

# RECURSIVE COMPUTATION OF COMPOUND RANDOM VARIABLES WITH A FINITE-MIXTURE COUNT DISTRIBUTION (POISSON, BINOMIAL, AND NEGATIVE BINOMIAL COMPONENTS)

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## 1. INTRODUCTION

Compound random variables have been studied extensively in probability theory and applied fields, as they provide a natural framework for modeling aggregate quantities subject to two layers of randomness: the distribution of the summands and the distribution of the count variable. Classical treatments can be found in [?], where compound distributions are developed in the context of actuarial mathematics and discrete distributions.

While the concept is straightforward, calculating the exact probability distribution of a compound random variable can be a significant computational challenge. Direct methods involving multiple convolutions are often cumbersome and inefficient. A more elegant and powerful approach is the use of recursive methods, which allow for the efficient calculation of the probability  $P(S_N = k)$  based on the probabilities of preceding values. This thesis explores one such recursive technique, which is particularly effective when the compounding distribution belongs to a specific class of distributions.

The primary contribution of this work is the extension and simplification of this recursive method to the case where the compounding distribution is a finite mixture of Poisson, Binomial and Negative Binomial distributions. A direct application of the recursive formula to this case leads to a computationally intensive, nested problem, where the mixing weights of the distribution must be recalculated at each step. Additionally, we derive a closed-form expression that computes these recursive weights directly, thereby eliminating some of the nested recursion and simplifying the calculation.

## 2. BACKGROUND

**Definition 1.** The expected value of a discrete random variable,  $X$ , is denoted by  $E(X)$ , and given by

$$E[X] = \sum_{x=1}^{\infty} x \cdot f(x),$$

where  $f(x)$  is the probability mass function of  $X$ .

**Definition 2.** Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed random variables. Let  $N$  be a nonnegative integer valued random variable that is independent of the sequence  $X_1, X_2, \dots, X_n$ . Then,

$$S_N = \sum_{i=1}^N X_i$$

is a *compound random variable*.

**Definition 3.** In a compound random variable, the distribution of  $N$  is called the *compounding distribution*.

**Theorem 1** (Compound Random Variable Identity). *Let  $M$  be a random variable that is independent of the sequence of random variables  $X_1, X_2, \dots, X_n$  that forms a compound random variable*

and  $M$  satisfies

$$P(M = n) = \frac{n P(N = n)}{E[N]}, \quad n = 1, 2, 3, \dots$$

Then, for any function  $h$ ,

$$E(S_N \cdot h(S_N)) = E[N] \cdot E(X_1 \cdot h(S_M)).$$

*Proof.*

$$\begin{aligned} E(S_N \cdot h(S_N)) &= E\left[\sum_{i=1}^N X_i h(S_N)\right] \\ &= E\left[E\left(\sum_{i=1}^N X_i h(S_N) \mid N = n\right)\right] \\ &= \sum_{n=0}^{\infty} E\left(\sum_{i=1}^n X_i h(S_n) \mid N = n\right) P(N = n) \\ &= \sum_{n=0}^{\infty} E\left(\sum_{i=1}^n X_i h(S_n)\right) P(N = n) \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^n E(X_i h(S_n)) P(N = n). \end{aligned}$$

Since  $X_1, X_2, \dots$  are independent and identically distributed, we have

$$E(X_i \cdot h(S_N)) = E(X_j \cdot h(S_N)), \quad \forall i, j \in \{1, 2, \dots, n\}.$$

So,

$$\begin{aligned} E(S_N \cdot h(S_N)) &= \sum_{n=0}^{\infty} \sum_{i=1}^n E(X_1 h(S_n)) P(N = n) \\ &= \sum_{n=1}^{\infty} n \cdot E(X_1 h(S_n)) P(N = n). \end{aligned}$$

Then, by substitution of  $P(M = n) = \frac{n P(N = n)}{E[N]}$ , we have

$$\begin{aligned} &= \sum_{n=0}^{\infty} E[N] \cdot E(X_1 h(S_n)) P(M = n) \\ &= E[N] \sum_{n=0}^{\infty} E(X_1 h(S_n)) P(M = n) \\ &= E[N] \sum_{n=0}^{\infty} E(X_1 h(S_n) \mid M = n) P(M = n) \\ &= E[N] \sum_{n=0}^{\infty} E(X_1 h(S_M) \mid M = n) P(M = n) \\ &= E[N] E(E(X_1 h(S_M) \mid M = n)) \\ &= E[N] E(X_1 h(S_M)). \end{aligned}$$

□

**Corollary 1.** Let  $S_N$  be a compound random variable and suppose  $X_1, X_2, \dots, X_n$  are positive integer valued random variables. Suppose  $P(X_1 = i) = \alpha_i$ ,  $i > 0$ , and  $M$  is a random variable that

is independent of the sequence  $X_1, X_2, \dots, X_n$  and satisfies

$$P(M = n) = \frac{n P(N = n)}{E[N]}.$$

Then,

$$P(S_N = 0) = P(N = 0) \quad \text{and} \quad P(S_N = k) = \frac{1}{k} E[N] \sum_{i=1}^k i \alpha_i P(S_{M-1} = k - i).$$

*Proof.* For a fixed  $k$ , let

$$h(x) = \begin{cases} 1 & \text{if } x = k, \\ 0 & \text{otherwise,} \end{cases}$$

and note that

$$S_N \cdot h(S_N) = \begin{cases} k & \text{if } S_N = k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(S_N \cdot h(S_N)) = k \cdot P(S_N = k).$$

Now, by the Compound Random Variable Identity,  $E(S_N \cdot h(S_N)) = E[N] \cdot E(X_1 \cdot h(S_M))$ . So,

$$\begin{aligned} k \cdot P(S_N = k) &= E[N] \cdot E(X_1 \cdot h(S_M)) \\ &= E[N] \cdot E(E(X_1 \cdot h(S_M) \mid X_1 = j)) \\ &= E[N] \sum_{j=1}^{\infty} E(X_1 \cdot h(S_M) \mid X_1 = j) \cdot P(X_1 = j) \\ &= E[N] \sum_{j=1}^{\infty} E(X_1 \cdot h(S_M) \mid X_1 = j) \cdot \alpha_j \\ &= E[N] \sum_{j=1}^{\infty} E(j \cdot h(S_M) \mid X_1 = j) \cdot \alpha_j \\ &= E[N] \sum_{j=1}^{\infty} j \cdot E(h(S_M) \mid X_1 = j) \cdot \alpha_j. \end{aligned}$$

Now,

$$E(h(S_M) \mid X_1 = j) = \sum_{S'_M} h(S'_M) f_{S_M \mid X_1}(S'_M \mid j) = P(S_M = k \mid X_1 = j).$$

And,

$$\begin{aligned} P(S_M = k \mid X_1 = j) &= P\left(\sum_{i=1}^m X_i = k \mid X_1 = j\right) \\ &= P\left(j + \sum_{i=2}^m X_i = k\right) = P\left(\sum_{i=2}^m X_i = k - j\right). \end{aligned}$$

Letting  $\ell = i - 1$ ,

$$= P\left(\sum_{\ell=1}^{m-1} X_\ell = k - j\right) = P(S_{M-1} = k - j).$$

So,

$$k \cdot P(S_N = k) = E[N] \sum_{j=1}^{\infty} j \cdot \alpha_j \cdot P(S_{M-1} = k - j).$$

Note, when  $j > k$ ,  $P(S_{M-1} = k - j) = 0$ . So,

$$k \cdot P(S_N = k) = E[N] \sum_{j=1}^k j \cdot \alpha_j \cdot P(S_{M-1} = k - j),$$

and therefore,

$$P(S_N = k) = \frac{1}{k} \cdot E[N] \sum_{j=1}^k j \cdot \alpha_j \cdot P(S_{M-1} = k - j).$$

Notice, by definition, for  $N \geq 1$ ,

$$P(S_N = 0) = P\left(\sum_{j=1}^N X_i = 0\right) = 0,$$

so  $P(S_N = 0) = P(N = 0)$ .  $\square$

### 3. EXAMPLES

For each of the distributions below, let  $S_N = \sum_{i=1}^N X_i$  be a compound random variable where

$$P(X_1 = 1) = 0.05, \quad P(X_1 = 2) = 0.4, \quad P(X_1 = 3) = 0.1,$$

$$P(X_1 = 4) = 0.25, \quad P(X_1 = 5) = 0.2.$$

Find  $P(S_N = 4)$ .

**Example 1** (N in the Poisson Distribution). Let  $N$  be a random variable of the Poisson distribution with parameter  $\lambda = 3$ . So the probability mass function for the distribution of  $N$  is

$$P^{(s)}(N = n) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2, \dots$$

We claim that  $E[N^{(s)}] = \lambda$ .

