MATH50013 - Probability and Statistics for JMC

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Definition. A **histogram** partitions the range of a sample into **bins** and shows what number of data points in each bin. Rather than frequency, the amount shown can also be relative frequency or density.

3.1.2 Empirical CDF

Definition. The **indicator function** is defined as $I(\text{false}) := 0$ and $I(\text{true}) = 1$.

Definition. The **empirical cumulative distribution function** of a sample is

$$F_n(x) := \frac{1}{n}\sum_{i=1}^{n} I(x_i \leq x)$$

3.2 Summary Statistics

3.2.1 Measures of Location

Definition. The **arithmetic mean** is $\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x_i$.

Definition. The **geometric mean** is $x_G := (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$.

Definition. The **harmonic mean** is $x_H := n\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}$

Definition. The i**th order statistic**, written $x_{(i)}$, is the ith smallest value of the sample. For non-integer values of the form $i + \alpha$ with $\alpha \in (0, 1)$, we define

$$x_{(i+\alpha)} := (1-\alpha)x_{(i)} + \alpha x_{(i+1)}$$

Definition. The **median** is $x_{(\frac{n+1}{2})}$.

Definition. The **mode** is the most frequently occurring value. If there are multiple then the sample is **multimodal**.

3.2.2 Measures of Dispersion

Definition. The **mean square** or **sample variance** is

$$s_x^2 := \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Definition. The **root mean square** or **sample standard deviation** is

$$s_x := \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Definition. The **range** is $x_{(n)} - x_{(1)}$.

Definition. The **first quartile** is $x_{\left(\frac{1}{4}(n+1)\right)}$. The **third quartile** is $x_{\left(\frac{3}{4}(n+1)\right)}$. The **interquartile range** is the difference between the third and first quartiles.

3.2.3 Covariance and Correlation

Definition. For a sample where each data point is an (x_i, y_i) pair, the **covariance** is

$$s_{xy} := \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{\sum_{i=1}^{n} x_i y_i}{n} - \bar{x}\bar{y}$$

.

Definition. For a sample as above, the **correlation** is

$$r_{xy} := \frac{s_{xy}}{s_x s_y}$$

3.2.4 Skewness

Definition. The **skewness** is $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{s}\right)^3$.

3.3 One more visualization: the box-and-whisker plot

Definition. A **box-and-whisker plot** shows the median, first and third quartiles, points within $\frac{3}{2} \times IQR$ of the quartiles, and any outliers.

4 Probability

4.1 The formal structure

4.1.1 σ-algebras

Definition 4.1.1. A **σ-algebra associated with** S is a set $\mathcal{F}$ of subsets of S where $S \in \mathcal{F}$, $\mathcal{F}$ is closed under complements with respect to S, and $\mathcal{F}$ is closed under countable unions.

Proposition. *$\emptyset \in \mathcal{F}$. $\mathcal{F}$ is also closed under countable intersections.*

4.1.2 Probability measure

Definition 4.1.2. A **probability measure** is a function $P : \mathcal{F} \to \mathbb{R}$ where $P(E) \geq 0$ for any E, $P(S) = 1$, and for countably many disjoint sets E_i,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

. A triple $(S, \mathcal{F}, P)$ as previously defined is a **probability space**.

4.2 Interpretations of the probability space

4.3 Interpretation of the σ-algebra

4.3.1 The sample space (S)

Definition. The **sample space** S is the set of all possible outcomes of an experiment.

4.3.2 The event space ($\mathcal{F}$)

Definition. An **event** is a subset $E \subset S$. $\mathcal{F}$ is the set of all possible events being considered (which may not include all possible combinations of outcomes).

Definition. E_1 and E_2 are **mutually exclusive** iff $E_1 \cap E_2 = \emptyset$ i.e. they cannot both happen at once.

4.4 Interpretations of the probability measure (P)

4.4.1 Classical interpretation

Definition. In the **classical interpretiation**, S consists of finitely many equally likely **elementary events** and $P(E) = \frac{|E|}{|S|}$. For an infinite S, this can still be applied by replacing cardinality above with a different measure.

