

This document describes the domains and ranges of functions by using  $\rightarrow$  as a right-associative binary operator on sets, like in Haskell. Additionally, ?s are used to indicate how the arguments are written. For instance, if a function is applied as  $g_x(y)$  then we write  $g_?(?) : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$ . The function declarations are always written fully curried even if the applications are uncurried. The order of the ?s corresponds to the order of the sets in the declaration.

## 4 Probability

Let  $\mathcal{F} \subseteq \mathcal{P}(S)$  be a  $\sigma$ -algebra

Let  $P : \mathcal{F} \rightarrow \mathbb{R}$  be a probability measure.

$$P_?(?) : \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathbb{R}$$

$$P(E \mid F) := \frac{P(E \cap F)}{P(F)}$$

## 5 Discrete Random Variables

Let  $S \xrightarrow{\text{RV}} \mathbb{X}$  be the set of all random variables with sample space  $S$  and range  $\mathbb{X}$ .

$$P_?(?) : (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$$

$$P_X(X \in B) \equiv P_X(B)$$

$$P_X(X \in B) := P(\{s \in S : X(s) \in B\})$$

$$\sim \subseteq (\mathbf{RV}_{\mathbb{X}})^2$$

$$X \sim Y \iff P_X = P_Y$$

Most of the functions in this course are written as if they depend on  $X$ , but actually depend only on  $P_X$ . To make this clear, I will write  $g : \mathbf{RV}_{\mathbb{X}} \rightarrow B$  instead of  $g : (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow B$  whenever  $X \sim Y \implies g(X) = g(Y)$ . The symbols  $\xrightarrow{\text{DRV}}$ ,  $\mathbf{DRV}$ ,  $\xrightarrow{\text{CRV}}$ ,  $\mathbf{CRV}$  are defined similarly but with the added constraint of discreteness/continuity.

$$F_?(?) : \mathbf{RV}_{\mathbb{X}} \rightarrow \mathbb{R} \rightarrow [0, 1]$$

$$F_X(x) := P_X(X \leq x)$$

$$p_?(x) : \mathbf{DRV}_{\mathbb{X}} \rightarrow \mathbb{R} \rightarrow [0, 1]$$

$$p_X(x) := \begin{cases} P_X(X = x) & x \in \mathbb{X} \\ 0 & x \notin \mathbb{X} \end{cases}$$

$$E(?, ?) : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbf{DRV}_{\mathbb{X}} \rightarrow \mathbb{R}$$

$$E(g, X) := \sum_{x \in \mathbb{X}} g(x) p_X(x)$$

$$E(g(X)) \equiv E_X(g(X)) \equiv E(g, X)$$

$$E(X) \equiv E_X(X) \equiv E(\text{id}, X)$$

$$\text{Bernoulli}(?) : [0, 1] \rightarrow \mathbf{DRV}_{\{0,1\}}$$

$$\text{Binomial}(?, ?) : \mathbb{N} \rightarrow [0, 1] \rightarrow \mathbf{DRV}_{\{0, \dots, n\}}$$

$$\text{Geometric}(?) : [0, 1] \rightarrow \mathbf{DRV}_{\mathbb{Z}^+}$$

$$\text{Poisson}(?) : \mathbb{R}^+ \rightarrow \mathbf{DRV}_{\mathbb{N}}$$

$$\text{U}(?) : A \rightarrow \mathbf{DRV}_A$$

where  $A$  is any finite set

## 6 Continuous Random Variables

$$f_?(?) : \mathbf{CRV}_{\mathbb{X}} : \mathbb{X} \rightarrow \mathbb{R}$$

$$\forall B \subseteq \mathbb{R}, P_X(X \in B) := \int_{x \in B} f_X(x) dx$$

$$E(?, ?) : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbf{CRV}_{\mathbb{X}} \rightarrow \mathbb{R}$$

$$E(g, X) := \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

$$E(g(X)) \equiv E_X(f(X)) \equiv E(g, X)$$

$$E(X) \equiv E_X(X) \equiv E(\text{id}, X)$$

$$(\text{Uniform}(?, ?) : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbf{CRV}_{\mathbb{R}})$$

$$\text{Uniform}(a, b) \in \mathbf{CRV}_{(a,b)}$$

$$\text{Exp}(?) : \mathbb{R}^+ \rightarrow \mathbf{CRV}_{\{0\} \cup \mathbb{R}^+}$$

$$\text{N}(?, ?) : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbf{CRV}_{\mathbb{R}}$$

$$\Phi \equiv \text{N}(0, 1)$$

## 7 Joint Random Variables

$$P_{??}(\cdot, \cdot) : (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{RV}} \mathbb{Y}) \rightarrow \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R}) \rightarrow [0, 1]$$

$$P_{XY}(B_X, B_Y) := P(X^{-1}(B_X) \cap Y^{-1}(B_Y))$$

$$F_{??}(\cdot, \cdot) : (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{RV}} \mathbb{Y}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow [0, 1]$$

$$F_{XY}(x, y) := P_{XY}(X \leq x, Y \leq y)$$

$$p_{??}(\cdot, \cdot) : (S \xrightarrow{\text{DRV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{DRV}} \mathbb{Y}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow [0, 1]$$

$$p_{XY}(x, y) := P_{XY}(X = x, Y = y)$$

$$f_{??}(\cdot, \cdot) : (S \xrightarrow{\text{CRV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{CRV}} \mathbb{Y}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$$

$$\forall B_X Y \subseteq \mathbb{R} \times \mathbb{R}, P_{XY}(B_X Y) := \int_{(x,y) \in B_{XY}} f_{XY}(x, y) dx dy$$

