MATH50001 Analysis 2 Term 1

also known as

MATH50017 Real Analysis and Topology for JMC

Notes by Robert Weingart

Contents

1 Differentiation in higher dimensions **2**

1.1 Euclidean spaces . . . 2

1.1.1 Preliminaries from analysis 1 . . . 2

1.1.2 Euclidean space of dimension n . . . 3

1.1.3 Convergence of sequences in Euclidean spaces . . . 3

1.2 Continuity . . . 3

1.2.1 Open sets in Euclidean spaces . . . 3

1.2.2 Continuity at a point, and continuity on an open set . . . 4

1.3 Derivative of a map of Euclidean spaces . . . 4

1.3.1 Derivative as a linear map . . . 4

1.3.2 Chain rule . . . 4

1.4 Directional derivatives . . . 5

1.4.1 Rates of change and partial derivatives . . . 5

1.4.2 Relation between partial derivatives and differentiability . . . 5

1.5 Higher derivatives . . . 5

1.5.1 Higher derivatives as linear maps . . . 5

1.5.2 Symmetry of mixed partial derivatives . . . 5

1.5.3 Taylor's theorem . . . 6

1.6 Inverse and Implicit function theorems . . . 6

1.6.1 Inverse function theorem . . . 6

1.6.2 Implicit Function Theorem . . . 6

1.6.3 * Sketch of the proof of the Implicit Function Theorem . . . 6

1.6.4 The general from of the Implicit Function Theorem . . . 7

1.6.5 * Equivalence of the two theorems . . . 7

2 Metric and topological spaces **7**

2.1 Metric spaces . . . 7

2.1.1 Motivation and definition . . . 7

2.1.2 Examples of metric spaces . . . 7

2.1.3 Normed vector spaces . . . 8

2.1.4 Open sets in metric spaces . . . 8

2.1.5 Convergence in metric spaces . . . 9

2.1.6 Closed sets in metric spaces . . . 9

2.1.7 Interior, isolated, limit, and boundary points in metric spaces . . . 9

2.1.8 Continuous maps of metric spaces 10
2.2 Topological spaces 10
2.2.1 Motivation 10
2.2.2 Topology on a set 11
2.2.3 Convergence, and the Hausdorff property 12
2.2.4 Closed sets in topological spaces 12
2.2.5 Continuous maps on topological spaces 12
2.3 Connectedness 13
2.3.1 Connected sets 13
2.3.2 Continuous maps and connected sets 13
2.3.3 Path connected sets 13
2.4 Compactness 14
2.4.1 Compactness by covers 14
2.4.2 Sequential compactness 14
2.4.3 Continuous maps and compact sets 15
2.5 Completeness 15
2.5.1 Complete metric spaces and Banach space 15
2.5.2 Arzelà-Ascoli 16
2.5.3 Fixed point theorem 16

General notes

This document is a concise summary of the official course notes. It includes every theorem, lemma, proposition, corollary and definition, without any proofs or exercises. The numbering should match the official notes, but notation has been changed in many places to be more consistent.

Notation which is used throughout an entire section is defined only once at the start rather than in every statement. When a symbol has the same meaning in multiple consecutive statements, the definition is not repeated as it should be clear from the context.

Unnumbered definitions correspond to bolded terms that are not given numbers. Unnumbered propositions are results from exercises that I though were important enough to include.

1 Differentiation in higher dimensions

1.1 Euclidean spaces

1.1.1 Preliminaries from analysis 1

Definition. Some definitions from year 1:

$$\mathbb{N} := \{1, 2, 3, \dots\}$$
$$\mathbb{Z} := \{\dots,\}$$

1.1.2 Euclidean space of dimension n

Definition. The **n-dimensional Euclidean space** $\mathbb{R}^n$ is the set of all ordered n-tuples of members (**coordinates**) of $\mathbb{R}$. In this module, the ith coordinate of $x \in \mathbb{R}^n$ is denoted x^i. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, with addition and scalar multiplication applied to each element.