4.4.2 Frequentist interpretation

Definition. In the **frequentist interpretation**, when an experiment is repeated infinitely many times, the proportion of trials in which E occurs approaches $P(E)$.

4.4.3 Subjective interpretation

Definition. In the **subjective interpretation**, $P(E)$ is the degree of belief a person has that E occurs.

4.5 A few derivations from the axioms

Proposition. *For $E, F \in \mathcal{F}$,*

- $P(\emptyset) = 0$
- $P(E) \leq 1$
- $P(\overline{E}) = 1 - P(E)$
- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
- $P(E \cap \overline{F}) = P(E) - P(E \cap F)$
- $E \subseteq F \implies P(E) \leq P(F)$

4.6 Conditional Probability

Definition 4.6.1. For $P(F) > 0$ the **conditional probability of E given F** is

$$P(E \mid F) := \frac{P(E \cap F)}{P(F)}$$

Proposition. *For $P(F) > 0$,*

- *For any $E \in \mathcal{F}$, $P(E \mid F) \geq 0$*
- $P(F \mid F) = 1$
- *For $E_1, \ldots, E_n \in \mathcal{F}$ pairwise disjoint, $P\left(\bigcup_{i=1}^n E_i \mid F\right) = \sum_{i=1}^n P(E_i \mid F)$*

4.7 Independent Events

Definition 4.7.1. $E, F \in \mathcal{F}$ are **independent** iff $P(E \cap F) = P(E)P(F)$. $E_1, \dots E_n$ are **independent** iff for any subset $E_{i_1}, \dots, E_{i_l}$ we have $P\left(\bigcap_{j=1}^{l} E_{i_j}\right) = \prod_{j=1}^{l} P(E_{i_j})$.

Proposition. *E and F are independent $\implies$ E and $\overline{F}$ are independent.*

Proposition. *E and F are independent $\iff$ $P(E \mid F) = P(E)$.*

4.7.1 More Examples

4.7.2 Conditional Independence

Definition. For $E_1, E_2, F \in \mathcal{F}$, E_1 and E_2 are **conditionally independent given** F iff $P(E_1 \cap E_2 \cap F) = P(E_1 \mid F)P(E_2 \mid F)$.

4.7.3 Joint Events

Definition. When combining multiple independent experiments, a **probability table** can be used to show the probabilities of all elementary events (i.e. combinations of an elementary event in each experiment).

4.8 Bayes's Theorem

Theorem 4.9. *(Bayes's) For $E, F \in \mathcal{F}$ with $P(E) > 0$ and $P(F) > 0$,*

$$P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)}$$

Theorem 4.10. *(The Law of Total Probability) For a partition $E_1, \dots$ of S, and any $F \in \mathcal{F}$, $P(F) = \sum_i P(F \mid E_i)P(E_i)$.*

Theorem 4.11. *(Bayes's applied to a partition) For a partition $E_1, \dots$ of S with $P(E_i) > 0$ for all i and $F \in \mathcal{F}$ with $P(F) > 0$,*

$$P(E_i \mid F) = \frac{P(F \mid E_i)P(E_i)}{\sum_j P(F \mid E_j)P(E_j)}$$

4.12 More Examples

5 Discrete Random Variables

5.1 Random Variables

Definition 5.1.1. A **random variable** is a measurable mapping $X : S \to \mathbb{R}$ where $\forall x \in \mathbb{R}$, $\{s \in S : X(s) \leq x\} \in \mathcal{F}$.

Definition 5.1.2. The **range** of X is $\mathbb{X}$, the image of S under X.

Definition. The **probability distribution** of X is

$$P_X(X \in B) := P(\{s \in S : X(S) \in B\})$$

where $B \subseteq \mathbb{R}$.

Notation. For brevity we write $\{X \in B\} := \{s \in S : X(s) \in B\}$ (TODO: doesn't this make P and P_X interchangeable?) and $\{a < X \leq b\} := \{X \in (a, b]\}$ etc.

