X_1 \cdot h(S_M) \mid X_1 = j)) \\ &= E[N] \sum_{j=1}^{\infty} E(X_1 \cdot h(S_M) \mid X_1 = j) \cdot P(X_1 = j) \\ &= E[N] \sum_{j=1}^{\infty} E(X_1 \cdot h(S_M) \mid X_1 = j) \cdot \alpha_j \\ &= E[N] \sum_{j=1}^{\infty} E(j \cdot h(S_M) \mid X_1 = j) \cdot \alpha_j \\ &= E[N] \sum_{j=1}^{\infty} j \cdot E(h(S_M) \mid X_1 = j) \cdot \alpha_j. \end{aligned}$$

Now,

$$E(h(S_M) \mid X_1 = j) = \sum_{S'_M} h(S'_M)\, f_{S_M|X_1}(S'_M \mid j) = P(S_M = k \mid X_1 = j).$$

And,

$$\begin{aligned} P(S_M = k \mid X_1 = j) &= P\left(\sum_{i=1}^{m} X_i = k \,\middle|\, X_1 = j \right) \\ &= P\left(j + \sum_{i=2}^{m} X_i = k \right) = P\left(\sum_{i=2}^{m} X_i = k - j \right). \end{aligned}$$

Letting $\ell = i - 1$,

$$= P\left(\sum_{\ell=1}^{m-1} X_\ell = k - j \right) = P(S_{M-1} = k - j).$$

So,

$$k \cdot P(S_N = k) = E[N] \sum_{j=1}^{\infty} j \cdot \alpha_j \cdot P(S_{M-1} = k - j).$$

Note, when $j > k$, $P(S_{M-1} = k - j) = 0$. So,

$$k \cdot P(S_N = k) = E[N] \sum_{j=1}^{k} j \cdot \alpha_j \cdot P(S_{M-1} = k - j),$$

and therefore,

$$P(S_N = k) = \frac{1}{k} \cdot E[N] \sum_{j=1}^{k} j \cdot \alpha_j \cdot P(S_{M-1} = k - j).$$

Notice, by definition, for $N \geq 1$,

$$P(S_N = 0) = P\left(\sum_{j=i}^{N} X_i = 0 \right) = 0,$$

so $P(S_N = 0) = P(N = 0)$. □

3. Examples

For each of the distributions below, let $S_N = \sum_{i=1}^{N} X_i$ be a compound random variable where

$$P(X_1 = 1) = 0.05, \quad P(X_1 = 2) = 0.4, \quad P(X_1 = 3) = 0.1,$$

$$P(X_1 = 4) = 0.25, \quad P(X_1 = 5) = 0.2.$$

Find $P(S_N = 4)$.

Example 1 (N in the Poisson Distribution)**.** Let N be a random variable of the Poisson distribution with parameter $\lambda = 3$. So the probability mass function for the distribution of N is

$$P^{(s)}(N = n) = \frac{e^{-\lambda}\, \lambda^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

We claim that $E[N^{(s)}] = \lambda$.

Proof.

$$E[N^{(s)}] = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\, \lambda^k}{k!} = 0 + \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda}\, \lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda}\, \lambda^k}{(k-1)!}$$
$$= e^{-\lambda} \cdot \lambda \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}.$$

Now let $j = k - 1$. So when $k = 1$, $j = 0$, and as $k \to \infty$, $j \to \infty$. So,

$$E[N^{(s)}] = \lambda \cdot e^{-\lambda} \cdot \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}.$$

We recognize $\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$ as the Taylor series for e^{λ}. So,

$$E[N^{(s)}] = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda.$$ □

Therefore, $E[N^{(s)}] = \lambda = 3$. So we have,

$$P(M^{(s)} - 1 = n) = P(M^{(s)} = n+1) = \frac{(n+1)\,P^{(s)}(N = n+1)}{E[N^{(s)}]}$$
$$= \frac{n+1}{\lambda} \cdot \frac{e^{-\lambda}\,\lambda^{n+1}}{(n+1)!} = \frac{e^{-\lambda}\,\lambda^{n}}{n!}.$$

Therefore, we see that $P^{(s+1)}(N = n)$ is also the Poisson distribution with parameter λ. Since the distribution is unchanged at each recursion level, we have

$$P^{(s)}(S_N = k) = \frac{\lambda}{k}\sum_{i=1}^{k} i\,\alpha_i \cdot P^{(s+1)}(S_N = k-i),$$

which, because $P^{(s)} = P^{(s+1)}$ for all $s \geq 0$, simplifies to

$$P(S_N = k) = \frac{\lambda}{k}\sum_{i=1}^{k} i\,\alpha_i \cdot P(S_N = k-i).$$

Now we calculate $P(S_N = 4)$.

$$P(S_N = 0) = P(N = 0) = \frac{e^{-\lambda}\,\lambda^0}{0!} = e^{-\lambda} = e^{-3}.$$
$$P(S_N = 1) = \frac{3}{1}\big[1 \cdot (0.05) \cdot P(S_N = 0)\big] = 3 \cdot (0.05) \cdot e^{-3} = 0.15\,e^{-3}.$$
$$P(S_N = 2) = \frac{3}{2}\big[(0.05) \cdot P(S_N = 1) + 2 \cdot (0.4) \cdot P(S_N = 0)\big]$$
$$= \frac{3}{2}\big[(0.05)(0.15\,e^{-3}) + (0.8)\,e^{-3}\big] = 1.21125\,e^{-3}.$$
$$P(S_N = 3) = \frac{3}{3}\big[(0.05) \cdot P(S_N = 2) + (0.8) \cdot P(S_N = 1) + 3 \cdot (0.1) \cdot P(S_N = 0)\big]$$
$$= (0.05)(1.21125\,e^{-3}) + (0.8)(0.15\,e^{-3}) + (0.3)\,e^{-3} = 0.4805625\,e^{-3}.$$
$$P(S_N = 4) = \frac{3}{4}\big[0.05 \cdot P(S_N = 3) + 0.8 \cdot P(S_N = 2) + 0.3 \cdot P(S_N = 1) + 4 \cdot (0.25) \cdot P(S_N = 0)\big]$$
$$= \frac{3}{4}\big[0.05(0.4805625\,e^{-3}) + 0.8(1.21125\,e^{-3}) + 0.3(0.15\,e^{-3}) + e^{-3}\big]$$
$$= 1.528521094\,e^{-3}.$$

Example 2 (N in the Negative Binomial Distribution)**.** Let N be a random variable of the negative binomial distribution with parameters $r = 6$ and $p = 0.6$. The probability mass function for the distribution of $N^{(s)}$ at recursion level s is

$$P^{(s)}(N = n) = \binom{n+r+s-1}{n} p^{r+s}(1-p)^n, \qquad n \geq 0.$$

We claim that $E[N^{(s)}] = \dfrac{(r+s)(1-p)}{p}$.

Proof.

$$E[N^{(s)}] = \sum_{k=0}^{\infty} k\binom{k+r+s-1}{k} p^{r+s}(1-p)^k = \sum_{k=1}^{\infty} k\binom{k+r+s-1}{k} p^{r+s}(1-p)^k.$$

Recalling that $\binom{a}{b} = \dfrac{a!}{b!\,(a-b)!}$, we see that

$$k\binom{k+r+s-1}{k} = k\cdot\frac{(k+r+s-1)!}{k!\,(r+s-1)!} = \frac{(k+r+s-1)!}{(k-1)!\,(r+s-1)!} = (r+s)\binom{k+r+s-1}{k-1}.$$

So we have,

$$E[N^{(s)}] = (r+s)(1-p)\,p^{r+s}\sum_{k=1}^{\infty}\binom{k+r+s-1}{k-1}(1-p)^{k-1}.$$

Let $m = k-1$. By substitution,

$$E[N^{(s)}] = (r+s)(1-p)\,p^{r+s}\sum_{m=0}^{\infty}\binom{m+(r+s)}{m}(1-p)^{m}.$$

Recalling the negative binomial theorem, $(1+x)^{-t} = \sum_{i=0}^{\infty}\binom{-t}{i}x^i$, and applying it, we see that

$$\sum_{m=0}^{\infty}\binom{m+r+s}{m}(1-p)^m = p^{-(r+s+1)}.$$

So by substitution,

$$E[N^{(s)}] = (r+s)(1-p)\,p^{r+s}\cdot p^{-(r+s+1)} = \frac{(r+s)(1-p)}{p}. \qquad \square$$

So we have,

$$\begin{aligned}
P^{(s)}(M^{(s)}-1=n) &= P^{(s)}(M^{(s)}=n+1)\\
&= \frac{(n+1)\,P^{(s)}(N=n+1)}{E[N^{(s)}]}\\
&= \binom{n+r+s}{n+1}p^{r+s}(1-p)^{n+1}\cdot\frac{(n+1)}{(r+s)(1-p)/p}\\
&= \frac{(n+1)\,p}{(r+s)(1-p)}\cdot\frac{(n+r+s)!}{(n+1)!\,(r+s-1)!}\cdot p^{r+s}(1-p)^{n+1}\\
&= \frac{(n+r+s)!}{n!\,(r+s)!}\cdot p^{r+s+1}(1-p)^n\\
&= \binom{n+(r+s+1)-1}{n}p^{r+s+1}(1-p)^n = P^{(s+1)}(N=n).
\end{aligned}$$

