Robust investments under risk and ambiguity

Harold Zurcher's robust replacement policy

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We incorporate techniques from distributionally robust optimization into a dynamic investment model. This allows us to explicitly account for ambiguity in the decision-making process. We outline an economic, mathematical, and computational model to study the seminal bus replacement problem (Rust, 1987) under potential model misspecification. We specify ambiguity sets for the transition dynamics of the model. These are based on empirical estimates, statistically meaningful, and computationally tractable. We analyze alternative policies in a series of computational experiments. We find that, given the structure of the model and the available data on past transitions, a policy simply ignoring model misspecification often outperforms its alternatives that are designed to explicitly account for it.

JEL Codes: D81, C44, D25

Keywords: investment decision, risk, ambiguity, robust Markov decision process

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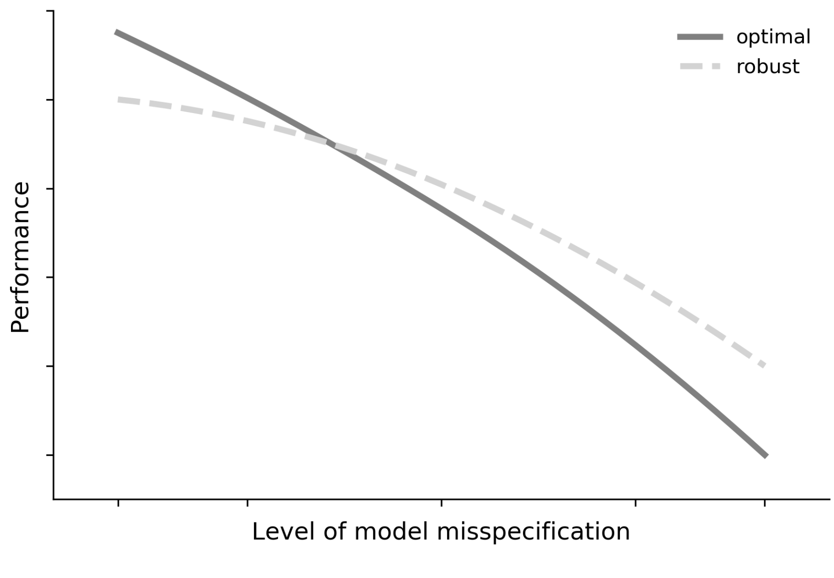
1 Introduction

Individuals face ubiquitous uncertainties when faced with important decisions. Policy makers vote for climate change mitigation efforts facing uncertainty about future costs and benefits (Barnett, Brock, & Hansen, 2019), while doctors decide on the timing of an organ transplant in light of uncertainty about future patient health (Kaufman, Schaefer, & Roberts, 2017). Economic models formalize the objectives, trade-offs, and uncertainties for such decisions. In these models, the treatment of uncertainty is often limited to risk as the model induces a unique probability distribution over sequences of possible futures. There is no role for ambiguity about the true model (Knight, 1921; Arrow, 1951) and thus no fear of model misspecification. However, limits to knowledge lead to considerable ambiguity about how the future unfolds (Hayek, 1975; Hansen, 2015).

This creates the need for robust decision rules that work well over a whole range of different models, instead of a decision rule that is optimal for one particular model. Figure 1 clarifies the notion of robust decision-making. It shows the performance of the optimal and a robust decision rule for different levels of model misspecification. The optimal decision rule is designed without any fear of misspecification, using a single model to inform decisions. It thus performs very well if that model turns out to be true. However, its performance is very sensitive and deteriorates rapidly in light of model misspecification. Robust decision rules explicitly account for such a possibility and their performance is less affected. At some point it actually outperforms the optimal decision rule.

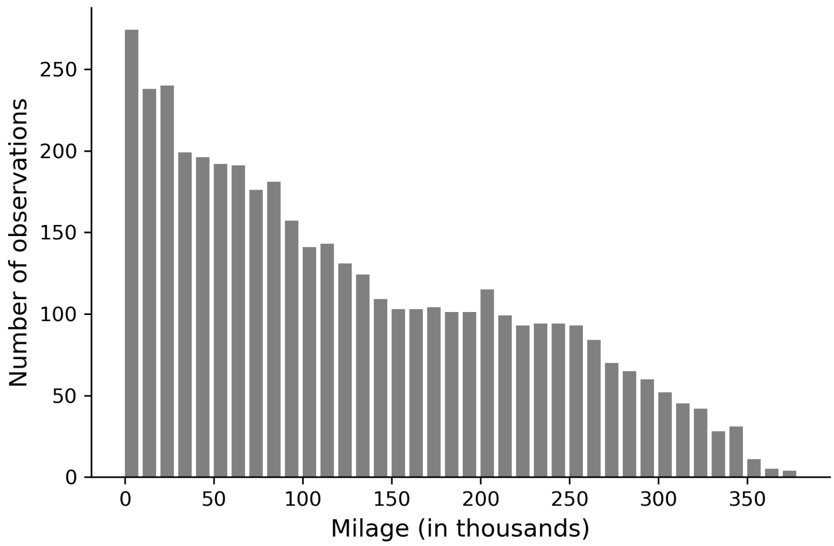
Figure 1: Models of decision-making



We revisit Rust (1987)'s seminal bus replacement problem in light of potential model misspecification.[1] In the model, a maintenance manager, Harold Zurcher, seeks to implement a maintenance plan for a fleet of buses in light of uncertainty about their future mileage utilization. He assumes it follows some exogenous distribution and uses data on past utilization to estimate it. In the standard analysis, the estimated distribution serves as a plug-in for the true distribution. Any remaining ambiguity about future mileage utilization is simply ignored. This creates an artificial separation between capturing risk by an empirical estimate and the subsequent calculation of an optimal decision based on it. Such recommendations are potentially very sensitive to model misspecification resulting in a serious degradation in performance (Michaud, 1989; Smith & Winkler, 2006; Mannor, Simester, Sun, & Tsitsiklis, 2007).

