MATH50001 Analysis 2 Term 2

also known as

MATH50018 Complex Analysis for JMC

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Contents

1 Holomorphic functions

1.1 Basic properties

A complex number has the form $z = x + iy$ with $x, y \in \mathbb{R}$ and $i^2 = -1$.

$$\begin{aligned} \operatorname{Re}(x+iy) &:= x \\ \operatorname{Im}(x+iy) &:= y \\ \overline{x+iy} &:= x - iy \end{aligned}$$

They can also be written in polar form:

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) =: r\operatorname{cis}\theta \\ |z| := r &= \sqrt{x^2+y^2} \\ \arg z &:= \theta \end{aligned}$$

Note that the argument is not unique; it can be increased or decreased by any integer multiple of 2π with no effect.

Definition 1.1. The **principal argument** of $z = r\operatorname{cis}\theta$ is $\operatorname{Arg} z := \theta, \theta \in (-\pi, \pi]$

Theorem 1.1. $r_1 \operatorname{cis}\theta_1 r_2 \operatorname{cis}\theta_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

Corollary 1.2. *(De Moivre's formula)* $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$

Note that $\arg z_1 + \arg z_2 = \arg z_1 z_2$ but $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \neq \operatorname{Arg} z_1 z_2$

1.2 Sets in the complex plane

Definition 1.2. For $z_0 \in \mathbb{C}, r > 0$, we have the **open disc** $D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ and the **circle** $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$.

Definition 1.3. The **unit disc** is $\mathbb{D} := D_1(0)$.

Definition 1.4. For $\Omega \subset \mathbb{C}$, $z \in \Omega$ is an **interior point** of $\Omega \iff \exists r > 0 : D_r(z) \subset \Omega$. The **interior** of Ω is the set of its interior points.

Definition 1.5. Ω is **open** $\iff$ all points in Ω are interior.

Definition 1.6. Ω is **closed** $\iff$ $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

Definition 1.7. The **closure** of Ω is $\overline{\Omega} = \Omega \cup$ limit points of Ω.

Definition 1.8. The **boundary** of Ω is $\partial\Omega = \overline{\Omega} \setminus$ interior of Ω.

Definition 1.9. Ω is **bounded** $\iff \exists M > 0 : \forall z \in \Omega,\ |z| < M$.

Definition 1.10. The **diameter** of Ω bounded is $\operatorname{diam}\Omega = \sup_{z,w\in\Omega} |z - w|$.

Definition 1.11. Ω is **compact** $\iff$ Ω is closed and bounded.

Theorem 1.3. *Ω is compact $\iff$ all sequences in Ω have a subsequence converging in Ω.*

Definition 1.12. An **open covering** of Ω is a family of open sets whose union contains Ω.

Theorem 1.4. *Ω is compact $\iff$ every open covering of Ω has a finite subcovering.*

Theorem 1.5. *For $\Omega_1 \supset \Omega_2 \supset \dots$ all non-empty with $\operatorname{diam} \Omega_n \to 0$, $\exists! w \in \mathbb{C} : \forall n,\ w \in \Omega_n$.*

Definition 1.13. Ω open is **connected** $\iff$ all pairs of points in Ω are joined by a curve contained in Ω.

1.3 Complex functions

A function $f : \Omega_1 \to \Omega_2$ can be written as $f(x+iy) = u(x,y)+iv(x,y)$ with $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Definition 1.14. $f : \Omega \to \mathbb{C}$ is **continuous at** $z_0 \in \Omega \iff \forall \epsilon > 0 \exists \delta > 0 : \forall z \in \Omega,\ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$.

Definition 1.15. f is **continuous** $\iff$ it is continuous at all points.

1.4 Complex derivative

Definition 1.16. f is **differentiable** or **holomorphic at** $z \in \Omega \iff$ the derivative

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. Note that $h \in \mathbb{C}$ so it can approach 0 from any direction.

Definition 1.17. f is **holomorphic on** Ω open $\iff$ f is holomorphic at all $z \in \Omega$.