Therefore, we see that $P^{(s)}(M^{(s)}-1=n)$ is also the negative binomial distribution but at the next recursion level $(s+1)$. So, we have

$$P^{(s)}(S_N=k) = \frac{(r+s)(1-p)}{k\,p}\sum_{i=1}^{k} i\,\alpha_i\cdot P^{(s+1)}(S_N=k-i).$$

Now we can begin the computation. With $r=6$, $p=0.6$ (so $1-p=0.4$), at level $s=0$ the coefficient is $\dfrac{r(1-p)}{p} = 4$, giving

$$P^{(0)}(S_N=k) = \frac{4}{k}\sum_{i=1}^{k} i\,\alpha_i\cdot P^{(1)}(S_N=k-i).$$

Base cases $P^{(s)}(S_N = 0) = p^{r+s}$:

$$P^{(0)}(S_N = 0) = (0.6)^6 = 0.046656.$$
$$P^{(1)}(S_N = 0) = (0.6)^7 = 0.0279936.$$
$$P^{(2)}(S_N = 0) = (0.6)^8 = 0.01679616.$$
$$P^{(3)}(S_N = 0) = (0.6)^9 = 0.010077696.$$
$$P^{(4)}(S_N = 0) = (0.6)^{10} = 0.0060466176.$$

Computing $P^{(s)}(S_N = 1)$:

$$P^{(0)}(S_N = 1) = \frac{(r)(1-p)}{1 \cdot p}\big[0.05 \cdot P^{(1)}(S_N = 0)\big] = (0.2)(0.0279936) = 0.00559872.$$
$$P^{(1)}(S_N = 1) = \frac{(r+1)(1-p)}{1 \cdot p}\big[0.05 \cdot P^{(2)}(S_N = 0)\big] = \tfrac{7 \cdot 0.4}{0.6}(0.05)(0.01679616) = 0.003919104.$$
$$P^{(2)}(S_N = 1) = \frac{(r+2)(1-p)}{1 \cdot p}\big[0.05 \cdot P^{(3)}(S_N = 0)\big] = \tfrac{8 \cdot 0.4}{0.6}(0.05)(0.010077696) = 0.0026873856.$$
$$P^{(3)}(S_N = 1) = \frac{(r+3)(1-p)}{1 \cdot p}\big[0.05 \cdot P^{(4)}(S_N = 0)\big] = \tfrac{9 \cdot 0.4}{0.6}(0.05)(0.0060466176) = 0.00181398528.$$

Computing $P^{(s)}(S_N = 2)$:

$$\begin{aligned}
P^{(1)}(S_N = 2) &= \frac{(r+1)(1-p)}{2\,p}\big[(0.05) \cdot P^{(2)}(S_N = 1) + (0.8) \cdot P^{(2)}(S_N = 0)\big] \\
&= \frac{7 \cdot 0.4}{2 \cdot 0.6}\big[(0.05)(0.0026873856) + (0.8)(0.01679616)\big] = 0.03166636032. \\
P^{(2)}(S_N = 2) &= \frac{(r+2)(1-p)}{2\,p}\big[(0.05) \cdot P^{(3)}(S_N = 1) + (0.8) \cdot P^{(3)}(S_N = 0)\big] \\
&= \frac{8 \cdot 0.4}{2 \cdot 0.6}\big[(0.05)(0.00181398528) + (0.8)(0.010077696)\big] = 0.021740949504.
\end{aligned}$$

Computing $P^{(1)}(S_N = 3)$:

$$\begin{aligned}
P^{(1)}(S_N = 3) &= \frac{(r+1)(1-p)}{3\,p}\big[(0.05) \cdot P^{(2)}(S_N = 2) + (0.8) \cdot P^{(2)}(S_N = 1) + (0.3) \cdot P^{(2)}(S_N = 0)\big] \\
&= \frac{7 \cdot 0.4}{3 \cdot 0.6}\big[(0.05)(0.021740949504) + (0.8)(0.0026873856) + (0.3)(0.01679616)\big] \\
&= 0.01287347282.
\end{aligned}$$

Computing $P^{(0)}(S_N = 4)$:

$$\begin{aligned}
P^{(0)}(S_N = 4) &= \frac{r(1-p)}{4\,p}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\big] \\
&= \frac{6 \cdot 0.4}{4 \cdot 0.6}\big[(0.05)(0.01287347282) + (0.8)(0.03166636032) \\
&\qquad + (0.3)(0.003919104) + 0.0279936\big] \\
&= \boxed{0.05514609310}.
\end{aligned}$$

Example 3 (N in the Binomial Distribution)**.** Let N be a random variable of the Binomial distribution with parameters $r = 6$ and $p = 0.6$. The probability mass function for the distribution of $N^{(s)}$ at recursion level s is

$$P^{(s)}(N = n) = \binom{r-s}{n} p^n (1-p)^{(r-s)-n}, \qquad n \geq 0.$$

We claim that $E[N^{(s)}] = (r - s)\, p$.

Proof.

$$E[N^{(s)}] = \sum_{k=0}^{\infty} k \binom{r-s}{k} p^k (1-p)^{(r-s)-k} = \sum_{k=1}^{\infty} k \binom{r-s}{k} p^k (1-p)^{(r-s)-k}.$$

Notice that

$$k\binom{r-s}{k} = k \cdot \frac{(r-s)!}{k!\,(r-s-k)!} = \frac{(r-s)!}{(k-1)!\,(r-s-k)!} = (r-s)\binom{r-s-1}{k-1}.$$

By substitution,

$$E[N^{(s)}] = (r-s)\, p \sum_{k=1}^{\infty} \binom{r-s-1}{k-1} p^{k-1} (1-p)^{(r-s)-k}.$$

We let $m = k - 1$. So we have,

$$E[N^{(s)}] = (r-s)\, p \sum_{m=0}^{\infty} \binom{r-s-1}{m} p^{m} (1-p)^{(r-s-1)-m}.$$

And by the binomial theorem,

$$\sum_{m=0}^{\infty} \binom{r-s-1}{m} p^{m} (1-p)^{(r-s-1)-m} = \big[p + (1-p)\big]^{r-s-1} = 1.$$

By substitution we have $E[N^{(s)}] = (r - s)\, p$. □

So, for this example at $s = 0$, $E[N^{(0)}] = r\, p = 3.6$. We have,

$$\begin{aligned}
P^{(s)}(M^{(s)} - 1 = n) = P^{(s)}(M^{(s)} = n+1) &= \frac{(n+1)\, P^{(s)}(N = n+1)}{E[N^{(s)}]} \\
&= \binom{r-s}{n+1} p^{n+1} (1-p)^{(r-s)-n-1} \cdot \frac{n+1}{(r-s)\, p} \\
&= \frac{(r-s)!}{(n+1)!\,(r-s-n-1)!}\, p (1-p)^{(r-s-1)-n} \cdot \frac{n+1}{(r-s)\, p} \\
&= \frac{(r-s-1)!}{n!\,(r-s-1-n)!} \cdot p^n (1-p)^{(r-s-1)-n} \\
&= \binom{r-s-1}{n} p^n (1-p)^{(r-s-1)-n} = P^{(s+1)}(N = n).
\end{aligned}$$

Therefore, we see that $P^{(s)}(M^{(s)} - 1 = n)$ is also the binomial distribution but at the next recursion level $(s + 1)$, which corresponds to a parameter change to $(r - s - 1)$. So, we have

$$P^{(s)}(S_N = k) = \frac{(r-s)\, p}{k} \sum_{i=1}^{k} i\, \alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

Now we can begin the computation to find $P(S_N = 4)$.

Base cases $P^{(s)}(S_N = 0) = (1-p)^{r-s}$:

$$P^{(1)}(S_N = 0) = P^{(1)}(N = 0) = (0.4)^5 = 0.01024.$$
$$P^{(2)}(S_N = 0) = P^{(2)}(N = 0) = (0.4)^4 = 0.0256.$$
$$P^{(3)}(S_N = 0) = P^{(3)}(N = 0) = (0.4)^3 = 0.064.$$

From Corollary 1, $P^{(0)}(S_N = 0) = P^{(0)}(N = 0) = (0.4)^6 = 0.004096$.

$$\begin{aligned}
P^{(0)}(S_N = 1) &= \frac{r\,p}{1}\big[1 \cdot (0.05) \cdot P^{(1)}(S_N = 0)\big] = 6(0.6)(0.05)(0.01024) = 0.0018432.\\
P^{(3)}(S_N = 1) &= \frac{(r-3)\,p}{1}\big[0.05 \cdot P^{(4)}(S_N = 0)\big] = 3(0.6)(0.05)(0.4)^2 = 0.0144.\\
P^{(2)}(S_N = 1) &= \frac{(r-2)\,p}{1}\big[0.05 \cdot P^{(3)}(S_N = 0)\big] = 4(0.6)(0.05)(0.064) = 0.00768.\\
P^{(1)}(S_N = 1) &= \frac{(r-1)\,p}{1}\big[0.05 \cdot P^{(2)}(S_N = 0)\big] = 5(0.6)(0.05)(0.0256) = 0.00384.
\end{aligned}$$

$$\begin{aligned}
P^{(2)}(S_N = 2) &= \frac{(r-2)\,p}{2}\big[0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)\big]\\
&= \frac{4(0.6)}{2}\big[0.05(0.0144) + 0.8(0.064)\big] = 0.062304.\\
P^{(1)}(S_N = 2) &= \frac{(r-1)\,p}{2}\big[0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)\big]\\
&= \frac{5(0.6)}{2}\big[0.05(0.00768) + 0.8(0.0256)\big] = 0.031296.
\end{aligned}$$

$$\begin{aligned}
P^{(1)}(S_N = 3) &= \frac{(r-1)\,p}{3}\big[(0.05) \cdot P^{(2)}(S_N = 2) + (0.8) \cdot P^{(2)}(S_N = 1) + (0.3) \cdot P^{(2)}(S_N = 0)\big]\\
&= \frac{5(0.6)}{3}\big[(0.05)(0.062304) + (0.8)(0.00768) + (0.3)(0.0256)\big] = 0.016937200.
\end{aligned}$$

$$\begin{aligned}
P^{(0)}(S_N = 4) &= \frac{r\,p}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\big]\\
&= \frac{6(0.6)}{4}\big[(0.05)(0.016937200) + (0.8)(0.031296) + (0.3)(0.00384) + 0.01024\big]\\
&= \boxed{0.033548184}.
\end{aligned}$$

4. Computation with a Finite Mixture of Poisson Distributions

We will now consider the computation of compound random variable probabilities when the counting distribution is a mixture of Poisson distributions.

