# Robust investments under risk and ambiguity

## Harold Zurcher's robust replacement policy

Maximilian Blesch & Philipp Eisenhauer

April 8, 2020

Open Source Economics Bonn

# Table of Content

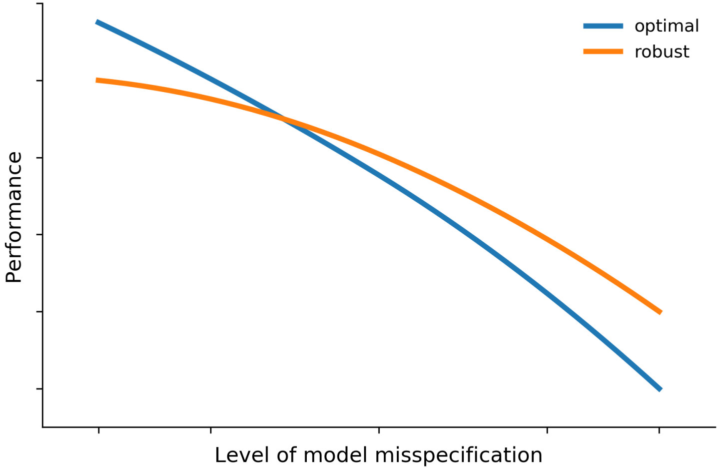

1. Introduction
2. Decision making under risk: Revisiting Rust (1987)
3. Assessing the ambiguity
4. Decision making under risk and ambiguity
5. Application
6. Conclusion

# Introduction

# Introduction

- Policy makers vote for climate change mitigation efforts facing uncertainty about future costs and benefits (Barnett, Brock, & Hansen, 2020).
- Investors trade with assets uncertain about future returns (Epstein & Schneider, 2008).
- Young adults decide on their careers in light of uncertainty about future job outcomes (Keane & Wolpin, 1997).

# Notion of Robust decision rule

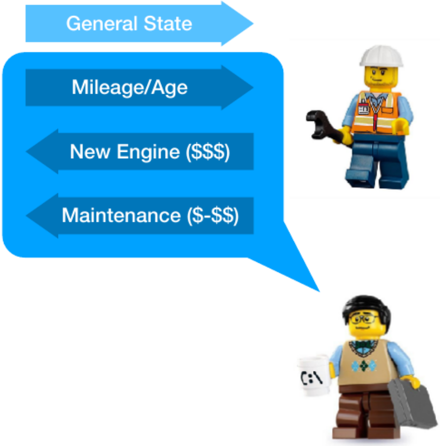


# Questions addressed in our research

- How sensitive is the performance of the optimal decision rule?
- When does a robust decision rule perform better?
- How to construct a robust decision rule?

# Decision making under risk: Revisiting Rust (1987)

# A brief summary

# The framework

- Single agent Markov decision problem over infinite horizon.
- The agent decides each month $t$ to maintain $i_t = 0$ or replace $i_t = 1$ the bus engine.
- Mileage $x_t$ is discretized in states of length 5000 miles.
- If the engine is replaced, $x_t$ is set to 0.

# Optimal decision

The optimal decision in each period $t$ can be inferred from the Bellman equation (Bellman (1954)):

$$V_\theta(x_t) = \max_{i_t \in \{0,1\}} \Big[ u(x_t, i_t, \theta_1) + \epsilon_t(i_t) + \beta\, EV_\theta(x_t, i_t) \Big]$$

with

$$u(x_t, i_t, \theta_1) = \begin{cases} -c(x_t, \theta_1) & \textbf{if} \quad i_t = 0 \\ -RC & \textbf{if} \quad i_t = 1 \end{cases}$$

and unobserved utility shock $\epsilon_t(i_t)$.

# Estimation strategy

Rust (1987) assumes Conditional Independence (CI):

$$p(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t, i_t, \theta_2, \theta_3) = q(\epsilon_{t+1} | \theta_2)\, p(x_{t+1} | x_t, i_t, \theta_3)$$

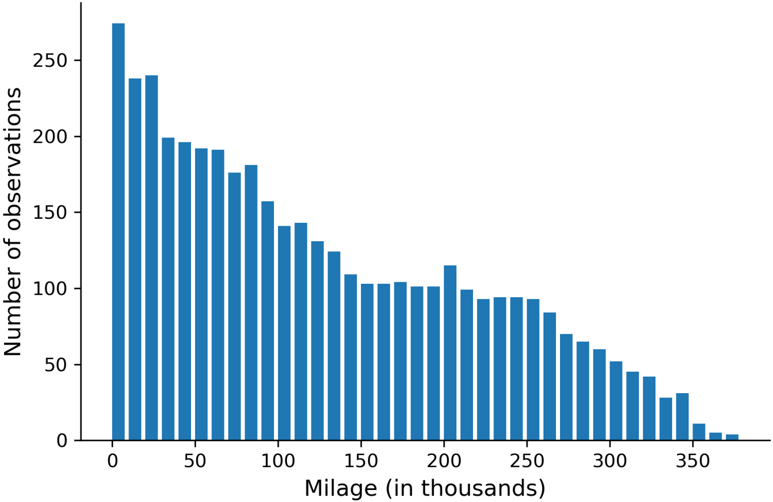
This allows to estimate the model in two steps:

$$\max_{\theta_3} \prod_{t=1}^{T} p(x_t | x_{t-1}, i_{t-1}, \theta_3)$$

Second:

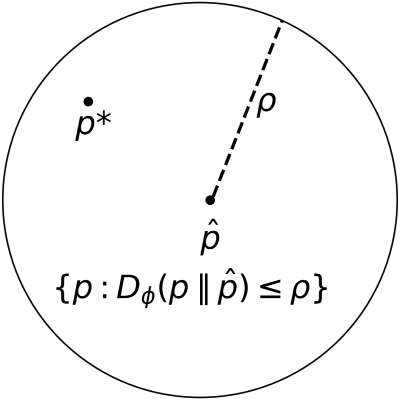
$$\max_{RC, \theta_1} \prod_{t=1}^{T} P(i_t | x_t, \theta)$$

# Differences in estimation quality



# Assessing the ambiguity

# Constructing the ambiguity set



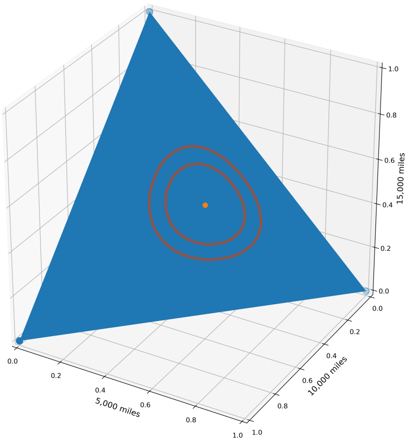
# Calibrating the ambiguity set

Following Ben-Tal et al. (2013), we can calibrate the size of the ambiguity set with the number of observations $N_x$ for state $x \in X$:

$$\begin{aligned}
\omega &= \mathbb{P}\{p_x^* \in \mathcal{P}_x\} \\
&= \mathbb{P}\{p_x^* \in \{p : D(p||\hat{p}_x) \leq \rho_x\}\} \\
&\approx \mathbb{P}\{\chi^2_{|X|-1} \leq 2N_x\rho_x\} \\
&= F_{|X|-1}(2N_x\rho_x)
\end{aligned}$$

$$\Rightarrow \rho_x = F^{-1}_{|X|-1}(\omega)/2N_x$$

# Probability simplex



# Decision making under risk and ambiguity

# Theory

Ben-Tal, El Ghaoui, and Nemirowski (2009) develop, building on Wald (1950), the following idea of robust decision making:

- Robust decision making can be modeled as a game agent vs. nature.
- First the agent chooses his action to maximize his present value.
- Then nature chooses the transition probabilities accordingly to minimize the agent's value.
- As the agent and nature have common information, the agent chooses in the first step the alternative with the highest (max) minimal (min) value.

# Rectangularity

The set of all transition probabilities associated with decision rule $d$:

$$\mathcal{F}^d = \{\mathbf{p} : \forall x \in X, p(x) \in \mathcal{P}_x^{d(x)}\}.$$

We assume that the set of probability distributions on the set of all possible histories $\mathcal{H}$ is rectangular:

$$\left\{\mathbf{P} : \forall\, h \in \mathcal{H},\ \mathbf{P}(h) = \prod_{t=0}^{\infty} p(x_t), p(x_t) \in \mathcal{F}^d, t = 0, 1, \dots \right\} = \prod_{t=0}^{\infty} \mathcal{F}^d$$

$\Rightarrow$ **nature's choices are uncoupled over time and states.**

# Theory

Rectangularity allows to carry over the results of the classic dynamic programming problem. The robust Bellman equation in the model setup of Rust (1987):

$$V_\theta(x_t) = \max_{i_t \in \{0,1\}} \left[ u(x_t, i_t, \theta_1) + \epsilon_t(i_t) + \beta \min_{p_x \in \mathcal{P}_x^{i_t}} EV_\theta(x_t, i_t) \right]$$

In comparison, the standard Bellman equation from before:

$$V_\theta(x_t) = \max_{i_t \in \{0,1\}} \left[ u(x_t, i_t, \theta_1) + \epsilon_t(i_t) + \beta EV_\theta(x_t, i_t) \right]$$

