

MATH 304

Linear algebra

**Lecture 37:**  
**Rotations in space.**

## Orthogonal matrices

*Definition.* An  $n \times n$  matrix  $A$  is called **orthogonal** if  $AA^T = A^T A = I$ , i.e.,  $A^T = A^{-1}$ .

*Example.*  $A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ .

- $A_\phi$  is orthogonal.
- Eigenvalues:  $\lambda_1 = \cos \phi + i \sin \phi = e^{i\phi}$ ,  
 $\lambda_2 = \cos \phi - i \sin \phi = e^{-i\phi}$ .
- $|\lambda_1| = |\lambda_2| = 1$ ,  $\overline{\lambda_1} = \lambda_2$ .
- Associated eigenvectors:  $\mathbf{v}_1 = (1, -i)$ ,  
 $\mathbf{v}_2 = (1, i)$ .
- Vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  form a basis for  $\mathbb{C}^2$ .

Consider a linear operator  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $L(\mathbf{x}) = A\mathbf{x}$ , where  $A$  is an  $n \times n$  orthogonal matrix.

**Theorem** There exists an orthonormal basis for  $\mathbb{R}^n$  such that the matrix of  $L$  relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where  $D_{\pm 1}$  is a diagonal matrix whose diagonal entries are equal to 1 or  $-1$ , and

$$R_j = \begin{pmatrix} \cos \phi_j & -\sin \phi_j \\ \sin \phi_j & \cos \phi_j \end{pmatrix}, \quad \phi_j \in \mathbb{R}.$$

## Why are orthogonal matrices called so?

**Theorem** Given an  $n \times n$  matrix  $A$ , the following conditions are equivalent:

- (i)  $A$  is orthogonal:  $A^T = A^{-1}$ ;
- (ii) columns of  $A$  form an orthonormal basis for  $\mathbb{R}^n$ ;
- (iii) rows of  $A$  form an orthonormal basis for  $\mathbb{R}^n$ .

*Proof:* Entries of the matrix  $A^T A$  are dot products of columns of  $A$ . Entries of  $AA^T$  are dot products of rows of  $A$ .

Thus an orthogonal matrix is the transition matrix from one orthonormal basis to another.

Consider a linear operator  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $L(\mathbf{x}) = A\mathbf{x}$ , where  $A$  is an  $n \times n$  matrix.

**Theorem** The following conditions are equivalent:

- (i)  $|L(\mathbf{x})| = |\mathbf{x}|$  for all  $\mathbf{x} \in \mathbb{R}^n$ ;
- (ii)  $L(\mathbf{x}) \cdot L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$  for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ;
- (iii) the matrix  $A$  is orthogonal.

*Definition.* A transformation  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called an **isometry** if it preserves distances between points:  $|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$ .

**Theorem** Any isometry  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  can be represented as  $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$ , where  $\mathbf{x}_0 \in \mathbb{R}^n$  and  $A$  is an orthogonal matrix.

*Classification of  $3 \times 3$  orthogonal matrices:*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

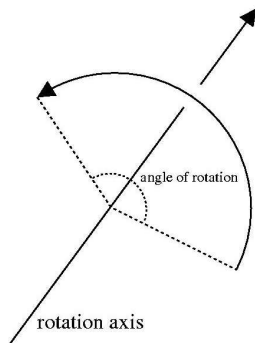
$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}.$$

$A$  = rotation about a line;  $B$  = reflection in a plane;  $C$  = rotation about a line combined with reflection in the orthogonal plane.

$$\det A = 1, \quad \det B = \det C = -1.$$

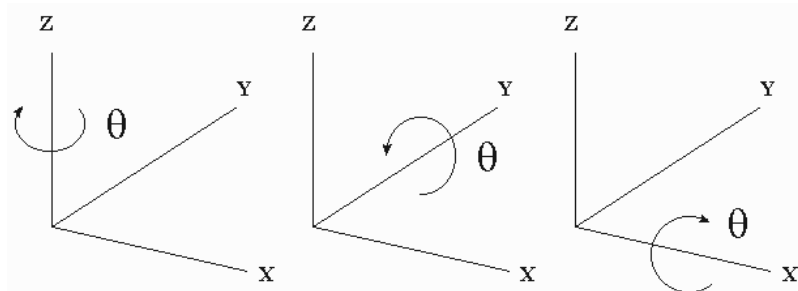
$A$  has eigenvalues  $1, e^{i\phi}, e^{-i\phi}$ .  $B$  has eigenvalues  $-1, 1, 1$ .  $C$  has eigenvalues  $-1, e^{i\phi}, e^{-i\phi}$ .