Figure 2 previews our empirical application and highlights the limited information about the true distribution of mileage utilization. It shows the number of observations that is available to estimate next month's utilization for different levels of accumulated mileage. While there are more than 1,150 observations on buses with less than 50,000 miles, there are only about 220 with more than 300,000.

Figure 2: Distributions of observations



We explicitly account for the remaining ambiguity using methods from distributionally robust optimization (Ben-Tal, El Ghaoui, & Nemirowski, 2009; Wiesemann, Kuhn, & Sim, 2014; Rahimian & Mehrotra, 2019) to construct decision rules that explicitly take potential model misspecification into account. We use the available data on past utilization to generate an ambiguity set using the Kullback-Leibler divergence (Kullback & Leibler, 1951) that is anchored

[1] This model serves as a computational sandbox in a variety of settings, e.g. Su & Judd (2012), Iskhakov, Lee, Rust, Schjerning, & Seo (2016), and Reich (2018).

in empirical estimates, statistically meaningful, and computationally tractable (Ben-Tal, den Hertog, De Waegenaere, Melenberg, & Rennen, 2013). We assume maxmin expected utility preferences (Gilboa & Schmeidler, 1989; Epstein & Schneider, 2003b) and build on recent results for robust Markov decision processes (Iyengar, 2005; Nilim & El Ghaoui, 2005; Wiesemann, Kuhn, & Rustem, 2013) to set up a robust version of the bus replacement problem.[2] We conduct numerous computational experiments to explore the differences between the optimal and robust decision rules and assess their relative performance in light of potential model misspecification.

The structure of the remaining analysis is as follows. We provide the conceptual framework for our analysis in Section 2. We clearly delineate the economic, mathematical, and computational model with a focus on challenges to account for ambiguity about the true model. Section 3 presents our analysis of the robust bus replacement problem and we explore the implications of robust decision-making in a variety of scenarios. Our final section concludes.

2 Conceptual framework

We now outline the conceptual framework for our analysis. We first present the canonical economic model of dynamic decision making under uncertainty. We then turn to its corresponding mathematical model of a (robust) Markov decision process and present the computational model for our robust bus replacement problem. We always focus on the new challenges introduced into the analysis when studying decision making under ambiguity.

In line with our empirical application, we discuss an infinite horizon model in discrete time, stationary utility and transition probabilities, and discrete states and actions.

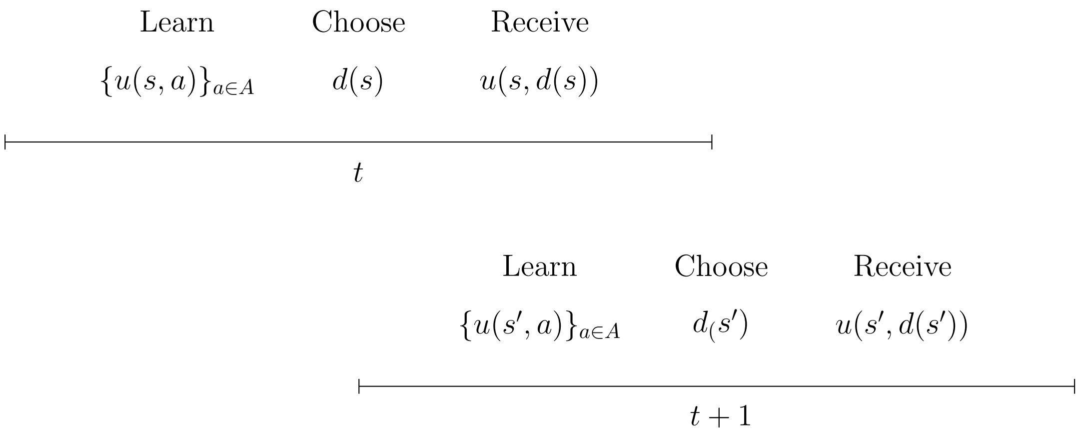
2.1 Economic model

At time $t = 0, 1, 2, \ldots$ an individual observes the state of the economic environment $s \in \mathcal{S}$ and chooses an action a from the set of admissible actions $\mathcal{A}$. The decision has two consequences. It creates an immediate utility $u(s, a)$ and the environment evolves to a new state s'. The transition from s to s' is affected by the action, at least partly unknown, and governed by transition probability distribution $p(s, a)$.

Individuals are forward-looking, thus they do not simply choose the alternative with the high-

[2]While still limited in number, recent successful applications of robust optimization in operations research include portfolio allocation (Zymler, Kuhn, & Rustem, 2013), elective admissions to hospitals (Meng et al., 2015), and scheduling of liver transplantations (Kaufman et al., 2017).

Figure 3: Timing of events



est immediate utility. Instead, they take the future consequences of their current action into account. A decision rule d specifies the planned action for all possible states within one period. A policy π is a collection of decision rules and specifies all actions for all future periods.

Figure 3 depicts the timing of events in the model for two time periods. At the beginning of period t an individual learns about the utility of each alternative, chooses one of them, and receives its immediate utility. Then the state evolves from s to s' and the process is repeated in $t+1$.

In the standard model, there exists a unique probability distribution for the evolution of state s to s' that is known to the individual. However, limits of knowledge or scarcity of data on past experiences often lead to numerous distributions $p(s,a)$ that are equally reasonable descriptions of the economic environment. All candidate distributions are collected in an ambiguity set $\mathcal{P}(s,a)$. Individuals now face risk for a given distribution and ambiguity about the true distribution. In the case of decision-making under risk, the ambiguity set is a singleton.