Definition 1.18. f is **holomorphic on** C closed $\iff$ f is holomorphic on an open superset of C.

Definition 1.19. f is **entire** $\iff$ f is holomorphic on $\mathbb{C}$.

Proposition 1.6. *f is holomorphic at $z \iff \exists \alpha \in \mathbb{C} : f(z+h) - f(z) - ah = h\psi(h)$ with $\lim_{h \to 0} \psi(h) = 0$. In this case $\alpha = f'(z)$.*

Corollary 1.7. *f holomorphic $\implies$ f continuous.*

Proposition 1.8. *The sum, product, quotient and chain rules for differentiation of real functions hold for complex derivatives as well.*

1.5 Cauchy-Riemann equations

Definition 1.20. For $f(x,y) = u(x,y) + iv(x,y)$, the **Cauchy-Riemann Equations** are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Proposition 1.9. *f is holomorphic $\implies$ the CREs hold.*

Definition 1.21. The **partial derivative with respect to a complex number** $z = x + iy$ is $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$.

Theorem 1.10. *If f is holomorphic at z_0 then $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$ and $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$*

Theorem 1.11. *If u and v are continuously differentiable and the CREs hold on Ω then f is holomorphic on Ω with $f'(z) = \frac{\partial f(z)}{\partial z}$*

1.6 Cauchy-Riemann equations in polar coordinates

Proposition 1.12. *In polar coordinates, the CREs become* $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ *and* $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

1.7 Power series

Definition 1.22. A **power series** has the form $S(z) := \sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \mathbb{C}$. It exists at some z iff the sum converges there. It is **absolutely convergent** at z iff $\sum_{n=0}^{\infty} |a_n||z^n|$ converges.

Definition 1.23. The **radius of convergence** of a series is $R \in [0, \infty]$ defined by

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$$

The **disc of convergence** is $D_R(0)$.

Theorem 1.13. *If the power series* $S(z)$ *has a radius of convergence* R *then the series converges absolutely if* $|z| < R$ *and diverges if* $|z| > R$*.*

Theorem 1.14. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is holomorphic in its disc of convergence with* $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$*.* f' *has the same radius of convergence as* f*.*

Corollary 1.15. *Power series are infinitely differentiable.*

1.8 Elementary functions

Definition 1.24. The **exponential function** is $e^{x+iy} := e^x \operatorname{cis} y$.

Proposition 1.16. e^z *is entire.*

Definition 1.25. The **sine** and **cosine** of a complex number are

$$\sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$
$$\cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Proposition 1.17. $\sin z$ *and* $\cos z$ *are entire and obey the usual identities for their real-valued counterparts.*

Definition 1.26. A **complex logarithm** of z is $\log z := \ln|z| + i \arg z = \log r + i(\theta + 2\pi k)$. Note that $e^{\log z} = z$, and that $\log z$ is multi-valued.

Definition 1.27. The **principal value** of the complex logarithm is $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$.

Proposition 1.18. $\log z_1 z_2 = \log z_1 \log z_2$ *but* $\operatorname{Log} z_1 z_2 \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$.

Proposition 1.19. $\operatorname{Log} z$ *is holomorphic on* $\mathbb{C} \setminus (-\infty, 0]$

Definition 1.28. For $\alpha \in \mathbb{C}$ we have the multi-valued function $z^\alpha = e^{\alpha \log z}$ and its principal value $e^{\alpha \operatorname{Log} z}$.

Proposition 1.20. $z^\alpha z^\beta = z^{\alpha+\beta}$

2 Cauchy's integral formulae

2.1 Parametrised curves

Definition 2.1. A **parametrised curve** is a function $\gamma : [a,b] \to \mathbb{C}$. We use the symbol γ also to refer to the set of values of γ.

Definition 2.2. γ is **smooth** $\iff$ it is continuously differentiable with $\gamma' \neq 0$ anywhere, where $\gamma'(a)$ and $\gamma'(b)$ are defined using one-sided limits.

