3 Img in 2D Signals

Wednesday, December 4, 2024 3:00 PM

Impulse

$\delta(n_1,n_2) = \begin{cases} 1 & (n_1,n_2) = (0,0) \\ 0 & (n_1,n_2) \neq (0,0) \end{cases}$

2D signal $x_1 = x_2$ of shifted impulses

$x(n_1,n_2) = 3\delta(n_1-1,n_2-2)$

$\rightarrow$ 3 at $(1,2)$ (shifted 1 up, shift right 2, + take the 3)

$\rightarrow$ 0 elsewhere

Delta Seq Signal

Step Function: Unit δ impulse

$u(n_1,n_2) = \begin{cases} 1 & n_1 \geq 0, n_2 \geq 0 \\ 0 & \text{else} \end{cases}$ $\rightarrow$ 1 in 1st quadrant

$u(n_1,n_2) \neq \delta(n_1,n_2)$

Separable: $u(n_1,n_2) = u(n_1)u(n_2)$

$= \delta(n_1)\delta(n_2)$

$\rightarrow$ separable and shift invariant functions

Separable: $\delta(n_1,n_2) = \delta(n_1)\delta(n_2)$

$\delta(n_1,n_2) \neq \delta(n_1)\delta(n_2)$

$= \delta(n_1-1)\delta(n_2-2)$

$\rightarrow$ $\delta(n_1,n_2)$ is separable, $\delta(n_1-1)\delta(n_2-2)$ is not separable

Periodic Sequences

$\tilde{x}(n_1,n_2)$ is periodic if $\tilde{x}(n_1,n_2) = \tilde{x}(n_1+N_1, n_2+N_2)$

$\rightarrow$ $x(n_1,n_2) = \tilde{x}(n_1,n_2)$ for all (n_1,n_2)

$\rightarrow$ $x(n_1,n_2) \neq \tilde{x}(n_1,n_2)$

LSI (Linear Shift Invariant Systems)

any processing $\rightarrow$ signal output (LSI) system

Convolution

$y(n_1,n_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\,h(n_1-k_1, n_2-k_2)$

$= x(n_1,n_2) * h(n_1,n_2)$

- Input $x(n_1,n_2)$
- Impulse response $h(n_1,n_2)$
- Output $y(n_1,n_2)$

Separable Impulse Response

$h(n_1,n_2) = h(n_1)h(n_2)$

$\rightarrow$ Output 2D convolution, take the 1D convolution

$\rightarrow$ Then output, take the output, output convolution

$\rightarrow$ Row output, column output (shift invariant)

$y(n_1,n_2) = x(n_1,n_2) * h(n_1,n_2)$

$= x(n_1,n_2) * h(n_1)h(n_2)$

$= x(n_1,n_2) * h(n_1) * h(n_2)$

Row output, column output, row output, column output, row output, column output (shift invariant) $\rightarrow$ same output

Basic Properties

Commutativity

$x(n_1,n_2) * y(n_1,n_2) = y(n_1,n_2) * x(n_1,n_2)$

Associativity

$[x(n_1,n_2) * y(n_1,n_2)] * z(n_1,n_2) = x(n_1,n_2) * [y(n_1,n_2) * z(n_1,n_2)]$

$\rightarrow$ order doesn't matter

Distributivity

$x * (y+z) = (x*y) + (x*z)$

Conv w/ Shifted Impulse

$x(n_1,n_2) * \delta(n_1-n_1, n_2-n_2) = x(n_1-n_1, n_2-n_2)$

- Conv w/ shifted impulse $\rightarrow$ shifts signal
- Properties are important $\rightarrow$ $y(n) = x(n)$

Conv Examples

if $x(n_1,n_2) \rightarrow$ $N \times N$ $\rightarrow$ input sig 2D signal

if $h(n_1,n_2) \rightarrow$ $M \times M$ $\rightarrow$ impulse response (filter)