Definition. The **inner product** on $\mathbb{R}^n$ is

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$\langle x, y \rangle := \sum_{i=1}^{n} x^i y^i$$

Definition. The **length** or **norm** on $\mathbb{R}^n$ is

$$\|\cdot\| : \mathbb{R}^n \to [0, \infty)$$
$$\|x\| := \sqrt{\langle x, x \rangle}$$

Proposition. *For all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$*

- $\|x\| \geq 0$
- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- *Triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$

1.1.3 Convergence of sequences in Euclidean spaces

Definition 1.1. A sequence $(x_i) \subset \mathbb{R}^n$ **converges to** $x \in \mathbb{R}$ iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall i \geq N,\ \|x_i - x\| < \epsilon$$

Proposition 1.1. $x_i \to x \iff \forall j \in \{1, \ldots, n\},\ x_i^j \to x^j$ *(note the use of superscript to mean component extraction, not exponentiation)*

1.2 Continuity

1.2.1 Open sets in Euclidean spaces

Definition. The **open ball** of radius $r \in \mathbb{R}$ about $x \in \mathbb{R}^n$ is

$$B_r(x) := \{y \in \mathbb{R} : \|y - x\| < r\}$$

Definition 1.2. $U \subseteq \mathbb{R}^n$ is **open in** $\mathbb{R}^n$ $\iff \forall x \in U,\ \exists r > 0 : B_r(x) \subseteq U$.

1.2.2 Continuity at a point, and continuity on an open set

Definition 1.3. For A open, $f : A \to \mathbb{R}$ is **continuous at** $p \in A$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 : \forall x \in A,\ \|x - p\| < \delta \implies \|f(x) - f(p)\| < \epsilon$$

f is **continuous on** $A \iff f$ is continuous at all $p \in A$.

Theorem 1.2. *For $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ both open, if $f : A \to B$ is continuous at p and $g : B \to \mathbb{R}^l$ is continuous at $f(p)$ then $g \circ f$ is continuous at p.*

Definition 1.4. For A open and $f : A \to \mathbb{R}^m$, the **limit of** f **as** x **tends to** $p \in A$ is $q \in \mathbb{R}^m$ where

$$\forall \epsilon > 0,\ \exists \delta > 0 : \forall x \in A,\ 0 < \|x - p\| < \delta \implies \|f(x) - q\| < \epsilon$$

which we write $\lim_{x \to p} f(x) = q$.

Proposition. *f is continuous at $p \iff \lim_{x \to p} f(x) = f(p)$.*

Theorem 1.3. *The limit can be distributed over addition and multiplication, and also over division if the limit of the denominator is not 0.*

Corollary 1.4. *If $f : A \to \mathbb{R}$ and $g : A \to R$ are continuous at p then so are $f + g$ and fg, and so is $\frac{f}{g}$ assuming $g(p) \neq 0$.*

1.3 Derivative of a map of Euclidean spaces

1.3.1 Derivative as a linear map

Lemma 1.5. *$f : (a, b) \to \mathbb{R}$ is differentiable at $p \in (a, b) \iff$ there exists a map $A_\lambda(x) = \lambda(x - p) + f(p)$ with $\lambda \in \mathbb{R}$ such that*

$$\lim_{x \to p} \frac{|f(x) - A_\lambda(x)|}{|x - p|} = 0$$

Definition 1.5. For Ω open, $f : \Omega \to \mathbb{R}^m$ is **differentiable at** $p \in \Omega$ iff there is a linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to p} \frac{\|f(x) - (\Lambda(x - p) + f(p))\|}{\|x - p\|} = 0$$

Then the **derivative of** f **at** p or (**total derivative** or **differential**) is $Df(p) := \Lambda$.

Lemma 1.6. *f is differentiable at $p \implies f$ is continuous at p.*

Theorem 1.7. *If the derivative exists, it is unique.*

1.3.2 Chain rule

Theorem 1.8. *For $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^m$ both open, $g : A \to B$ differentiable at p and $f : B \to \mathbb{R}^l$ differentiable at $g(p)$, $f \circ g$ is differentiable at p with*

$$D(f \circ g)(p) = Df(g(p)) \circ Dg(p)$$

1.4 Directional derivatives

1.4.1 Rates of change and partial derivatives

Definition 1.6. For f differentiable at p and $\|v\| = 1$, the **directional derivative** of f at p in the direction v is

$$\frac{\partial f}{\partial v}(p) := Df(p)(v) = \lim_{t \to 0} \frac{f(p + vt) - f(p)}{t}$$

The **partial derivatives** of f at p are $D_i f(p) := \frac{\partial f}{\partial e_i}(p)$ for $i = 1, \ldots, n$.

