This document describes the domains and ranges of functions by using $\to$ as a right-associative binary operator on sets, like in Haskell. Additionally, ?s are used to indicate how the arguments are written. For instance, if a function is applied as $g_x(y)$ then we write $g_?(?) : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$. The function declarations are always written fully curried even if the applications are uncurried. The order of the ?s corresponds to the order of the sets in the declaration.

4 Probability

Let $\mathcal{F} \subseteq \mathcal{P}(S)$ be a σ-algebra
Let $P : \mathcal{F} \to \mathbb{R}$ be a probability measure.

$$\begin{aligned} P(? \mid ?) &: \mathcal{F} \to \mathcal{F} \to \mathbb{R} \\ P(E \mid F) &:= \frac{P(E \cap F)}{P(F)} \end{aligned}$$

5 Discrete Random Variables

Let $S \xrightarrow{\text{RV}} \mathbb{X}$ be the set of all random variables with sample space S and range $\mathbb{X}$.

$$\begin{aligned} P_?(?) &: (S \xrightarrow{\text{RV}} \mathbb{X}) \to \mathcal{P}(\mathbb{R}) \to \mathbb{R} \\ P_X(X \in B) &\equiv P_X(B) \\ P_X(X \in B) &:= P(\{s \in S : X(S) \in B\}) \end{aligned}$$

$$\begin{aligned} \sim &\subseteq (\mathbf{RV}_{\mathbb{X}})^2 \\ X \sim Y &\iff P_X = P_Y \end{aligned}$$

Most of the functions in this course are written as if they depend on X, but actually depend only on P_X. To make this clear, I will write $g : \mathbf{RV}_{\mathbb{X}} \to B$ instead of $g : (S \xrightarrow{\text{RV}} \mathbb{X}) \to B$ whenever $X \sim Y \implies g(X) = g(Y)$. The symbols $\xrightarrow{\text{DRV}}$, $\mathbf{DRV}$, $\xrightarrow{\text{CRV}}$, $\mathbf{CRV}$ are defined similarly but with the added constraint of discreteness/continuity.

$$\begin{aligned} F_?(?) &: \mathbf{RV}_{\mathbb{X}} \to \mathbb{R} \to [0, 1] \\ F_X(x) &:= P_X(X \le x) \end{aligned}$$

$$
\begin{aligned}
p_?(x) &: \mathbf{DRV}_{\mathbb{X}} \to \mathbb{R} \to [0,1] \\
p_X(x) &:= \begin{cases} P_X(X = x) & x \in \mathbb{X} \\ 0 & x \notin \mathbb{X} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
E(?,?) &: (\mathbb{R} \to \mathbb{R}) \to \mathbf{DRV}_{\mathbb{X}} \to \mathbb{R} \\
E(g, X) &:= \sum_{x \in \mathbb{X}} g(x) p_X(x) \\
E(g(X)) &\equiv E_X(g(X)) \equiv E(g, X) \\
E(X) &\equiv E_X(X) \equiv E(\mathrm{id}, X)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Bernoulli}(?) &: [0,1] \to \mathbf{DRV}_{\{0,1\}} \\
\mathrm{Binomial}(?,?) &: \mathbb{N} \to [0,1] \to \mathbf{DRV}_{\{0,\ldots,n\}} \\
\mathrm{Geometric}(?) &: [0,1] \to \mathbf{DRV}_{\mathbb{Z}^+} \\
\mathrm{Poisson}(?) &: \mathbb{R}^+ \to \mathbf{DRV}_{\mathbb{N}} \\
\mathrm{U}(?) &: A \to \mathbf{DRV}_A \\
&\quad \text{where } A \text{ is any finite set}
\end{aligned}
$$