Definition 2.3. γ is **piecewise-smooth** $\iff$ it is continuous and there are finitely many points in $[a,b]$ such that γ is smooth on the intervals between them.

Definition 2.4. $\gamma : [a,b] \to \mathbb{C}$ and $\tilde{\gamma} : [c,d] \to \mathbb{C}$ are **equivalent** $\iff$ there exists $t : [c,d] \to [a,b]$ bijective and continuously differentiable such that $t'(s) > 0$ and $\tilde{\gamma} = \gamma \circ t$. This is an equivalence relation.

In all the definitions in this module, if two curves are equivalent then they are interchangeable.

Definition 2.5. For $\gamma : [a,b] \to \mathbb{C}$ smooth and f continuous on γ, the **integral of f along** γ is $\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$.

Definition 2.6. The **integral along** a piecewise-smooth curve is the sum of the integrals along the smooth intervals.

Definition 2.7. For any $\gamma : [a,b] \to \mathbb{C}$ we have $\gamma^-(t) := \gamma(b + a - t)$.

Definition 2.8. γ smooth or piecewise-smooth is **closed** $\iff$ $\gamma(a) = \gamma(b)$.

Notation. We write the integral along a closed curve using $\oint$ instead of $\int$.

Definition 2.9. γ smooth or piecewise-smooth is **simple** $\iff$ $\forall s,t \in (a,b),\ s \neq t \implies \gamma(s) \neq \gamma(t)$.

2.2 Integration along curves

Definition 2.10. The **length** of γ smooth is $\operatorname{length}\gamma := \int_a^b |\gamma'(t)|dt$.

Theorem 2.1. *Integration along curves is linear.* $\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$. *ML-inequality:* $|\int_\gamma f(z)dz| \leq \sup_{z\in\gamma}|f(z)| \operatorname{length}\gamma$.

2.3 Primitive functions

Definition 2.11. A **primitive** for f on Ω is F holomorphic on Ω with $F' = f$.

Theorem 2.2. *If f continuous has a primitive F in an open set Ω containing $\gamma : [a,b] \to \mathbb{C}$ then* $\int_\gamma f(z)dz = F(b) - F(a)$.

Corollary 2.3. *If γ is closed and f has a primitive then* $\oint_\gamma f(z)dz = 0$.

Corollary 2.4. *If f is holomorphic in an open connected set with $f' = 0$ then f is constant.*

2.4 Properties of holomorphic functions

Theorem 2.5. *For Ω open, T a triangle contained (along with its interior) in Ω, and f holomorphic in Ω, $\oint_T f(z)dz = 0$.*

Corollary 2.6. *The above theorem also holds for rectangles.*

2.5 Local existence of primitives and Cauchy-Goursat theorem in a disc

Theorem 2.7. *A function which is holomorphic in an open disc has a primitive in that disc.*

Corollary 2.8. *(Cauchy-Goursat Theorem for a disc) If f is holomorphic in a disc and γ is a closed curve in that disc then $\oint_\gamma f(z)dz = 0$*

Corollary 2.9. *For Ω open, C a circle contained (along with its interior) in Ω, and f holomorphic in Ω, $\oint_C f(z)dz = 0$.*

2.6 Homotopies and simply connected domains

Definition 2.12. $\gamma_0 : [a,b] \to \Omega$ and $\gamma_1 : [a,b] \to \Omega$ are **homotopic** in $\Omega \iff \forall s \in (a,b)$, $\exists \gamma_s : [a,b] \to \Omega$ such that all the $\gamma_s(t)$ are jointly continuous in $s \in [a,b]$ and $t \in [a,b]$ (including the two original curves!)

Theorem 2.10. *If γ_0 and γ_1 are homotopic and f is holomorphic in Ω then $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$.*

Definition 2.13. An open set Ω is **simply connected** if all pairs of curves in Ω with the same endpoints are homotopic.

