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BARRIER TRAJECTORIES OF A REALISTIC MISSILE/TARGET PURSUIT–EVASION GAME

Breitner, M.[1)], Grimm, W.[2)], Pesch, H.J.[3)]

1. Introduction

Presently a computer based pilot's decision aid is developed for future fighter aircraft. The objective is to increase the probability of success in the case of own attack and to improve the chance of survival in a situation of hostile attack. In the first case the firing range of a missile must be estimated by the onboard computer to pick the right launch time. In the second case the aircraft becomes the target of an adversary missile. Now, a favourable evasive maneuver must be initiated in time and controlled by an autonomous guidance algorithm.

For both purposes the pursuit–evasion game concept (Isaacs [8]) applied to the missile/target encounter provides the suitable mathematical framework. The game solution indicates both the firing range of the missile and the optimal evasive maneuver of the target.

First, game solutions were obtained for simplified models (e.g. linearized equations of motion (Shinar, Gutman [14]), simplified dynamics in the vertical plane (Guelman et al. [7])) or approximated with the help of singular perturbation technique (Shinar, Gazit [13]).

The game solution also shows the optimal missile guidance. If one is only interested in this aspect a one sided optimal control formulation is sufficient. This approach was applied to the complete point mass model in the vertical plane to maximize the missile's range subject to the condition that there remains enough energy for the final pursuit of the target (Kumar et al. [11]). The final pursuit phase itself was not explicitly considered in the study. The optimal trajectories of [11] served as reference flight paths in a closed–loop missile guidance law (Kumar et al. [10]).

The intention of the present paper is to resume the differential game approach and to combine it with a dynamic model containing realistic approximations for thrust and drag. The objective is to determine barrier trajectories in the vertical plane under the assumption that complete state and model information is available to each vehicle. Initial speed and altitude of the target aircraft are systematically varied within the flight envelope. For the missile two different launch positions are considered. The barrier trajectories are computed numerically by solving multipoint boundary value problems derived from the necessary conditions for the barrier. As a by–product, the dependence of the firing range on the altering initial values is obtained. Thus, the results draw a detailed picture of the firing envelope for the underlying vehicle models.

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2. Dynamic Model

The flight path of both vehicles (missile and target) is the solution of the following ODE–system:

$$\dot{x} = V \cos\gamma \tag{1}$$

$$\dot{h} = V \sin\gamma \tag{2}$$

$$\dot{V} = (T - D_o - \cos^2\gamma\ D_i) / m - g \sin\gamma \tag{3}$$

For abbreviation let

$$\dot{z} = f(z, \gamma, t) \tag{4}$$

with $z := (x, h, V)^T$ denote the ODE–system above. (1) – (3) describes the motion of a vehicle in the vertical plane with the flight path angle γ as control variable. This is already a model reduction in the sense of singular perturbation theory, since γ is a state variable in reality. The authors are aware of the falsifying effects of the simplification, as described by Katzir et al. [9].
The symbols in (1) – (3) have the usual meaning:

x:	range	h:	altitude
V:	velocity	γ:	flight path angle
T:	thrust	D_o:	zero–lift drag
D_i:	induced drag	m:	mass
g:	gravitational acceleration (constant)		

The structure of the drag components is

$$D_o(h, V) = q\, S\, C_{D0}(M), \qquad D_i(h, V) = C_{Di}(M)\, (m\, g)^2 / (q\, S)$$

with $q(h, V) = \rho(h)\, V^2 / 2$ and the following meaning of the additional symbols:

q:	dynamic pressure	M:	Mach number, $= V$ / speed of sound
S:	reference wing area	C_{D0}:	zero–lift drag coefficient
ρ:	air density	C_{Di}:	induced drag coefficient

$\rho(h)$ is a realistic model function for air density.
In the generic <u>missile</u> model thrust is a decreasing step function of time. It is divided into the boost phase (the first 3 sec), the march phase (the following 12 sec) and the coasting phase ($T = 0$). In accordance with the thrust profile the mass is a piecewise linearly decreasing function of time which remains constant in the coasting phase. Therefore, the system (4) is explicitly time dependent.
In the <u>aircraft</u> model $T(h, M)$, $C_{D0}(M)$ and $C_{Di}(M)$ are analytic functions which are carefully adapted to realistic tabular data of a fighter aircraft. We assume maximum thrust to be optimal for the aircraft – an assumption to be verified a posteriori. Mass is assumed to be constant. The aircraft is constrained by its dynamic pressure limit q_{max}:

$$Q(z) := q - q_{max} \leq 0 \tag{5}$$

Differentiation of $Q(z) = 0$ w.r.t. time yields the control along the constraint (5):

$$0 = Q^{(1)}(z, \gamma) := \frac{d}{dt} Q(z) \qquad \Rightarrow \qquad \gamma = \gamma^q(h) \tag{6}$$

Since h and V are coupled by $Q(z) = 0$ along the q_{max}–boundary, h is actually the only argument of γ^q.