*Proof.*

$$\begin{aligned} E[N^{(s)}] &= \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = 0 + \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{(k-1)!} \\ &= e^{-\lambda} \cdot \lambda \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}. \end{aligned}$$

Now let  $j = k - 1$ . So when  $k = 1$ ,  $j = 0$ , and as  $k \rightarrow \infty$ ,  $j \rightarrow \infty$ . So,

$$E[N^{(s)}] = \lambda \cdot e^{-\lambda} \cdot \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}.$$

We recognize  $\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$  as the Taylor series for  $e^\lambda$ . So,

$$E[N^{(s)}] = \lambda \cdot e^{-\lambda} \cdot e^\lambda = \lambda. \quad \square$$

Therefore,  $E[N^{(s)}] = \lambda = 3$ . So we have,

$$\begin{aligned} P(M^{(s)} - 1 = n) &= P(M^{(s)} = n + 1) = \frac{(n+1) P^{(s)}(N = n+1)}{E[N^{(s)}]} \\ &= \frac{n+1}{\lambda} \cdot \frac{e^{-\lambda} \lambda^{n+1}}{(n+1)!} = \frac{e^{-\lambda} \lambda^n}{n!}. \end{aligned}$$

Therefore, we see that  $P^{(s+1)}(N = n)$  is also the Poisson distribution with parameter  $\lambda$ . Since the distribution is unchanged at each recursion level, we have

$$P^{(s)}(S_N = k) = \frac{\lambda}{k} \sum_{i=1}^k i \alpha_i \cdot P^{(s+1)}(S_N = k - i),$$

which, because  $P^{(s)} = P^{(s+1)}$  for all  $s \geq 0$ , simplifies to

$$P(S_N = k) = \frac{\lambda}{k} \sum_{i=1}^k i \alpha_i \cdot P(S_N = k - i).$$

Now we calculate  $P(S_N = 4)$ .

$$\begin{aligned} P(S_N = 0) &= P(N = 0) = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda} = e^{-3}. \\ P(S_N = 1) &= \frac{3}{1} [1 \cdot (0.05) \cdot P(S_N = 0)] = 3 \cdot (0.05) \cdot e^{-3} = 0.15 e^{-3}. \\ P(S_N = 2) &= \frac{3}{2} [(0.05) \cdot P(S_N = 1) + 2 \cdot (0.4) \cdot P(S_N = 0)] \\ &= \frac{3}{2} [(0.05)(0.15 e^{-3}) + (0.8) e^{-3}] = 1.21125 e^{-3}. \\ P(S_N = 3) &= \frac{3}{3} [(0.05) \cdot P(S_N = 2) + (0.8) \cdot P(S_N = 1) + 3 \cdot (0.1) \cdot P(S_N = 0)] \\ &= (0.05)(1.21125 e^{-3}) + (0.8)(0.15 e^{-3}) + (0.3) e^{-3} = 0.4805625 e^{-3}. \\ P(S_N = 4) &= \frac{3}{4} [0.05 \cdot P(S_N = 3) + 0.8 \cdot P(S_N = 2) + 0.3 \cdot P(S_N = 1) + 4 \cdot (0.25) \cdot P(S_N = 0)] \\ &= \frac{3}{4} [0.05(0.4805625 e^{-3}) + 0.8(1.21125 e^{-3}) + 0.3(0.15 e^{-3}) + e^{-3}] \\ &= 1.528521094 e^{-3}. \end{aligned}$$

**Example 2** (N in the Negative Binomial Distribution). Let  $N$  be a random variable of the negative binomial distribution with parameters  $r = 6$  and  $p = 0.6$ . The probability mass function for the distribution of  $N^{(s)}$  at recursion level  $s$  is

$$P^{(s)}(N = n) = \binom{n+r+s-1}{n} p^{r+s} (1-p)^n, \quad n \geq 0.$$

We claim that  $E[N^{(s)}] = \frac{(r+s)(1-p)}{p}$ .

*Proof.*

$$E[N^{(s)}] = \sum_{k=0}^{\infty} k \binom{k+r+s-1}{k} p^{r+s} (1-p)^k = \sum_{k=1}^{\infty} k \binom{k+r+s-1}{k} p^{r+s} (1-p)^k.$$

Recalling that  $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ , we see that

$$k \binom{k+r+s-1}{k} = k \cdot \frac{(k+r+s-1)!}{k! (r+s-1)!} = \frac{(k+r+s-1)!}{(k-1)! (r+s-1)!} = (r+s) \binom{k+r+s-1}{k-1}.$$

So we have,

$$E[N^{(s)}] = (r+s)(1-p) p^{r+s} \sum_{k=1}^{\infty} \binom{k+r+s-1}{k-1} (1-p)^{k-1}.$$

Let  $m = k - 1$ . By substitution,

$$E[N^{(s)}] = (r + s)(1 - p)p^{r+s} \sum_{m=0}^{\infty} \binom{m + (r + s)}{m} (1 - p)^m.$$

Recalling the negative binomial theorem,  $(1 + x)^{-t} = \sum_{i=0}^{\infty} \binom{-t}{i} x^i$ , and applying it, we see that

$$\sum_{m=0}^{\infty} \binom{m + r + s}{m} (1 - p)^m = p^{-(r+s+1)}.$$

So by substitution,

$$E[N^{(s)}] = (r + s)(1 - p)p^{r+s} \cdot p^{-(r+s+1)} = \frac{(r + s)(1 - p)}{p}. \quad \square$$

So we have,

$$\begin{aligned} P^{(s)}(M^{(s)} - 1 = n) &= P^{(s)}(M^{(s)} = n + 1) \\ &= \frac{(n + 1) P^{(s)}(N = n + 1)}{E[N^{(s)}]} \\ &= \binom{n + r + s}{n + 1} p^{r+s} (1 - p)^{n+1} \cdot \frac{(n + 1)}{(r + s)(1 - p)/p} \\ &= \frac{(n + 1) p}{(r + s)(1 - p)} \cdot \frac{(n + r + s)!}{(n + 1)! (r + s - 1)!} \cdot p^{r+s} (1 - p)^{n+1} \\ &= \frac{(n + r + s)!}{n! (r + s)!} \cdot p^{r+s+1} (1 - p)^n \\ &= \binom{n + (r + s + 1) - 1}{n} p^{r+s+1} (1 - p)^n = P^{(s+1)}(N = n). \end{aligned}$$

Therefore, we see that  $P^{(s)}(M^{(s)} - 1 = n)$  is also the negative binomial distribution but at the next recursion level  $(s + 1)$ . So, we have

$$P^{(s)}(S_N = k) = \frac{(r + s)(1 - p)}{k p} \sum_{i=1}^k i \alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

Now we can begin the computation. With  $r = 6$ ,  $p = 0.6$  (so  $1 - p = 0.4$ ), at level  $s = 0$  the coefficient is  $\frac{r(1 - p)}{p} = 4$ , giving

$$P^{(0)}(S_N = k) = \frac{4}{k} \sum_{i=1}^k i \alpha_i \cdot P^{(1)}(S_N = k - i).$$

*Base cases*  $P^{(s)}(S_N = 0) = p^{r+s}$ :

$$P^{(0)}(S_N = 0) = (0.6)^6 = 0.046656.$$

$$P^{(1)}(S_N = 0) = (0.6)^7 = 0.0279936.$$

$$P^{(2)}(S_N = 0) = (0.6)^8 = 0.01679616.$$

$$P^{(3)}(S_N = 0) = (0.6)^9 = 0.010077696.$$

$$P^{(4)}(S_N = 0) = (0.6)^{10} = 0.0060466176.$$

*Computing*  $P^{(s)}(S_N = 1)$ :

$$P^{(0)}(S_N = 1) = \frac{(r)(1 - p)}{1 \cdot p} [0.05 \cdot P^{(1)}(S_N = 0)] = (0.2)(0.0279936) = 0.00559872.$$

$$P^{(1)}(S_N = 1) = \frac{(r+1)(1-p)}{1 \cdot p} [0.05 \cdot P^{(2)}(S_N = 0)] = \frac{7 \cdot 0.4}{0.6} (0.05)(0.01679616) = 0.003919104.$$

$$P^{(2)}(S_N = 1) = \frac{(r+2)(1-p)}{1 \cdot p} [0.05 \cdot P^{(3)}(S_N = 0)] = \frac{8 \cdot 0.4}{0.6} (0.05)(0.010077696) = 0.0026873856.$$

$$P^{(3)}(S_N = 1) = \frac{(r+3)(1-p)}{1 \cdot p} [0.05 \cdot P^{(4)}(S_N = 0)] = \frac{9 \cdot 0.4}{0.6} (0.05)(0.0060466176) = 0.00181398528.$$

Computing  $P^{(s)}(S_N = 2)$ :

$$\begin{aligned} P^{(1)}(S_N = 2) &= \frac{(r+1)(1-p)}{2p} [(0.05) \cdot P^{(2)}(S_N = 1) + (0.8) \cdot P^{(2)}(S_N = 0)] \\ &= \frac{7 \cdot 0.4}{2 \cdot 0.6} [(0.05)(0.0026873856) + (0.8)(0.01679616)] = 0.03166636032. \end{aligned}$$

$$\begin{aligned} P^{(2)}(S_N = 2) &= \frac{(r+2)(1-p)}{2p} [(0.05) \cdot P^{(3)}(S_N = 1) + (0.8) \cdot P^{(3)}(S_N = 0)] \\ &= \frac{8 \cdot 0.4}{2 \cdot 0.6} [(0.05)(0.00181398528) + (0.8)(0.010077696)] = 0.021740949504. \end{aligned}$$

Computing  $P^{(1)}(S_N = 3)$ :

$$\begin{aligned} P^{(1)}(S_N = 3) &= \frac{(r+1)(1-p)}{3p} [(0.05) \cdot P^{(2)}(S_N = 2) + (0.8) \cdot P^{(2)}(S_N = 1) + (0.3) \cdot P^{(2)}(S_N = 0)] \\ &= \frac{7 \cdot 0.4}{3 \cdot 0.6} [(0.05)(0.021740949504) + (0.8)(0.0026873856) + (0.3)(0.01679616)] \\ &= 0.01287347282. \end{aligned}$$

Computing  $P^{(0)}(S_N = 4)$ :

$$\begin{aligned} P^{(0)}(S_N = 4) &= \frac{r(1-p)}{4p} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)] \\ &= \frac{6 \cdot 0.4}{4 \cdot 0.6} [(0.05)(0.01287347282) + (0.8)(0.03166636032) \\ &\quad + (0.3)(0.003919104) + 0.0279936] \\ &= \boxed{0.05514609310}. \end{aligned}$$

**Example 3** (N in the Binomial Distribution). Let  $N$  be a random variable of the Binomial distribution with parameters  $r = 6$  and  $p = 0.6$ . The probability mass function for the distribution of  $N^{(s)}$  at recursion level  $s$  is

$$P^{(s)}(N = n) = \binom{r-s}{n} p^n (1-p)^{(r-s)-n}, \quad n \geq 0.$$

We claim that  $E[N^{(s)}] = (r-s)p$ .