Example 4 (N in a Finite Mixture of Two Poisson Distributions)**.** Consider a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Poisson distribution with mean λ_1, or process two, which follows a Poisson distribution with mean λ_2. The probability of process one is β_1, and the probability of process two is β_2, where $\beta_1 + \beta_2 = 1$. Let the parameters of this distribution be $\lambda_1, \lambda_2, \beta_1, \beta_2$.

So, we have,

$$P^{(s)}(N = n) = \beta_1^{(s)} \cdot \frac{e^{-\lambda_1}\,\lambda_1^n}{n!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2}\,\lambda_2^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

We claim that $E[N^{(s)}] = \beta_1^{(s)}\lambda_1 + \beta_2^{(s)}\lambda_2$.

Proof.

$$\begin{aligned} E[N^{(s)}] &= \sum_{k=0}^{\infty} k \left[\beta_1^{(s)} \cdot \frac{e^{-\lambda_1}\,\lambda_1^k}{k!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2}\,\lambda_2^k}{k!}\right] \\ &= \beta_1^{(s)} \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda_1}\,\lambda_1^k}{k!} + \beta_2^{(s)} \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda_2}\,\lambda_2^k}{k!}. \end{aligned}$$

Recall from Example 1 that we showed $\sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda_i}\,\lambda_i^k}{k!} = \lambda_i$. So, by substitution we have $E[N^{(s)}] = \beta_1^{(s)}\lambda_1 + \beta_2^{(s)}\lambda_2$. □

So, we have,

$$\begin{aligned} P(M^{(s)} - 1 = n) = P(M^{(s)} = n + 1) &= \frac{(n+1)\,P^{(s)}(N = n+1)}{E[N^{(s)}]} \\ &= \frac{(n+1)\left[\beta_1^{(s)} \cdot \frac{e^{-\lambda_1}\lambda_1^{n+1}}{(n+1)!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2}\lambda_2^{n+1}}{(n+1)!}\right]}{\beta_1^{(s)}\lambda_1 + \beta_2^{(s)}\lambda_2} \\ &= \frac{\beta_1^{(s)}\lambda_1 \cdot \frac{e^{-\lambda_1}\lambda_1^{n}}{n!} + \beta_2^{(s)}\lambda_2 \cdot \frac{e^{-\lambda_2}\lambda_2^{n}}{n!}}{\beta_1^{(s)}\lambda_1 + \beta_2^{(s)}\lambda_2} \\ &= \frac{\beta_1^{(s)}\lambda_1}{\beta_1^{(s)}\lambda_1 + \beta_2^{(s)}\lambda_2} \cdot \frac{e^{-\lambda_1}\lambda_1^{n}}{n!} + \frac{\beta_2^{(s)}\lambda_2}{\beta_1^{(s)}\lambda_1 + \beta_2^{(s)}\lambda_2} \cdot \frac{e^{-\lambda_2}\lambda_2^{n}}{n!}. \end{aligned}$$

Notice, this is also a finite mixture distribution where both processes follow the Poisson distribution with λ_1, λ_2 respectively. However, we have updated values for $\beta_1^{(s)}$ and $\beta_2^{(s)}$. The weight update rule is

$$\beta_j^{(s+1)} = \frac{\beta_j^{(s)}\,\lambda_j}{\beta_1^{(s)}\lambda_1 + \beta_2^{(s)}\lambda_2}.$$

So, we have

$$P^{(s)}(S_N = k) = \frac{E[N^{(s)}]}{k} \sum_{i=1}^{k} i\,\alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

Now we can begin the computation. Let $\lambda_1 = 3$, $\lambda_2 = 4$, $\beta_1^{(0)} = 0.6$, $\beta_2^{(0)} = 0.4$. We want to find $P^{(0)}(S_N = 4)$.

Computing the mixing weights at each level:

$$\beta_1^{(0)} = 0.6, \quad \beta_2^{(0)} = 0.4, \quad E[N^{(0)}] = (0.6)(3) + (0.4)(4) = 3.4.$$

$$\beta_1^{(1)} = \frac{(0.6)(3)}{3.4} = \frac{1.8}{3.4}, \quad \beta_2^{(1)} = \frac{(0.4)(4)}{3.4} = \frac{1.6}{3.4}, \quad E[N^{(1)}] = \frac{1.8(3) + 1.6(4)}{3.4} = \frac{11.8}{3.4}.$$

$$\beta_1^{(2)} = \frac{5.4}{11.8}, \quad \beta_2^{(2)} = \frac{6.4}{11.8}, \quad E[N^{(2)}] = \frac{5.4(3)+6.4(4)}{11.8} = \frac{41.8}{11.8}.$$
$$\beta_1^{(3)} = \frac{16.2}{41.8}, \quad \beta_2^{(3)} = \frac{25.6}{41.8}, \quad E[N^{(3)}] = \frac{16.2(3)+25.6(4)}{41.8} = \frac{151}{41.8}.$$
$$\beta_1^{(4)} = \frac{48.6}{151}, \quad \beta_2^{(4)} = \frac{102.4}{151}.$$

Base cases $P^{(s)}(S_N=0) = \beta_1^{(s)} e^{-\lambda_1} + \beta_2^{(s)} e^{-\lambda_2}$:

$$P^{(1)}(S_N=0) = \tfrac{1.8}{3.4}\, e^{-3} + \tfrac{1.6}{3.4}\, e^{-4}.$$
$$P^{(2)}(S_N=0) = \tfrac{5.4}{11.8}\, e^{-3} + \tfrac{6.4}{11.8}\, e^{-4}.$$
$$P^{(3)}(S_N=0) = \tfrac{16.2}{41.8}\, e^{-3} + \tfrac{25.6}{41.8}\, e^{-4}.$$
$$P^{(4)}(S_N=0) = \tfrac{48.6}{151}\, e^{-3} + \tfrac{102.4}{151}\, e^{-4}.$$

Computing $P^{(s)}(S_N=1)$:

$$\begin{aligned} P^{(1)}(S_N=1) &= \frac{E[N^{(1)}]}{1}\big[1\cdot 0.05\cdot P^{(2)}(S_N=0)\big] \\ &= \frac{11.8}{3.4}\left[(0.05)\left(\frac{5.4}{11.8}\, e^{-3} + \frac{6.4}{11.8}\, e^{-4}\right)\right] \\ &= 0.07941176471\, e^{-3} + 0.09411764706\, e^{-4}. \end{aligned}$$

$$\begin{aligned} P^{(2)}(S_N=1) &= \frac{E[N^{(2)}]}{1}\big[0.05\cdot P^{(3)}(S_N=0)\big] \\ &= 0.06864406780\, e^{-3} + 0.10847457627\, e^{-4}. \end{aligned}$$

$$\begin{aligned} P^{(3)}(S_N=1) &= \frac{E[N^{(3)}]}{1}\big[0.05\cdot P^{(4)}(S_N=0)\big] \\ &= 0.05813397129\, e^{-3} + 0.12248803828\, e^{-4}. \end{aligned}$$

Computing $P^{(s)}(S_N=2)$:

$$\begin{aligned} P^{(1)}(S_N=2) &= \frac{E[N^{(1)}]}{2}\big[(0.05)\cdot P^{(2)}(S_N=1) + (0.8)\cdot P^{(2)}(S_N=0)\big] \\ &= 0.64125\, e^{-3} + 0.76235294118\, e^{-4}. \end{aligned}$$

$$\begin{aligned} P^{(2)}(S_N=2) &= \frac{E[N^{(2)}]}{2}\big[(0.05)\cdot P^{(3)}(S_N=1) + (0.8)\cdot P^{(3)}(S_N=0)\big] \\ &= 0.55430084746\, e^{-3} + 0.87864406780\, e^{-4}. \end{aligned}$$

Computing $P^{(1)}(S_N=3)$:

$$\begin{aligned} P^{(1)}(S_N=3) &= \frac{E[N^{(1)}]}{3}\big[(0.05)\cdot P^{(2)}(S_N=2) + (0.8)\cdot P^{(2)}(S_N=1) + (0.3)\cdot P^{(2)}(S_N=0)\big] \\ &= 0.25441544118\, e^{-3} + 0.33945098039\, e^{-4}. \end{aligned}$$