6 Continuous Random Variables

$$
\begin{aligned}
f_?(?) &: \mathbf{CRV}_{\mathbb{X}} : \mathbb{X} \to \mathbb{R} \\
\forall B \subseteq \mathbb{R},\ P_X(X \in B) &:= \int_{x \in B} f_X(x) dx
\end{aligned}
$$

$$
\begin{aligned}
E(?,?) &: (\mathbb{R} \to \mathbb{R}) \to \mathbf{CRV}_{\mathbb{X}} \to \mathbb{R} \\
E(g, X) &:= \int_{-\infty}^{\infty} g(x) f_X(x) dx \\
E(g(X)) &\equiv E_X(f(X)) \equiv E(g, X) \\
E(X) &\equiv E_X(X) \equiv E(\mathrm{id}, X)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{Uniform}(?,?) &: \mathbb{R} \to \mathbb{R} \to \mathbf{CRV}_{\mathbb{R}}) \\
\mathrm{Uniform}(a, b) &\in \mathbf{CRV}_{(a,b)} \\
\mathrm{Exp}(?) &: \mathbb{R}^+ \to \mathbf{CRV}_{\{0\} \cup \mathbb{R}^+} \\
\mathrm{N}(?,?) &: \mathbb{R} \to \mathbb{R} \to \mathbf{CRV}_{\mathbb{R}} \\
\Phi &\equiv \mathrm{N}(0,1)
\end{aligned}
$$

7 Joint Random Variables

$$\begin{aligned}
P_{??}(?,?) &: (S \xrightarrow{\text{RV}} \mathbb{X}) \to (S \xrightarrow{\text{RV}} \mathbb{Y}) \to \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}) \to [0,1] \\
P_{XY}(B_X, B_Y) &:= P(X^{-1}(B_X) \cap Y^{-1}(B_Y))
\end{aligned}$$

$$\begin{aligned}
F_{??}(?,?) &: (S \xrightarrow{\text{RV}} \mathbb{X}) \to (S \xrightarrow{\text{RV}} \mathbb{Y}) \to \mathbb{R} \to \mathbb{R} \to [0,1] \\
F_{XY}(x,y) &:= P_{XY}(X \leq x, Y \leq y)
\end{aligned}$$

$$\begin{aligned}
p_{??}(?,?) &: (S \xrightarrow{\text{DRV}} \mathbb{X}) \to (S \xrightarrow{\text{DRV}} \mathbb{Y}) \to \mathbb{R} \to \mathbb{R} \to [0,1] \\
p_{XY}(x,y) &:= P_{XY}(X = x, Y = y)
\end{aligned}$$

$$\begin{aligned}
f_{??}(?,?) &: (S \xrightarrow{\text{CRV}} \mathbb{X}) \to (S \xrightarrow{\text{CRV}} \mathbb{Y}) \to \mathbb{R} \to R \to \mathbb{R} \\
\forall B_X Y \subseteq \mathbb{R} \times \mathbb{R},\ P_{XY}(B_X Y) &:= \int_{(x,y) \in B_{XY}} f_{XY}(x,y) dx dy \\
&\quad \text{(exists iff } X \text{ and } Y \text{ are jointly continuous)}
\end{aligned}$$

$$\begin{aligned}
P_{?|?}(?\ |?) &: (S \xrightarrow{\text{RV}} \mathbb{Y}) \to (S \xrightarrow{\text{RV}} \mathbb{X}) \to \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}) \to \mathbb{R} \\
P_{Y|X}(B_Y \mid B_X) &:= \frac{P_{XY}(B_X, B_Y)}{P_X(B_X)}
\end{aligned}$$

$$\begin{aligned}
f_{?|?}(?\ |?) &: (S \xrightarrow{\text{RV}} \mathbb{Y}) \to (S \xrightarrow{\text{RV}} \mathbb{X}) \to \mathbb{R} \to \mathbb{R} \to \mathbb{R} \\
f_{Y|X}(y \mid x) &:= \frac{f_{XY}(x,y)}{f_X(x)}
\end{aligned}$$

$$\begin{aligned}
E(?,?,?) &: (\mathbb{R} \to \mathbb{R} \to \mathbb{R}) \to (S \xrightarrow{\text{DRV}} \mathbb{X}) \to (S \xrightarrow{\text{DRV}} \mathbb{Y}) \to \mathbb{R} \\
E(g, X, Y) &:= \sum_{y \in \mathbb{Y}} \sum_{x \in \mathbb{X}} g(x,y) p_{XY}(x,y)
\end{aligned}$$