5.1.1 Cumulative Distribution Function

Definition 5.1.3. The **cumulative distribution function** of X is $F_X : \mathbb{R} \to [0,1]$ where $F_X(x) = P_X(X \leq x)$.

Definition. A function f is **right-continuous** iff for any decreasing sequence $x_i \to x$ we have $f(x_i) \to f(x)$.

Proposition. *A CDF is right-continuous.*

Proposition. *F_X is a CDF iff all the following hold:*

- *F_X is right-continuous*
- $F_X(\mathbb{R}) \subseteq [0,1]$
- *F_X is monotonically increasing*
- $\lim_{x\to-\infty} F_X(x) = 0$
- $\lim_{x\to\infty} F_X(x) = 1$

5.2 Discrete Random Variables

Definition 5.2.1. A random variable is **discrete** iff its range is finite or countably infinite.

Definition 5.2.2. For a DRV X, the **probability mass function** $p_X : \mathbb{R} \to [0,1]$ is $p_X(x) = P_X(X = x)$ for $x \in \mathbb{X}$ and $p_X(x) = 0$ for $x \notin \mathbb{X}$.

Definition. The **support** of X is $\{x \in \mathbb{R} : p_X(x) > 0\}$. Usually this is $\mathbb{X}$.

5.2.1 Properties of Mass Function p_X

Proposition. *An arbitrary function p_X can be a PMF for X iff $\forall x \in \mathbb{X},\ p_X(x) \geq 0$ and $\sum_{x\in\mathbb{X}} p_X(x) = 1$.*

5.2.2 Discrete Cumulative Distribution Function

Definition. The **cumulative distribution function** of a DRV X is $F_X(x) = P(X \leq x)$ (TODO: is this not what it always is?).

5.2.3 Connection between F_X and p_X

Proposition. *For $\mathbb{X} = \{x_1, \ldots\}$ with the $x_i \leq x_{i+1}$ for all i,*

$$F_X(x) = \sum_{x_i \leq x} p_X(x_i)$$

Equivalently,

$$\forall i \geq 1,\ p_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

5.2.4 Properties of Discrete CDF F_X

Proposition. *We have*

- $\lim_{x \to -\infty} F_X(x) = 0$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{h \to 0^+} F_X(x+h) = F_X(x)$
- $a < b \implies F_X(a) \leq F_X(b)$
- *For* $a < b$, $P(a < X \leq b) = F_X(b) - F_X(a)$

5.3 Functions of a discrete random variable

Proposition. *For a DRV* X *and* $g : \mathbb{X} \to \mathbb{R}$, $Y = g(X)$ *is also a DRV. We have*

$$p_Y(y) = \sum_{x \in \mathbb{X} : g(x) = y} p_X(x)$$

5.4 Mean and Variance

5.4.1 Expectation

Definition 5.4.1. The **expected value** or **mean** of a DRV X is

$$E_X(X) := \sum_{x \in \mathbb{X}} x p_X(x)$$

It is often abbreviated to $E(X)$.

Theorem 5.5. *For a **function of interest*** $g : \mathbb{R} \to \mathbb{R}$, *we have*

$$E(g(X)) = \sum_{x \in \mathbb{X}} g(x) p_X(x)$$

Proposition. E *is linear.*

Definition 5.5.1. For a DRV X, the **variance** of X is

$$\operatorname{Var}_X(X) := E_X\left((X - E_X(X))^2\right) = E(X^2) - E(X)^2$$

Proposition. *For* $a, b \in \mathbb{R}$, $\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$