Finally, $P^{(0)}(S_N=4)$:

$$\begin{aligned} P^{(0)}(S_N=4) = \frac{E[N^{(0)}]}{4}\big[&(0.05)\cdot P^{(1)}(S_N=3) + (0.8)\cdot P^{(1)}(S_N=2) \\ &+ (0.3)\cdot P^{(1)}(S_N=1) + P^{(1)}(S_N=0)\big] \end{aligned}$$

$$\begin{aligned}
&= \frac{3.4}{4}\big[(0.05)(0.25441544118\, e^{-3} + 0.33945098039\, e^{-4}) \\
&\qquad + (0.8)(0.64125\, e^{-3} + 0.76235294118\, e^{-4}) \\
&\qquad + (0.3)(0.07941176471\, e^{-3} + 0.09411764706\, e^{-4}) \\
&\qquad + \left(\tfrac{1.8}{3.4}\, e^{-3} + \tfrac{1.6}{3.4}\, e^{-4}\right)\big] \\
&= \boxed{0.91711265625\, e^{-3} + 0.95682666667\, e^{-4}}.
\end{aligned}$$

5. Finite Poisson Mixture Generalization

Next, we will describe a more general case of the Finite Mixture of Poisson Distributions. Consider a discrete mixture distribution where the event is generated by one or more random processes. The event is generated by one of t processes, which follows a Poisson distribution with mean λ_i for $i \in \mathbb{Z}$ with $0 < i \leq t$. The probability of process i is β_i for $i \in \mathbb{Z}$ with $0 < i \leq t$ and where $\beta_1 + \beta_2 + \cdots + \beta_t = 1$.

Now we are ready to define our probability mass function. Let $s \in \mathbb{Z}$ with $0 \leq s$, and let s represent the level of recursion. Let the parameters of this distribution be $s, \lambda_1, \ldots, \lambda_t$.

$$P^{(s)}(N = n) = \beta_1^{(s)} \cdot \frac{e^{-\lambda_1}\lambda_1^n}{n!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2}\lambda_2^n}{n!} + \cdots + \beta_t^{(s)} \cdot \frac{e^{-\lambda_t}\lambda_t^n}{n!}, \qquad n \geq 0.$$

Corollary 2 (Recursive Properties of the Generalized Finite Mixture of Poisson Distributions)**.** *Let $N^{(s)}$ be a mixed Poisson random variable at recursion level s with probability mass function*

$$P^{(s)}(N = n) = \sum_{j=1}^{t} \left(\frac{\beta_j \lambda_j^s}{\sum_{i=1}^t \beta_i \lambda_i^s} \right) \cdot \frac{e^{-\lambda_j}\lambda_j^n}{n!},$$

where $\sum_{i=1}^t \beta_i = 1$, $\beta_i > 0$, and $\lambda_i > 0$. Let $M^{(s)}$ be the random variable satisfying the compound identity condition associated with $N^{(s)}$. We claim that for all $s \geq 0$:

(1) $E[N^{(s)}] = \dfrac{\sum_{i=1}^t \beta_i \lambda_i^{s+1}}{\sum_{j=1}^t \beta_j \lambda_j^s}$.

(2) *The distribution of $M^{(s)} - 1$ is identical to the distribution of $N^{(s+1)}$.*

Proof (by induction). Base Case ($s = 0$): For $s = 0$, the weights are β_i. First, we compute the expected value of $E[N^{(0)}]$:

$$E[N^{(0)}] = \sum_{n=0}^{\infty} n \left[\sum_{i=1}^t \beta_i \cdot \frac{e^{-\lambda_i}\lambda_i^n}{n!} \right] = \sum_{i=1}^t \beta_i \lambda_i.$$

This matches the form of Claim 1 where the denominator $\sum_{i=1}^t \beta_i \lambda_i^0 = \sum_{i=1}^t \beta_i = 1$.

Next, we apply the compound random identity and see that,

$$\begin{aligned}
P(M^{(0)} - 1 = n) = P(M^{(0)} = n+1) &= \frac{(n+1)\, P^{(0)}(N = n+1)}{E[N^{(0)}]} \\
&= \frac{(n+1)\left[\beta_1 \cdot \dfrac{e^{-\lambda_1}\lambda_1^{n+1}}{(n+1)!} + \cdots + \beta_t \cdot \dfrac{e^{-\lambda_t}\lambda_t^{n+1}}{(n+1)!}\right]}{\beta_1\lambda_1 + \cdots + \beta_t\lambda_t} \\
&= \frac{n+1}{\sum_{j=1}^t \beta_j\lambda_j} \cdot \sum_{i=1}^t \beta_i \cdot \frac{e^{-\lambda_i}\lambda_i^{n+1}}{(n+1)!}
\end{aligned}$$

$$= \sum_{i=1}^{t} \frac{\beta_i \, \lambda_i}{\sum_{j=1}^{t} \beta_j \, \lambda_j} \cdot \frac{e^{-\lambda_i} \lambda_i^n}{n!}.$$

This is exactly the definition of $P^{(1)}(N = n)$. Thus, the claim holds for $s = 0$.

Inductive Step: Assume the statement is true for $s = k$. That is, $M^{(k)} - 1$ is equivalent to $N^{(k+1)}$ and $E[N^{(k)}]$ follows the claim. We need to show the statement is true for $s = k + 1$. We must determine the distribution of $M^{(k+1)} - 1$.

We have,

$$P(M^{(k+1)} - 1 = n) = P(M^{(k+1)} = n + 1) = \frac{(n + 1) \, P^{(k+1)}(N = n + 1)}{E[N^{(k+1)}]}.$$

First, we compute $E[N^{(k+1)}]$:

$$\begin{aligned} E[N^{(k+1)}] &= \sum_{n=0}^{\infty} n \cdot \sum_{i=1}^{t} \left(\frac{\beta_i \, \lambda_i^{k+1}}{\sum_{j=1}^{t} \beta_j \, \lambda_j^{k+1}} \right) \cdot \frac{e^{-\lambda_i} \lambda_i^n}{n!} \\ &= \sum_{i=1}^{t} \left(\frac{\beta_i \, \lambda_i^{k+1}}{\sum_{j=1}^{t} \beta_j \, \lambda_j^{k+1}} \right) \cdot \lambda_i = \frac{\sum_{i=1}^{t} \beta_i \, \lambda_i^{k+2}}{\sum_{j=1}^{t} \beta_j \, \lambda_j^{k+1}}. \end{aligned}$$

This confirms Claim 1 for $s = k + 1$. Next, we substitute this into the probability expression:

$$P(M^{(k+1)} - 1 = n) = \frac{n + 1}{\frac{\sum_{i=1}^{t} \beta_i \lambda_i^{k+2}}{\sum_{j=1}^{t} \beta_j \lambda_j^{k+1}}} \sum_{i=1}^{t} \left(\frac{\beta_i \lambda_i^{k+1}}{\sum_{j=1}^{t} \beta_j \lambda_j^{k+1}} \right) \cdot \frac{e^{-\lambda_i} \lambda_i^{n+1}}{(n + 1)!}.$$

We cancel the common denominator terms $\sum_{j=1}^{t} \beta_j \lambda_j^{k+1}$ and we have

$$\begin{aligned} &= \frac{n + 1}{\sum_{j=1}^{t} \beta_j \lambda_j^{k+2}} \sum_{i=1}^{t} \beta_i \lambda_i^{k+1} \cdot \frac{e^{-\lambda_i} \lambda_i^{n+1}}{(n + 1)!} \\ &= \sum_{i=1}^{t} \frac{\beta_i \lambda_i^{k+2}}{\sum_{j=1}^{t} \beta_j \lambda_j^{k+2}} \cdot \frac{e^{-\lambda_i} \lambda_i^n}{n!}. \end{aligned}$$

This expression is exactly the PMF for $N^{(k+2)}$ as desired.

So, by the principle of mathematical induction, for all $s \in \mathbb{N}$, the compound identity variable associated with $N^{(s)}$ shifts the distribution to $N^{(s+1)}$, and the expected value satisfies the derived closed form. □

Now we are ready to construct our new recursive expression by applying Corollary 1. From

$$P(S_N = k) = \frac{1}{k} \, E[N] \sum_{i=1}^{k} i \, \alpha_i \, P(S_{M-1} = k - i),$$

we arrive at the updated recursive equation

$$P^{(s)}(S_N = k) = \frac{1}{k} \cdot \frac{\sum_{i=1}^{t} \beta_i \, \lambda_i^{s+1}}{\sum_{i=1}^{t} \beta_i \, \lambda_i^{s}} \cdot \sum_{i=1}^{k} i \, \alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

Example 5 (Revisiting Example 1)**.** Now we can compute the same compound random variable from Example 1, but with our updated recursive equation. Like before, let N be a random variable of the Poisson distribution with parameter $\lambda_1 = 3$. So the probability mass function for the distribution of N is

$$P^{(s)}(N = n) = \beta_1 \cdot \frac{e^{-\lambda}\,\lambda_1^n}{n!}, \quad \text{where } \beta_1 = 1.$$

Furthermore,

$$E[N^{(s)}] = \frac{\sum_{i=1}^{1} \beta_i\,\lambda_i^{s+1}}{\sum_{i=1}^{1} \beta_i\,\lambda_i^{s}} = \frac{\beta_1\,\lambda_1^{s+1}}{\beta_1\,\lambda_1^{s}} = \frac{\lambda_1^{s+1}}{\lambda_1^{s}} = \lambda_1 = 3, \qquad \forall\, s \in \mathbb{Z},\ s \geq 0.$$