3. Problem Formulation

In the sequel the subscripts P and E denote pursuer (=missile) and evader (=aircraft), respectively. We consider a game of kind with the terminal manifold

$$0 = R(z_P, z_E) = d^2 - (x_P - x_E)^2 - (h_P - h_E)^2 . \tag{7}$$

There is a barrier separating the capture zone of the pursuer from the escape zone of the evader. Since the pursuer (=missile) has a limited amount of fuel the existence of the barrier is clear from physical intuition. The task is to find the barrier trajectory determined by prescribed initial values for x_P, h_P, V_P, h_E and V_E. Without loss of generality one can set $x_P(0) = 0$. The resulting value of $x_E(0)$ (≥ 0) may be thought of as the maximum range from which a hit can be achieved. $h_E(0)$ and $V_E(0)$ are varied all over the flight envelope of the evader aircraft. $z_P(0)$ is restricted to two cases:

case 1:	$h_P(0) = 5000$ m	$V_P(0) = 250$ m/s
case 2:	$h_P(0) = 12000$ m	$V_P(0) = 400$ m/s

The main interest of the present paper is to evaluate the optimal control along a barrier trajectory for a representative set of initial conditions.

4. Necessary Conditions for the Barrier Trajectories

The necessary conditions for the barrier involve the adjoint vector $\lambda_z := (\lambda_{z,P}, \lambda_{z,E})$ corresponding to the state vector (z_P, z_E). Together with the state vector they form the expression

$$H := H_P + H_E + \lambda_t$$

with $\qquad H_P := \lambda_{z,P}{}^T f(z_P, \gamma_P, t)$

and $\qquad H_E := \lambda_{z,E}{}^T f(z_E, \gamma_E, t) + \mu_E Q^{(1)}(z_E, \gamma_E) . \qquad (8)$

λ_t is an adjoint variable accounting for the fact that the dynamic system is explicitly time dependent. The multiplier μ_E vanishes as long as $Q(z_E) < 0$ and is nonpositive on constrained arcs:

$$\begin{aligned} \mu_E &= 0 \qquad \text{for} \quad Q(z_E) < 0 \\ \mu_E &\leq 0 \qquad \text{for} \quad Q(z_E) = 0 \end{aligned} \tag{9}$$

The optimal control along the barrier is determined by the semipermeability condition:

$$0 = \min_{\gamma_P} \max_{\gamma_E} H . \tag{10}$$

This interpretation is based on the fact that λ_z is the normal vector on the barrier surface. Since H is separable in γ_P and γ_E the controls can be determined by one sided optimization of H and the min– and max–operations in (10) can be exchanged (minmax–assumption):

$$\min_{\gamma_P} H_P , \ \max_{\gamma_E} H_E \quad \Longrightarrow \quad 0 = \partial H_i / \partial \gamma_i , \quad i = P, E. \tag{11}$$

Along the dynamic pressure boundary γ_E is determined by (6). In this case $0 = \partial H_E / \partial \gamma_E$ essentially serves to evaluate μ_E.
The terminal value of the adjoint vector is given by the transversality conditions

$$\lambda_{z_i}(t_f) = \partial R / \partial z_i^{\,T} , \quad i = P, E, \qquad\qquad \lambda_t(t_f) = 0 . \tag{12}$$

Along the barrier trajectories λ_z must satisfy the adjoint differential equations

$$\dot{\lambda}_{z_i} = - \partial H / \partial z_i^{\,T} , \quad i = P, E, \qquad\qquad \dot{\lambda}_t = - H_t . \tag{13}$$