*Proof.*

$$E[N^{(s)}] = \sum_{k=0}^{\infty} k \binom{r-s}{k} p^k (1-p)^{(r-s)-k} = \sum_{k=1}^{\infty} k \binom{r-s}{k} p^k (1-p)^{(r-s)-k}.$$

Notice that

$$k \binom{r-s}{k} = k \cdot \frac{(r-s)!}{k! (r-s-k)!} = \frac{(r-s)!}{(k-1)! (r-s-k)!} = (r-s) \binom{r-s-1}{k-1}.$$

By substitution,

$$E[N^{(s)}] = (r-s)p \sum_{k=1}^{\infty} \binom{r-s-1}{k-1} p^{k-1} (1-p)^{(r-s)-k}.$$

We let  $m = k - 1$ . So we have,

$$E[N^{(s)}] = (r - s)p \sum_{m=0}^{\infty} \binom{r - s - 1}{m} p^m (1 - p)^{(r-s-1)-m}.$$

And by the binomial theorem,

$$\sum_{m=0}^{\infty} \binom{r - s - 1}{m} p^m (1 - p)^{(r-s-1)-m} = [p + (1 - p)]^{r-s-1} = 1.$$

By substitution we have  $E[N^{(s)}] = (r - s)p$ . □

So, for this example at  $s = 0$ ,  $E[N^{(0)}] = r p = 3.6$ . We have,

$$\begin{aligned} P^{(s)}(M^{(s)} - 1 = n) &= P^{(s)}(M^{(s)} = n + 1) = \frac{(n + 1) P^{(s)}(N = n + 1)}{E[N^{(s)}]} \\ &= \binom{r - s}{n + 1} p^{n+1} (1 - p)^{(r-s)-n-1} \cdot \frac{n + 1}{(r - s)p} \\ &= \frac{(r - s)!}{(n + 1)! (r - s - n - 1)!} p (1 - p)^{(r-s-1)-n} \cdot \frac{n + 1}{(r - s)p} \\ &= \frac{(r - s - 1)!}{n! (r - s - 1 - n)!} \cdot p^n (1 - p)^{(r-s-1)-n} \\ &= \binom{r - s - 1}{n} p^n (1 - p)^{(r-s-1)-n} = P^{(s+1)}(N = n). \end{aligned}$$

Therefore, we see that  $P^{(s)}(M^{(s)} - 1 = n)$  is also the binomial distribution but at the next recursion level ( $s + 1$ ), which corresponds to a parameter change to  $(r - s - 1)$ . So, we have

$$P^{(s)}(S_N = k) = \frac{(r - s)p}{k} \sum_{i=1}^k i \alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

Now we can begin the computation to find  $P(S_N = 4)$ .

*Base cases*  $P^{(s)}(S_N = 0) = (1 - p)^{r-s}$ :

$$\begin{aligned} P^{(1)}(S_N = 0) &= P^{(1)}(N = 0) = (0.4)^5 = 0.01024. \\ P^{(2)}(S_N = 0) &= P^{(2)}(N = 0) = (0.4)^4 = 0.0256. \\ P^{(3)}(S_N = 0) &= P^{(3)}(N = 0) = (0.4)^3 = 0.064. \end{aligned}$$

From Corollary 1,  $P^{(0)}(S_N = 0) = P^{(0)}(N = 0) = (0.4)^6 = 0.004096$ .

$$\begin{aligned} P^{(0)}(S_N = 1) &= \frac{r p}{1} [1 \cdot (0.05) \cdot P^{(1)}(S_N = 0)] = 6(0.6)(0.05)(0.01024) = 0.0018432. \\ P^{(3)}(S_N = 1) &= \frac{(r - 3)p}{1} [0.05 \cdot P^{(4)}(S_N = 0)] = 3(0.6)(0.05)(0.4)^2 = 0.0144. \\ P^{(2)}(S_N = 1) &= \frac{(r - 2)p}{1} [0.05 \cdot P^{(3)}(S_N = 0)] = 4(0.6)(0.05)(0.064) = 0.00768. \\ P^{(1)}(S_N = 1) &= \frac{(r - 1)p}{1} [0.05 \cdot P^{(2)}(S_N = 0)] = 5(0.6)(0.05)(0.0256) = 0.00384. \\ P^{(2)}(S_N = 2) &= \frac{(r - 2)p}{2} [0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)] \\ &= \frac{4(0.6)}{2} [0.05(0.0144) + 0.8(0.064)] = 0.062304. \end{aligned}$$

$$\begin{aligned} P^{(1)}(S_N = 2) &= \frac{(r-1)p}{2} [0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)] \\ &= \frac{5(0.6)}{2} [0.05(0.00768) + 0.8(0.0256)] = 0.031296. \end{aligned}$$

$$\begin{aligned} P^{(1)}(S_N = 3) &= \frac{(r-1)p}{3} [(0.05) \cdot P^{(2)}(S_N = 2) + (0.8) \cdot P^{(2)}(S_N = 1) + (0.3) \cdot P^{(2)}(S_N = 0)] \\ &= \frac{5(0.6)}{3} [(0.05)(0.062304) + (0.8)(0.00768) + (0.3)(0.0256)] = 0.016937200. \end{aligned}$$

$$\begin{aligned} P^{(0)}(S_N = 4) &= \frac{rp}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)] \\ &= \frac{6(0.6)}{4} [(0.05)(0.016937200) + (0.8)(0.031296) + (0.3)(0.00384) + 0.01024] \\ &= \boxed{0.033548184}. \end{aligned}$$

#### 4. COMPUTATION WITH A FINITE MIXTURE OF POISSON DISTRIBUTIONS

We will now consider the computation of compound random variable probabilities when the counting distribution is a mixture of Poisson distributions.

**Example 4** (N in a Finite Mixture of Two Poisson Distributions). Consider a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Poisson distribution with mean  $\lambda_1$ , or process two, which follows a Poisson distribution with mean  $\lambda_2$ . The probability of process one is  $\beta_1$ , and the probability of process two is  $\beta_2$ , where  $\beta_1 + \beta_2 = 1$ . Let the parameters of this distribution be  $\lambda_1, \lambda_2, \beta_1, \beta_2$ .

So, we have,

$$P^{(s)}(N = n) = \beta_1^{(s)} \cdot \frac{e^{-\lambda_1} \lambda_1^n}{n!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2} \lambda_2^n}{n!}, \quad n = 0, 1, 2, \dots$$

We claim that  $E[N^{(s)}] = \beta_1^{(s)} \lambda_1 + \beta_2^{(s)} \lambda_2$ .

*Proof.*

$$\begin{aligned} E[N^{(s)}] &= \sum_{k=0}^{\infty} k \left[ \beta_1^{(s)} \cdot \frac{e^{-\lambda_1} \lambda_1^k}{k!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2} \lambda_2^k}{k!} \right] \\ &= \beta_1^{(s)} \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda_1} \lambda_1^k}{k!} + \beta_2^{(s)} \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda_2} \lambda_2^k}{k!}. \end{aligned}$$

Recall from Example 1 that we showed  $\sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda_i} \lambda_i^k}{k!} = \lambda_i$ . So, by substitution we have  $E[N^{(s)}] = \beta_1^{(s)} \lambda_1 + \beta_2^{(s)} \lambda_2$ .  $\square$

So, we have,

$$\begin{aligned} P(M^{(s)} - 1 = n) &= P(M^{(s)} = n + 1) = \frac{(n+1) P^{(s)}(N = n+1)}{E[N^{(s)}]} \\ &= \frac{(n+1) \left[ \beta_1^{(s)} \cdot \frac{e^{-\lambda_1} \lambda_1^{n+1}}{(n+1)!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2} \lambda_2^{n+1}}{(n+1)!} \right]}{\beta_1^{(s)} \lambda_1 + \beta_2^{(s)} \lambda_2} \\ &= \frac{\beta_1^{(s)} \lambda_1 \cdot \frac{e^{-\lambda_1} \lambda_1^n}{n!} + \beta_2^{(s)} \lambda_2 \cdot \frac{e^{-\lambda_2} \lambda_2^n}{n!}}{\beta_1^{(s)} \lambda_1 + \beta_2^{(s)} \lambda_2} \end{aligned}$$

$$= \frac{\beta_1^{(s)} \lambda_1}{\beta_1^{(s)} \lambda_1 + \beta_2^{(s)} \lambda_2} \cdot \frac{e^{-\lambda_1} \lambda_1^n}{n!} + \frac{\beta_2^{(s)} \lambda_2}{\beta_1^{(s)} \lambda_1 + \beta_2^{(s)} \lambda_2} \cdot \frac{e^{-\lambda_2} \lambda_2^n}{n!}.$$

Notice, this is also a finite mixture distribution where both processes follow the Poisson distribution with  $\lambda_1, \lambda_2$  respectively. However, we have updated values for  $\beta_1^{(s)}$  and  $\beta_2^{(s)}$ . The weight update rule is

$$\beta_j^{(s+1)} = \frac{\beta_j^{(s)} \lambda_j}{\beta_1^{(s)} \lambda_1 + \beta_2^{(s)} \lambda_2}.$$

So, we have

$$P^{(s)}(S_N = k) = \frac{E[N^{(s)}]}{k} \sum_{i=1}^k i \alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

Now we can begin the computation. Let  $\lambda_1 = 3, \lambda_2 = 4, \beta_1^{(0)} = 0.6, \beta_2^{(0)} = 0.4$ . We want to find  $P^{(0)}(S_N = 4)$ .