# Application

# Setup and structure

Setup:

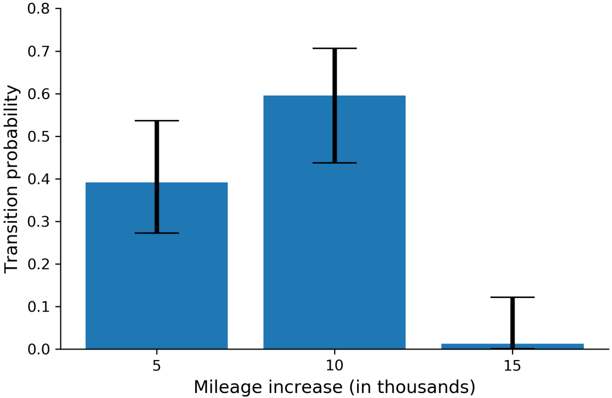
- Original transition probabilities, but mimicked state specific estimation.
- Increased cost of maintenance and engine replacement.

Structure of analysis:

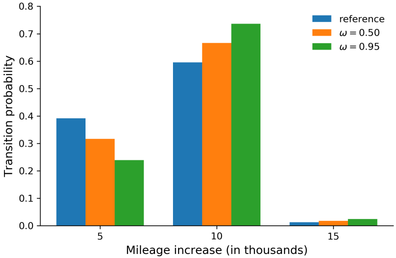
- Transition probabilities
- Policy features
- Policy performance