Combining (7) – (12) yields the condition for the intersection of the terminal manifold and the barrier, the "Boundary of the Usable Part":

$$\begin{aligned} Q(z_E(t_f)) < 0 \quad &\Longrightarrow \quad V_P(t_f) = V_E(t_f) \\ Q(z_E(t_f)) = 0 \quad &\Longrightarrow \quad V_P(t_f) = V_E(t_f) \, \cos(\theta(t_f) - \gamma^q(h_E(t_f))) \end{aligned} \tag{14}$$

where θ is the aspect angle from the pursuer on to the evader:

$$\theta = \arctan \frac{h_E - h_P}{x_E - x_P} \tag{15}$$

At the entry point t_{q1} on to a constrained evader subarc the jump condition

$$\lambda_{z,E}(t_{q1}^{+}) = \lambda_{z,E}(t_{q1}^{-}) - \mu_E(t_{q1}^{+}) \, \partial Q / \partial z_E^{\,T} \tag{16}$$

must be satisfied. An eventual exit point $t_{q2} < t_f$ is determined by

$$\mu_E(t_{q2}^{-}) = 0 . \tag{17}$$

The treatment of the constraint is tacitly adopted from optimal control theory (e.g. [4], [5]). State constrained differential games have not yet received much attention in literature. Some publications on this topic confirming the conditions above are Blaquiere et al. [2], Bernhard [1] and Taylor [15]. A solid justification for the present example is given by Breitner [3].

5. Formulation and Numerical Solution of Multipoint Boundary Value Problems

The conditions of the physical problem (section 3) and the necessary conditions for barrier trajectories (section 4) constitute a multipoint boundary value problem (MPBVP) for 13 dependent variables: z_P, z_E, $\lambda_{z,P}$, $\lambda_{z,E}$, t_f. In the unconstrained case (no interior point conditions) the number of boundary conditions matches the number of dependent variables:

at $t = 0$:	prescribed initial values for x_P, h_P, V_P, h_E, V_E	(5 conditions)
at $t = t_f$:	(7), (12), (14) (1 + 6 + 1 conditions)	

λ_t does not affect the other variables and is therefore omitted. To cope with the free final time the independent variable is normalized in the usual manner. The jump condition (16) suggests backward integration of the trajectory: The limit value of μ_E for $t \to t_{q1} + 0$ is known on backward integration; it would represent another unknown parameter in the opposite direction. Thus, the roles of initial and terminal conditions are reversed from the view of the MPBVP.

Dependent on the "switching structure", i.e. the sequence of constrained and unconstrained arcs, the problem is augmented by switching points and switching conditions:

> Switching structure A: $Q < 0$ / $Q = 0$. The unknown switching point t_{q1} is determined by $Q(t_{q1}) = 0$. Note that the jump (16) can be performed during numerical integration.
> Switching structure B: $Q < 0$ / $Q = 0$ / $Q < 0$. In addition to A the unknown exit point t_{q2} and the interior point condition (17) is added.
> Switching structure C: unconstrained case. MPBVP as described above.

The MPBVP's above are solved with the program package BNDSCO (see Bulirsch [6], Oberle, Grimm [12]), an implementation of the multiple shooting method taylored for the above kind of problem.

6. Discussion of the Numerical Results

The above problem is solved for case 1 and 2 for all pairs $(h_E(0), V_E(0))$ within the evader's flight envelope (see Fig. 1). Regions, where $h_E < 0$ or $\lambda_{V,E} < 0$ is encountered along the barrier trajectory, are omitted. Trajectories with $h_E < 0$ only occur for an evader starting in the right lower corner of his flight envelope (region I in Fig. 1). $\lambda_{V,E} < 0$ indicates that maximum thrust is not optimal for the evader; this happens for case 1 in a small domain along the upper right edge of the flight envelope (region II in Fig. 1). Switching structure A is found to be valid for most of the barrier trajectories (region A in Fig. 1). B and C only occur in case 1 (regions B and C in Fig. 1, respectively).

With the help of the multiple shooting algorithm also the boundaries of the different regions are computed. E.g. the boundary points of region I can be determined by solving a modified version of the MPBVP belonging to switching structure A: Let $h_E(0)$ be a free parameter of the problem and also the time t_h, where h_E takes its minimum. In return, two additional switching conditions are added: $h_E(t_h) = 0$ and $\gamma_E(t_h) = 0$. The resulting $h_E(0)$ together with the prescribed value $V_E(0)$ constitutes a point on the upper boundary of region I.