Theorem 2.11. *Any holomorphic function in a simply connected domain has a primitive.*

Corollary 2.12. *(Cauchy-Goursat Theorem) For Ω simply connected, $\gamma \subset \Omega$ closed and piecewise-smooth, and f holomorphic in Ω, $\oint_\gamma f(z)dz = 0$.*

Theorem 2.13. *(Deformation Theorem) For γ_1 and γ_2 simple, closed and piecewise-smooth with γ_2 wholly inside γ_1 and f holomorphic in the region between γ_1 and γ_2, $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$*

2.7 Cauchy's integral formulae

Theorem 2.14. *For γ simple, closed and piecewise-smooth, f holomorphic inside and on γ and z_0 inside γ*

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

Theorem 2.15. *(Generalised Cauchy's integral formula) For Ω open and f holomorphic in Ω, f has infinitely many complex derivatives in Ω. Furthermore, for $\gamma \subset \Omega$ simple, closed and piecewise-smooth, and z inside γ*

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}} d\eta$$

Corollary 2.16. *If f is holomorphic then so are all its derivatives.*

3 Applications of Cauchy's integral formulae

3.1 Applications of Cauchy's integral formulae

Corollary 3.1. *(Liouville's Theorem) A bounded entire function is constant.*

Theorem 3.2. *(Fundamental Theorem of Algebra) Any polynomial of degree greater than 0 with complex coefficients has at least one zero.*

Corollary 3.3. *Every polynomial $P(z)$ of degree $n > 0$ with complex coefficients has precisely n roots $w_1, w_2, \ldots, w_n \in \mathbb{C}$. Furthermore, $P(z) = a_n \prod_{k=1}^{n}(z - w_k)$ where a_n is the leading coefficient of $P(z)$.*

Theorem 3.4. *(Morera's Theorem) If D is an open disc and f is continuous in D such that for any triangle $T \subset D$ we have $\oint_T f(z)dz = 0$ then f is holomorphic.*

3.2 Taylor and Maclaurin series

Theorem 3.5. *(Taylor Expansion Theorem) For Ω open, f holomorphic in Ω, $z_0 \in \Omega$, and $z \in D_r(z_0) \subset \Omega$ for some r*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

Definition 3.1. The sum above is the **Taylor series** of f about z_0. If $z_0 = 0$ it is the **Maclaurin series** of f.

3.3 Sequences of holomorphic functions

Theorem 3.6. *If a sequence of holomorphic functions converges uniformly to f in every compact subset of Ω then f is holomorphic in Ω.*

Theorem 3.7. *If a sequence of holomorphic functions converges uniformly to f in every compact subset of Ω then the sequence of their derivatives converges uniformly to f'.*

Corollary 3.8. *If f_n is a sequence of holomorphic functions and $F(z) := \sum_{n=1}^{\infty} f_n(z)$ converges uniformly in all compact subsets of Ω then f is holomorphic in Ω.*

3.4 Holomorphic functions defined in terms of integrals

Theorem 3.9. *Given Ω open and $F : \Omega \times [0, 1] \to \mathbb{C}$ where F is continuous on $\Omega \times [0, 1]$ and $z \mapsto F(z, s)$ is holomorphic in Ω for any $s \in [0, 1]$, the function*

$$f(z) = \int_0^1 F(z, s)ds$$

is holomorphic in Ω.

3.5 Schwarz reflection principle

Notation. In this section, let Ω be open with $z \in \Omega \iff \overline{z} \in \Omega$. Then define:

$$\begin{aligned} \Omega^+ &:= \{z \in \Omega : \operatorname{Im} z > 0\} \\ I &:= \{z \in \Omega : \operatorname{Im} z = 0\} \\ \Omega^- &:= \{z \in \Omega : \operatorname{Im} z < 0\} \end{aligned}$$