$y = x(n_1,n_2) * h(n_1,n_2)$

output size $= (N+M-1)$

labelled from $0 \rightarrow$ $\rightarrow$ for range: $[0, N+M-2]$

2D Fourier Transform

Analysis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(n_1,n_2)\, e^{-j2\pi(un_1+vn_2)}\, dn_1 dn_2$$

$f(n_1,n_2)$ $\rightarrow$ continuous 2D func

$F(u,v)$ $\rightarrow$ 2D freq domain

u,v $\rightarrow$ frequencies (radians) $\rightarrow$ $-\pi, \pi$

$e^{-j2\pi(un_1+vn_2)}$ $\rightarrow$ 2D complex sinusoid (basis)

Synthesis

$$f(n_1,n_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(u,v)\, e^{j2\pi(un_1+vn_2)}\, du\, dv$$

Separable (1D x 1D, 1D x 1D)

$$F(u,v) = \int\left[\int f(n_1,n_2)\, e^{-j2\pi un_1} dn_1\right] e^{-j2\pi vn_2} dn_2$$

$\rightarrow$ Inner: horizontal (row)

$\rightarrow$ Outer: vertical (col)

Synthesis (1D x 1D, 1D x 1D)

$$f(n_1,n_2) = \int\left[\int F(u,v)\, e^{j2\pi un_1} du\right] e^{j2\pi vn_2} dv$$

$\rightarrow$ Inner: vertical (col)

$\rightarrow$ Outer: horizontal (row)

$\rightarrow$ u = row, v = column

$\rightarrow$ u = horizontal, v = vertical

$\rightarrow$ $f(n_1,n_2)$ is separable or $F(u,v)$ is separable

$f(n_1,n_2) = f_1(n_1)f_2(n_2)$ $\rightarrow$ $F(u,v) = F_1(u)F_2(v)$

$F(u,v) = F_1(u)F_2(v)$ $\rightarrow$ $f(n_1,n_2) = f_1(n_1)f_2(n_2)$

for freq

$F(u,v) = \int F_1(u)\left[\int F_2(v)\, e^{-j2\pi vn_2} dv\right] e^{-j2\pi un_1} du$

$\rightarrow$ $F_1(u)$ $\rightarrow$ $f_1(n_1)$

$\rightarrow$ $F_2(v)$ $\rightarrow$ $f_2(n_2)$

$\rightarrow$ $F_1(u)F_2(v)$ $\rightarrow$ $f_1(n_1)f_2(n_2)$

2D Fourier Basis Func

for 1D: $\{e^{j2\pi nk}, n \in \mathbb{Z}\}$

for 2D: $\{e^{j2\pi(un_1+vn_2)}, n \in \mathbb{Z}\}$

$\rightarrow$ u = horizontal freq

$\rightarrow$ v = vertical freq

$\rightarrow$ n_1 = horizontal pixel

$\rightarrow$ n_2 = vertical pixel

$\rightarrow$ $F(u,v)$ = coefficient for $e^{j2\pi(un_1+vn_2)}$ basis function

2D Separable FT: Basis signal is separable

$F(u,v) = \frac{1}{\sqrt{2\pi}}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} f(n_1,n_2)\, e^{-j2\pi(un_1+vn_2)}$

FT(u,v): Fourier domain coefficients (how much of each 2D basis is present)

$f(n_1,n_2)$: discrete image samples (spatial domain)

n_1, n_2: pixel indices

N: image dimension

$\frac{1}{\sqrt{2\pi}}$: normalization factor

$F(u,v) = \frac{1}{\sqrt{2\pi}}\sum_{n_1=0}^{N-1}\left[\sum_{n_2=0}^{N-1} f(n_1,n_2)\, e^{-j2\pi vn_2}\right] e^{-j2\pi un_1}$

Separable 2D DFT

$F(u,v) = \underbrace{\frac{1}{\sqrt{2\pi}}\sum_{n_1=0}^{N-1}\left[\frac{1}{\sqrt{2\pi}}\sum_{n_2=0}^{N-1} f(n_1,n_2)\, e^{-j2\pi vn_2}\right] e^{-j2\pi un_1}}_{\text{Fourier domain coefficients (frequency domain)}}$