Definition 5.5.2. For a DRV X, the **standard deviation** of X is

$$\operatorname{sd}(X) := \sqrt{\operatorname{Var}_X(X)}$$

Definition 5.5.3. For a DRV X, the **skewness** of X is

$$\gamma_1 := \frac{E_X((X - E_X(X))^3)}{\operatorname{sd}_X(X)^3}$$

5.5.1 Sums of Random Variables

Proposition. *For $X_1, \dots X_n$ (possibly with different distributions, not necessarily independent) with sum S_n, we have*

$$E(S_n) = \sum_{i=1}^{n} E(X_i)$$

and

$$E\left(\frac{S_n}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

Proposition. *For $X_1, \dots X_n$ independent with sum S_n, we have*

$$\text{Var}(S_n) = \sum_{i=1}^{n} \text{Var}(X_i)$$

and

$$\text{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i)$$

Proposition. *For $X_1, \dots X_n$ independent and identically distributed with sum S_n, $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$, we have*

$$E\left(\frac{S_n}{n}\right) = \mu$$

and $\text{Var}\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}$

5.6 Some Important Discrete Random Variables

X	$\mathbb{X}$	$p_X(x)$	$E(X)$	$\text{Var}(X)$	γ_1
$X \sim \text{Bernoulli}(p)$	$\{0, 1\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$	$\frac{1-2p}{\sqrt{p(1-p)}}*$
$X \sim \text{Binomial}(n, p)$	$\{0, \dots n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$\frac{1-2p}{\sqrt{np(1-p)}}$
$X \sim \text{Geometric}(p)$	$\{1, 2, \dots\}$	$p(1-p)^{x-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{2-p}{\sqrt{1-p}}$
$X \sim \text{Poisson}(\lambda)$	$\{0, 1, \dots\}$	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ	$\frac{1}{\sqrt{\lambda}}$
$X \sim \text{U}(\{1, \dots, n\})$	$\{1, \dots, n\}$	$\frac{1}{n}$	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$	0

$*$: The skewness of the Bernoulli distribution is not given in the official notes.

5.6.1 Bernoulli Distribution

$X \sim \text{Bernoulli}(p)$ chooses between 1 and 0 where $P(X = 1) = p$.

5.6.2 Binomial Distribution

$X \sim \text{Binomial}(n, p)$ is the total number of successes after n Bernoulli trials with probability p.

5.6.3 Geometric Distribution

$X \sim \text{Geometric}(p)$ is the number of Bernoulli trials with probability p it will take to have the first success.

5.6.4 Poisson Distribution

$X \sim \text{Poisson}(\lambda)$ is the number of occurrences of an event that occurs at a rate of λ.

5.6.5 Discrete Uniform Distribution

$X \sim \text{U}(\{1, \ldots, n\})$ is a random value out of $\{1, \ldots n\}$.

6 Continuous Random Variables

Definition 6.0.1. A random variable X is absolutely **continuous** iff there exists a measurable non-negative function $f_X : \mathbb{R} \to \mathbb{R}$ (the **probability density function**) where

$$\forall B \subseteq \mathbb{R},\ P(X \in B) = \int_{x \in B} f_X(x) dx$$

6.0.1 Continuous Cumulative Distribution Function

Definition 6.0.2. The **cumulative distribution function** of a CRV X is $F_X(x) = P(X \leq x)$ (as for any RV).

Proposition. *For a CRV X, $F_X(x) = \int_{-\infty}^{x} f_X(x')dx'$*

6.0.2 Properties of Continuous F_X and f_X

Proposition. *For a CRV X,*

- $\lim_{x \to -\infty} F_X(x) = 0$
- $\lim_{x \to \infty} F_X(x) = 1$
- *If F_X is differentiable at x then $f_X(x) = F'_X(x)$*
- $\forall a \in \mathbb{R},\ P(X = a) = 0$
- *For $a < b$, $P(a < X \leq b) = F_X(b) - F_X(a)$*
- *$f_X(X)$ is not a probability, so we do not require $f_X(x) \leq 1$*
- *X is uniquely defined by f_X*

Proposition. *An arbitrary function f_X is a PDF for a CRV iff $\forall x \in \mathbb{R},\ f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$ (f_X is **normalised**).*