(exists iff  $X$  and  $Y$  are jointly continuous)

$$P_{?|?}(\cdot | \cdot) : (S \xrightarrow{\text{RV}} \mathbb{Y}) \rightarrow (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$$

$$P_{Y|X}(B_Y | B_X) := \frac{P_{XY}(B_X, B_Y)}{P_X(B_X)}$$

$$f_{?|?}(\cdot | \cdot) : (S \xrightarrow{\text{RV}} \mathbb{Y}) \rightarrow (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$$

$$f_{Y|X}(y | x) := \frac{f_{XY}(x, y)}{f_X(x)}$$

$$E(\cdot, \cdot, \cdot) : (\mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}) \rightarrow (S \xrightarrow{\text{DRV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{DRV}} \mathbb{Y}) \rightarrow \mathbb{R}$$

$$E(g, X, Y) := \sum_{y \in \mathbb{Y}} \sum_{x \in \mathbb{X}} g(x, y) p_{XY}(x, y)$$

$$E(\cdot, \cdot, \cdot) : (\mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}) \rightarrow (S \xrightarrow{\text{CRV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{CRV}} \mathbb{Y}) \rightarrow \mathbb{R}$$

$$E(g, X, Y) := \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} g(x, y) f_{XY}(x, y) dx dy$$

$$E_{XY}(g(X, Y)) \equiv E(g, X, Y)$$

$$E(? \mid ? = ?) : (S \xrightarrow{\text{DRV}} \mathbb{Y}) \rightarrow (S \xrightarrow{\text{DRV}} \mathbb{X}) \rightarrow \mathbb{R} \rightarrow \mathbb{R}$$

$$E(Y \mid X = x) := \sum_{y \in \mathbb{Y}} y p_{Y|X}(y \mid x)$$

$$E(? \mid ? = ?) : (S \xrightarrow{\text{CRV}} \mathbb{Y}) \rightarrow (S \xrightarrow{\text{CRV}} \mathbb{X}) \rightarrow \mathbb{R} \rightarrow \mathbb{R}$$

$$E(Y \mid X = x) := \int_{y=-\infty}^{\infty} y f_{Y|X}(y \mid x) dy$$

$$\text{Cov}(?, ?) : (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{RV}} \mathbb{Y}) \rightarrow \mathbb{R}$$

$$\text{Cov}(X, Y) := E_{XY}((X - E_X(X))(Y - E_Y(Y)))$$

$$\sigma_{XY} \equiv \text{Cov}(X, Y)$$

$$\text{Cor}(?, ?) : (S \xrightarrow{\text{RV}} \mathbb{X}) \rightarrow (S \xrightarrow{\text{RV}} \mathbb{Y}) \rightarrow \mathbb{R}$$

$$\text{Cor}(X, Y) := \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$\rho_{XY} \equiv \text{Cor}(X, Y)$$

$$\Gamma : \mathbb{R}^+ \rightarrow \mathbb{R}$$

$$\Gamma(\alpha) := \int_0^{\infty} t^{\alpha-1} e^{-t} dt$$

$$\text{Gamma}(?, ?) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \rightarrow \mathbf{CRV}_{\mathbb{R}^+}$$

$$B(?, ?) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \rightarrow \mathbb{R}$$

$$B(\alpha, \beta) := \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\text{Beta}(?, ?) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \rightarrow \mathbf{CRV}_{(0,1)}$$

## 8 Estimation

$$\underline{X} \in (\Theta \rightarrow \mathbf{RV}_{\mathbb{X}})^n \text{ i.i.d}$$

$$t : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$T : \Theta \rightarrow \mathbf{RV}_{\mathbb{R}}$$

$$T := t \circ \underline{X}$$

$$T \mid \theta \equiv T(\theta)$$

for some definitions  $T$  is a “rule” that works for any  $n$ , so

$$t : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R} \text{ (finite subsets only)}$$

$$\text{bias}(?, ?) : (\Theta \rightarrow \mathbf{RV}_{\mathbb{R}}) \rightarrow \Theta \rightarrow \mathbb{R}$$

$$\text{bias}(T, \theta) := E(T - \theta \mid \theta) = E(T \mid \theta) - \theta$$

$$L_?(? \mid ?) : \mathbf{DRV}_{\mathbb{X}} \rightarrow \Theta \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}$$

$$L_X(\theta \mid \underline{x}) := \prod_{i=1}^n p_{X|\theta}(x_i)$$

$$L_?(? \mid ?) : \mathbf{CRV}_{\mathbb{X}} \rightarrow \Theta \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}$$

$$L_X(\theta \mid \underline{x}) := \prod_{i=1}^n f_{X|\theta}(x_i)$$

$$\ell = \log \circ L$$

## 9 Hypothesis Testing

## 10 Convergence Concepts