*Computing the mixing weights at each level:*

$$\begin{aligned} \beta_1^{(0)} &= 0.6, \quad \beta_2^{(0)} = 0.4, \quad E[N^{(0)}] = (0.6)(3) + (0.4)(4) = 3.4. \\ \beta_1^{(1)} &= \frac{(0.6)(3)}{3.4} = \frac{1.8}{3.4}, \quad \beta_2^{(1)} = \frac{(0.4)(4)}{3.4} = \frac{1.6}{3.4}, \quad E[N^{(1)}] = \frac{1.8(3) + 1.6(4)}{3.4} = \frac{11.8}{3.4}. \\ \beta_1^{(2)} &= \frac{5.4}{11.8}, \quad \beta_2^{(2)} = \frac{6.4}{11.8}, \quad E[N^{(2)}] = \frac{5.4(3) + 6.4(4)}{11.8} = \frac{41.8}{11.8}. \\ \beta_1^{(3)} &= \frac{16.2}{41.8}, \quad \beta_2^{(3)} = \frac{25.6}{41.8}, \quad E[N^{(3)}] = \frac{16.2(3) + 25.6(4)}{41.8} = \frac{151}{41.8}. \\ \beta_1^{(4)} &= \frac{48.6}{151}, \quad \beta_2^{(4)} = \frac{102.4}{151}. \end{aligned}$$

*Base cases*  $P^{(s)}(S_N = 0) = \beta_1^{(s)} e^{-\lambda_1} + \beta_2^{(s)} e^{-\lambda_2}$ :

$$\begin{aligned} P^{(1)}(S_N = 0) &= \frac{1.8}{3.4} e^{-3} + \frac{1.6}{3.4} e^{-4}. \\ P^{(2)}(S_N = 0) &= \frac{5.4}{11.8} e^{-3} + \frac{6.4}{11.8} e^{-4}. \\ P^{(3)}(S_N = 0) &= \frac{16.2}{41.8} e^{-3} + \frac{25.6}{41.8} e^{-4}. \\ P^{(4)}(S_N = 0) &= \frac{48.6}{151} e^{-3} + \frac{102.4}{151} e^{-4}. \end{aligned}$$

*Computing*  $P^{(s)}(S_N = 1)$ :

$$\begin{aligned} P^{(1)}(S_N = 1) &= \frac{E[N^{(1)}]}{1} [1 \cdot 0.05 \cdot P^{(2)}(S_N = 0)] \\ &= \frac{11.8}{3.4} \left[ (0.05) \left( \frac{5.4}{11.8} e^{-3} + \frac{6.4}{11.8} e^{-4} \right) \right] \\ &= 0.07941176471 e^{-3} + 0.09411764706 e^{-4}. \end{aligned}$$

$$\begin{aligned} P^{(2)}(S_N = 1) &= \frac{E[N^{(2)}]}{1} [0.05 \cdot P^{(3)}(S_N = 0)] \\ &= 0.06864406780 e^{-3} + 0.10847457627 e^{-4}. \end{aligned}$$

$$\begin{aligned} P^{(3)}(S_N = 1) &= \frac{E[N^{(3)}]}{1} [0.05 \cdot P^{(4)}(S_N = 0)] \\ &= 0.05813397129 e^{-3} + 0.12248803828 e^{-4}. \end{aligned}$$

Computing  $P^{(s)}(S_N = 2)$ :

$$\begin{aligned} P^{(1)}(S_N = 2) &= \frac{E[N^{(1)}]}{2} [(0.05) \cdot P^{(2)}(S_N = 1) + (0.8) \cdot P^{(2)}(S_N = 0)] \\ &= 0.64125 e^{-3} + 0.76235294118 e^{-4}. \\ P^{(2)}(S_N = 2) &= \frac{E[N^{(2)}]}{2} [(0.05) \cdot P^{(3)}(S_N = 1) + (0.8) \cdot P^{(3)}(S_N = 0)] \\ &= 0.55430084746 e^{-3} + 0.87864406780 e^{-4}. \end{aligned}$$

Computing  $P^{(1)}(S_N = 3)$ :

$$\begin{aligned} P^{(1)}(S_N = 3) &= \frac{E[N^{(1)}]}{3} [(0.05) \cdot P^{(2)}(S_N = 2) + (0.8) \cdot P^{(2)}(S_N = 1) + (0.3) \cdot P^{(2)}(S_N = 0)] \\ &= 0.25441544118 e^{-3} + 0.33945098039 e^{-4}. \end{aligned}$$

Finally,  $P^{(0)}(S_N = 4)$ :

$$\begin{aligned} P^{(0)}(S_N = 4) &= \frac{E[N^{(0)}]}{4} [(0.05) \cdot P^{(1)}(S_N = 3) + (0.8) \cdot P^{(1)}(S_N = 2) \\ &\quad + (0.3) \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)] \\ &= \frac{3.4}{4} [(0.05)(0.25441544118 e^{-3} + 0.33945098039 e^{-4}) \\ &\quad + (0.8)(0.64125 e^{-3} + 0.76235294118 e^{-4}) \\ &\quad + (0.3)(0.07941176471 e^{-3} + 0.09411764706 e^{-4}) \\ &\quad + (\frac{1.8}{3.4} e^{-3} + \frac{1.6}{3.4} e^{-4})] \\ &= \boxed{0.91711265625 e^{-3} + 0.95682666667 e^{-4}}. \end{aligned}$$

## 5. FINITE POISSON MIXTURE GENERALIZATION

Next, we will describe a more general case of the Finite Mixture of Poisson Distributions. Consider a discrete mixture distribution where the event is generated by one or more random processes. The event is generated by one of  $t$  processes, which follows a Poisson distribution with mean  $\lambda_i$  for  $i \in \mathbb{Z}$  with  $0 < i \leq t$ . The probability of process  $i$  is  $\beta_i$  for  $i \in \mathbb{Z}$  with  $0 < i \leq t$  and where  $\beta_1 + \beta_2 + \dots + \beta_t = 1$ .

Now we are ready to define our probability mass function. Let  $s \in \mathbb{Z}$  with  $0 \leq s$ , and let  $s$  represent the level of recursion. Let the parameters of this distribution be  $s, \lambda_1, \dots, \lambda_t$ .

$$P^{(s)}(N = n) = \beta_1^{(s)} \cdot \frac{e^{-\lambda_1} \lambda_1^n}{n!} + \beta_2^{(s)} \cdot \frac{e^{-\lambda_2} \lambda_2^n}{n!} + \dots + \beta_t^{(s)} \cdot \frac{e^{-\lambda_t} \lambda_t^n}{n!}, \quad n \geq 0.$$

**Corollary 2** (Recursive Properties of the Generalized Finite Mixture of Poisson Distributions). *Let  $N^{(s)}$  be a mixed Poisson random variable at recursion level  $s$  with probability mass function*

$$P^{(s)}(N = n) = \sum_{j=1}^t \left( \frac{\beta_j \lambda_j^s}{\sum_{i=1}^t \beta_i \lambda_i^s} \right) \cdot \frac{e^{-\lambda_j} \lambda_j^n}{n!},$$

where  $\sum_{i=1}^t \beta_i = 1$ ,  $\beta_i > 0$ , and  $\lambda_i > 0$ . Let  $M^{(s)}$  be the random variable satisfying the compound identity condition associated with  $N^{(s)}$ . We claim that for all  $s \geq 0$ :

$$(1) \quad E[N^{(s)}] = \frac{\sum_{i=1}^t \beta_i \lambda_i^{s+1}}{\sum_{j=1}^t \beta_j \lambda_j^s}.$$

(2) The distribution of  $M^{(s)} - 1$  is identical to the distribution of  $N^{(s+1)}$ .

*Proof (by induction).* *Base Case* ( $s = 0$ ): For  $s = 0$ , the weights are  $\beta_i$ . First, we compute the expected value of  $E[N^{(0)}]$ :

$$E[N^{(0)}] = \sum_{n=0}^{\infty} n \left[ \sum_{i=1}^t \beta_i \cdot \frac{e^{-\lambda_i} \lambda_i^n}{n!} \right] = \sum_{i=1}^t \beta_i \lambda_i.$$

This matches the form of Claim 1 where the denominator  $\sum_{i=1}^t \beta_i \lambda_i^0 = \sum_{i=1}^t \beta_i = 1$ .

Next, we apply the compound random identity and see that,

$$\begin{aligned} P(M^{(0)} - 1 = n) &= P(M^{(0)} = n + 1) = \frac{(n+1) P^{(0)}(N = n+1)}{E[N^{(0)}]} \\ &= \frac{(n+1) \left[ \beta_1 \cdot \frac{e^{-\lambda_1} \lambda_1^{n+1}}{(n+1)!} + \cdots + \beta_t \cdot \frac{e^{-\lambda_t} \lambda_t^{n+1}}{(n+1)!} \right]}{\beta_1 \lambda_1 + \cdots + \beta_t \lambda_t} \\ &= \frac{n+1}{\sum_{j=1}^t \beta_j \lambda_j} \cdot \sum_{i=1}^t \beta_i \cdot \frac{e^{-\lambda_i} \lambda_i^{n+1}}{(n+1)!} \\ &= \sum_{i=1}^t \frac{\beta_i \lambda_i}{\sum_{j=1}^t \beta_j \lambda_j} \cdot \frac{e^{-\lambda_i} \lambda_i^n}{n!}. \end{aligned}$$

This is exactly the definition of  $P^{(1)}(N = n)$ . Thus, the claim holds for  $s = 0$ .

*Inductive Step:* Assume the statement is true for  $s = k$ . That is,  $M^{(k)} - 1$  is equivalent to  $N^{(k+1)}$  and  $E[N^{(k)}]$  follows the claim. We need to show the statement is true for  $s = k + 1$ . We must determine the distribution of  $M^{(k+1)} - 1$ .

We have,

$$P(M^{(k+1)} - 1 = n) = P(M^{(k+1)} = n + 1) = \frac{(n+1) P^{(k+1)}(N = n+1)}{E[N^{(k+1)}]}.$$

First, we compute  $E[N^{(k+1)}]$ :

$$\begin{aligned} E[N^{(k+1)}] &= \sum_{n=0}^{\infty} n \cdot \sum_{i=1}^t \left( \frac{\beta_i \lambda_i^{k+1}}{\sum_{j=1}^t \beta_j \lambda_j^{k+1}} \right) \cdot \frac{e^{-\lambda_i} \lambda_i^n}{n!} \\ &= \sum_{i=1}^t \left( \frac{\beta_i \lambda_i^{k+1}}{\sum_{j=1}^t \beta_j \lambda_j^{k+1}} \right) \cdot \lambda_i = \frac{\sum_{i=1}^t \beta_i \lambda_i^{k+2}}{\sum_{j=1}^t \beta_j \lambda_j^{k+1}}. \end{aligned}$$

This confirms Claim 1 for  $s = k + 1$ . Next, we substitute this into the probability expression:

$$P(M^{(k+1)} - 1 = n) = \frac{n+1}{\sum_{i=1}^t \beta_i \lambda_i^{k+2}} \sum_{i=1}^t \left( \frac{\beta_i \lambda_i^{k+1}}{\sum_{j=1}^t \beta_j \lambda_j^{k+1}} \right) \cdot \frac{e^{-\lambda_i} \lambda_i^{n+1}}{(n+1)!}.$$

We cancel the common denominator terms  $\sum_{j=1}^t \beta_j \lambda_j^{k+1}$  and we have

$$\begin{aligned} &= \frac{n+1}{\sum_{j=1}^t \beta_j \lambda_j^{k+2}} \sum_{i=1}^t \beta_i \lambda_i^{k+1} \cdot \frac{e^{-\lambda_i} \lambda_i^{n+1}}{(n+1)!} \\ &= \sum_{i=1}^t \frac{\beta_i \lambda_i^{k+2}}{\sum_{j=1}^t \beta_j \lambda_j^{k+2}} \cdot \frac{e^{-\lambda_i} \lambda_i^n}{n!}. \end{aligned}$$

This expression is exactly the PMF for  $N^{(k+2)}$  as desired.