$$\begin{aligned}
E(?,?,?) &: (\mathbb{R} \to \mathbb{R} \to \mathbb{R}) \to (S \xrightarrow{\text{CRV}} \mathbb{X}) \to (S \xrightarrow{\text{CRV}} \mathbb{Y}) \to \mathbb{R} \\
E(g, X, Y) &:= \int_{y=-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy
\end{aligned}$$

$$E_{XY}(g(X,Y)) \equiv E(g, X, Y)$$

$$
\begin{aligned}
E(?\mid ?=?) &: (S \xrightarrow{\text{DRV}} \mathbb{Y}) \to (S \xrightarrow{\text{DRV}} \mathbb{X}) \to \mathbb{R} \to \mathbb{R} \\
E(Y \mid X = x) &:= \sum_{y\in\mathbb{Y}} y p_{Y|X}(y \mid x)
\end{aligned}
$$

$$
\begin{aligned}
E(?\mid ?=?) &: (S \xrightarrow{\text{CRV}} \mathbb{Y}) \to (S \xrightarrow{\text{CRV}} \mathbb{X}) \to \mathbb{R} \to \mathbb{R} \\
E(Y \mid X = x) &:= \int_{y=-\infty}^{\infty} y f_{Y|X}(y \mid x) dy
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(?,?) &: (S \xrightarrow{\text{RV}} \mathbb{X}) \to (S \xrightarrow{\text{RV}} \mathbb{Y}) \to \mathbb{R} \\
\mathrm{Cov}(X,Y) &:= E_{XY}((X - E_X(X))(Y - E_Y(Y))) \\
\sigma_{XY} &\equiv \mathrm{Cov}(X,Y)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cor}(?,?) &: (S \xrightarrow{\text{RV}} \mathbb{X}) \to (S \xrightarrow{\text{RV}} \mathbb{Y}) \to \mathbb{R} \\
\mathrm{Cor}(X,Y) &:= \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \\
\rho_{XY} &\equiv \mathrm{Cor}(X,Y) \\
\Gamma &: \mathbb{R}^+ \to \mathbb{R} \\
\Gamma(\alpha) &:= \int_0^\infty t^{\alpha-1} e^{-t} dt
\end{aligned}
$$

$$
\mathrm{Gamma}(?,?) : \mathbb{R}^+ \to \mathbb{R}^+ \to \mathbf{CRV}_{\mathbb{R}^+}
$$

$$
\begin{aligned}
&B(?,?) : \mathbb{R}^+ \to \mathbb{R}^+ \to \mathbb{R} \\
&B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}
\end{aligned}
$$

$$
\mathrm{Beta}(?,?) : \mathbb{R}^+ \to \mathbb{R}^+ \to \mathbf{CRV}_{(0,1)}
$$

8 Estimation

$$
\begin{aligned}
\underline{X} &\in (\Theta \to \mathbf{RV}_{\mathbb{X}})^n \text{ i.i.d} \\
t &: \mathbb{R}^n \to \mathbb{R} \\
T &: \Theta \to \mathbf{RV}_{\mathbb{R}} \\
T &:= t \circ \underline{X}
\end{aligned}
$$

$$T \mid \theta \equiv T(\theta)$$

for some definitions T is a "rule" that works for any n, so

$$t : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \text{ (finite subsets only)}$$

$$
\begin{aligned}
\text{bias}(?,?) &: (\Theta \to \mathbf{RV}_{\mathbb{R}}) \to \Theta \to \mathbb{R} \\
\text{bias}(T,\theta) &:= E(T - \theta \mid \theta) = E(T \mid \theta) - \theta \\
L_?(? \mid ?) &: \mathbf{DRV}_{\mathbb{X}} \to \Theta \to \mathbb{R}^n \to \mathbb{R} \\
L_X(\theta \mid \underline{x}) &:= \prod_{i=1}^{n} p_{X|\theta}(x_i)
\end{aligned}
$$

$$
\begin{aligned}
L_?(? \mid ?) &: \mathbf{CRV}_{\mathbb{X}} \to \Theta \to \mathbb{R}^n \to \mathbb{R} \\
L_X(\theta \mid \underline{x}) &:= \prod_{i=1}^{n} f_{X|\theta}(x_i)
\end{aligned}
$$

$$\ell = \log \circ L$$