

# Video Signals

## 3. Images as 2D Signals & Systems

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### 1 Fundamentals of 2D Signals

#### 1.1 Basic Signals and Sequences

An image can be modeled as a 2D signal  $f(x, y)$ . We define several fundamental discrete sequences used in analysis:

- **Unit Impulse Function ( $\delta[n]$ ):** The most basic signal, non-zero only at the origin.

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} \quad (1)$$

- **Unit Step Function ( $u[n]$ ):** Represents a signal that is "on" for positive indices.

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \quad (2)$$

- **Periodic Sequences:** A sequence  $x(n)$  is periodic with period  $N$  if it repeats itself indefinitely:

$$x(n) = x(n + N) \quad \text{for all } n \quad (3)$$

- **Image Function Range:** A labeled image function typically maps spatial coordinates to an intensity range, e.g.,  $[0, L - 1]$  (where  $L = 256$  for 8-bit images).

### 2 Linear Shift Invariant (LSI) Systems

An LSI system is the foundation of classical image processing (filtering). It is characterized by two main properties:

#### 2.1 System Properties

1. **Linearity (Superposition):** If an input is the sum of two signals, the output is the sum of their individual responses.

$$T\{a \cdot x_1 + b \cdot x_2\} = a \cdot T\{x_1\} + b \cdot T\{x_2\} \quad (4)$$

2. **Shift Invariance (Spatial Invariance):** A shift in the input signal coordinates results in an identical shift in the output signal. The system behavior does not change depending on *where* in the image we are processing.

$$x(n - n_0) \longrightarrow y(n - n_0) \quad (5)$$

#### 2.2 Impulse Response and Convolution

Any LSI system is completely characterized by its **Impulse Response**, denoted as  $h(n)$ .

- **Definition:**  $h(n)$  is the output of the system when the input is a unit impulse  $\delta(n)$ .
- **Convolution Sum:** The output  $y(n)$  for any arbitrary input  $x(n)$  is calculated by convolving  $x$  with  $h$ .

## 1D Convolution

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (6)$$

*Key Insight:* Convolution involves flipping the filter  $h$  and sliding it across the input signal  $x$ .

## 2D Convolution (Image Filtering)

For images, we extend this to two dimensions. The pixel value at  $(n_1, n_2)$  depends on the neighborhood weighted by the filter  $h$ .

$$y(n_1, n_2) = \sum_{k_1} \sum_{k_2} x(k_1, k_2)h(n_1 - k_1, n_2 - k_2) \quad (7)$$

## 2.3 Properties of Convolution

- **Commutativity:**  $x * h = h * x$  (Order does not matter).
- **Shift Property:** Convolution with a shifted impulse shifts the entire signal.
- **Output Size Calculation:** If input  $x$  has length  $N$  and filter  $h$  has length  $M$ , the resulting valid convolution output has a size of:

$$\text{Size} = N + M - 1 \quad (8)$$

## 3 Fourier Transform (DFT) & Frequency Analysis

The Discrete Fourier Transform (DFT) allows us to analyze the image in the frequency domain, revealing patterns of change (smoothness vs. edges).

### 3.1 DFT Formulas

- **Analysis Equation (Forward DFT):** Converts spatial data  $f(x, y)$  into frequency coefficients  $F(u, v)$ .

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})} \quad (9)$$

- **Synthesis Equation (Inverse DFT):** Reconstructs the image from frequency coefficients.

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(\frac{ux}{M} + \frac{vy}{N})} \quad (10)$$

### 3.2 Visual Interpretation of the Spectrum

When we visualize the magnitude of the Fourier Transform  $|F(u, v)|$ :

- **DC Component (Center):** Corresponds to zero frequency ( $u = 0, v = 0$ ). It represents the average brightness of the image.
- **Low Frequencies (Near Center):** Represent smooth regions, gradual transitions, and "bland" areas. Most of an image's energy is usually concentrated here.
- **High Frequencies (Edges):** Represent sharp details, edges, and noise. These values are typically smaller in magnitude and located further from the center.

## 4 Filtering in the Frequency Domain

**The Convolution Theorem:** Convolution in the spatial domain is equivalent to point-wise multiplication in the frequency domain. This is computationally efficient for large filters.

$$y(n) = x(n) * h(n) \longleftrightarrow Y(\omega) = X(\omega)H(\omega) \quad (11)$$

## 4.1 Separable Filters

A 2D filter  $h(n_1, n_2)$  is **separable** if it can be decomposed into two 1D filters (vertical and horizontal).

$$h(n_1, n_2) = h_1(n_1) \cdot h_2(n_2) \implies H(\omega_1, \omega_2) = H_1(\omega_1) \cdot H_2(\omega_2) \quad (12)$$

*Benefit:* Allows filtering rows first, then columns, reducing computational complexity.

## 4.2 Types of Filters

### 4.2.1 Low-Pass Filter (Smoothing)

- **Function:** Attenuates (removes) high frequencies while passing low frequencies.
- **Effect:** Blurs the image, reduces noise, and smooths edges.
- **Ideal Low-Pass:** In the frequency domain, this is a box function. In the spatial domain, its impulse response is the **Sinc function**:

$$\text{sinc}(n) = \frac{\sin(\pi n)}{\pi n} \quad (13)$$

- *Note:* After low-pass filtering, the energy spectrum is concentrated at the center.

### 4.2.2 High-Pass Filter (Edge Detection)

- **Function:** Attenuates low frequencies; passes high frequencies.
- **Effect:** Enhances edges and fine details; removes slowly varying brightness (like shadows).
- **Construction:** A high-pass filter can be created by subtracting a low-pass filter from an "Identity" (All-Pass) filter:

$$\text{HighPass} = \text{Identity} - \text{LowPass} \quad (14)$$

### 4.2.3 Band-Pass Filter

- Retains frequencies within a specific range (band). Useful for texture analysis.

## 5 Sampling and Reconstruction

To convert a continuous analog image into a digital signal, we must sample it. This process is governed by the Sampling Theorem.

### 5.1 Nyquist-Shannon Sampling Theorem

For a continuous signal with a maximum frequency bandwidth of  $B$ , perfect reconstruction is possible **if and only if** the sampling frequency  $f_s$  satisfies:

$$f_s \geq 2B \quad (15)$$

- **Nyquist Rate:** The minimum required sampling rate,  $2B$ .
- This ensures that the spectral replicas created by sampling do not overlap.

### 5.2 Aliasing

**Aliasing** occurs when the sampling condition is violated ( $f_s < 2B$ ).

- **Definition:** High-frequency components "fold over" and appear as false low-frequency components in the sampled signal.
- **Visual Effect:** Moiré patterns, jagged edges ("jaggies"), and distortion that cannot be removed after sampling.
- **Prevention:** Apply an anti-aliasing (low-pass) filter *before* sampling to limit the bandwidth to  $B < f_s/2$ .