Theorem 3.10. *(Symmetry principle) Suppose f^+ is holomorphic in Ω^+, f^- is holomorphic in Ω^-, and they extend continuously to I such that $\forall x \in I\, f^+(x) = f^-(x)$. Then*

$$f(z) := \begin{cases} f^+(z) & z \in \Omega^+ \\ f^+(z) & z \in I \\ f^-(z) & z \in \Omega^- \end{cases}$$

Theorem 3.11. *For f holomorphic on Ω^+ and extended continuously to I such that $f(I) \subset \mathbb{R}$, there exists F holomorphic on Ω such that $\forall z \in \Omega^+,\ F(z) = f(z)$.*

3.6 The complex logarithm

Theorem 3.12. *For Ω simply connected with $1 \in \Omega$ and $0 \notin \Omega$, there exists a branch of the logarithm $F(z) = \log_\Omega(z)$ which is holomorphic in Ω with $e^{F(z)} = z\ \forall z \in \Omega$ and $F(x) = \log x$ for $x \in \mathbb{R}$ close to 1.*

4 Meromorphic Functions

4.1 Zeros of holomorphic functions

Definition 4.1. f has a **zero of order** m at $z_0 \in \mathbb{C} \iff f(z_0) = f'(z_0) = f''(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.

Theorem 4.1. *f holomorphic has a zero of order m at $z_0 \iff f(z) = (z - z_0)^m g(z)$ for some g holomorphic at z_0 with $g(z_0) \neq 0$.*

Corollary 4.2. *For every zero of a non-constant holomorphic function, there is a neighbourhood inside of which it is the only zero.*

4.2 Laurent series

Definition 4.2. The **Laurent series** for f at z_0 is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

(if it converges)

Theorem 4.3. *(Laurent Expansion Theorem) If f is holomorphic in $D = \{z : r < |z - z_0| < R\}$ then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z_0)^{n+1}} d\eta$$

For any simple, closed and piecewise $\gamma \subset D$ whose interior contains z_0.

4.3 Poles of holomorphic functions

Definition 4.3. A **singularity** of f is z_0 such that f is not holomorphic at z_0 but every neighbourhood of z_0 contains at least one point at which f is holomorphic.

Definition 4.4. A singularity is **isolated** if there is a neighbourhood in which it is the only singularity.

Definition 4.5. For f holomorphic with an isolated singularity at z_0 and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in \{z : 0 < |z - z_0| < R\}$,

- z_0 is a **removable singularity** $\iff \forall n < 0,\ a_n = 0$
- z_0 is a **pole of order** $m \iff \forall n < -m,\ a_n = 0$ and $a_{-m} \neq 0$
- z_0 is an **essential singularity** $\iff a_n \neq 0$ for infinitely many $n < 0$.

Definition 4.6. A **simple** pole is a pole of order 1.

Definition 4.7. f is **meromorphic** $\iff$ f is holomorphic except at a set of isolated poles. This definition isn't in the lectures, but it's the section title according to Blackboard so I thought it should be included.

Theorem 4.4. *f has a pole of order m at z_0 $\iff$ $f(z) = \frac{g(z)}{(z-z_0)^m}$ with g holomorphic at z_0 with $g(z_0) \neq 0$.*

4.4 Residue theory

Definition 4.8. The **residue** of f at z_0 is $\text{Res}[f, z_0] = a_{-1}$ where

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in \{z : 0 < |z - z_0| < R\}$,

Theorem 4.5. *Suppose*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in \{z : 0 < |z - z_0| < R\}$. For $\gamma \subset \{z : 0 < |z - z_0| < R\}$ simple, closed, piecewise-smooth and containing z_0 in its interior,

$$\text{Res}[f, z_0] = \frac{1}{2\pi i} \oint_\gamma f(z) dz$$

Furthermore, if m is the largest integer such that $a_{-m} \neq 0$ then

$$\text{Res}\,[f, z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z))$$

Note that if z_0 is a pole of f then m is its order. In particular if z_0 is a simple pole ($\iff m = 1$) then

$$\text{Res}\,[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

Theorem 4.6. *For γ simple, closed and piecewise-smooth and f holomorphic inside and on γ except for singularities $z_1, \ldots z_n$ in the interior of γ,*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}\,[f, z_j]$$