Now we calculate $P^{(0)}(S_N = 4)$. We want to find

$$P^{(0)}(S_N = 4) = \frac{3}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + 4(0.25) \cdot P^{(1)}(S_N = 0)\big].$$

We can solve each of these recursive expressions for all $s \in \mathbb{Z}$, $s \geq 0$. So we have,

$$\begin{aligned}
P^{(s)}(S_N = 0) &= P(N^{(s)} = 0) = 1 \cdot \frac{e^{-\lambda}\,\lambda^0}{0!} = e^{-\lambda} = e^{-3}, \qquad \forall\, s \geq 0.\\
P^{(s)}(S_N = 1) &= \frac{3}{1}\big[1 \cdot (0.05) \cdot P(S_N = 0)\big] = 3(0.05)\,e^{-3} = 0.15\,e^{-3}, \qquad \forall\, s \geq 0.\\
P^{(s)}(S_N = 2) &= \frac{3}{2}\big[(0.05) \cdot P(S_N = 1) + 2(0.4) \cdot P(S_N = 0)\big]\\
&= \frac{3}{2}\big[(0.05)(0.15\,e^{-3}) + (0.8)\,e^{-3}\big] = 1.21125\,e^{-3}, \qquad \forall\, s \geq 0.\\
P^{(s)}(S_N = 3) &= \frac{3}{3}\big[(0.05) \cdot P(S_N = 2) + (0.8) \cdot P(S_N = 1) + 3(0.1) \cdot P(S_N = 0)\big]\\
&= (0.05)(1.21125\,e^{-3}) + (0.8)(0.15\,e^{-3}) + (0.3)\,e^{-3} = 0.4805625\,e^{-3}, \qquad \forall\, s \geq 0.
\end{aligned}$$

Now we have everything we need to compute $P^{(0)}(S_N = 4)$. We have,

$$\begin{aligned}
P^{(0)}(S_N = 4) &= \frac{3}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + 4(0.25) \cdot P^{(1)}(S_N = 0)\big]\\
&= \frac{3}{4}\big[0.05(0.4805625\,e^{-3}) + 0.8(1.21125\,e^{-3}) + 0.3(0.15\,e^{-3}) + e^{-3}\big]\\
&= 1.528521094\,e^{-3}.
\end{aligned}$$

Notice, this is the same as our result from Example 1.

Example 6 (Revisiting Example 4)**.** We will consider again the case of a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Poisson distribution with mean λ_1, or process two, which follows a Poisson distribution with mean λ_2. Like in Example 4, we let $\lambda_1 = 3$, $\lambda_2 = 4$, $\beta_1^{(0)} = 0.6$, $\beta_2^{(0)} = 0.4$. We want to find $P^{(0)}(S_N = 4)$.

So we have,

$$P^{(s)}(S_N = k) = \frac{1}{k}\left[\frac{(0.6)(3)^{s+1} + (0.4)(4)^{s+1}}{(0.6)(3)^{s} + (0.4)(4)^{s}}\right] \sum_{i=1}^{k} i\,\alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

$$P^{(0)}(S_N = 4) = \frac{1}{4}\left[\frac{(0.6)(3)^{1} + (0.4)(4)^{1}}{(0.6)(3)^{0} + (0.4)(4)^{0}}\right] \sum_{i=1}^{4} i\,\alpha_i \cdot P^{(1)}(S_N = 4 - i).$$

Base cases:

$$P^{(1)}(S_N=0)=\frac{(0.6)(3)^1}{(0.6)(3)^1+(0.4)(4)^1}\,e^{-3}+\frac{(0.4)(4)^1}{(0.6)(3)^1+(0.4)(4)^1}\,e^{-4}=\frac{1.8}{3.4}\,e^{-3}+\frac{1.6}{3.4}\,e^{-4}.$$

$$P^{(2)}(S_N=0)=\frac{(0.6)(3)^2}{(0.6)(3)^2+(0.4)(4)^2}\,e^{-3}+\frac{(0.4)(4)^2}{(0.6)(3)^2+(0.4)(4)^2}\,e^{-4}=\frac{5.4}{11.8}\,e^{-3}+\frac{6.4}{11.8}\,e^{-4}.$$

$$P^{(3)}(S_N=0)=\frac{16.2}{41.8}\,e^{-3}+\frac{25.6}{41.8}\,e^{-4}.$$

$$P^{(4)}(S_N=0)=\frac{48.6}{151}\,e^{-3}+\frac{102.4}{151}\,e^{-4}.$$

Computing $P^{(s)}(S_N=1)$:

$$\begin{aligned}P^{(1)}(S_N=1)&=\left[\frac{(0.6)(3)^2+(0.4)(4)^2}{(0.6)(3)^1+(0.4)(4)^1}\right]\left[1\cdot 0.05\cdot P^{(2)}(S_N=0)\right]\\&=\frac{11.8}{3.4}\left[0.05\left(\frac{5.4}{11.8}\,e^{-3}+\frac{6.4}{11.8}\,e^{-4}\right)\right]=0.07941176471\,e^{-3}+0.09411764706\,e^{-4}.\end{aligned}$$

$$P^{(2)}(S_N=1)=\frac{41.8}{11.8}\left[0.05\cdot P^{(3)}(S_N=0)\right]=0.06864406780\,e^{-3}+0.10847457627\,e^{-4}.$$

$$P^{(3)}(S_N=1)=\frac{151}{41.8}\left[0.05\cdot P^{(4)}(S_N=0)\right]=0.05813397129\,e^{-3}+0.12248803828\,e^{-4}.$$

Computing $P^{(s)}(S_N=2)$:

$$\begin{aligned}P^{(1)}(S_N=2)&=\frac{1}{2}\cdot\frac{11.8}{3.4}\left[(0.05)\cdot P^{(2)}(S_N=1)+(0.8)\cdot P^{(2)}(S_N=0)\right]\\&=0.64125\,e^{-3}+0.76235294118\,e^{-4}.\\P^{(2)}(S_N=2)&=\frac{1}{2}\cdot\frac{41.8}{11.8}\left[(0.05)\cdot P^{(3)}(S_N=1)+(0.8)\cdot P^{(3)}(S_N=0)\right]\\&=0.55430084746\,e^{-3}+0.87864406780\,e^{-4}.\end{aligned}$$

Computing $P^{(1)}(S_N=3)$:

$$\begin{aligned}P^{(1)}(S_N=3)&=\frac{1}{3}\cdot\frac{11.8}{3.4}\left[(0.05)\cdot P^{(2)}(S_N=2)+(0.8)\cdot P^{(2)}(S_N=1)+(0.3)\cdot P^{(2)}(S_N=0)\right]\\&=0.25441544118\,e^{-3}+0.33945098039\,e^{-4}.\end{aligned}$$

Finally, $P^{(0)}(S_N=4)$:

$$\begin{aligned}P^{(0)}(S_N=4)&=\frac{1}{4}\cdot\frac{3.4}{1}\big[(0.05)\cdot P^{(1)}(S_N=3)+(0.8)\cdot P^{(1)}(S_N=2)\\&\qquad+(0.3)\cdot P^{(1)}(S_N=1)+P^{(1)}(S_N=0)\big]\\&=\boxed{0.91711265625\,e^{-3}+0.95682666667\,e^{-4}}.\end{aligned}$$

6. Generalized Computation with a Finite Mixture of Distributions

We have shown in Example 1 with N in the Poisson distribution that

$$P^{(s+1)}(M^{(s)}-1=n)=P^{(s)}(N=n).$$

We have shown in Example 2 with N in the negative binomial distribution that

$$P^{(s)}(M^{(s)}-1=n)=P^{(s+1)}(N=n),$$

where the negative binomial parameter shifts from $r+s$ to $r+s+1$. Similarly, we have shown in Example 3 with N in the binomial distribution that

$$P^{(s)}(M^{(s)}-1=n)=P^{(s+1)}(N=n),$$

where the binomial parameter shifts from $r-s$ to $r-s-1$.

Notice that for each of these distributions, when calculating $P(M^{(s)}-1=n)$, at a given level of recursion, we arrive at either $P^{(s)}(N=n)$ or a version of $P^{(s)}(N=n)$ with a parameter value change that is a function of the level of recursion.

We will now show that by moving the level of recursion to a parameter of the probability mass function of the finite mixture of distributions we can compute a compound random variable with the counting distribution in any finite mixture of any of these three distributions.

Corollary 3 (Recursive Properties of the Finite Mixture Distributions)**.** *Let $s \in \mathbb{Z}$ with $0 \le s$, and let s represent the level of recursion. Let $N^{(s)}$ be a mixed random variable of Binomial or Negative Binomial distribution at recursion level s with probability mass function:*

$$P^{(s)}(N=n)=\sum_{j=1}^{t}\beta_j^{(s)}\,P_{r_j}(N^{(s)}=n), \qquad n\ge 0,$$

where $\sum_{i=1}^{t}\beta_i=1$. Let $M^{(s)}$ be the random variable satisfying the compound identity condition associated with $N^{(s)}$. We claim that for all $s\ge 0$:

(1) $E[N^{(s)}]=\sum_{j=1}^{t}\beta_j^{(s)}\,E[N_j^{(0)}]$.