Economic decision theory offers guidance on desirable principles in settings with uncertainty (Gilboa, 2009). If individuals face risk, then there is broad agreement that they should maximize their expected utility (Bernoulli, 1738; von Neumann & Morgenstern, 1944, 1947). In a dynamic setting, individuals in state s choose the optimal policy π^* from the set of all possible policies Π that maximizes their expected total discounted utility $\tilde{v}^{\pi^*}(s)$ (Samuelson, 1937; Koopmans, 1960) as formalized in Equation (2.1):

$$\tilde{v}^{\pi^*}(s) \equiv \max_{\pi\in\Pi} \mathrm{E}^{\pi}\left[\sum_{t=0}^{\infty} \delta^t u(X_t, d(X_t))\right]. \tag{2.1}$$

The exponential discount factor δ captures a preference for immediate over future utilities. X_t is a random variable for the state at time t and the superscript of the expectation π emphasizes that each policy induces a different probability distribution over sequences of possible futures.

No such consensus exists when individuals face ambiguity (Marinacci, 2015). We assume maxmin expected utility preferences (Gilboa & Schmeidler, 1989; Epstein & Schneider, 2003b). This allows us to maintain many of the other core features of the original bus replacement problem such as exponential discounting, the absence of learning, and the indifference to the temporal resolution of uncertainty (Strzalecki, 2013). This provides a clean benchmark to assess the impact of ambiguity on bus replacement decisions. In addition, the close relationship of maxmin preferences to Wald (1950)'s maximin model allows to draw on the mathematical model of a robust Markov decision process (RMDP) to analyze individual decision-making in light of ambiguity.

In the dynamic setting, the objective of the individual is to implement the optimal policy π^* that maximizes the expected total discounted utility under a worst-case scenario. Each policy π induces a whole set of probabilities over sequences of possible futures $\mathcal{F}^\pi$ and its worst-case realization determines its ranking. The formal representation of the individual's objective is Equation (2.2):

$$v^{\pi^*}(s) \equiv \max_{\pi \in \Pi} \left\{ \min_{\mathbf{P} \in F^\pi} \mathrm{E}^{\mathbf{P}} \left[\sum_{t=0}^{\infty} \delta^t u(X_t, d(X_t)) \right] \right\}. \tag{2.2}$$

Epstein & Schneider (2003b) axiomatize a recursive approach for maxmin preferences that ensures dynamic consistency, i.e. actual decisions agree with their planned for contingencies. This generates the requirement that the ambiguity sets are rectangular and revised according to full Bayesian updating (Epstein & Schneider, 2003b; Cerreia-Vioglio, Maccheroni, Marinacci, & Montrucchio, 2011). It restricts the structure of information and is best interpreted in a game-theoretic setup, where the individual views himself as playing a zero-sum game against a malevolent opponent called nature. Rectangularity is a form of an independence assumption where the choice of any particular distribution in a state-action pair does not limit nature's other choices, i.e. its choices are uncoupled. Uncertainty is time-invariant (Epstein & Schneider, 2003a) and the individual thus gains nothing from having future actions depend explicitly on past realizations of uncertainty. This rules out any kind of learning about the future from past experiences. We provide a formal definition of rectangularity when introducing the mathematical model next.

2.2 Mathematical model

The economic model of decision-making under risk is set up as a standard Markov decision process (MDP) with a unique transition probability distribution $p(s,a)$ for each state and action pair.[3] However, when analyzing decisions under ambiguity there is a whole set of distributions $\mathcal{P}(s,a)$ to consider. Recent work in operations research on robust Markov decision processes (RMDP) extends the standard MDP and allows for ambiguous transitions (Iyengar, 2005; Nilim & El Ghaoui, 2005; Xu & Mannor, 2012; Yang, 2017; Chen, Yu, & Haskell, 2018). The MDP remains a special case of the RMDP where $\mathcal{P}(s,a)$ is a singleton.

In the case of decision making under risk, the goal is to maximize the expected total discounted utility as formalized in Equation (2.1). This requires to evaluate the performance of all policies based on all possible sequences of utilities and the probability that each occurs. However, stationarity implies that the future looks the same whether the individual is in state s at time t or k. Thus the optimal policy is stationary as well (Blackwell, 1965) and the same decision rule is used in every period. The value function is the fix point to the following Bellman equation:

$$\tilde{v}(s) = \max_{a \in A} \Big[u(s,a) + \delta \, \mathrm{E}^p[\tilde{v}(X_{t+1})] \Big].$$

The optimal decision rule is recovered from $\tilde{v}(s)$ by finding the value $a \in A$ that attains a maximum for each $s \in S$.

The contraction mapping property of the Bellman operator $\Gamma(\tilde{v})(s)$ allows to compute the solution to the value function $\tilde{v}(s)$ (Denardo, 1967).

$$\Gamma(\tilde{v})(s) = \max_{a \in A} \Big[u(s,a) + \delta \, \mathrm{E}^p[\tilde{v}(X_{t+1})] \Big].$$

Iyengar (2005) and Nilim & El Ghaoui (2005) establish that, given the assumption of rectangularity, the contraction mapping property of the Bellman operator and the optimality of a stationary deterministic Markovian decision rule is preserved under ambiguous transition probabilities.

We first discuss the formal definition of rectangularity in more detail. Let the set of all transition probability distributions associated with decision rule d be given by:

$$\mathcal{F}^d = \{\mathbf{p} : \forall s \in S, p(s) \in P(s, d(s))\}.$$

[3] See Puterman (1994) for a textbook introduction to the standard MDP and Rust (1994, 1996) for a review of MDPs in economics and structural estimation.

For every state $s \in S$, the next state can be determined by any $p(s) \in P(s, d(s))$.

A policy π now induces a set of probability distributions $\mathcal{F}^\pi$ on the set of all possible histories $\mathcal{H}$. Any particular history $h = (s_0, a_0, s_1, a_1, \ldots)$ can be the result of many possible combinations of transition probabilities. The key assumption is the rectangularity of $\mathcal{F}^\pi$ which is stated below.