6.0.3 Transformations

Proposition. *For $Y = g(X)$ with g strictly monotonically increasing, we have*

$$F_Y(y) = F_X(g^{-1}(y))$$

and

$$f_Y(y) = f_X\left(g^{-1}(y)\right) g^{-1'}(y)$$

Proposition. *For $Y = g(X)$ with g strictly monotonically decreasing, we have*

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

and

$$f_Y(y) = -f_X\left(g^{-1}(y)\right) g^{-1'}(y)$$

6.1 Mean, Variance and Quantiles

6.1.1 Expectation

Definition 6.1.1. The **mean** or **expectation** of a CRV X is

$$E(X) := \int_{-\infty}^{\infty} x f_X(x) dx$$

Definition. For any measurable **function of interest** $g : \mathbb{R} \to \mathbb{R}$ we have

$$E(g(X)) := \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Proposition. *E is linear.*

6.1.2 Variance

Definition 6.1.2. The **variance** of a CRV X is

$$\mathrm{Var}_X(X) = E((X - E(X))^2) = E(X^2) - E(X)^2$$

Proposition. *For $a, b \in \mathbb{R}$, $\mathrm{Var}(aX + b) = a^2 \,\mathrm{Var}(X)$*

6.1.3 Quantiles

Definition 6.1.3. For $\alpha \in [0, 1]$, we α-**quantile** of a CRV X is

$$Q_X(\alpha) := F_X^{-1}(\alpha)$$

so that $P(X \leq Q_X(\alpha)) = \alpha$.

6.2 Some Important Continuous Random Variables

X	$\mathbb{X}$	$f_X(x)$	$F_X(x)$	$E(X)$	$\mathrm{Var}(X)$
$X \sim \mathrm{U}(a, b)$	(a, b)	$\begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$X \sim \mathrm{Exp}(\lambda)$	$[0, \infty)$	$\lambda e^{-\lambda x}$	$1 - e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$X \sim \mathbb{N}(\mu, \sigma^2)$	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$	μ	σ^2

6.2.1 Continuous Uniform Distribution

$X \sim \mathrm{U}(a,b)$ or $X \sim \mathrm{Uniform}(a,b)$ is uniformly distributed on the interval (a,b) and 0 elsewhere.

Definition. Tht **standard uniform** is $\mathrm{Uniform}(0,1)$.

Proposition. *$X \sim \mathrm{Uniform}(0,1) \implies (a+(b-a)X) \sim \mathrm{Uniform}(a,b)$.*

6.2.2 Exponential Distribution

$X \sim \mathrm{Exp}(\lambda)$ is the time until an event occurring at rate λ occurs.

Proposition. *$X \sim \mathrm{Exp}(\lambda)$ exhibits the **Lack of Memory Property**:*

$$\forall x,t > 0,\ P(X > t+x \mid X > t) = P(X > x)$$

Proposition. *If the number of events occurring in an interval of size x is $N_x \sim \mathrm{Poisson}(\lambda x)$ then the separation between two events is $X \sim \mathrm{Exp}(\lambda)$.*

6.2.3 Normal (Gaussian) Distribution

$X \sim N(\mu,\sigma^2)$ has no obvious interpretation.

Definition. $X \sim N(0,1)$ is the **standard normal distribution** or **unit normal distribution**. It has the PDF

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

and the CDF

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$$

Proposition. *$X \sim N(0,1) \implies (\sigma X + \mu) \sim N(\mu,\sigma^2)$*

Theorem 6.3. *(Central Limit Theorem) For $X_1,\ldots,X_n$ independent and identically distributed with mean μ and variance σ^2,*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$$

6.4 Further examples

7 Joint Random Variables

Definition 7.0.1. For RVs X and Y with the same sample space, the **joint probability distribution** is $P_{XY}(B_X,B_Y) := P(X^{-1}(B_X) \cap Y^{-1}(B_Y))$ where $B_X, B_Y \subseteq \mathbb{R}$.

7.0.1 Joint Cumulative Distribution Function

Definition 7.0.2. The **joint cumulative distribution function** is $F_{xy}(x,y) := P_{XY}(X \leq x, Y \leq y)$.