(2) *The distribution of $M^{(s)}-1$ is identical to the distribution of $N^{(s+1)}$, where the new weights are updated such that*

$$\beta_j^{(s+1)}=\beta_j^{(s)}\cdot\frac{E[N_j^{(s)}]}{E[N^{(s)}]}.$$

Proof (by induction). Base Case ($s=0$): For $s=0$, the weights are β_i. First, we compute the expected value of $E[N^{(0)}]$:

$$E[N^{(0)}]=\sum_{n=0}^{\infty}n\cdot\left[\sum_{j=1}^{t}\beta_j^{(0)}\,P_{r_j}(n)\right]=\sum_{j=1}^{t}\beta_j^{(0)}\,E[N_j^{(0)}].$$

Thus, Claim 1 holds for case $s=0$.

Next, we apply the compound random identity and see that,

$$\begin{aligned}
P(M^{(0)}-1=n)=P(M^{(0)}=n+1)&=\frac{(n+1)\,P(N^{(0)}=n+1)}{E[N^{(0)}]}\\
&=\frac{(n+1)\left[\sum_{j=1}^{t}\beta_j^{(0)}\,P_{r_j}(N^{(0)}=n+1)\right]}{E[N^{(0)}]}\\
&=\frac{1}{E[N^{(0)}]}\sum_{j=1}^{t}\beta_j^{(0)}\left[(n+1)\cdot P_{r_j}(N^{(0)}=n+1)\right].
\end{aligned}$$

Notice that for the negative binomial distribution, $P^{(s)}(M^{(s)}-1=n)=P^{(s+1)}(N=n)$ implies

$$(n+1)\,P_r(N=n+1)=E[N]\,P_{r+1}(N=n).$$

So by substitution we have,

$$\begin{aligned} P(M^{(0)} - 1 = n) &= \frac{1}{E[N^{(0)}]} \sum_{j=1}^{t} \beta_j^{(0)} \, E[N_j^{(0)}] \cdot P_{r_j+1}(N^{(1)} = n) \\ &= \sum_{j=1}^{t} \frac{\beta_j \, E[N_j^{(0)}]}{E[N^{(0)}]} \cdot P_{r_j+1}(N^{(1)} = n). \end{aligned}$$

Similarly, for the binomial distribution, $P^{(s)}(M^{(s)} - 1 = n) = P^{(s+1)}(N = n)$ implies

$$(n+1) \, P_r(N = n+1) = E[N] \, P_{r-1}(N = n).$$

So by substitution we have,

$$\begin{aligned} P(M^{(0)} - 1 = n) &= \frac{1}{E[N^{(0)}]} \sum_{j=1}^{t} \beta_j^{(0)} \, E[N_j^{(0)}] \cdot P_{r_j-1}(N^{(1)} = n) \\ &= \sum_{j=1}^{t} \frac{\beta_j \, E[N_j^{(0)}]}{E[N^{(0)}]} \cdot P_{r_j-1}(N^{(1)} = n). \end{aligned}$$

These match the definition of $N^{(1)}$, a mixture with shifted parameters according to the distribution. Thus, the claim holds for $s = 0$.

Inductive Step: Assume the statement is true for $s = k$. That is, $M^{(k)} - 1$ is equivalent to $N^{(k+1)}$ and $E[N^{(k)}]$ follows the claim. We need to show the statement is true for $s = k+1$. We must determine the distribution of $M^{(k+1)} - 1$.

We have,

$$P(M^{(k+1)} - 1 = n) = P(M^{(k+1)} = n+1) = \frac{(n+1) \, P(N^{(k+1)} = n+1)}{E[N^{(k+1)}]}.$$

Substituting the mixture sum for $N^{(k+1)}$,

$$P(M^{(k+1)} - 1 = n) = \frac{1}{E[N^{(k+1)}]} \sum_{j=1}^{t} \beta_j^{(k+1)} \Big[(n+1) \cdot P_{r_j+(k+1)}(N^{(0)} = n+1) \Big].$$

So, for the negative binomial distribution, by substitution we have,

$$\begin{aligned} P(M^{(k+1)} - 1 = n) &= \frac{1}{E[N^{(k+1)}]} \sum_{j=1}^{t} \beta_j^{(k+1)} \, E[N_j^{(k+1)}] \cdot P_{r_j+k+2}(N^{(1)} = n) \\ &= \sum_{j=1}^{t} \frac{\beta_j^{k+1} \, E[N_j^{(k+1)}]}{E[N^{(k+1)}]} \cdot P_{r_j+k+2}(N^{(1)} = n). \end{aligned}$$

Similarly, for the binomial distribution, by substitution we have,

$$\begin{aligned} P(M^{(k+1)} - 1 = n) &= \frac{1}{E[N^{(k+1)}]} \sum_{j=1}^{t} \beta_j^{(k+1)} \, E[N_j^{(k+1)}] \cdot P_{r_j-(k+2)}(N^{(1)} = n) \\ &= \sum_{j=1}^{t} \frac{\beta_j^{k+1} \, E[N_j^{(k+1)}]}{E[N^{(k+1)}]} \cdot P_{r_j-(k+2)}(N^{(1)} = n). \end{aligned}$$

This expression is exactly the PMF for $N^{(k+2)}$ with weights β_j^{k+1} as desired.

So, by the principle of mathematical induction, for all $s \in \mathbb{N}$, the compound identity variable associated with $N^{(s)}$ shifts the distribution to $N^{(s+1)}$, and the expected value satisfies the derived closed form. $\square$

Now we are ready to construct our new recursive expression by applying Corollary 1. From

$$P(S_N = k) = \frac{1}{k}\, E[N] \sum_{i=1}^{k} i\, \alpha_i\, P(S_{M-1} = k - i),$$

we arrive at the updated recursive equation

$$P^{(s)}(S_N = k) = \frac{E[N^{(s)}]}{k} \cdot \sum_{i=1}^{k} i\, \alpha_i \cdot P^{(s+1)}(S_N = k - i),$$

where $E[N^{(s)}] = \sum_{j=1}^{t} \beta_j^{(s)}\, E[N_j^{(0)}]$ and the computation of $P^{(s+1)}$ uses $\beta^{(s+1)}$ and parameters appropriate to the distribution of P_i, that is, $(r - s)$ for the binomial distribution, $(r + s)$ for the negative binomial distribution and no parameter change for the Poisson distribution.

Example 7 (N in a Finite Mixture of Binomial Distributions)**.** Consider a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Binomial distribution with $r_1 = 8$ and $p_1 = 0.5$, or process two, which follows a Binomial distribution with $r_2 = 10$ and $p_2 = 0.7$. The probability of process one is $\beta_1^{(0)} = 0.6$. The probability of process two is $\beta_2^{(0)} = 0.4$. Notice that $\beta_1^{(0)} + \beta_2^{(0)} = 1$. We want to find $P(S_N = 4)$.

So, the probability mass function for this mixed distribution is

$$P^{(s)}(N = n) = \sum_{j=1}^{2} \beta_j^{(s)} \binom{r_j - s}{n} p_j^n (1 - p_j)^{(r_j - s) - n}, \quad 0 \le n \le \max(r_1 - s,\, r_2 - s).$$

Computing the mixing weights at each level:

$$\begin{aligned}
E[N^{(0)}] &= \beta_1^{(0)}(r_1 - 0)p_1 + \beta_2^{(0)}(r_2 - 0)p_2 = 0.6 \cdot 8 \cdot 0.5 + 0.4 \cdot 10 \cdot 0.7 = 5.2.\\
\beta_1^{(1)} &= \beta_1^{(0)} \cdot \frac{E[N_1^{(0)}]}{E[N^{(0)}]} = \frac{2.4}{5.2} = 0.46153846154.\\
\beta_2^{(1)} &= \beta_2^{(0)} \cdot \frac{E[N_2^{(0)}]}{E[N^{(0)}]} = \frac{2.8}{5.2} = 0.53846153846.\\
E[N^{(1)}] &= \beta_1^{(1)}(r_1 - 1)p_1 + \beta_2^{(1)}(r_2 - 1)p_2\\
&= 0.46153846154 \cdot 7 \cdot 0.5 + 0.53846153846 \cdot 9 \cdot 0.7 = 5.00769230769.\\
\beta_1^{(2)} &= \beta_1^{(1)} \cdot \frac{E[N_1^{(1)}]}{E[N^{(1)}]} = \frac{1.61538461538}{5.00769230769} = 0.32258064516.\\
\beta_2^{(2)} &= \beta_2^{(1)} \cdot \frac{E[N_2^{(1)}]}{E[N^{(1)}]} = \frac{3.39230769231}{5.00769230769} = 0.67741935484.\\
E[N^{(2)}] &= \beta_1^{(2)}(r_1 - 2)p_1 + \beta_2^{(2)}(r_2 - 2)p_2\\
&= 0.32258064516 \cdot 6 \cdot 0.5 + 0.67741935484 \cdot 8 \cdot 0.7 = 4.76129032258.\\
\beta_1^{(3)} &= \beta_1^{(2)} \cdot \frac{E[N_1^{(2)}]}{E[N^{(2)}]} = \frac{0.96774193548}{4.76129032258} = 0.20325203252.\\
\beta_2^{(3)} &= \beta_2^{(2)} \cdot \frac{E[N_2^{(2)}]}{E[N^{(2)}]} = \frac{3.79354838710}{4.76129032258} = 0.79674796748.\\
E[N^{(3)}] &= \beta_1^{(3)}(r_1 - 3)p_1 + \beta_2^{(3)}(r_2 - 3)p_2
\end{aligned}$$

$$= 0.20325203252 \cdot 5 \cdot 0.5 + 0.79674796748 \cdot 7 \cdot 0.7 = 4.41219512195.$$

$$\beta_1^{(4)} = \beta_1^{(3)} \cdot \frac{E[N_1^{(3)}]}{E[N^{(3)}]} = \frac{0.50813008130}{4.41219512195} = 0.11516491616.$$

$$\beta_2^{(4)} = \beta_2^{(3)} \cdot \frac{E[N_2^{(3)}]}{E[N^{(3)}]} = \frac{3.90406504065}{4.41219512195} = 0.88483508384.$$

$$E[N^{(4)}] = \beta_1^{(4)}(r_1 - 4)p_1 + \beta_2^{(4)}(r_2 - 4)p_2$$
$$= 0.11516491616 \cdot 4 \cdot 0.5 + 0.88483508384 \cdot 6 \cdot 0.7 = 3.94663718445.$$

