Numerical Integration:
Composite Simpson's Rule

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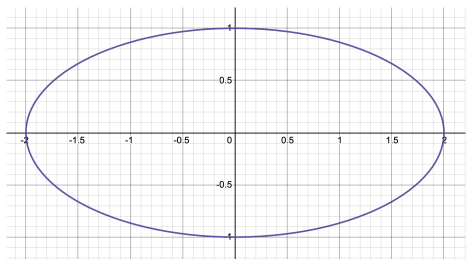
1 Introduction

Let's find the arc length of an ellipse. When learning how to integrate with polar coordinates, you will probably come across the integration needed to find the circumference of a circle. This is given by: $\int_0^{2\pi} r\,d\theta = 2\pi r$, where r is the radius of the circle. You may think that finding the arc length of an ellipse would be this simple, however, it turns out that finding an analytical solution to this problem is quite difficult. Finding the arc length of an ellipse is not just a geometrical problem. Ellipses can be found in various physics and engineering applications such as elliptical orbits or construction. Suppose we want to find the arc length of the following ellipse: $\frac{x^2}{4} + y^2 = 1$.



First, let's change this equation into parametric form:

$$\frac{x^2}{4} + y^2 = 1 \Rightarrow x = 2\cos(t), y = \sin(t), t \in [0, 2\pi]$$

Let L be the arc length of our ellipse, this is given by:

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt \ [1]$$

$$\text{where } x' = -2\sin(t), y' = \cos(t)$$

So we get:

$$L = \int_0^{2\pi} \sqrt{4\sin^2(t) + \cos^2(t)}\,dt$$

If you try to analytically evaluate this integral, you will quickly find that it is nearly impossible. Thus, numerical integration is key to finding a solution. Let's use composite Simpson's rule to find the length of this ellipse with an accuracy of 10^{-9}.

2 Methodology

According to section 4.4 in Numerical Analysis, *"The Newton-Cotes formulas are generally unsuitable for use over large integration intervals."* Also, with regards to composite numerical integration, it is said that *"These are the techniques most often applied"*[2]. Hence, the application of composite Simpson's rule for this problem.

2.1 Composite Simpson's Rule

First, lets look at the closed Newton-Cotes version of Simpson's rule:

$$\int_{x_0}^{x_2} f(x)\,dx \approx \tfrac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \tfrac{h^5}{90} f^{(4)}(\xi), \text{ where } x_0 < \xi < x_2. \text{ [3]}$$

Here, $h = \frac{x_2 - x_0}{n}$, where n is the number of subintervals, so $n = 2$. What if we wanted more subintervals for a better approximation? This is where the composite Simpson's rule comes from:

$$\int_a^b f(x)\,dx \approx \tfrac{h}{3}[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)] - \frac{b-a}{180} h^4 f^{(4)}(\mu).$$

Where the last term is the error function [4]. Notice that the upper limits of the summations in the equation above involve an $n/2$ term. This implies that there must be an even number of subintervals, n.

2.2 Accuracy/Error

We need to find h, n such that we reach our desired accuracy of 10^{-9}. To do this, we can refer to the error term in our equation above. First, let's find h. Consider the following:

$$\tfrac{2\pi}{180} h^4 \left[\max_{\mu \in [0,2\pi]} |f^{(4)}(\mu)|\right] < 10^{-9}$$

$$f^{(4)}(t) = \frac{336\sin^8(t) + 816\cos^2(t)\sin^6(t) + 936\cos^4(t)\sin^4(t) + 336\cos^6(t)\sin^2(t) - 39\cos^8(t)}{(4\sin^2(t) + \cos^2(t))^{\frac{7}{2}}}$$

Using MATLAB, we find that: $\max_{\mu \in [0,2\pi]} |f^{(4)}(\mu)| = 39$, which gives us:

$$\tfrac{2\pi}{180} h^4 \,(39) < 10^{-9}$$

Solving for h, we get:

$$h < \left(\tfrac{90 * 10^{-9}}{39\pi}\right)^{\frac{1}{4}}$$

Recall that $h = \frac{b-a}{n}$. In our case, we get $h = \frac{2\pi}{n}$. To solve for n, we can substitute our inequality from above. Observe the following:

$$\frac{2\pi}{n} < \left(\frac{90*10^{-9}}{39\pi}\right)^{\frac{1}{4}}$$

$$\Rightarrow n > 2\pi\left(\frac{39\pi}{90*10^{-9}}\right)^{\frac{1}{4}}$$

$$\Rightarrow n > 1206.9$$

Since we need an even number of subintervals, we let $n = 1208$, to achieve an accuracy within 10^{-9}.

3 Results

Using $n = 1208$ subintervals, we find that the arc length of the ellipse, $\frac{x^2}{4} + y^2 = 1$, is $\boxed{9.688448220}$, which is accurate within 10^{-9}. Compare this result with the closed Newton-Cotes formula where $n = 2$. In this case, our result is 6.283185307, so we can see that using more subintervals can be crucial in finding an accurate answer.

4 Conclusion

Using numerical integration is useful when finding an analytical solution is not ideal. On most computers, this numerical method can find an accurate solution with high efficiency. The MATLAB code developed for this project can be found in the Appendix section or on GitHub.

References

[1] Rogawski J., Adams C. 2015, Calculus: Early Transcendentals 3E, 590

[2] Burden R.L., Faires D.J.,& Burden A.M. 2018, Numerical Analysis 10E, 202

[3] Burden R.L., Faires D.J.,& Burden A.M. 2018, Numerical Analysis 10E, 197

[4] Burden R.L., Faires D.J.,& Burden A.M. 2018, Numerical Analysis 10E, 204

[5] Burden R.L., Faires D.J.,& Burden A.M. 2018, Numerical Analysis 10E, 205

Appendix: MATLAB implementation

```
% Algorithm 4.1 - Composite Simpson's Rule, [5]
format long
f=@(x) sqrt(4*sin(x)^2+cos(x)^2); % Function to integrate
a=0; % Lower limit
b=2*pi;  % Upper limit
n=1208; % Subintervals, must be an even positive integer.

h=(b-a)/n;
XI0=f(a)+f(b);
XI1=0;
XI2=0;

for i=1:n-1
    X=a+i*h;
    if mod(i,2)==0
        XI2=XI2+f(X);
    else
        XI1=XI1+f(X);
    end
end

XI=h*(XI0+2*XI2+4*XI1)/3;
solution = XI
```

```
%Find the max value of a function, within an interval.
format rat
x1 = 0;
x2 = 2*pi;
f= @(x) abs((336*sin(x)^8+816*cos(x)^2*sin(x)^6+936*cos(x)^4*sin(x)^4 ...
+336*cos(x)^6*sin(x)^2-39*cos(x)^8)/(4*sin(x)^2+cos(x)^2)^(7/2)); %This is the function in which we want to find the max
[x,max] = fminbnd(@(x) -f(x),x1,x2);
-max % Here is the max of the function
x % Here is the location at which the max occurs
```