So, by the principle of mathematical induction, for all  $s \in \mathbb{N}$ , the compound identity variable associated with  $N^{(s)}$  shifts the distribution to  $N^{(s+1)}$ , and the expected value satisfies the derived closed form.  $\square$

Now we are ready to construct our new recursive expression by applying Corollary 1. From

$$P(S_N = k) = \frac{1}{k} E[N] \sum_{i=1}^k i \alpha_i P(S_{M-1} = k-i),$$

we arrive at the updated recursive equation

$$P^{(s)}(S_N = k) = \frac{1}{k} \cdot \frac{\sum_{i=1}^t \beta_i \lambda_i^{s+1}}{\sum_{i=1}^t \beta_i \lambda_i^s} \cdot \sum_{i=1}^k i \alpha_i \cdot P^{(s+1)}(S_N = k-i).$$

**Example 5** (Revisiting Example 1). Now we can compute the same compound random variable from Example 1, but with our updated recursive equation. Like before, let  $N$  be a random variable of the Poisson distribution with parameter  $\lambda_1 = 3$ . So the probability mass function for the distribution of  $N$  is

$$P^{(s)}(N = n) = \beta_1 \cdot \frac{e^{-\lambda} \lambda^n}{n!}, \quad \text{where } \beta_1 = 1.$$

Furthermore,

$$E[N^{(s)}] = \frac{\sum_{i=1}^1 \beta_i \lambda_i^{s+1}}{\sum_{i=1}^1 \beta_i \lambda_i^s} = \frac{\beta_1 \lambda_1^{s+1}}{\beta_1 \lambda_1^s} = \frac{\lambda_1^{s+1}}{\lambda_1^s} = \lambda_1 = 3, \quad \forall s \in \mathbb{Z}, s \geq 0.$$

Now we calculate  $P^{(0)}(S_N = 4)$ . We want to find

$$P^{(0)}(S_N = 4) = \frac{3}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + 4(0.25) \cdot P^{(1)}(S_N = 0)].$$

We can solve each of these recursive expressions for all  $s \in \mathbb{Z}$ ,  $s \geq 0$ . So we have,

$$P^{(s)}(S_N = 0) = P(N^{(s)} = 0) = 1 \cdot \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda} = e^{-3}, \quad \forall s \geq 0.$$

$$P^{(s)}(S_N = 1) = \frac{3}{1} [1 \cdot (0.05) \cdot P(S_N = 0)] = 3(0.05) e^{-3} = 0.15 e^{-3}, \quad \forall s \geq 0.$$

$$\begin{aligned} P^{(s)}(S_N = 2) &= \frac{3}{2} [(0.05) \cdot P(S_N = 1) + 2(0.4) \cdot P(S_N = 0)] \\ &= \frac{3}{2} [(0.05)(0.15 e^{-3}) + (0.8) e^{-3}] = 1.21125 e^{-3}, \quad \forall s \geq 0. \end{aligned}$$

$$\begin{aligned} P^{(s)}(S_N = 3) &= \frac{3}{3} [(0.05) \cdot P(S_N = 2) + (0.8) \cdot P(S_N = 1) + 3(0.1) \cdot P(S_N = 0)] \\ &= (0.05)(1.21125 e^{-3}) + (0.8)(0.15 e^{-3}) + (0.3) e^{-3} = 0.4805625 e^{-3}, \quad \forall s \geq 0. \end{aligned}$$

Now we have everything we need to compute  $P^{(0)}(S_N = 4)$ . We have,

$$\begin{aligned} P^{(0)}(S_N = 4) &= \frac{3}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + 4(0.25) \cdot P^{(1)}(S_N = 0)] \\ &= \frac{3}{4} [0.05(0.4805625 e^{-3}) + 0.8(1.21125 e^{-3}) + 0.3(0.15 e^{-3}) + e^{-3}] \\ &= 1.528521094 e^{-3}. \end{aligned}$$

Notice, this is the same as our result from Example 1.

**Example 6** (Revisiting Example 4). We will consider again the case of a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Poisson distribution with mean  $\lambda_1$ , or process two, which follows a

Poisson distribution with mean  $\lambda_2$ . Like in Example 4, we let  $\lambda_1 = 3$ ,  $\lambda_2 = 4$ ,  $\beta_1^{(0)} = 0.6$ ,  $\beta_2^{(0)} = 0.4$ . We want to find  $P^{(0)}(S_N = 4)$ .

So we have,

$$P^{(s)}(S_N = k) = \frac{1}{k} \left[ \frac{(0.6)(3)^{s+1} + (0.4)(4)^{s+1}}{(0.6)(3)^s + (0.4)(4)^s} \right] \sum_{i=1}^k i \alpha_i \cdot P^{(s+1)}(S_N = k - i).$$

$$P^{(0)}(S_N = 4) = \frac{1}{4} \left[ \frac{(0.6)(3)^1 + (0.4)(4)^1}{(0.6)(3)^0 + (0.4)(4)^0} \right] \sum_{i=1}^4 i \alpha_i \cdot P^{(1)}(S_N = 4 - i).$$

*Base cases:*

$$\begin{aligned} P^{(1)}(S_N = 0) &= \frac{(0.6)(3)^1}{(0.6)(3)^1 + (0.4)(4)^1} e^{-3} + \frac{(0.4)(4)^1}{(0.6)(3)^1 + (0.4)(4)^1} e^{-4} = \frac{1.8}{3.4} e^{-3} + \frac{1.6}{3.4} e^{-4}. \\ P^{(2)}(S_N = 0) &= \frac{(0.6)(3)^2}{(0.6)(3)^2 + (0.4)(4)^2} e^{-3} + \frac{(0.4)(4)^2}{(0.6)(3)^2 + (0.4)(4)^2} e^{-4} = \frac{5.4}{11.8} e^{-3} + \frac{6.4}{11.8} e^{-4}. \\ P^{(3)}(S_N = 0) &= \frac{16.2}{41.8} e^{-3} + \frac{25.6}{41.8} e^{-4}. \\ P^{(4)}(S_N = 0) &= \frac{48.6}{151} e^{-3} + \frac{102.4}{151} e^{-4}. \end{aligned}$$

*Computing  $P^{(s)}(S_N = 1)$ :*

$$\begin{aligned} P^{(1)}(S_N = 1) &= \left[ \frac{(0.6)(3)^2 + (0.4)(4)^2}{(0.6)(3)^1 + (0.4)(4)^1} \right] [1 \cdot 0.05 \cdot P^{(2)}(S_N = 0)] \\ &= \frac{11.8}{3.4} \left[ 0.05 \left( \frac{5.4}{11.8} e^{-3} + \frac{6.4}{11.8} e^{-4} \right) \right] = 0.07941176471 e^{-3} + 0.09411764706 e^{-4}. \\ P^{(2)}(S_N = 1) &= \frac{41.8}{11.8} [0.05 \cdot P^{(3)}(S_N = 0)] = 0.06864406780 e^{-3} + 0.10847457627 e^{-4}. \\ P^{(3)}(S_N = 1) &= \frac{151}{41.8} [0.05 \cdot P^{(4)}(S_N = 0)] = 0.05813397129 e^{-3} + 0.12248803828 e^{-4}. \end{aligned}$$

*Computing  $P^{(s)}(S_N = 2)$ :*

$$\begin{aligned} P^{(1)}(S_N = 2) &= \frac{1}{2} \cdot \frac{11.8}{3.4} [(0.05) \cdot P^{(2)}(S_N = 1) + (0.8) \cdot P^{(2)}(S_N = 0)] \\ &= 0.64125 e^{-3} + 0.76235294118 e^{-4}. \\ P^{(2)}(S_N = 2) &= \frac{1}{2} \cdot \frac{41.8}{11.8} [(0.05) \cdot P^{(3)}(S_N = 1) + (0.8) \cdot P^{(3)}(S_N = 0)] \\ &= 0.55430084746 e^{-3} + 0.87864406780 e^{-4}. \end{aligned}$$

*Computing  $P^{(1)}(S_N = 3)$ :*

$$\begin{aligned} P^{(1)}(S_N = 3) &= \frac{1}{3} \cdot \frac{11.8}{3.4} [(0.05) \cdot P^{(2)}(S_N = 2) + (0.8) \cdot P^{(2)}(S_N = 1) + (0.3) \cdot P^{(2)}(S_N = 0)] \\ &= 0.25441544118 e^{-3} + 0.33945098039 e^{-4}. \end{aligned}$$

*Finally,  $P^{(0)}(S_N = 4)$ :*

$$\begin{aligned} P^{(0)}(S_N = 4) &= \frac{1}{4} \cdot \frac{3.4}{1} [(0.05) \cdot P^{(1)}(S_N = 3) + (0.8) \cdot P^{(1)}(S_N = 2) \\ &\quad + (0.3) \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)] \\ &= \boxed{0.91711265625 e^{-3} + 0.95682666667 e^{-4}}. \end{aligned}$$

## 6. GENERALIZED COMPUTATION WITH A FINITE MIXTURE OF DISTRIBUTIONS

We have shown in Example 1 with  $N$  in the Poisson distribution that

$$P^{(s+1)}(M^{(s)} - 1 = n) = P^{(s)}(N = n).$$

We have shown in Example 2 with  $N$  in the negative binomial distribution that

$$P^{(s)}(M^{(s)} - 1 = n) = P^{(s+1)}(N = n),$$

where the negative binomial parameter shifts from  $r + s$  to  $r + s + 1$ . Similarly, we have shown in Example 3 with  $N$  in the binomial distribution that

$$P^{(s)}(M^{(s)} - 1 = n) = P^{(s+1)}(N = n),$$

where the binomial parameter shifts from  $r - s$  to  $r - s - 1$ .