Proposition. *$F_X(x) = F_{XY}(x,\infty)$ and $F_Y(y) = F_{XY}(\infty,y)$.*

7.0.2 Properties of Joint CDF F_{XY}

Proposition. *And arbitrary function F_{XY} is a valid joint CDF iff the following hold:*

- $\forall x, y \in \mathbb{R},\ F_{XY}(x,y) \in [0,1]$
- $\forall x_1, x_2, y \in \mathbb{R},\ x_1 < x_2 \implies F_{XY}(x_1, y) \leq F_{XY}(x_2, y)$
- $\forall x, y_1, y_2 \in \mathbb{R},\ y_1 < y_2 \implies F_{XY}(x, y_1) \leq F_{XY}(x, y_2)$
- $\forall x, y \in \mathbb{R},\ F_{XY}(x, -\infty) = F_{XY}(-\infty, y) = 0$
- $F_{XY}(\infty, \infty) = 1$

7.0.3 Joint Probability Mass Functions

Definition 7.0.3. For DRVs X, Y, the **joint probability mass function** is $p_{XY}(x,y) := P_{XY}(X = x, Y = y)$.

Proposition. $p_X(x) = \sum_{y \in \mathbb{Y}} p_{XY}(x,y)$ *and* $p_Y(y) = \sum_{x \in \mathbb{X}} p_{XY}(x,y)$

Proposition. *An arbitrary function p_{XY} is a valid joint PMF iff $\forall x, y \in \mathbb{R}, p_{XY}(x,y) \in [0,1]$ and $\sum_{y \in \mathbb{Y}} \sum_{x \in \mathbb{X}} p_{XY}(x,y) = 1$.*

7.0.4 Joint Probability Density Functions

Definition. CRVs X and Y are **jointly continuous** iff $\exists f_{XY} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where

$$\forall B_{XY} \subseteq \mathbb{R} \times \mathbb{R},\ P_{XY}(B_{XY}) = \int_{(x,y) \in B_{XY}} f_{XY}(x,y)dxdy$$

Then f_{XY} is the **joint probability density function** of X and Y.

Proposition. *For jointly continuous CRVs, we have*

$$F_{XY}(x,y) = \int_{t=-\infty}^{y} \int_{s=-\infty}^{x} f_{XY}(s,t)dsdt$$

Definition 7.0.4. (Not actually a definition) The joint PDF is

$$f_{XY} = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Proposition. $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)dy$ *and* $f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)dx$

Proposition. *An arbitrary function f_{XY} is a valid joint PDF iff $\forall x, y \in \mathbb{R},\ f_{XY}(x,y) \geq 0$ and $\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$.*

7.1 Independence, Conditional Probability, Expectation

7.1.1 Independence and conditional probability

Definition. RVs X and Y are **independent** iff $\forall B_X, B_Y \subseteq \mathbb{R},\ P_{XY}(B_X, B_Y) = P_X(B_X)P_Y(B_Y)$.

Definition 7.1.1. CRVs X and Y are **independent** iff $\forall x, y \in \mathbb{R},\ f_{XY}(x, y) = f_X(x)f_Y(y)$.

Definition 7.1.2. For RVs X and Y, the **conditional probability distribution** is

$$P_{Y|X}(B_Y \mid B_X) := \frac{P_{XY}(B_X, B_Y)}{P_X(B_X)}$$

Proposition. *X and Y are independent* $\iff \forall B_X, B_Y \subseteq \mathbb{R},\ P_{Y|X}(B_Y \mid B_X) = P_Y(B_Y)$.

Definition 7.1.3. For CRVs X and Y, the **conditional probability density function** is

$$f_{Y|X}(y \mid x) := \frac{f_{XY}(x, y)}{f_X(x)}$$

Proposition. *X and Y are independent* $\iff \forall x, y \in \mathbb{R},\ f_{Y|X}(y \mid x) = f_Y(y)$.