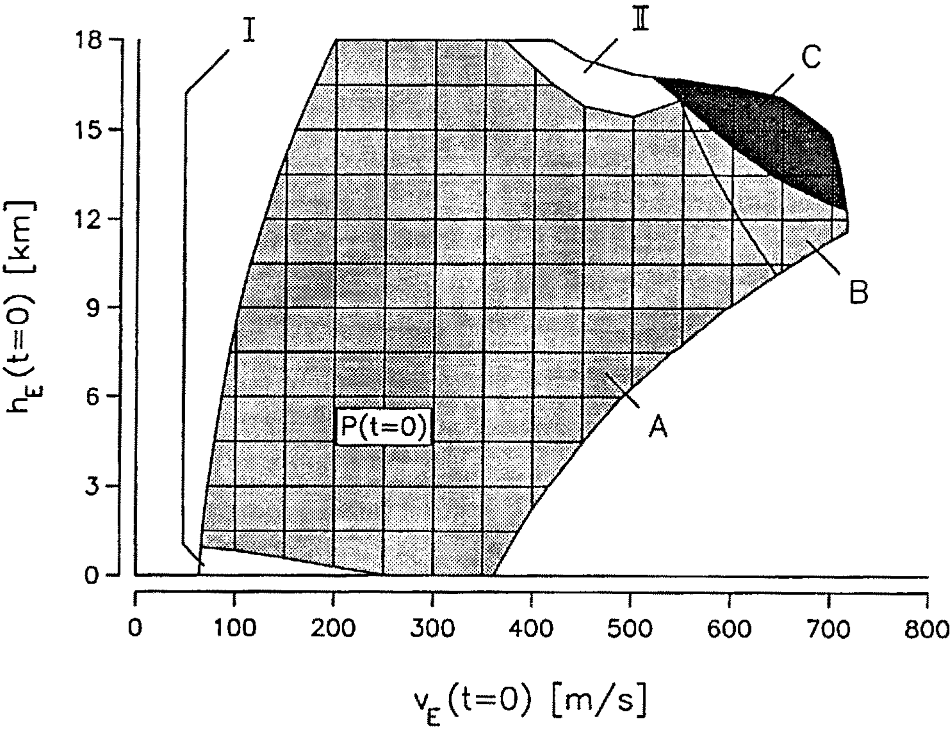

Figure 1: Switching structures of the evader's control in case 1. Regions A, B, C are the validity domains of switching structures A, B and C, respectively. In region I the evader would dive below sea level. In region II maximum thrust is not optimal for the evader. P(t = 0) indicates the initial state of the pursuer ($h_P(0) = 5$ km, $V_P(0) = 250$ m/s).

The whole set of MPBVP's is divided in suitable continuation series. Within each series the computational process is accelerated by generating start values extrapolated from already obtained solutions. Due to the high accuracy of the solutions this method causes an enormous speedup.

Figure 2 shows the the influence of the q_{max}–constraint on the game solution. The trajectories are an example for case 1. The dashed flight paths result on neglecting the constraint, leading to the typical dive of the evader ($q_E(t_f) \approx 150$ kN/m^2 , $q_{max,E} = 80$ kN/m^2). The constraint prevents the dive and extends the flight time considerably ($t_f \approx 33$ sec $\longrightarrow$ $t_f \approx 86$ sec). Of course, the firing range $x_E(0)$ increases (from about 7.2 km to about 11.0 km) when the evader is constrained by his q_{max}–boundary. The pursuer uses the longer duration for a climb up to about 18.5 km starting with $\gamma_P(0) > 60^0$. The small drag at this altitude brings him a distinct gain of range. This "lofting" is typical for all examples, where t_f is not too small. However, the flight path requires a missile with an all aspect seeker. Note that the "look angle" $\omega_P := \gamma_P - \theta$ (for θ see Equ. (15)) increases from -80^0 to 0^0 within a few seconds at the end of the trajectory. This is similar for all solutions obtained. In Fig. 3 the trajectories are depicted in the altitude/velocity diagram of the evader. The constrained subarc is a small portion of the whole flight path only – a typical feature of all solutions which do not start too close to the dynamic pressure boundary. Nevertheless, the constraint is essential for the character of the solution.