Notice that for each of these distributions, when calculating  $P(M^{(s)} - 1 = n)$ , at a given level of recursion, we arrive at either  $P^{(s)}(N = n)$  or a version of  $P^{(s)}(N = n)$  with a parameter value change that is a function of the level of recursion.

We will now show that by moving the level of recursion to a parameter of the probability mass function of the finite mixture of distributions we can compute a compound random variable with the counting distribution in any finite mixture of any of these three distributions.

**Corollary 3** (Recursive Properties of the Finite Mixture Distributions). *Let  $s \in \mathbb{Z}$  with  $0 \leq s$ , and let  $s$  represent the level of recursion. Let  $N^{(s)}$  be a mixed random variable of Binomial or Negative Binomial distribution at recursion level  $s$  with probability mass function:*

$$P^{(s)}(N = n) = \sum_{j=1}^t \beta_j^{(s)} P_{r_j}(N^{(s)} = n), \quad n \geq 0,$$

where  $\sum_{i=1}^t \beta_i = 1$ . Let  $M^{(s)}$  be the random variable satisfying the compound identity condition associated with  $N^{(s)}$ . We claim that for all  $s \geq 0$ :

$$(1) \quad E[N^{(s)}] = \sum_{j=1}^t \beta_j^{(s)} E[N_j^{(0)}].$$

(2) The distribution of  $M^{(s)} - 1$  is identical to the distribution of  $N^{(s+1)}$ , where the new weights are updated such that

$$\beta_j^{(s+1)} = \beta_j^{(s)} \cdot \frac{E[N_j^{(s)}]}{E[N^{(s)}]}.$$

*Proof (by induction).* Base Case ( $s = 0$ ): For  $s = 0$ , the weights are  $\beta_i$ . First, we compute the expected value of  $E[N^{(0)}]$ :

$$E[N^{(0)}] = \sum_{n=0}^{\infty} n \cdot \left[ \sum_{j=1}^t \beta_j^{(0)} P_{r_j}(n) \right] = \sum_{j=1}^t \beta_j^{(0)} E[N_j^{(0)}].$$

Thus, Claim 1 holds for case  $s = 0$ .

Next, we apply the compound random identity and see that,

$$\begin{aligned} P(M^{(0)} - 1 = n) &= P(M^{(0)} = n + 1) = \frac{(n + 1) P(N^{(0)} = n + 1)}{E[N^{(0)}]} \\ &= \frac{(n + 1) \left[ \sum_{j=1}^t \beta_j^{(0)} P_{r_j}(N^{(0)} = n + 1) \right]}{E[N^{(0)}]} \\ &= \frac{1}{E[N^{(0)}]} \sum_{j=1}^t \beta_j^{(0)} \left[ (n + 1) \cdot P_{r_j}(N^{(0)} = n + 1) \right]. \end{aligned}$$

Notice that for the negative binomial distribution,  $P^{(s)}(M^{(s)} - 1 = n) = P^{(s+1)}(N = n)$  implies

$$(n + 1) P_r(N = n + 1) = E[N] P_{r+1}(N = n).$$

So by substitution we have,

$$\begin{aligned} P(M^{(0)} - 1 = n) &= \frac{1}{E[N^{(0)}]} \sum_{j=1}^t \beta_j^{(0)} E[N_j^{(0)}] \cdot P_{r_j+1}(N^{(1)} = n) \\ &= \sum_{j=1}^t \frac{\beta_j E[N_j^{(0)}]}{E[N^{(0)}]} \cdot P_{r_j+1}(N^{(1)} = n). \end{aligned}$$

Similarly, for the binomial distribution,  $P^{(s)}(M^{(s)} - 1 = n) = P^{(s+1)}(N = n)$  implies

$$(n + 1) P_r(N = n + 1) = E[N] P_{r-1}(N = n).$$

So by substitution we have,

$$\begin{aligned} P(M^{(0)} - 1 = n) &= \frac{1}{E[N^{(0)}]} \sum_{j=1}^t \beta_j^{(0)} E[N_j^{(0)}] \cdot P_{r_j-1}(N^{(1)} = n) \\ &= \sum_{j=1}^t \frac{\beta_j E[N_j^{(0)}]}{E[N^{(0)}]} \cdot P_{r_j-1}(N^{(1)} = n). \end{aligned}$$

These match the definition of  $N^{(1)}$ , a mixture with shifted parameters according to the distribution. Thus, the claim holds for  $s = 0$ .

*Inductive Step:* Assume the statement is true for  $s = k$ . That is,  $M^{(k)} - 1$  is equivalent to  $N^{(k+1)}$  and  $E[N^{(k)}]$  follows the claim. We need to show the statement is true for  $s = k + 1$ . We must determine the distribution of  $M^{(k+1)} - 1$ .

We have,

$$P(M^{(k+1)} - 1 = n) = P(M^{(k+1)} = n + 1) = \frac{(n + 1) P(N^{(k+1)} = n + 1)}{E[N^{(k+1)}]}.$$

Substituting the mixture sum for  $N^{(k+1)}$ ,

$$P(M^{(k+1)} - 1 = n) = \frac{1}{E[N^{(k+1)}]} \sum_{j=1}^t \beta_j^{(k+1)} \left[ (n + 1) \cdot P_{r_j+(k+1)}(N^{(0)} = n + 1) \right].$$

So, for the negative binomial distribution, by substitution we have,

$$\begin{aligned} P(M^{(k+1)} - 1 = n) &= \frac{1}{E[N^{(k+1)}]} \sum_{j=1}^t \beta_j^{(k+1)} E[N_j^{(k+1)}] \cdot P_{r_j+k+2}(N^{(1)} = n) \\ &= \sum_{j=1}^t \frac{\beta_j^{k+1} E[N_j^{(k+1)}]}{E[N^{(k+1)}]} \cdot P_{r_j+k+2}(N^{(1)} = n). \end{aligned}$$

Similarly, for the binomial distribution, by substitution we have,

$$\begin{aligned} P(M^{(k+1)} - 1 = n) &= \frac{1}{E[N^{(k+1)}]} \sum_{j=1}^t \beta_j^{(k+1)} E[N_j^{(k+1)}] \cdot P_{r_j-(k+2)}(N^{(1)} = n) \\ &= \sum_{j=1}^t \frac{\beta_j^{k+1} E[N_j^{(k+1)}]}{E[N^{(k+1)}]} \cdot P_{r_j-(k+2)}(N^{(1)} = n). \end{aligned}$$

This expression is exactly the PMF for  $N^{(k+2)}$  with weights  $\beta_j^{k+1}$  as desired.

So, by the principle of mathematical induction, for all  $s \in \mathbb{N}$ , the compound identity variable associated with  $N^{(s)}$  shifts the distribution to  $N^{(s+1)}$ , and the expected value satisfies the derived closed form.  $\square$

Now we are ready to construct our new recursive expression by applying Corollary 1. From

$$P(S_N = k) = \frac{1}{k} E[N] \sum_{i=1}^k i \alpha_i P(S_{M-1} = k-i),$$

we arrive at the updated recursive equation

$$P^{(s)}(S_N = k) = \frac{E[N^{(s)}]}{k} \cdot \sum_{i=1}^k i \alpha_i \cdot P^{(s+1)}(S_N = k-i),$$

where  $E[N^{(s)}] = \sum_{j=1}^t \beta_j^{(s)} E[N_j^{(0)}]$  and the computation of  $P^{(s+1)}$  uses  $\beta^{(s+1)}$  and parameters appropriate to the distribution of  $P_i$ , that is,  $(r-s)$  for the binomial distribution,  $(r+s)$  for the negative binomial distribution and no parameter change for the Poisson distribution.

**Example 7** (N in a Finite Mixture of Binomial Distributions). Consider a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Binomial distribution with  $r_1 = 8$  and  $p_1 = 0.5$ , or process two, which follows a Binomial distribution with  $r_2 = 10$  and  $p_2 = 0.7$ . The probability of process one is  $\beta_1^{(0)} = 0.6$ . The probability of process two is  $\beta_2^{(0)} = 0.4$ . Notice that  $\beta_1^{(0)} + \beta_2^{(0)} = 1$ . We want to find  $P(S_N = 4)$ .

So, the probability mass function for this mixed distribution is

$$P^{(s)}(N = n) = \sum_{j=1}^2 \beta_j^{(s)} \binom{r_j - s}{n} p_j^n (1 - p_j)^{(r_j - s) - n}, \quad 0 \leq n \leq \max(r_1 - s, r_2 - s).$$