7.1.2 Expectation

Definition 7.1.4. For DRVs X and Y:

$$E_{XY}(g(X, Y)) := \sum_{y \in \mathbb{Y}} \sum_{x \in \mathbb{X}} g(x, y) p_{XY}(x, y)$$

Definition 7.1.5. For CRVs X and Y:

$$E_{XY}(g(X, Y)) := \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} g(x, y) f_{XY}(x, y) dx dy$$

Proposition. *Both versions of E are linear.*

Proposition. *$E_{XY}(g_1(X) + g_2(Y)) = E_X(g_1(X)) + E_Y(g_2(y))$. If X and Y are independent then $E_{XY}(g_1(x)g_2(y)) = E_X(g_1(x))E_Y(g_2(y))$.*

7.1.3 Conditional Expectiation

Definition 7.1.6. The **conditional expectation** of Y given $X = x$ is

$$E_{Y|X}(Y \mid X = x) := \sum_{y \in \mathbb{Y}} y p(y \mid x)$$

or

$$E_{Y|X}(Y \mid X = x) := \int_{y=-\infty}^{\infty} y f(y \mid x) dy$$

Definition. The **covariance** of X and Y is

$$\sigma_{XY} = \mathrm{Cov}(X, Y) := E_{XY}((X - E_X(X))(Y - E_Y(Y)))$$

Definition 7.1.7. The **correlation** of X and Y is

$$\rho_{XY} = \mathrm{Cor}(X, Y) := \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Proposition. *X and Y are independent* $\implies \sigma_{XY} = \rho_{XY} = 0$.

7.2 Examples

7.3 Multivariate Transformations

7.3.1 Convolutions (sums of random variables)

Theorem 7.4. *(Convolution Theorem) For independent RVs X and Y and $Z = X + Y$,*

$$p_Z(z) = \sum_{x \in \mathbb{X}} p_X(x) p_Y(z - x)$$

or

$$p_Z(z) = \int_{\mathbb{R}} f_X(x) f_Y(z - x) dx$$

Theorem 7.5. *If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent then $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.*

7.5.1 General Bivariate Transformations

Proposition. *For DRVs X and Y with $U = g_1(X, Y)$ and $V = g_2(X, Y)$,*

$$p_{UV}(u, v) = \sum_{(x,y) \in A} p_{XY}(x, y)$$

where

$$A := \{(x, y) : (g_1(x, y), g_2(x, y)) = (u, v)\}$$

Proposition. *For CRVs X and Y with $U = g_1(X, Y)$ and $V = g_2(X, Y)$, and given $u := g_1(x, y)$ and $v := g_2(x, y)$,*

$$f_{UV}(u, v) = f_{XY}(x, y) \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right|$$

where

$$A := \{(x, y) : (g_1(x, y), g_2(x, y)) = (u, v)\}$$

Definition. The **Gamma function** is $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, defined for $\alpha \in (0, \infty)$.

Proposition. *We have:*

- $\forall \alpha > 1,\ \Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha)$
- $\Gamma(1) = 1$
- $\forall n \in \mathbb{N},\ \Gamma(n) = (n - 1)!$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Definition. The **Gamma distribution** $X \sim \mathrm{Gamma}(\alpha, \beta)$ with $\alpha, \beta > 0$ has the following properties:

- $f_X(x) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$
- $\mathbb{X} = (0, \infty)$

- $E(X) = \frac{\alpha}{\beta}$
- $\mathrm{Var}(X) = \frac{\alpha}{\beta^2}$

Definition. The **Beta function** is $B(\alpha, \beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Definition. The **Beta distribution** $X \sim \mathrm{Beta}(\alpha, \beta)$ has PDF $f_X(x) := \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ and $\mathbb{X} = (0, 1)$

Theorem 7.6. *If* $X \sim \mathrm{Gamma}(\lambda, \beta)$ *and* $Y \sim \mathrm{Gamma}(\xi, \beta)$ *are independent then* $X + Y \sim \mathrm{Gamma}(\lambda + \xi, \beta)$

8 Estimation

9 Hypothesis Testing

10 Convergence Concepts