$\rightarrow$ Inner: vertical (col) step

$\rightarrow$ Outer: horizontal (row) step

2D DFT Step 1

$F = (WF)W^T$

$\rightarrow$ F = 2D DFT

$\rightarrow$ W = 1D DFT matrix

$\rightarrow$ F = 2D DFT matrix

$\rightarrow$ W^T = transpose of 1D DFT matrix

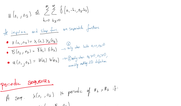
$F = (WF)W$

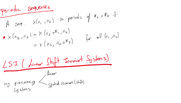
Transform Examples

* Bright center $\rightarrow$ low freq $\rightarrow$ big smooth regions
* Bright thin lines $\rightarrow$ strong edges in 1D
  - diagonal edges produce strong vertical and horizontal components

1) Smooth (blurred image) $\rightarrow$ energy in center (low freq)
2) Sharp edges $\rightarrow$ energy spread out (high freq)
3) Checkerboard $\rightarrow$ strong diagonal pattern (high freq in both directions)

DFT: Sin + Filters





$$h(n_1,n_2) = h_1(n_1)\,h_2(n_2)$$

$$H(\omega_1,\omega_2) = H_1(\omega_1)\,H_2(\omega_2)$$

$$= \frac{\sin\omega_1 n_1}{\pi n_1}\cdot\frac{\sin\omega_2 n_2}{\pi n_2}$$

Low-Pass of sig

* Aliasing high freqs
* After low pass, signal is mostly in center
  - low freqs dominate image

Hi-Pass Filter

1) Start w/ identity filter (everything is unchanged)
2) Subtract LP filter

$h_{HP}(n_1,n_2) = \delta(n_1,n_2) - h_{LP}(n_1,n_2)$

$H_{HP}(\omega_1,\omega_2) = 1 - H_{LP}(\omega_1,\omega_2)$

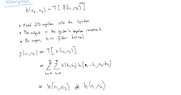
Band-Pass

$H_{BP} = H_{LP} - H_{LP}$

Circular Filter

Circular $\rightarrow$ Cross $\rightarrow$ Low Pass

$\rightarrow$ Ideal



Ideal Simply Thresh Filter

Blurring: If input is sampled low enough ($\frac{\omega_s}{2}$)

1) Low pass cutoff, upper sampling rate

2) Low pass cutoff, lower sampling rate

$\rightarrow$ Low pass cutoff, lower sampling rate

Aliasing: If sampled at original cutoff or low

1) Low pass cutoff

2) Upper sampling rate

$\rightarrow$ Low pass cutoff, upper sampling rate

Upscaling: Interpolation

1) Zero stuffing

2) Low pass cutoff

$\rightarrow$ Zero stuffing, low pass cutoff, low pass cutoff

1) Zero stuffing $\rightarrow$ low pass cutoff

2) Low pass cutoff

$\rightarrow$ Zero stuffing, low pass cutoff, low pass cutoff

$\rightarrow$ Zero stuffing, low pass cutoff, low pass cutoff

Downsampling

1) Low pass cutoff $\rightarrow$ downsample

2) Low pass cutoff, downsample

$\rightarrow$ Low pass cutoff, downsample, low pass cutoff

$\rightarrow$ Low pass cutoff, downsample, low pass cutoff

$\rightarrow$ Low pass cutoff, downsample, low pass cutoff

2) Only conv (6-5)

1) Cutoff, low pass cutoff

2) Conv downsampling

3) Low pass cutoff, low pass cutoff, low pass cutoff

$\rightarrow$ Low pass cutoff, low pass cutoff, low pass cutoff

$\rightarrow$ Low pass cutoff, low pass cutoff, low pass cutoff

$\rightarrow$ Low pass cutoff, low pass cutoff, low pass cutoff