Definition 1 ***Rectangularity*** *The set $\mathcal{F}^\pi$ of probability distributions associated with a policy π is given by*

$$\mathcal{F}^\pi = \left\{ \mathbf{P} : \forall\, h \in \mathcal{H},\, \mathbf{P}(h) = \prod_{t=0}^{\infty} p(s_t), p(s_t) \in \mathcal{F}^d, t = 0, 1, \ldots \right\} = \prod_{t=0}^{\infty} F^d,$$

where the notation simply denotes that each element in $\mathcal{F}^\pi$ is a product of $p(s) \in \mathcal{F}^d$, and vice versa (Iyengar, 2005).

Definition 1 formalizes the idea that ambiguity about the transition probability distribution is uncoupled across states and time. All $p(s)$ can be freely combined to generate a particular history.[4]

The objective is to implement a policy π^* that maximizes the expected total discounted utility under a worst-case scenario as presented in Equation (2.2). Equation (2.3) presents the robust Bellman equation, where the future value is evaluated using the worst-case element in the ambiguity set.

$$v(s) = \max_{a \in A} \left[u(s, a) + \delta \min_{p \in \mathcal{P}(s,a)} \mathrm{E}^p[v(X_{t+1})] \right]. \tag{2.3}$$

The robust Bellman operator follows directly:

$$\Gamma(v)(s) = \max_{a \in A} \left[u(s, a) + \delta \min_{p \in \mathcal{P}(s,a)} \mathrm{E}^p[v(X_{t+1})] \right]. \tag{2.4}$$

Algorithm 1 allows to solve the RMDP by a robust version of the value iteration algorithm (Iyengar, 2005) where $\mathbb{V}$ denotes the set of bounded real valued functions on S and κ a convergence threshold. The calculation of future values under the worst-case scenario is the key difference to the standard approach.

[4]See Wiesemann et al. (2013) and Mannor, Mebel, & Xu (2016) for recent attempts to introduce milder rectangularity conditions that allow for some coupling of uncertainties.

Algorithm 1 Robust Value Iteration Algorithm

Input: $v \in \mathbb{V}, \kappa > 0$

For each $s \in S$, set $\hat{v}(s) = \max_{a \in A} \left\{ u(s,a) + \min_{p \in \mathcal{P}(s,a)} \mathrm{E}^p[v(X_{t+1})] \right\}$.

while $||\, v - \hat{v}\, || > \kappa$ **do**

$v = \hat{v}$

$\forall\, s \in S$, set $\hat{v}(s) = \max_{a \in A} \left\{ u(s,a) + \min_{p \in \mathcal{P}(s,a)} \mathrm{E}^p[v(X_{t+1})] \right\}$

end while

2.3 Computational model

We now present the computational details of our robust bus replacement problem and start with its general setup. We then discuss the distribution of the state variables, where we focus on the construction of the ambiguity sets. We follow Rust (1987) throughout unless otherwise noted.

The model is set up as a regenerative optimal stopping problem. We consider the dynamic decisions by a maintenance manger, Harold Zurcher, for a fleet of buses. He makes repeated decisions about their maintenance in order to maximize his expected total discounted utility under a worst-case scenario. Each month, a bus arrives at the bus depot in state $s = (x, \epsilon)$ described by its mileage since last engine replacement x and other signs of wear and tear ϵ. He is faced with the decision to either conduct a complete engine replacement ($a = 1$) or perform basic maintenance work ($a = 0$). The cost of maintenance $c(x)$ increases with the mileage state, while the cost of replacement RC remains constant. In the case of an engine replacement, the mileage state is reset to zero.

The immediate utility of each action is given by:

$$u(a,x) + \epsilon(a) \quad \text{with} \quad u(a,x) = \begin{cases} -RC & a = 1 \\ -c(x) & a = 0. \end{cases}$$

Harold Zurcher makes his decision in light of uncertainty about next month's state variables captured by their conditional distribution $p(x, \epsilon, a)$.

We impose conditional independence between the observable and unobservable state variables, i.e. $p(x, \epsilon, a) = p(x, a)\, p(\epsilon)$, and assume that the unobservables $\epsilon(a)$ are independent and identically distributed according to an extreme value distribution with mean zero and scale parameter one. Rust (1988) shows that these two assumptions, together with the additive separability between the observed and unobserved state variables in the immediate utilities, allow to compute the value function as the unique fix point of a contraction mapping on the observable states x only. In addition, the conditional choice probabilities $p(a \mid x)$ have a closed-form solution (McFadden, 1973).

The monthly mileage transitions $p(x, a)$ are represented by action-specific $K \times K$ transition matrices M^a. The entry in the k-th row and the l-th column represents $p_l(k, a)$ which is the probability to observe state l in the next month given current state k and action a.

In the analysis of the original bus replacement problem, the elements of M^a are estimated in a first step and used as plug-in components for the subsequent analysis. We extend the original setup and explicitly account for the ambiguity in the estimation. We construct an ambiguity set $\mathcal{P}(x, a)$ based on the Kullback-Leibler divergence D_{KL} (Kullback & Leibler, 1951) that is anchored in the empirical estimates of $\hat{p}(x, a)$, related to statistical confidence sets, and computationally tractable.

Our ambiguity set takes the following form for each row of the transition matrix:

$$\hat{\mathcal{P}}(x_k, a; \rho) = \left\{ q \in \Delta_{J_k} : D_{KL}(q \parallel \hat{p}(x_k, a)) = \sum_{j=1}^{J_k} q_j \ln \left(\frac{q_j}{\hat{p}_j(x_k, a)} \right) \leq \rho \right\},$$

where J_k denotes the number of all nonzero probabilities in the k-th row, Δ_{J_k} is the J_k - dimensional probability simplex, and ρ the size of the set.