Computing the mixing weights at each level:

$$E[N^{(0)}] = \beta_1^{(0)}(r_1 - 0)p_1 + \beta_2^{(0)}(r_2 - 0)p_2 = 0.6 \cdot 8 \cdot 0.5 + 0.4 \cdot 10 \cdot 0.7 = 5.2.$$

$$\beta_1^{(1)} = \beta_1^{(0)} \cdot \frac{E[N_1^{(0)}]}{E[N^{(0)}]} = \frac{2.4}{5.2} = 0.46153846154.$$

$$\beta_2^{(1)} = \beta_2^{(0)} \cdot \frac{E[N_2^{(0)}]}{E[N^{(0)}]} = \frac{2.8}{5.2} = 0.53846153846.$$

$$\begin{aligned} E[N^{(1)}] &= \beta_1^{(1)}(r_1 - 1)p_1 + \beta_2^{(1)}(r_2 - 1)p_2 \\ &= 0.46153846154 \cdot 7 \cdot 0.5 + 0.53846153846 \cdot 9 \cdot 0.7 = 5.00769230769. \end{aligned}$$

$$\beta_1^{(2)} = \beta_1^{(1)} \cdot \frac{E[N_1^{(1)}]}{E[N^{(1)}]} = \frac{1.61538461538}{5.00769230769} = 0.32258064516.$$

$$\beta_2^{(2)} = \beta_2^{(1)} \cdot \frac{E[N_2^{(1)}]}{E[N^{(1)}]} = \frac{3.39230769231}{5.00769230769} = 0.67741935484.$$

$$\begin{aligned} E[N^{(2)}] &= \beta_1^{(2)}(r_1 - 2)p_1 + \beta_2^{(2)}(r_2 - 2)p_2 \\ &= 0.32258064516 \cdot 6 \cdot 0.5 + 0.67741935484 \cdot 8 \cdot 0.7 = 4.76129032258. \end{aligned}$$

$$\beta_1^{(3)} = \beta_1^{(2)} \cdot \frac{E[N_1^{(2)}]}{E[N^{(2)}]} = \frac{0.96774193548}{4.76129032258} = 0.20325203252.$$

$$\beta_2^{(3)} = \beta_2^{(2)} \cdot \frac{E[N_2^{(2)}]}{E[N^{(2)}]} = \frac{3.79354838710}{4.76129032258} = 0.79674796748.$$

$$E[N^{(3)}] = \beta_1^{(3)}(r_1 - 3)p_1 + \beta_2^{(3)}(r_2 - 3)p_2$$

$$= 0.20325203252 \cdot 5 \cdot 0.5 + 0.79674796748 \cdot 7 \cdot 0.7 = 4.41219512195.$$

$$\beta_1^{(4)} = \beta_1^{(3)} \cdot \frac{E[N_1^{(3)}]}{E[N^{(3)}]} = \frac{0.50813008130}{4.41219512195} = 0.11516491616.$$

$$\beta_2^{(4)} = \beta_2^{(3)} \cdot \frac{E[N_2^{(3)}]}{E[N^{(3)}]} = \frac{3.90406504065}{4.41219512195} = 0.88483508384.$$

$$\begin{aligned} E[N^{(4)}] &= \beta_1^{(4)}(r_1 - 4)p_1 + \beta_2^{(4)}(r_2 - 4)p_2 \\ &= 0.11516491616 \cdot 4 \cdot 0.5 + 0.88483508384 \cdot 6 \cdot 0.7 = 3.94663718445. \end{aligned}$$

Now we can begin the computation to find  $P^{(0)}(S_N = 4)$ . We want to find,

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)].$$

*Base cases:*

$$P^{(s)}(S_N = 0) = \beta_1^{(s)} \cdot (1 - p_1)^{r_1 - s} + \beta_2^{(s)} \cdot (1 - p_2)^{r_2 - s} = \beta_1^{(s)} \cdot (0.5)^{8-s} + \beta_2^{(s)} \cdot (0.3)^{10-s}.$$

$$P^{(1)}(S_N = 0) = 0.46153846154 (0.5)^7 + 0.53846153846 (0.3)^9 = 0.00361636777.$$

$$P^{(2)}(S_N = 0) = 0.32258064516 (0.5)^6 + 0.67741935484 (0.3)^8 = 0.00508476806.$$

$$P^{(3)}(S_N = 0) = 0.20325203252 (0.5)^5 + 0.79674796748 (0.3)^7 = 0.00652587480.$$

$$P^{(4)}(S_N = 0) = 0.11516491616 (0.5)^4 + 0.88483508384 (0.3)^6 = 0.00784285204.$$

*Computing  $P^{(s)}(S_N = 1)$ :*

$$P^{(1)}(S_N = 1) = \frac{E[N^{(1)}]}{1} [0.05 \cdot P^{(2)}(S_N = 0)] = 5.00769230769(0.05)(0.00508476806) = 0.00127314770.$$

$$P^{(2)}(S_N = 1) = \frac{E[N^{(2)}]}{1} [0.05 \cdot P^{(3)}(S_N = 0)] = 4.76129032258(0.05)(0.00652587480) = 0.00155357923.$$

$$P^{(3)}(S_N = 1) = \frac{E[N^{(3)}]}{1} [0.05 \cdot P^{(4)}(S_N = 0)] = 4.41219512195(0.05)(0.00784285204) = 0.00173020967.$$

*Computing  $P^{(s)}(S_N = 2)$ :*

$$\begin{aligned} P^{(1)}(S_N = 2) &= \frac{E[N^{(1)}]}{2} [0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)] \\ &= \frac{5.00769230769}{2} [0.05(0.00155357923) + 0.8(0.00508476806)] = 0.01037967774. \end{aligned}$$

$$\begin{aligned} P^{(2)}(S_N = 2) &= \frac{E[N^{(2)}]}{2} [0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)] \\ &= \frac{4.76129032258}{2} [0.05(0.00173020967) + 0.8(0.00652587480)] = 0.01263458457. \end{aligned}$$

*Computing  $P^{(1)}(S_N = 3)$ :*

$$\begin{aligned} P^{(1)}(S_N = 3) &= \frac{E[N^{(1)}]}{3} [0.05 \cdot P^{(2)}(S_N = 2) + 0.8 \cdot P^{(2)}(S_N = 1) + 0.3 \cdot P^{(2)}(S_N = 0)] \\ &= \frac{5.00769230769}{3} [0.05(0.01263458457) + 0.8(0.00155357923) + 0.3(0.00508476806)] \\ &= 0.00567542306. \end{aligned}$$

*Computing  $P^{(0)}(S_N = 4)$ :*

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)]$$

$$\begin{aligned}
&= \frac{5.2}{4} [0.05(0.00567542306) + 0.8(0.01037967774) \\
&\quad + 0.3(0.00127314770) + 0.00361636777] \\
&= \boxed{0.01636157305}.
\end{aligned}$$

**Example 8** (N in a Finite Mixture of Negative Binomial Distributions). Consider a discrete mixture distribution where the event is generated by one of two random processes. The event is either generated by process one, which follows a Negative Binomial distribution with  $r_1 = 4$  and  $p_1 = 0.6$ , or process two, which follows a Negative Binomial distribution with  $r_2 = 6$  and  $p_2 = 0.5$ . The probability of process one is  $\beta_1^{(0)} = 0.5$ . The probability of process two is  $\beta_2^{(0)} = 0.5$ . Notice that  $\beta_1^{(0)} + \beta_2^{(0)} = 1$ . We want to find  $P(S_N = 4)$ .

So, the probability mass function for this mixed distribution is

$$P^{(s)}(N = n) = \sum_{j=1}^2 \beta_j^{(s)} \binom{n + r_j + s - 1}{n} p_j^{r_j+s} (1 - p_j)^n, \quad 0 \leq n \leq \max(r_1 + s, r_2 + s).$$

Computing the mixing weights at each level:

$$E[N^{(0)}] = 0.5 \left( \frac{(4+0)(0.4)}{0.6} \right) + 0.5 \left( \frac{(6+0)(0.5)}{0.5} \right) = 4.33333333333.$$

$$\beta_1^{(1)} = 0.5 \cdot \frac{1.33333333333}{4.33333333333} = 0.30769230769.$$

$$\beta_2^{(1)} = 0.5 \cdot \frac{3}{4.33333333333} = 0.69230769231.$$

$$E[N^{(1)}] = 0.30769230769 \cdot \frac{5 \cdot 0.4}{0.6} + 0.69230769231 \cdot \frac{7 \cdot 0.5}{0.5} = 5.87179487179.$$

$$\beta_1^{(2)} = 0.30769230769 \cdot \frac{1.02564102564}{5.87179487179} = 0.17467248908.$$

$$\beta_2^{(2)} = 0.69230769231 \cdot \frac{4.84615384615}{5.87179487179} = 0.82532751092.$$

$$E[N^{(2)}] = 0.17467248908 \cdot \frac{6 \cdot 0.4}{0.6} + 0.82532751092 \cdot \frac{8 \cdot 0.5}{0.5} = 7.30131004367.$$

$$\beta_1^{(3)} = 0.17467248908 \cdot \frac{0.69868995633}{7.30131004367} = 0.09569377990.$$

$$\beta_2^{(3)} = 0.82532751092 \cdot \frac{6.60262008734}{7.30131004367} = 0.90430622010.$$

$$E[N^{(3)}] = 0.09569377990 \cdot \frac{7 \cdot 0.4}{0.6} + 0.90430622010 \cdot \frac{9 \cdot 0.5}{0.5} = 8.58532695375.$$

$$\beta_1^{(4)} = 0.09569377990 \cdot \frac{0.44664430295}{8.58532695375} = 0.05201560468.$$

$$\beta_2^{(4)} = 0.90430622010 \cdot \frac{8.13875598086}{8.58532695375} = 0.94798439532.$$

$$E[N^{(4)}] = 0.05201560468 \cdot \frac{8 \cdot 0.4}{0.6} + 0.94798439532 \cdot \frac{10 \cdot 0.5}{0.5} = 9.75726051149.$$

Now we can begin the computation to find  $P^{(0)}(S_N = 4)$ . We want to find,

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)].$$

$$\text{Base cases } P^{(s)}(S_N = 0) = \beta_1^{(s)} p_1^{r_1+s} + \beta_2^{(s)} p_2^{r_2+s};$$

$$P^{(0)}(S_N = 0) = 0.5(0.6)^4 + 0.5(0.5)^6 = 0.0726125.$$

$$P^{(1)}(S_N = 0) = 0.30769230769(0.6)^5 + 0.69230769231(0.5)^7 = 0.02933480769.$$

$$P^{(2)}(S_N = 0) = 0.17467248908(0.6)^6 + 0.82532751092(0.5)^8 = 0.01137345524.$$

$$P^{(3)}(S_N = 0) = 0.09569377990(0.6)^7 + 0.90430622010(0.5)^9 = 0.00444503648.$$

$$P^{(4)}(S_N = 0) = 0.05201560468(0.6)^8 + 0.94798439532(0.5)^{10} = 0.00179942843.$$

Computing  $P^{(s)}(S_N = 1)$ :

$$P^{(1)}(S_N = 1) = \frac{E[N^{(1)}]}{1} [0.05 \cdot P^{(2)}(S_N = 0)] = 5.87179487179(0.05)(0.01137345524) = 0.00333912981.$$

$$P^{(2)}(S_N = 1) = \frac{E[N^{(2)}]}{1} [0.05 \cdot P^{(3)}(S_N = 0)] = 7.30131004367(0.05)(0.00444503648) = 0.00162272948.$$

$$P^{(3)}(S_N = 1) = \frac{E[N^{(3)}]}{1} [0.05 \cdot P^{(2)}(S_N = 0)] = 8.58532695375(0.05)(0.01137345524) = 0.00488224159.$$