4.5 The argument principle

Theorem 4.7. *(Principle of the Argument) For*

- *Ω open*
- *f holomorphic in Ω except for a finite number of poles*
- *$\gamma \subset \Omega$ simple, closed, piecewise-smooth and not passing through any zeroes or poles of f*
- *$N =$ the sum of the orders of the zeroes of f inside γ*
- *$P =$ the sum of the orders of the poles of f inside γ*

we have

$$\oint_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i(N - P)$$

Notation. Let $N_\gamma(f)$ be the sum of the orders of the zeroes of f inside γ.

Theorem 4.8. *(Rouche's Theorem) For Ω open, $\gamma \subset \Omega$ simple, closed and piecewise-smooth with interior $\subset \Omega$ and f and g holomorphic in Ω with $|f(z)| > |g(z)|$ for $z \in \gamma$, we have $N_\gamma(f + g) = N_\gamma(f)$.*

5 Harmonic Functions

5.1 Open mapping theorem and maximum modulus principle

Definition 5.1. f is **open** $\iff$ (Ω open $\implies f(\Omega)$ open).

Theorem 5.1. *(Open Mapping Theorem) If Ω is open and f is holomorphic and non-constant in Ω then f is open.*

Theorem 5.2. *(Maximum Modulus Principle) If Ω is open and f is holomorphic and non-constant in Ω then $|f|$ does not attain a maximum in Ω.*

Corollary 5.3. *For Ω open with $\overline{\Omega}$ compact and f holomorphic on Ω and continuous on $\overline{\Omega}$,*

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \overline{\Omega} \setminus \Omega} |f(z)|$$

5.2 Evaluation of definite integrals

This section has no new content.

5.3 Harmonic functions

Definition 5.2. The **Laplace operator** Δ transforms a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ into

$$\Delta\varphi(x, y) := \frac{\partial^2\varphi}{\partial x^2}(x, y) + \frac{\partial^2\varphi}{\partial y^2}(x, y)$$

Definition 5.3. For $\Omega \subset \mathbb{R}^2$ open, $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is **harmonic** in $\Omega \iff \forall(x, y) \in \Omega,\ \Delta\phi(x, y) = 0$.

Theorem 5.4. *If $u(x, y) + iv(x, y)$ is holomorphic then u and v are harmonic.*

Theorem 5.5. *(Harmonic Conjugate) For an open disc $D \subset \mathbb{C}$, u and harmonic in D, there exists v harmonic such that $u + iv$ is holomorphic in D.*

Definition 5.4. In the situation above, v is the **harmonic conjugate** of u.

Proposition 5.6. *In a simply connected Ω, the harmonic conjugate of any harmonic u is*

$$v(x, y) = \int_\gamma \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right)$$

where γ is any curve starting at a fixed point (x_0, y_0) and ending at (x, y). The proof of this statement, in particular the fact that $v(x, y)$ is independent of γ, is out of the scope of the course.

5.4 Properties of real and imaginary parts of holomorphic functions

Theorem 5.7. *For:*

- *Ω open and connected*
- *$f = u + iv$ holomorphic in Ω*
- *$C, K \in \mathbb{R}$*
- *$\gamma_1 := \{x + iy : u(x, y) = C\}$*
- *$\gamma_2 := \{x + iy : v(x, y) = K\}$*
- *$x_0 + iy_0 \in \Omega$ where $u(x_0, y_0) = C$ and $v(x_0, y_0) = K$ (i.e $x_0 + iy_0 \in \gamma_1 \cap \gamma_2$) and $f'(x + iy_0) \neq 0$*

γ_1 and γ_2 are orthogonal at $x_0 + iy_0$.