Iyengar (2002) and Ben-Tal et al. (2013) provide the statistical foundation to calibrate ρ such that the true (but unknown) distribution p_0 is contained within the ambiguity set for a given level of confidence ω. Let χ^2_d denote a chi-squared random variable with d degrees of freedom and let $F_d(\cdot)$ denote its cumulative distribution function with inverse $F_d^{-1}(\cdot)$. Then, the following approximate relationship holds as the number of observations N_k for state k tends to infinity (Pardo, 2005):

$$\begin{aligned} \omega &= \Pr[\, p_0 \in \hat{\mathcal{P}}(x_k, a; \rho)\,] \\ &\approx \Pr[\, \chi^2_{K-1} \leq 2N_k\rho\,] \\ &= F_{K-1}(2N_k\rho). \end{aligned}$$

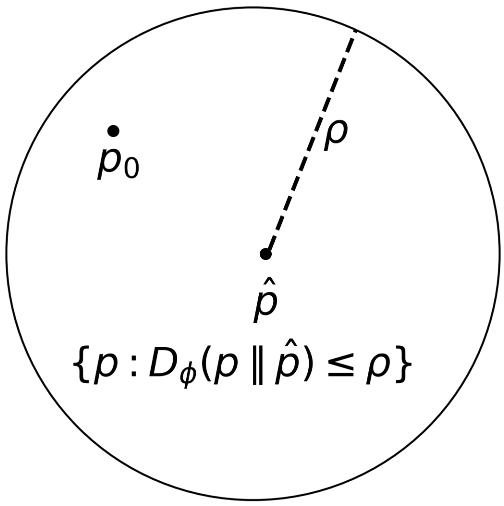
Therefore we can calibrate the size of the ambiguity set based on the following relationship:

$$\rho = \tfrac{1}{2N_k} F^{-1}_{K-1}(\omega). \tag{2.5}$$

Note that ω is approximately equal to the probability that an individual row falls within the ambiguity set as opposed to a collective confidence level for the entire transition matrix.

Figure 4 illustrates the construction of the ambiguity set for each row of the transition matrix. We consider all distributions q that are within a certain distance ρ measured by the divergence function $D_{KL}(q \parallel \hat{p})$ around a reference distribution $\hat{p}$. We ensure rectangularity as we construct the ambiguity set independently for each row of the transition matrix.

Figure 4: Calibration of ambiguity set



3 Application

We now apply the conceptual framework to study our robust bus replacement problem. We first introduce the general setup and present the reference and worst-case transition probabilities. We then construct optimal and robust policies with varying levels of confidence, discuss the resulting differences in maintenance decisions, and evaluate their relative performance under different scenarios.

We use Rust (1987)'s original data to inform our computational experiments. His data consists of monthly odometer readings x and engine replacement decisions a for 162 buses. The fleet consists of eight groups that differ in their manufacturer and model. We focus on the fourth group of 37 buses with a total of 4,292 monthly observations. We discretize mileage into $K = 78$ equally spaced bins of length $5,000$ and set the discount factor to $\delta = 0.9999$.

We analyze a specific example of Rust (1987)'s bus replacement problem. We do not use his reported estimates of the maintenance and replacement costs. Given these estimates, Harold Zurcher's decisions are mainly driven by the unobserved state variable ϵ and so ambiguity about the evolution of the observed state variable x does not have a substantial effect on decisions. We ensure that a bus' accumulated mileage has a considerable impact on the timing of engine replacements by increasing the maintenance and replacement costs compared to their empirical estimates. We specify the following cost function $c(x) = 0.4\,x$ and set the replacement costs RC to 50.

We solve the model using a modified version of the original inner fixed-point algorithm (FXP) (Rust, 1987, 1988, 2000). We determine the worst-case transition probabilities in each successive approximation of the fix point. Given our ambiguity set, we can determine the worst-case probabilities for a given ρ as the solution to a straightforward one-dimensional convex optimization problem (Iyengar, 2005; Nilim & El Ghaoui, 2005). The core routines are implemented in the ruspy (2019) and robupy (2019) packages.

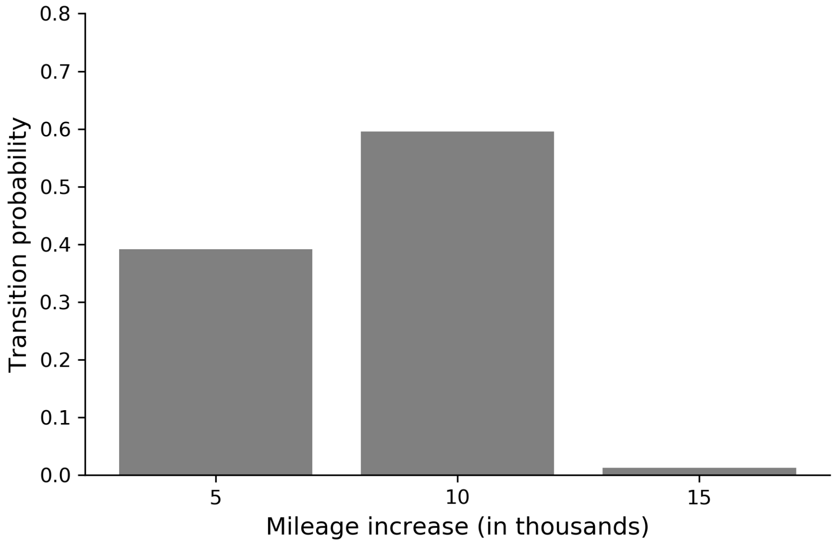
3.1 Transition probabilities

Figure 5 shows the estimates for the transition probabilities of monthly mileage usage that serve as our reference distribution $\hat{p}$. We pool all 4,292 observations to estimate this distribution by maximum likelihood and thus the transition matrix M^a has the same nonzero elements in each row. We only observe increases of at most $J = 3$ grid points per month. For about 60% of the sample, monthly bus utilization is between 5,000 to 10,000 miles. Very high usage of more than 10,000 miles amounts to only 1.2%.

The size of the ambiguity set is determined by the confidence level ω and the available number of observations N_k as outlined in Equation (2.5). Going forward, we mimic state-specific ambiguity sets by using the average number of observations per state of 55 for their construction. Note that while the reference distribution is the same for all levels of mileage, its worst-case realization is not. However, there are only minor differences across mileage levels and so we focus our next discussion on a bus with an odometer reading of 75,000.