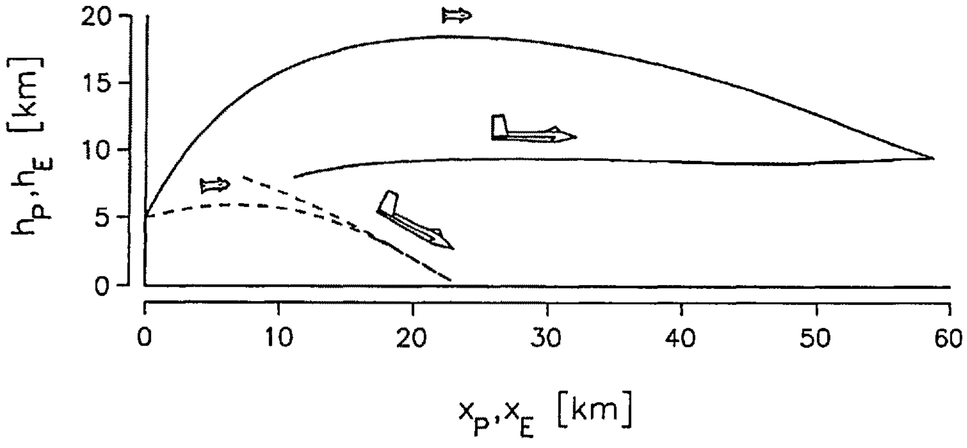

Figure 2: Barrier trajectory for case 1 in the (x, h)–plane. The picture shows the difference between the solutions with and without the dynamic pressure constraint for the evader (solid and dashed lines, respectively). $h_P(0) = 5$ km, $V_P(0) = 250$ m/s, $h_E(0) = 8$ km, $V_E(0) = 500$ m/s.

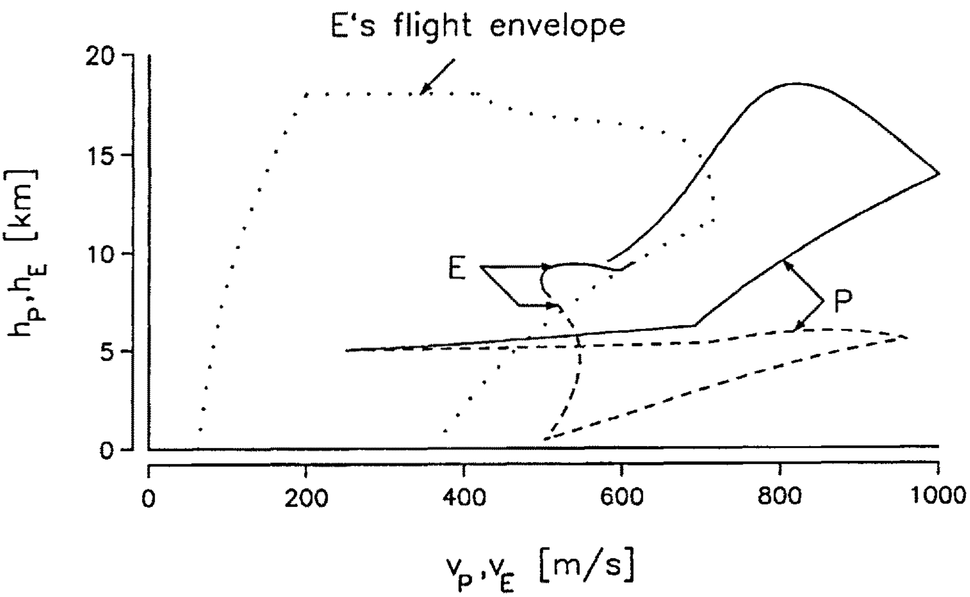

Figure 3: Barrier trajectory for case 1 in the (h, V)–plane. The picture shows the difference between the solutions with and without the dynamic pressure constraint for the evader (solid and dashed lines, respectively). $h_P(0) = 5$ km, $V_P(0) = 250$ m/s, $h_E(0) = 8$ km, $V_E(0) = 500$ m/s.

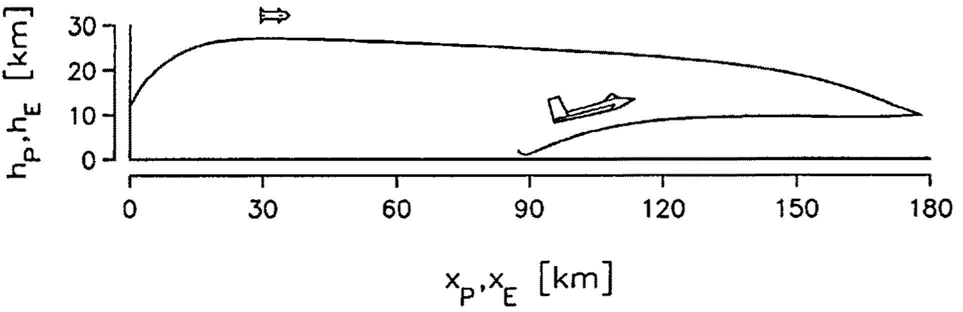

Figure 4: Barrier trajectory for case 2 in the (x, h)–plane.
$h_E(0) = 2$ km, $V_E(0) = 100$ m/s, $h_P(0) = 12$ km, $V_P(0) = 400$ m/s.