Now we can begin the computation to find $P^{(0)}(S_N = 4)$. We want to find,

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\big].$$

Base cases:

$$P^{(s)}(S_N = 0) = \beta_1^{(s)} \cdot (1 - p_1)^{r_1 - s} + \beta_2^{(s)} \cdot (1 - p_2)^{r_2 - s} = \beta_1^{(s)} \cdot (0.5)^{8-s} + \beta_2^{(s)} \cdot (0.3)^{10-s}.$$

$$P^{(1)}(S_N = 0) = 0.46153846154\,(0.5)^7 + 0.53846153846\,(0.3)^9 = 0.00361636777.$$
$$P^{(2)}(S_N = 0) = 0.32258064516\,(0.5)^6 + 0.67741935484\,(0.3)^8 = 0.00508476806.$$
$$P^{(3)}(S_N = 0) = 0.20325203252\,(0.5)^5 + 0.79674796748\,(0.3)^7 = 0.00652587480.$$
$$P^{(4)}(S_N = 0) = 0.11516491616\,(0.5)^4 + 0.88483508384\,(0.3)^6 = 0.00784285204.$$

Computing $P^{(s)}(S_N = 1)$:

$$P^{(1)}(S_N = 1) = \frac{E[N^{(1)}]}{1}\big[0.05 \cdot P^{(2)}(S_N = 0)\big] = 5.00769230769(0.05)(0.00508476806) = 0.00127314770.$$
$$P^{(2)}(S_N = 1) = \frac{E[N^{(2)}]}{1}\big[0.05 \cdot P^{(3)}(S_N = 0)\big] = 4.76129032258(0.05)(0.00652587480) = 0.00155357923.$$
$$P^{(3)}(S_N = 1) = \frac{E[N^{(3)}]}{1}\big[0.05 \cdot P^{(4)}(S_N = 0)\big] = 4.41219512195(0.05)(0.00784285204) = 0.00173020967.$$

Computing $P^{(s)}(S_N = 2)$:

$$P^{(1)}(S_N = 2) = \frac{E[N^{(1)}]}{2}\big[0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)\big]$$
$$= \frac{5.00769230769}{2}\big[0.05(0.00155357923) + 0.8(0.00508476806)\big] = 0.01037967774.$$
$$P^{(2)}(S_N = 2) = \frac{E[N^{(2)}]}{2}\big[0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)\big]$$
$$= \frac{4.76129032258}{2}\big[0.05(0.00173020967) + 0.8(0.00652587480)\big] = 0.01263458457.$$

Computing $P^{(1)}(S_N = 3)$:

$$P^{(1)}(S_N = 3) = \frac{E[N^{(1)}]}{3}\big[0.05 \cdot P^{(2)}(S_N = 2) + 0.8 \cdot P^{(2)}(S_N = 1) + 0.3 \cdot P^{(2)}(S_N = 0)\big]$$
$$= \frac{5.00769230769}{3}\big[0.05(0.01263458457) + 0.8(0.00155357923) + 0.3(0.00508476806)\big]$$
$$= 0.00567542306.$$

Computing $P^{(0)}(S_N = 4)$:

$$\begin{aligned}
P^{(0)}(S_N = 4) &= \frac{E[N^{(0)}]}{4}\left[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\right] \\
&= \frac{5.2}{4}\big[0.05(0.00567542306) + 0.8(0.01037967774) \\
&\qquad + 0.3(0.00127314770) + 0.00361636777\big] \\
&= \boxed{0.01636157305}.
\end{aligned}$$

Example 8 (N in a Finite Mixture of Negative Binomial Distributions)**.** Consider a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Negative Binomial distribution with $r_1 = 4$ and $p_1 = 0.6$, or process two, which follows a Negative Binomial distribution with $r_2 = 6$ and $p_2 = 0.5$. The probability of process one is $\beta_1^{(0)} = 0.5$. The probability of process two is $\beta_2^{(0)} = 0.5$. Notice that $\beta_1^{(0)} + \beta_2^{(0)} = 1$. We want to find $P(S_N = 4)$.

So, the probability mass function for this mixed distribution is

$$P^{(s)}(N = n) = \sum_{j=1}^{2} \beta_j^{(s)} \binom{n + r_j + s - 1}{n} p_j^{r_j + s}(1 - p_j)^n, \quad 0 \le n \le \max(r_1 + s,\, r_2 + s).$$

Computing the mixing weights at each level:

$$\begin{aligned}
E[N^{(0)}] &= 0.5\left(\frac{(4+0)(0.4)}{0.6}\right) + 0.5\left(\frac{(6+0)(0.5)}{0.5}\right) = 4.33333333333. \\
\beta_1^{(1)} &= 0.5 \cdot \frac{1.33333333333}{4.33333333333} = 0.30769230769. \\
\beta_2^{(1)} &= 0.5 \cdot \frac{3}{4.33333333333} = 0.69230769231. \\
E[N^{(1)}] &= 0.30769230769 \cdot \frac{5 \cdot 0.4}{0.6} + 0.69230769231 \cdot \frac{7 \cdot 0.5}{0.5} = 5.87179487179. \\
\beta_1^{(2)} &= 0.30769230769 \cdot \frac{1.02564102564}{5.87179487179} = 0.17467248908. \\
\beta_2^{(2)} &= 0.69230769231 \cdot \frac{4.84615384615}{5.87179487179} = 0.82532751092. \\
E[N^{(2)}] &= 0.17467248908 \cdot \frac{6 \cdot 0.4}{0.6} + 0.82532751092 \cdot \frac{8 \cdot 0.5}{0.5} = 7.30131004367. \\
\beta_1^{(3)} &= 0.17467248908 \cdot \frac{0.69868995633}{7.30131004367} = 0.09569377990. \\
\beta_2^{(3)} &= 0.82532751092 \cdot \frac{6.60262008734}{7.30131004367} = 0.90430622010. \\
E[N^{(3)}] &= 0.09569377990 \cdot \frac{7 \cdot 0.4}{0.6} + 0.90430622010 \cdot \frac{9 \cdot 0.5}{0.5} = 8.58532695375. \\
\beta_1^{(4)} &= 0.09569377990 \cdot \frac{0.44664430295}{8.58532695375} = 0.05201560468. \\
\beta_2^{(4)} &= 0.90430622010 \cdot \frac{8.13875598086}{8.58532695375} = 0.94798439532. \\
E[N^{(4)}] &= 0.05201560468 \cdot \frac{8 \cdot 0.4}{0.6} + 0.94798439532 \cdot \frac{10 \cdot 0.5}{0.5} = 9.75726051149.
\end{aligned}$$

Now we can begin the computation to find $P^{(0)}(S_N = 4)$. We want to find,

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\big].$$

Base cases $P^{(s)}(S_N = 0) = \beta_1^{(s)}\, p_1^{r_1+s} + \beta_2^{(s)}\, p_2^{r_2+s}$:

$$P^{(0)}(S_N = 0) = 0.5(0.6)^4 + 0.5(0.5)^6 = 0.0726125.$$

$$P^{(1)}(S_N = 0) = 0.30769230769(0.6)^5 + 0.69230769231(0.5)^7 = 0.02933480769.$$

$$P^{(2)}(S_N = 0) = 0.17467248908(0.6)^6 + 0.82532751092(0.5)^8 = 0.01137345524.$$

$$P^{(3)}(S_N = 0) = 0.09569377990(0.6)^7 + 0.90430622010(0.5)^9 = 0.00444503648.$$

$$P^{(4)}(S_N = 0) = 0.05201560468(0.6)^8 + 0.94798439532(0.5)^{10} = 0.00179942843.$$

Computing $P^{(s)}(S_N = 1)$:

$$P^{(1)}(S_N = 1) = \frac{E[N^{(1)}]}{1}\big[0.05 \cdot P^{(2)}(S_N = 0)\big] = 5.87179487179(0.05)(0.01137345524) = 0.00333912981.$$

$$P^{(2)}(S_N = 1) = \frac{E[N^{(2)}]}{1}\big[0.05 \cdot P^{(3)}(S_N = 0)\big] = 7.30131004367(0.05)(0.00444503648) = 0.00162272948.$$

$$P^{(3)}(S_N = 1) = \frac{E[N^{(3)}]}{1}\big[0.05 \cdot P^{(2)}(S_N = 0)\big] = 8.58532695375(0.05)(0.01137345524) = 0.00488224159.$$