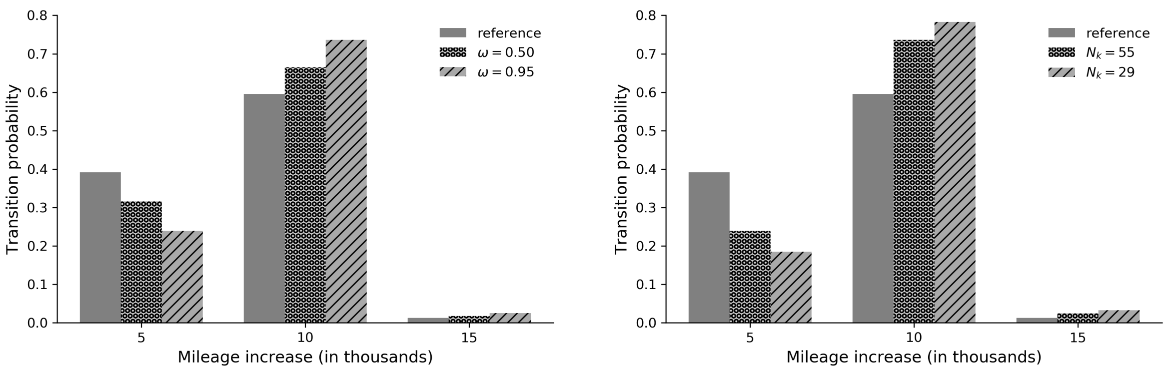
Figure 6 shows the transition probabilities for different sizes of the ambiguity set. We vary the level of confidence for the full number of observations ($N_k = 55$) on the left, while on the right the level of confidence remains fixed ($\omega = 0.95$) and we cut the number of observations

Figure 5: Estimated transition probabilities



roughly in half. The larger the ambiguity set, the more probability is attached to higher mileage utilization as this results in higher maintenance costs in the future. For example, while the probability of mileage increases of 10,000 or more is a very rare occurrence in the data, its probability increases first to 1.7% and then more than than doubles to 2.5% as we increase the confidence level. When only about half the data is available, this probability increases even further to 3.2%.

Figure 6: Worst-case transition probabilities



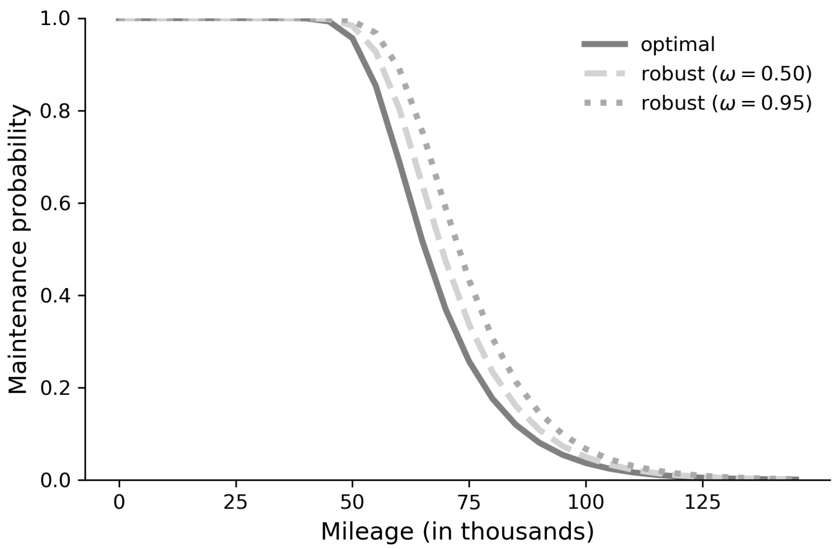
(a) Variation in ω, ($N_k = 55$) **(b)** Variation in N_k, ($\omega = 0.95$)

3.2 Policy features

Harold Zurcher chooses each month whether to perform regular maintenance work on a bus or replace its full engine. His decision is informed by the assumed transition probabilities. Given maxmin expected utility preferences, these correspond to the worst-case transitions within the ambiguity set. So any differences between the reference and worst-case distributions translate into different maintenance policies.

Figure 7 shows the maintenance probabilities for different levels of accumulated mileage and alternative policies. Overall, the maintenance probability decreases with accumulated mileage as maintenance gets more and more costly relative to an engine replacement. Both robust policies result in a higher probability of maintenance compared to the optimal. Under the worst-case transitions, a bus is more likely to experience higher usage compared to the reference case. As its cost is determined by the current mileage, maintenance becomes more attractive. For example, again considering a bus with 75,000 miles, the optimal maintenance probability is 25% while it is 33% ($\omega = 0.50$) and 43% ($\omega = 0.95$) following the robust policies.

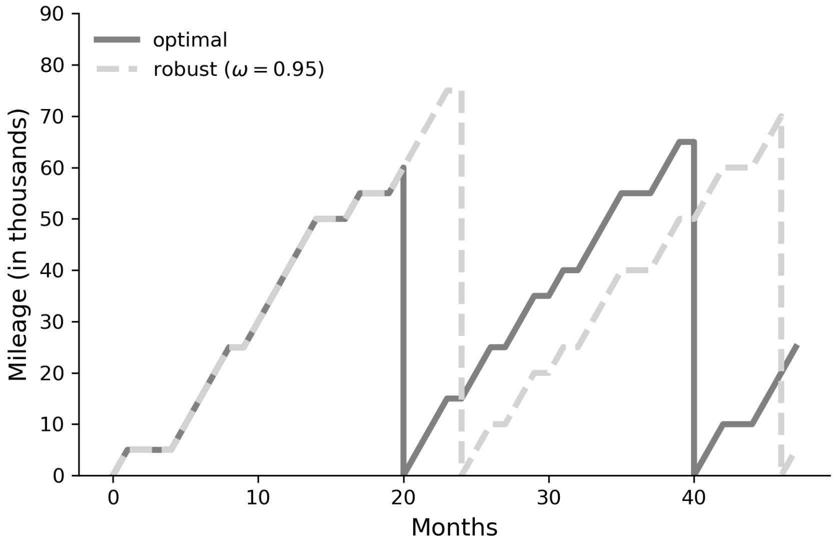
Figure 7: Maintenance probabilities



To gain further insights into the differences between the optimal and robust policies, we simulate a fleet of 1,000 buses for 100,000 months applying numerous alternative policies.