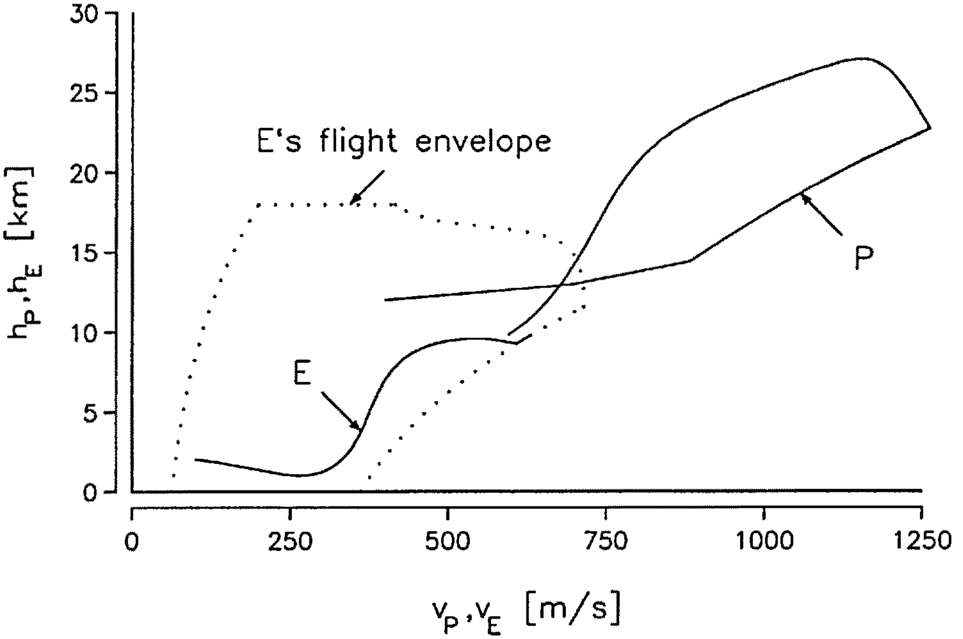

Figure 5: Barrier trajectory for case 2 in the (h, V)–plane.
$h_E(0) = 2$ km, $V_E(0) = 100$ m/s, $h_P(0) = 12$ km, $V_P(0) = 400$ m/s.

Figs. 4 and 5 show an example of case 2. The initial energy of the missile is much higher than in case 1 leading to a larger firing range ($x_E(0) \approx 87$ km) and a longer duration ($t_f \approx 201$ sec). The tendency of the missile to gain altitude is even more marked than in the first example. The maximum value of h_P is about 27 km. In this case, however, the model reduction probably has a falsifying effect on the solution. γ_E changes about 90^0 in the first 18 sec, γ_P alters about 60^0 in 30 sec. The induced drag in these portions is not accounted for. The flight along the q_{max}–constraint takes less than 9% of the flight time. One has the impression that the evader really wants to avoid the constraint as long as possible. Nevertheless, it is decisive for the solution. In spite of the low initial energy of the aircraft a dive below sea level is prevented. Altogether the evader's trajectory reminds one to maximum range extremals in former studies.

7. Conclusions

This paper presents barrier trajectories of a pursuit–evasion game formulation of a missile/target encounter. Two practically important observations are made:

- For sufficiently long flight time the missile climbs up to 20 km altitude or more to exploit the small drag for enlarging its range. Accordingly, the optimal flight path angle at launch time is extremely high (up to 70^0).
- The optimal target evasion maneuver generally ends on the dynamic pressure constraint, which prevents the aircraft from diving.

The multiple shooting algorithm turns out to be a useful tool for solving differential games:

- For the solution of (one sided) optimal control problems there are competing nonlinear programming algorithms. This is not true in the area of differential games.
- The solutions of the typically ill–conditioned multipoint boundary value problems are highly accurate and provide full information about the "switching structure" of the optimal control.
- The multiple shooting technique and the associated software package is an excellent tool to generate a series of solutions for continuously varying boundary conditions ("continuation" or "homotopy") and to demarcate the validity domains of switching structures.

The reduced vehicle model (flight path angle as control) permits a two dimensional tabular representation of the maximum firing range for each initial state of the missile. This property could be preserved in the complete model (flight path angle as state variable) if initial elevation was treated as a free parameter also subject to optimization.

Results are presented for a missile, which is guided optimally in the sense of differential game theory. To quantify the loss of firing range for a conventionally guided missile would be an interesting task.

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