## Rotations in space



If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

## Clockwise rotations about coordinate axes



$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$



**Problem.** Find the matrix of the rotation by  $90^\circ$  about the line spanned by the vector  $\mathbf{c} = (1, 2, 2)$ . The rotation is assumed to be counterclockwise when looking from the tip of  $\mathbf{c}$ .

$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is the matrix of (counterclockwise) rotation by  $90^\circ$  about the z-axis.

We need to find an orthonormal basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  such that  $\mathbf{v}_3$  has the same direction as  $\mathbf{c}$ . Also, the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  should obey the same hand rule as the standard basis. Then  $B$  is the matrix of the given rotation relative to the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

Let  $U$  denote the transition matrix from the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to the standard basis (columns of  $U$  are vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ). Then the desired matrix is  $A = UBU^{-1}$ .

Since  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is going to be an orthonormal basis, the matrix  $U$  will be orthogonal. Then  $U^{-1} = U^T$  and  $A = UBU^T$ .

*Remark.* The basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  obeys the same hand rule as the standard basis if and only if  $\det U > 0$ .

*Hint.* Vectors  $\mathbf{a} = (-2, -1, 2)$ ,  $\mathbf{b} = (2, -2, 1)$ , and  $\mathbf{c} = (1, 2, 2)$  are orthogonal.

We have  $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$ , hence  $\mathbf{v}_1 = \frac{1}{3}\mathbf{a}$ ,  $\mathbf{v}_2 = \frac{1}{3}\mathbf{b}$ ,  $\mathbf{v}_3 = \frac{1}{3}\mathbf{c}$  is an orthonormal basis.

Transition matrix:  $U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}.$

$$\det U = \frac{1}{27} \begin{vmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

(In the case  $\det U = -1$ , we should interchange vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .)

$$A = UBU^T$$

$$= \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}.$$

$U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$  is an orthogonal matrix.

$\det U = 1 \implies U$  is a rotation matrix.

**Problem.** (a) Find the axis of the rotation.  
(b) Find the angle of the rotation.

The axis is the set of points  $\mathbf{x} \in \mathbb{R}^n$  such that  $U\mathbf{x} = \mathbf{x} \iff (U - I)\mathbf{x} = \mathbf{0}$ . To find the axis, we apply row reduction to the matrix  $3(U - I)$ :

$$3U - 3I = \begin{pmatrix} -5 & 2 & 1 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 3 & 0 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned}
 &\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & -6 & 2 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \\
 &\begin{pmatrix} 1 & -1 & 0 \\ 0 & -6 & 2 \\ 0 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\
 &\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$\text{Thus } U\mathbf{x} = \mathbf{x} \iff \begin{cases} x - z/3 = 0 \\ y - z/3 = 0 \end{cases}$$

The general solution is  $x = y = t/3$ ,  $z = t$ ,  $t \in \mathbb{R}$ .

$\implies \mathbf{d} = (1, 1, 3)$  is the direction of the axis.

$$U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

Let  $\phi$  be the angle of rotation. Then the eigenvalues of  $U$  are 1,  $e^{i\phi}$ , and  $e^{-i\phi}$ . Therefore

$$\det(U - \lambda I) = (1 - \lambda)(e^{i\phi} - \lambda)(e^{-i\phi} - \lambda).$$

Besides,  $\det(U - \lambda I) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$ , where  $c_1 = \text{tr } U$  (the sum of diagonal entries).

It follows that

$$\text{tr } U = 1 + e^{i\phi} + e^{-i\phi} = 1 + 2\cos\phi.$$

$$\text{tr } U = -2/3 \implies \cos\phi = -5/6 \implies \phi \approx 146.44^\circ$$