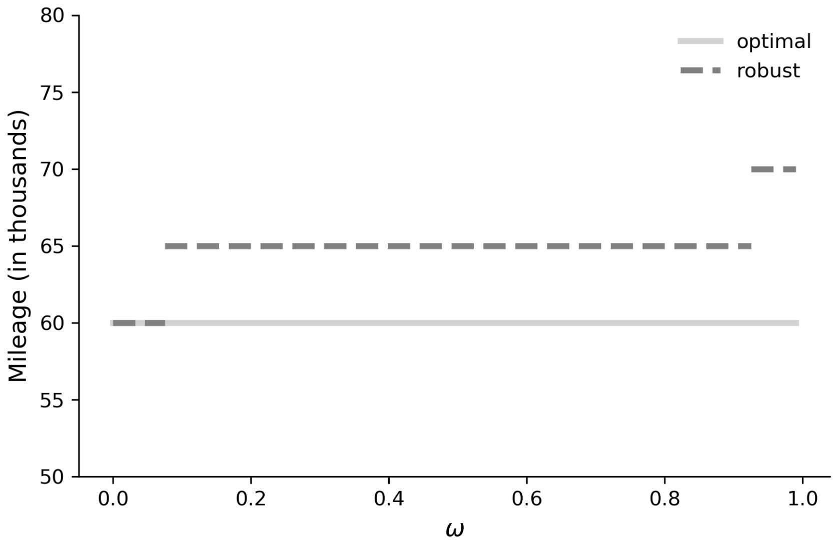
Figure 8 shows the level of accumulated mileage over time for a single bus under different policies. It clarifies our simulation setup, where we apply different policies to the same bus. The realizations of observed transitions and unobserved signs of wear and tear remain the same. The bus accumulates more and more mileage until Harold Zurcher decides to replace the full engine and the odometer is reset to zero. The first replacement happens after 20 months at 60,000 miles following the optimal policy, while it is delayed for another four months under the robust alternative ($\omega = 0.95$). As its timing differs, the odometer readings will start to diverge after 20 months even though monthly utilization remains the same.

Figure 8: Single bus under alternative policies



The changes in replacement probabilities translate into differences in observed behavior. Figure 9 shows the average observed mileage at replacement for the optimal and robust policies with varying confidence levels ω. The actual transitions are determined by the reference distribution. For very small ω, the optimal and robust policies both result in replacements at 60,000 miles. As ω increase further, average mileage first jumps to 65,000 and then 70,000. It remains unchanged over long stretches of ω.

Figure 9: Replacement thresholds

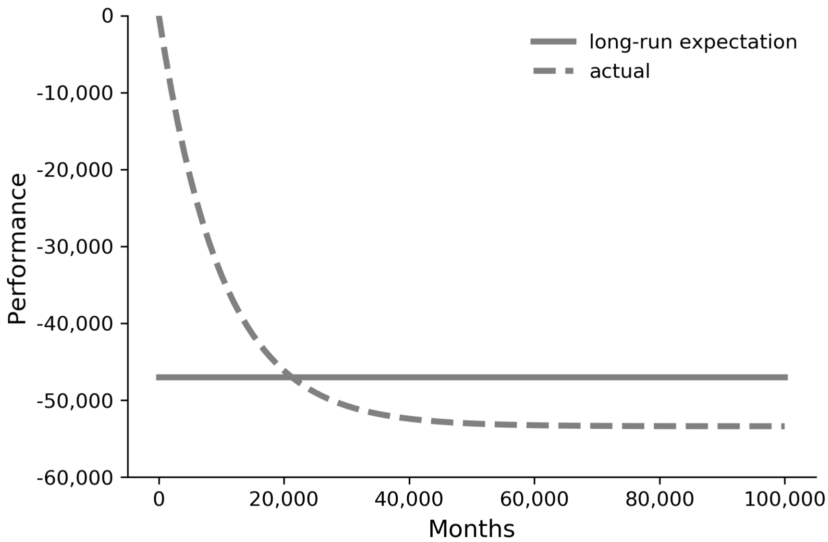


3.3 Policy performance

We now turn to the evaluation of the optimal and robust policies based on their total discounted utility under different combinations of assumed and actual mileage transitions.

Figure 10 shows the performance of the optimal policy over time when the actual transitions are governed by the worst-case distribution for a confidence level of 0.95. It illustrates the sensitivity of the optimal policy to perturbations in the transition probabilities. The solid line corresponds to its expected long-run performance in the absence of any model misspecification, while the dashed line indicates its observed performance. After about 20,000 months it already accumulates the cost of its expected long-run average and performs about 14% worse overall.