6 Conformal Mappings

6.1 Preservation of angles

Theorem 6.1. *(Angle Preservation Theorem) For Ω open, f holomorphic in Ω, $\gamma_1 : [0,1] \to \Omega$, $\gamma_2 : [0,1] \to \Omega$ and $z_0 \in \Omega$ with $z_0 = \gamma_1(0) = \gamma_2(0)$ where $\gamma_1'(0)$, $\gamma_2'(0)$ and $f'(z_0)$ are all non-zero,*

$$\arg \gamma_2'(0) - \arg \gamma_1'(0) \equiv \arg f(\gamma_2'(0)) - \arg f(\gamma_1'(0)) \pmod{2\pi}$$

Definition 6.1. f is **conformal** in Ω open $\iff$ f is holomorphic in Ω and $\forall z \in \Omega,\ f'(z) \neq 0$

Definition 6.2. f is a **local injection** on Ω open $\iff$ $\exists D_r(z_0) \in \Omega$ such that the restriction of f to D is injective.

Definition 6.3. If f is a holomorphic local injection on Ω then it is conformal on Ω. In particular, $f^{-1} : f(\Omega) \to \Omega$ is holomorphic, and in general the inverse of a conformal mapping is holomorphic.

Definition 6.4. Two open sets are **conformally equivalent** $\iff$ there is a bijective conformal mapping between them.

6.2 Möbius transformations

According to Blackboard this subsection and the next should be in section one, but in the lectures they're here.

Definition 6.5. A **Möbius** or **bilinear transformation** has the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Proposition 6.2. *A Möbius transformation is holomorphic and conformal everywhere except at the simple pole $-\frac{d}{c}$*

Theorem 6.3. *If f and g are Möbius transformations then so are f^{-1} and $f \circ g$.*

Proposition 6.4. *$f(z) = \frac{az+b}{cz+d}$ has a simple geometric interpretation in the following cases:*

1. *If $a = re^{i\theta}$, $b = 0$, $c = 0$ and $d = 1$ then $f(z) = az$ is expansion by r followed by anticlockwise rotation by θ*
2. *If $a = 1$, $c = 0$ and $d = 1$ then $f(z) = z + b$ is translation by b.*
3. *If $a = 0$, $b = 1$, $c = 1$ and $d = 0$ then $f(z) = \frac{1}{z}$ is inversion*

Theorem 6.5. *Every Möbius transformation is a composition of the transformations of the three forms above.*

Corollary 6.6. *A Möbius transformation transforms a circle into a circle and an interior point into an interior point (a line is a circle with infinite radius).*

6.3 Cross-ratios Möbius transformation

Theorem 6.7. *For a Möbius transformation f and $z_1, z_2, z_3 \in \mathbb{C}$ all distinct and $z \in \mathbb{C}$,*

$$\left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right) = \left(\frac{f(z) - f(z_1)}{f(z) - f(z_3)}\right)\left(\frac{f(z_2) - f(z_3)}{f(z_2) - f(z_1)}\right)$$

6.4 Conformal mapping of a half-plane to the unit disc

Theorem 6.8. *Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and recall $\mathbb{D} = D_1(0)$. Then $\mathbb{H}$ is conformally equivalent to $\mathbb{D}$. In particular, $f : \mathbb{H} \to \mathbb{D}$ defined by*

$$f(z) = \frac{i - z}{i + z}$$

is a conformal mapping with inverse $g : \mathbb{D} \to \mathbb{H}$ defined by

$$g(z) = i\frac{1 - z}{1 + z}$$

6.5 Riemann mapping theorem

Definition 6.6. $\Omega \subset \mathbb{C}$ is **proper** $\iff$ $\Omega \neq \emptyset$ and $\Omega \neq \mathbb{C}$.

Theorem 6.9. *For Ω proper and simply connected and $z_0 \in \Omega$, there is a unique conformal map $f : \Omega \to \mathbb{D}$ where $f(z_0) = 0$ and $f'(z_0) > 0$. The proof of this statement is out of the scope of the course.*

Corollary 6.10. *Any two proper simply connected open sets are conformally equivalent.*