Computing $P^{(s)}(S_N = 2)$:

$$\begin{aligned} P^{(1)}(S_N = 2) &= \frac{E[N^{(1)}]}{2}\big[0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)\big] \\ &= \frac{5.87179487179}{2}\big[0.05(0.00162272948) + 0.8(0.01137345524)\big] = 0.02695124683. \\ P^{(2)}(S_N = 2) &= \frac{E[N^{(2)}]}{2}\big[0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)\big] \\ &= \frac{7.30131004367}{2}\big[0.05(0.00488224159) + 0.8(0.00444503648)\big] = 0.01387300480. \end{aligned}$$

Computing $P^{(1)}(S_N = 3)$:

$$\begin{aligned} P^{(1)}(S_N = 3) &= \frac{E[N^{(1)}]}{3}\big[0.05 \cdot P^{(2)}(S_N = 2) + 0.8 \cdot P^{(2)}(S_N = 1) + 0.3 \cdot P^{(2)}(S_N = 0)\big] \\ &= \frac{5.87179487179}{3}\big[0.05(0.01387300480) + 0.8(0.00162272948) + 0.3(0.01137345524)\big] \\ &= 0.01057680615. \end{aligned}$$

Computing $P^{(0)}(S_N = 4)$:

$$\begin{aligned} P^{(0)}(S_N = 4) &= \frac{E[N^{(0)}]}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\big] \\ &= \frac{4.33333333333}{4}\big[0.05(0.01057680615) + 0.8(0.02695124683) \\ &\qquad + 0.3(0.00333912981) + 0.02933480769\big] \\ &= \boxed{0.05679524977}. \end{aligned}$$

Example 9 (N in a Finite Mixture of Poisson, Binomial and Negative Binomial Distributions)**.** Consider a discrete mixture distribution where the event is generated by one of three random processes. The event is either generated by process one, which follows a Poisson distribution with $\lambda = 3$, or process two, which follows a Binomial distribution with $r_2 = 6$ and $p_2 = 0.5$, or process three, which follows a Negative Binomial distribution with $r_3 = 2$ and $p_3 = 0.4$. The probability of process one is $\beta_1^{(0)} = 0.4$. The probability of process two is $\beta_2^{(0)} = 0.3$. The probability of process three is $\beta_3^{(0)} = 0.3$. Notice that $\beta_1^{(0)} + \beta_2^{(0)} + \beta_3^{(0)} = 1$. We want to find $P(S_N = 4)$.

So, the probability mass function for this mixed distribution is

$$P^{(s)}(N = n) = \beta_1^{(s)} \cdot \frac{e^{-\lambda}\lambda^n}{n!} + \beta_2^{(s)} \cdot \binom{r_2 - s}{n} p_2^n (1-p_2)^{(r_2-s)-n} + \beta_3^{(s)} \cdot \binom{n + r_3 + s - 1}{n} p_3^{r_3+s}(1-p_3)^n,$$

for $0 \le n \le \max(r_2 - s,\, r_3 + s)$.

So we have,

$$E[N_1^{(s)}] = \lambda = 3, \quad \text{for all } s.$$
$$E[N_2^{(s)}] = (r_2 - s)\, p_2 = (6 - s)(0.5).$$
$$E[N_3^{(s)}] = \frac{(r_3 + s)(1 - p_3)}{p_3} = \frac{(2 + s)(0.6)}{0.4} = (2 + s)(1.5).$$

Computing the mixing weights at each level:

$$\beta_1^{(0)} = 0.4, \quad \beta_2^{(0)} = 0.3, \quad \beta_3^{(0)} = 0.3.$$
$$E[N^{(0)}] = (0.4)(3) + (0.3)(3) + (0.3)(3) = 3.$$
$$\beta_1^{(1)} = 0.4 \cdot \frac{3}{3} = 0.4.$$
$$\beta_2^{(1)} = 0.3 \cdot \frac{3}{3} = 0.3.$$
$$\beta_3^{(1)} = 0.3 \cdot \frac{3}{3} = 0.3.$$
$$E[N^{(1)}] = 0.4(3) + 0.3(2.5) + 0.3(4.5) = 3.3.$$
$$\beta_1^{(2)} = 0.4 \cdot \frac{3}{3.3} = 0.36363636364.$$
$$\beta_2^{(2)} = 0.3 \cdot \frac{2.5}{3.3} = 0.22727272727.$$
$$\beta_3^{(2)} = 0.3 \cdot \frac{4.5}{3.3} = 0.40909090909.$$
$$E[N^{(2)}] = 0.36363636364(3) + 0.22727272727(2) + 0.40909090909(6) = 4.$$
$$\beta_1^{(3)} = 0.36363636364 \cdot \frac{3}{4} = 0.27272727273.$$
$$\beta_2^{(3)} = 0.22727272727 \cdot \frac{2}{4} = 0.11363636364.$$
$$\beta_3^{(3)} = 0.40909090909 \cdot \frac{6}{4} = 0.61363636364.$$
$$E[N^{(3)}] = 0.27272727273(3) + 0.11363636364(1.5) + 0.61363636364(7.5) = 5.59090909091.$$
$$\beta_1^{(4)} = 0.27272727273 \cdot \frac{3}{5.59090909091} = 0.14634146341.$$

$$\begin{aligned}
\beta_2^{(4)} &= 0.11363636364 \cdot \frac{1.5}{5.59090909091} = 0.03048780488. \\
\beta_3^{(4)} &= 0.61363636364 \cdot \frac{7.5}{5.59090909091} = 0.82317073171. \\
E[N^{(4)}] &= 0.14634146341(3) + 0.03048780488(1) \\
&\quad + 0.82317073171(9) = 7.87804878049.
\end{aligned}$$

Now we can begin the computation to find $P^{(0)}(S_N = 4)$. We want to find,

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\big].$$

Base cases:

$$P^{(s)}(N = 0) = \beta_1^{(s)}\, e^{-3} + \beta_2^{(s)}\, (0.5)^{6-s} + \beta_3^{(s)}\, (0.4)^{2+s}.$$

$$P^{(0)}(S_N = 0) = 0.4\, e^{-3} + 0.3(0.5)^6 + 0.3(0.4)^2 = 0.07260232735.$$

$$P^{(1)}(S_N = 0) = 0.4\, e^{-3} + 0.3(0.5)^5 + 0.3(0.4)^3 = 0.04848982735.$$

$$P^{(2)}(S_N = 0) = 0.36363636364\, e^{-3} + 0.22727272727(0.5)^4 + 0.40909090909(0.4)^4 = 0.04278166122.$$

$$P^{(3)}(S_N = 0) = 0.27272727273\, e^{-3} + 0.11363636364(0.5)^3 + 0.61363636364(0.4)^5 = 0.03406647319.$$

$$P^{(4)}(S_N = 0) = 0.14634146341\, e^{-3} + 0.03048780488(0.5)^2 + 0.82317073171(0.4)^6 = 0.01827957098.$$

Computing $P^{(s)}(S_N = 1)$:

$$P^{(1)}(S_N = 1) = \frac{E[N^{(1)}]}{1}\big[0.05 \cdot P^{(2)}(S_N = 0)\big] = 3.3(0.05)(0.04278166122) = 0.00705897410.$$

$$P^{(2)}(S_N = 1) = \frac{E[N^{(2)}]}{1}\big[0.05 \cdot P^{(3)}(S_N = 0)\big] = 4(0.05)(0.03406647319) = 0.00681329464.$$

$$P^{(3)}(S_N = 1) = \frac{E[N^{(3)}]}{1}\big[0.05 \cdot P^{(4)}(S_N = 0)\big] = 5.59090909091(0.05)(0.01827957098) = 0.00510997098.$$

Computing $P^{(s)}(S_N = 2)$:

$$\begin{aligned}
P^{(1)}(S_N = 2) &= \frac{E[N^{(1)}]}{2}\big[0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)\big] \\
&= \frac{3.3}{2}\big[0.05(0.00681329464) + 0.8(0.04278166122)\big] = 0.05703388962. \\
P^{(2)}(S_N = 2) &= \frac{E[N^{(2)}]}{2}\big[0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)\big] \\
&= \frac{4}{2}\big[0.05(0.00510997098) + 0.8(0.03406647319)\big] = 0.05501735420.
\end{aligned}$$

Computing $P^{(1)}(S_N = 3)$:

$$\begin{aligned}
P^{(1)}(S_N = 3) &= \frac{E[N^{(1)}]}{3}\big[0.05 \cdot P^{(2)}(S_N = 2) + 0.8 \cdot P^{(2)}(S_N = 1) + 0.3 \cdot P^{(2)}(S_N = 0)\big] \\
&= \frac{3.3}{3}\big[0.05(0.05501735420) + 0.8(0.00681329464) + 0.3(0.04278166122)\big] \\
&= 0.02313960197.
\end{aligned}$$

Computing $P^{(0)}(S_N = 4)$:

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4}\big[0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)\big]$$

$$
\begin{aligned}
&= \frac{3}{4}\big[0.05(0.02313960197) + 0.8(0.05703388962) + 0.3(0.00705897410) + 0.04848982735\big] \\
&= \boxed{0.07304370853}.
\end{aligned}
$$

References

[1] [TODO: Need the right list here. Potentials are Feller (1968), Grandell (1997), and Johnson, Kotz, and Kemp (1992).]