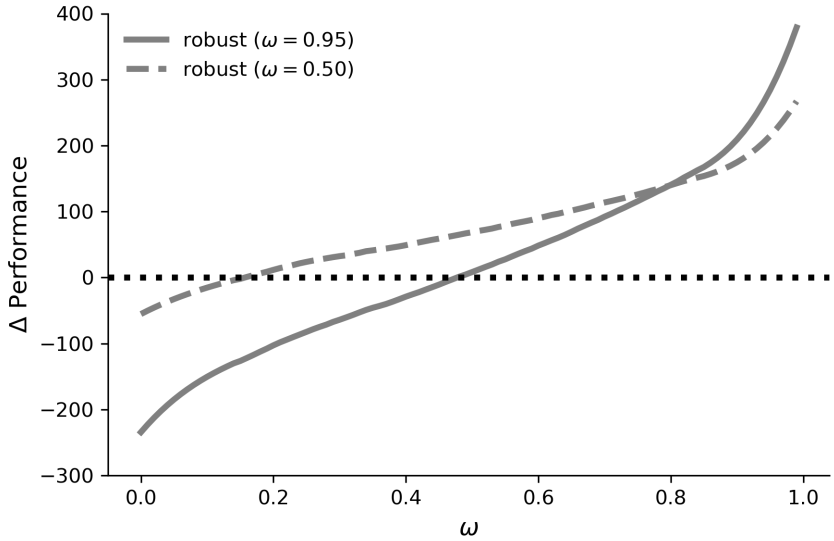
Figure 10: Performance of optimal policy



The deterioration in performance of the optimal policy by itself does not, however, provide any evidence on the benefits of adopting a robust policy.

Figure 11 shows the average difference in performance between the optimal and two robust policies with confidence level of 0.50 and 0.95. The actual transitions are determined by the worst-case distribution with varying ω. A positive value indicates that the robust policy outperforms the optimal policy. In the absence of any model misspecification, the optimal policy must defeat any other policy. The same is true for the robust policies when the actual transitions are governed by the same ω used for their construction. Nevertheless, the optimal policy continues to outperform both robust policies for moderate levels of ω. For worst-case distributions with ω larger than 0.2, the first robust policy ($\omega = 0.5$) starts to beat the optimal policy. For the other robust policy ($\omega = 0.95$), this is true for worst-case transitions of ω equal to 0.5.

Figure 11: Performance and model misspecification



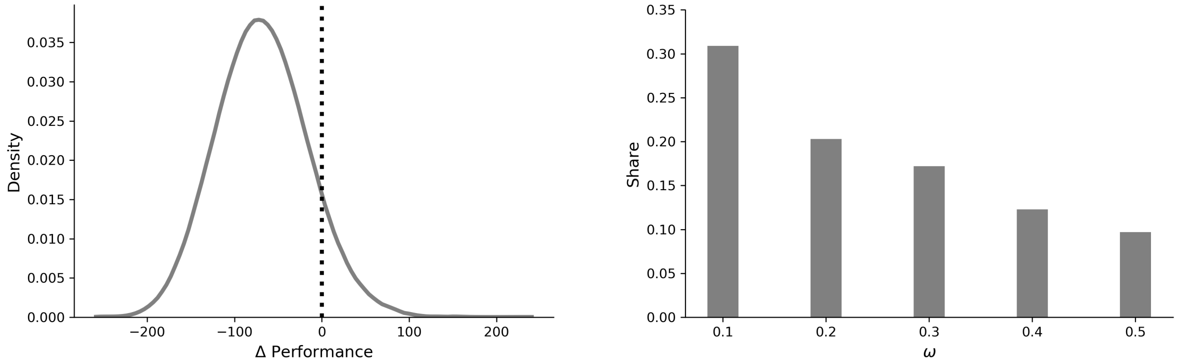
Notes: We apply a Savitzky-Golay filter (Savitzky & Golay, 1964) to smooth the simulation results.

Our analysis so far restricts the level of model misspecification to the worst-case transitions and provides insights for the performance at the boundary of the ambiguity sets. But given the maximum likelihood estimate of the reference probabilities, we can directly sample from their asymptotic distribution and thus investigate the performance over the whole set. We draw a sample of 10,000 actual transition matrices and determine the gain in performance when adopting a robust instead of an optimal policy for each draw.

Figure 12 plots the gains for various robust strategies. We show their whole distribution for one robust policy ($\omega = 0.50$) on the left. Adopting this policy results in better performance in less than 10% of draws. When it does, the gains from doing so are still smaller compared to the losses in the large majority of cases where the optimal policy performs better. The plot on the right summarizes the performance of a whole menu of robust policies in the same setup. It shows the share of draws where there are positive gains from adopting the robust policy. This share decreases with ω and never goes above 30%.

In summary, given this modeling setup and the available information on past transitions, the optimal policy still outperforms any robust policy even in the presence of model misspecification most of the time.

Figure 12: Gains in performance



(a) Distribution for given ω **(b)** Summary for alternative ω

Notes: We apply a Savitzky-Golay filter (Savitzky & Golay, 1964) to smooth the simulation results in the left figure.

4 Conclusion

We incorporate techniques from distributionally robust optimization (Ben-Tal et al., 2009; Rahimian & Mehrotra, 2019) into a dynamic investment model to account for ambiguity in the decision-making process. We outline an economic, mathematical, and computational model to study a version of the seminal bus replacement problem (Rust, 1987) that allows for ambiguity about the underlying dynamics. We instill individuals with a fear of model misspecification using maxmin expected utility preferences (Gilboa & Schmeidler, 1989; Epstein & Schneider, 2003b). We specify ambiguity sets that are based on empirical estimates, statistically meaningful, and computationally tractable (Ben-Tal et al., 2013). We conduct a series of computational experiments to study alternative decision policies. We find that, given the structure of model and the available data on past transitions, a policy that simply ignores potential model misspecification continues to perform very well compared to its alternatives that explicitly take this possibility into account.

Nevertheless, our work illustrates the feasibility and usefulness of techniques from distributionally robust optimization to explicitly account for ambiguity and a fear of model misspecification in economic models of decision-making. Going forward, we will examine settings with richer transition dynamics, more sparseness of data, and severe consequences of model misspecification. For example, ongoing research in labor economics, attempts to study the role of health in the accumulation of human capital and its impact on inequality in economic outcomes (Capatina, Keane, & Maruyama, 2018). A richer description of the health process comes at a price as less and less data is available for its estimation. The methods studied here could help to explicitly account for the remaining ambiguity about the true process.

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