Computing  $P^{(s)}(S_N = 2)$ :

$$\begin{aligned} P^{(1)}(S_N = 2) &= \frac{E[N^{(1)}]}{2} [0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)] \\ &= \frac{5.87179487179}{2} [0.05(0.00162272948) + 0.8(0.01137345524)] = 0.02695124683. \end{aligned}$$

$$\begin{aligned} P^{(2)}(S_N = 2) &= \frac{E[N^{(2)}]}{2} [0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)] \\ &= \frac{7.30131004367}{2} [0.05(0.00488224159) + 0.8(0.00444503648)] = 0.01387300480. \end{aligned}$$

Computing  $P^{(1)}(S_N = 3)$ :

$$\begin{aligned} P^{(1)}(S_N = 3) &= \frac{E[N^{(1)}]}{3} [0.05 \cdot P^{(2)}(S_N = 2) + 0.8 \cdot P^{(2)}(S_N = 1) + 0.3 \cdot P^{(2)}(S_N = 0)] \\ &= \frac{5.87179487179}{3} [0.05(0.01387300480) + 0.8(0.00162272948) + 0.3(0.01137345524)] \\ &= 0.01057680615. \end{aligned}$$

Computing  $P^{(0)}(S_N = 4)$ :

$$\begin{aligned} P^{(0)}(S_N = 4) &= \frac{E[N^{(0)}]}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)] \\ &= \frac{4.33333333333}{4} [0.05(0.01057680615) + 0.8(0.02695124683) \\ &\quad + 0.3(0.00333912981) + 0.02933480769] \\ &= \boxed{0.05679524977}. \end{aligned}$$

**Example 9** (N in a Finite Mixture of Poisson, Binomial and Negative Binomial Distributions). Consider a discrete mixture distribution where the event is generated by one of three random processes. The event is either generated by process one, which follows a Poisson distribution with  $\lambda = 3$ , or process two, which follows a Binomial distribution with  $r_2 = 6$  and  $p_2 = 0.5$ , or process three, which follows a Negative Binomial distribution with  $r_3 = 2$  and  $p_3 = 0.4$ . The probability of process one is  $\beta_1^{(0)} = 0.4$ . The probability of process two is  $\beta_2^{(0)} = 0.3$ . The probability of process three is  $\beta_3^{(0)} = 0.3$ . Notice that  $\beta_1^{(0)} + \beta_2^{(0)} + \beta_3^{(0)} = 1$ . We want to find  $P(S_N = 4)$ .

So, the probability mass function for this mixed distribution is

$$P^{(s)}(N = n) = \beta_1^{(s)} \cdot \frac{e^{-\lambda} \lambda^n}{n!} + \beta_2^{(s)} \cdot \binom{r_2 - s}{n} p_2^n (1 - p_2)^{(r_2 - s) - n} + \beta_3^{(s)} \cdot \binom{n + r_3 + s - 1}{n} p_3^{r_3 + s} (1 - p_3)^n,$$

for  $0 \leq n \leq \max(r_2 - s, r_3 + s)$ .

So we have,

$$\begin{aligned} E[N_1^{(s)}] &= \lambda = 3, \quad \text{for all } s. \\ E[N_2^{(s)}] &= (r_2 - s)p_2 = (6 - s)(0.5). \\ E[N_3^{(s)}] &= \frac{(r_3 + s)(1 - p_3)}{p_3} = \frac{(2 + s)(0.6)}{0.4} = (2 + s)(1.5). \end{aligned}$$

Computing the mixing weights at each level:

$$\begin{aligned} \beta_1^{(0)} &= 0.4, \quad \beta_2^{(0)} = 0.3, \quad \beta_3^{(0)} = 0.3. \\ E[N^{(0)}] &= (0.4)(3) + (0.3)(3) + (0.3)(3) = 3. \\ \beta_1^{(1)} &= 0.4 \cdot \frac{3}{3} = 0.4. \\ \beta_2^{(1)} &= 0.3 \cdot \frac{3}{3} = 0.3. \\ \beta_3^{(1)} &= 0.3 \cdot \frac{3}{3} = 0.3. \\ E[N^{(1)}] &= 0.4(3) + 0.3(2.5) + 0.3(4.5) = 3.3. \\ \beta_1^{(2)} &= 0.4 \cdot \frac{3}{3.3} = 0.36363636364. \\ \beta_2^{(2)} &= 0.3 \cdot \frac{2.5}{3.3} = 0.22727272727. \\ \beta_3^{(2)} &= 0.3 \cdot \frac{4.5}{3.3} = 0.40909090909. \\ E[N^{(2)}] &= 0.36363636364(3) + 0.22727272727(2) + 0.40909090909(6) = 4. \\ \beta_1^{(3)} &= 0.36363636364 \cdot \frac{3}{4} = 0.27272727273. \\ \beta_2^{(3)} &= 0.22727272727 \cdot \frac{2}{4} = 0.11363636364. \\ \beta_3^{(3)} &= 0.40909090909 \cdot \frac{6}{4} = 0.61363636364. \\ E[N^{(3)}] &= 0.27272727273(3) + 0.11363636364(1.5) \\ &\quad + 0.61363636364(7.5) = 5.59090909091. \\ \beta_1^{(4)} &= 0.27272727273 \cdot \frac{3}{5.59090909091} = 0.14634146341. \\ \beta_2^{(4)} &= 0.11363636364 \cdot \frac{1.5}{5.59090909091} = 0.03048780488. \\ \beta_3^{(4)} &= 0.61363636364 \cdot \frac{7.5}{5.59090909091} = 0.82317073171. \\ E[N^{(4)}] &= 0.14634146341(3) + 0.03048780488(1) \\ &\quad + 0.82317073171(9) = 7.87804878049. \end{aligned}$$

Now we can begin the computation to find  $P^{(0)}(S_N = 4)$ . We want to find,

$$P^{(0)}(S_N = 4) = \frac{E[N^{(0)}]}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)].$$

Base cases:

$$P^{(s)}(N = 0) = \beta_1^{(s)} e^{-3} + \beta_2^{(s)} (0.5)^{6-s} + \beta_3^{(s)} (0.4)^{2+s}.$$

$$P^{(0)}(S_N = 0) = 0.4 e^{-3} + 0.3(0.5)^6 + 0.3(0.4)^2 = 0.07260232735.$$

$$\begin{aligned}
P^{(1)}(S_N = 0) &= 0.4e^{-3} + 0.3(0.5)^5 + 0.3(0.4)^3 = 0.04848982735. \\
P^{(2)}(S_N = 0) &= 0.36363636364e^{-3} + 0.22727272727(0.5)^4 + 0.40909090909(0.4)^4 = 0.04278166122. \\
P^{(3)}(S_N = 0) &= 0.27272727273e^{-3} + 0.11363636364(0.5)^3 + 0.61363636364(0.4)^5 = 0.03406647319. \\
P^{(4)}(S_N = 0) &= 0.14634146341e^{-3} + 0.03048780488(0.5)^2 + 0.82317073171(0.4)^6 = 0.01827957098.
\end{aligned}$$

Computing  $P^{(s)}(S_N = 1)$ :

$$\begin{aligned}
P^{(1)}(S_N = 1) &= \frac{E[N^{(1)}]}{1} [0.05 \cdot P^{(2)}(S_N = 0)] = 3.3(0.05)(0.04278166122) = 0.00705897410. \\
P^{(2)}(S_N = 1) &= \frac{E[N^{(2)}]}{1} [0.05 \cdot P^{(3)}(S_N = 0)] = 4(0.05)(0.03406647319) = 0.00681329464. \\
P^{(3)}(S_N = 1) &= \frac{E[N^{(3)}]}{1} [0.05 \cdot P^{(4)}(S_N = 0)] = 5.59090909091(0.05)(0.01827957098) = 0.00510997098.
\end{aligned}$$

Computing  $P^{(s)}(S_N = 2)$ :

$$\begin{aligned}
P^{(1)}(S_N = 2) &= \frac{E[N^{(1)}]}{2} [0.05 \cdot P^{(2)}(S_N = 1) + 0.8 \cdot P^{(2)}(S_N = 0)] \\
&= \frac{3.3}{2} [0.05(0.00681329464) + 0.8(0.04278166122)] = 0.05703388962. \\
P^{(2)}(S_N = 2) &= \frac{E[N^{(2)}]}{2} [0.05 \cdot P^{(3)}(S_N = 1) + 0.8 \cdot P^{(3)}(S_N = 0)] \\
&= \frac{4}{2} [0.05(0.00510997098) + 0.8(0.03406647319)] = 0.05501735420.
\end{aligned}$$

Computing  $P^{(1)}(S_N = 3)$ :

$$\begin{aligned}
P^{(1)}(S_N = 3) &= \frac{E[N^{(1)}]}{3} [0.05 \cdot P^{(2)}(S_N = 2) + 0.8 \cdot P^{(2)}(S_N = 1) + 0.3 \cdot P^{(2)}(S_N = 0)] \\
&= \frac{3.3}{3} [0.05(0.05501735420) + 0.8(0.00681329464) + 0.3(0.04278166122)] \\
&= 0.02313960197.
\end{aligned}$$

Computing  $P^{(0)}(S_N = 4)$ :

$$\begin{aligned}
P^{(0)}(S_N = 4) &= \frac{E[N^{(0)}]}{4} [0.05 \cdot P^{(1)}(S_N = 3) + 0.8 \cdot P^{(1)}(S_N = 2) + 0.3 \cdot P^{(1)}(S_N = 1) + P^{(1)}(S_N = 0)] \\
&= \frac{3}{4} [0.05(0.02313960197) + 0.8(0.05703388962) + 0.3(0.00705897410) + 0.04848982735] \\
&= \boxed{0.07304370853}.
\end{aligned}$$

#### REFERENCES

- [1] [TODO: Need the right list here. Potentials are Feller (1968), Grandell (1997), and Johnson, Kotz, and Kemp (1992).]