# Transition probabilities

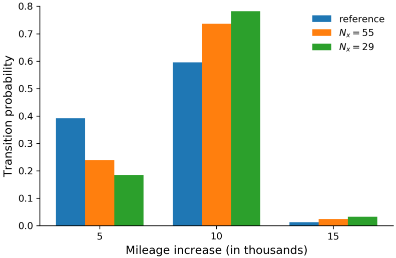
# Estimated transition probabilities



# Worst case distribution



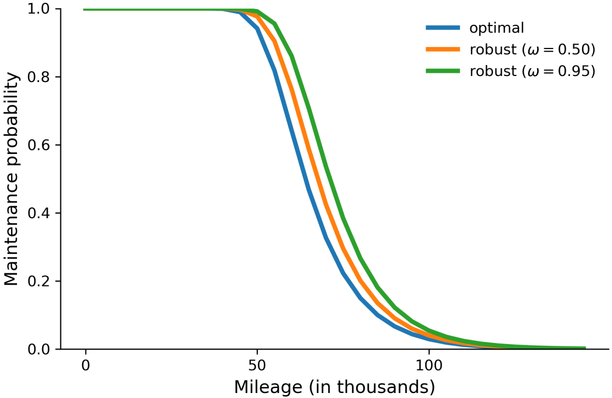

Variation in $\omega$, $(N_x = 55)$



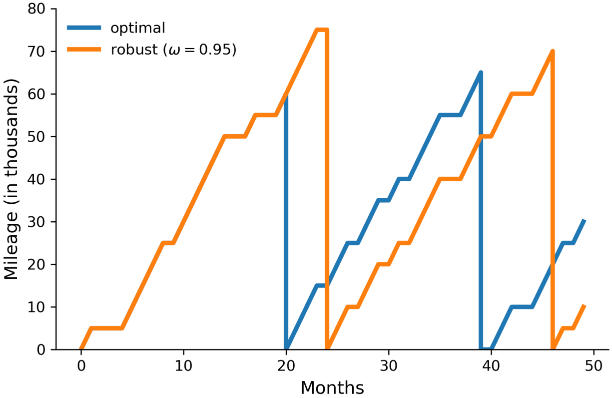

Variation in $N_x$, $(\omega = 0.95)$

# Policy features

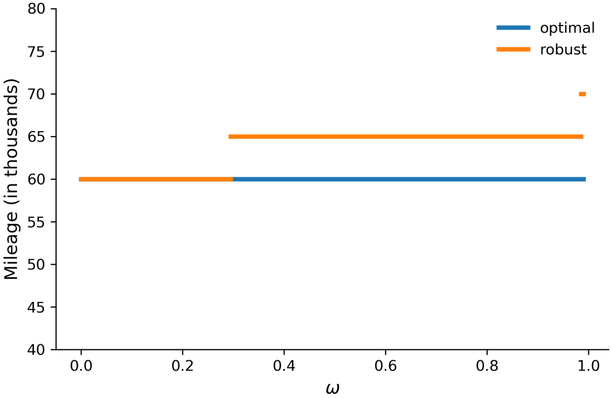
# Shift in maintenance probabilities



# Simulation setup

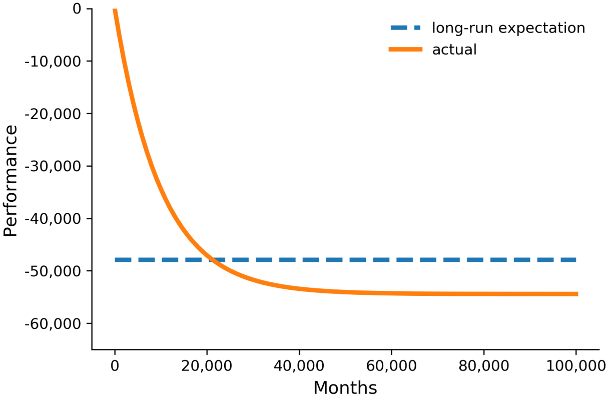


# Mean mileage at replacement

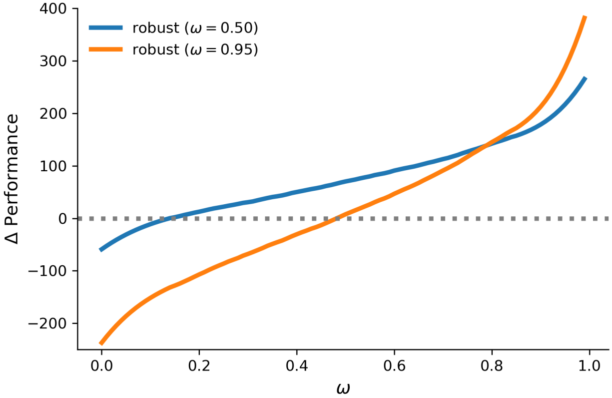


# Policy performance

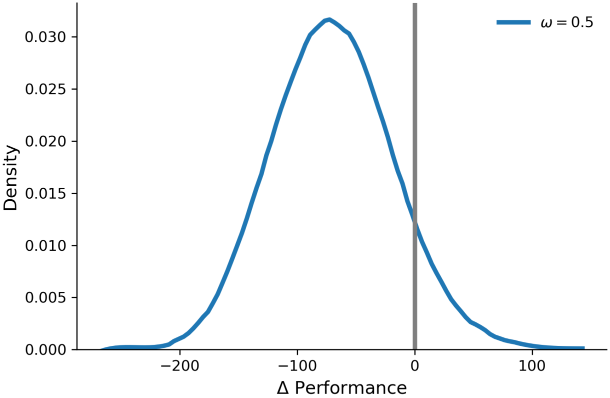
# Performance of optimal policy



# Performance on the boundary

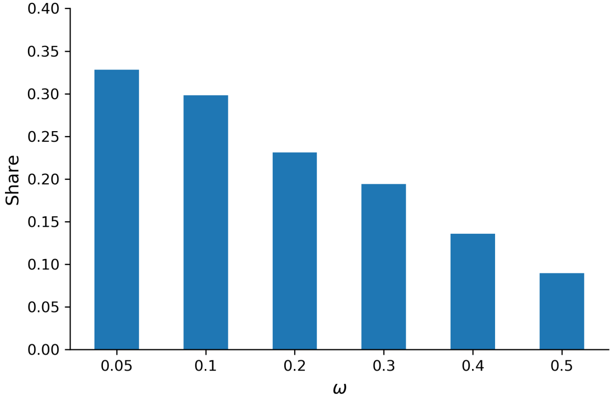


# Validation



ω = 0.5
Density
0.030
0.025
0.020
0.015
0.010
0.005
0.000
−200
−100
0
100
Δ Performance

# Validation



# Conclusion

# Thank you for your attention

# References I

Barnett, M., Brock, W., & Hansen, L. P. (2020, 02). Pricing uncertainty induced by climate change. *The Review of Financial Studies*, *33*(3), 1024-1066.

Bellman, R. E. (1954). The theory of dynamic programming. *Bulletin of the American Mathematical Society*, *60*(6), 503–515.

Ben-Tal, A., den Hertog, D., De Waegenaere, A., Melenberg, B., & Rennen, G. (2013). Robust solutions of optimization problems affected by uncertain probabilities. *Management Science*, *59*(2), 341–357.

Ben-Tal, A., El Ghaoui, L., & Nemirowski, A. (2009). *Robust optimization*. Princeton, NJ: Princeton University Press.

Epstein, L. G., & Schneider, M. (2008). Ambiguity, information quality, and asset pricing. *Journal of Finance*, *63*(1), 197-228.

# References II

Keane, M. P., & Wolpin, K. I. (1997). The career decisions of young men. *Journal of Political Economy*, *105*(3), 473–522.

Rust, J. (1987). Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher. *Econometrica*, *55*(5), 999–1033.

Wald, A. (1950). *Statistical decision functions*. New York City, NY: John Wiley & Sons.