Theorem 1.9. *For f differentiable at p,*

$$Df(p) = \begin{pmatrix} D_1 f^1(p) & \ldots & D_n f^1(p) \\ \vdots & \ddots & \vdots \\ D_1 f^m(p) & \ldots & D_n f^m(p) \end{pmatrix}$$

Corollary 1.10. *If $g : A \to B$ is differentiable at p and $f : B \to \mathbb{R}^l$ is differentiable at $g(p)$ then*

$$D(f \circ g)(p) = \begin{pmatrix} D_1 f^1(g(p)) & \ldots & D_m f^1(g(p)) \\ \vdots & \ddots & \vdots \\ D_1 f^l(g(p)) & \ldots & D_m f^l(g(p)) \end{pmatrix} \begin{pmatrix} D_1 g^1(p) & \ldots & D_n g^1(p) \\ \vdots & \ddots & \vdots \\ D_1 g^m(p) & \ldots & D_n g^m(p) \end{pmatrix}$$

Lemma 1.11. *If f is differentiable in Ω open and has a local minimum or maximum at $p \in \Omega$ then $Df(p) = 0$.*

1.4.2 Relation between partial derivatives and differentiability

Theorem 1.12. *If the partial derivatives of $f : \Omega \to \mathbb{R}^m$ all exist in Ω and are all continuous at p then f is differentiable at p.*

1.5 Higher derivatives

1.5.1 Higher derivatives as linear maps

Definition 1.7. f is **continuously differentiable** iff $Df : \Omega \to \mathbb{R}^{mn}$ is continuous, where the $m \times n$ matrix representing $Df(p)$ is converted into an mn-dimensional vector.

Definition 1.8. The **second derivative** of f is $DDf : \Omega \to (\mathbb{R}^n \to \mathbb{R}^{m \times n})$ (equivalently, $DDf : \Omega \to \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^m$). It is the derivative of $Df : \Omega \to \mathbb{R}^{mn}$. For $p \in \Omega$, $DDf(p)$ is a linear map from n-dimensional vectors to $m \times n$ matrices. This extends to higher derivatives.

1.5.2 Symmetry of mixed partial derivatives

Theorem 1.13. *(Schwartz) If f is differentiable in Ω and $D_i D_j f$ and $D_j D_i f$ exist and are continuous in Ω then $D_i D_j f(p) = D_j D_i f(p)$ for all $p \in \Omega$.*

1.5.3 Taylor's theorem

Notation. In this section α is a **multi-index**, an n-tuple of non-negative integers. We define:

$$\begin{aligned}
|\alpha| &:= \alpha_1 + \alpha_2 + \ldots + \alpha_n \\
D^\alpha f &:= D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} f \\
v^\alpha &:= (v^1)^{\alpha_1} (v^2)^{\alpha_2} \ldots (v^n)^{\alpha_n} \\
\alpha! &:= \alpha_1! \alpha_2! \ldots \alpha_n!
\end{aligned}$$

Notice that the nested superscripts in the third equation have different meanings; each factor is the ith component raised to the power of α_i.

Theorem 1.14. *(Taylor) For $p \in \mathbb{R}^n$, $h \in B_r(0)$, and $f : B_r(p) \to \mathbb{R}$ which is k-times continuously differentiable on $B_r(p)$,*

$$f(p+h) = \sum_{|\alpha| \leq k-1} \frac{h^\alpha}{\alpha!} D^\alpha f(p) + R_k(p,h)$$

where

$$R_k(p,h) = \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} D^\alpha f(x)$$

for some x with $0 < \|x-p\| < \|h\|$. Each sum is over all n-multi-indices α with the specified property.

1.6 Inverse and Implicit function theorems

1.6.1 Inverse function theorem

Theorem 1.15. *(Inverse Function) For f continuously differentiable on Ω and $q \in \Omega$ such that $Df(q)$ is invertible, there exist $U \subset \Omega$ and $V \subset \mathbb{R}^n$ both open such that $q \in U$, $f(q) \in V$, $f : U \to V$ is bijective, $f^{-1} : V \to U$ is continuously differentiable, and for all $y \in V$,*

$$Df^{-1}(y) = \left(Df(f^{-1}(y))\right)^{-1}$$

Definition. $f : \Omega \to \Omega'$ is a **C^1-diffeomorphism** iff f is bijective, continuously differentiable, and $Df(x)$ is invertible for all $x \in \Omega$.

1.6.2 Implicit Function Theorem

Theorem 1.16. *(Implicit Function simplified) For $\Omega \subset \mathbb{R}^2$ open, $f : \Omega \to \mathbb{R}$ continuously differentiable, and $(x', y') \in \Omega$ with $f(x', y') = 0$ and $D_2 f(x', y') \neq 0$, there exist $A, B \subset \mathbb{R}$ both open and $g : A \to B$ continuously differentiable where $x' \in A$, $y' \in B$, and*

$$\forall y \in B,\ (\exists x \in A : f(x,y) = 0) \iff (\exists x \in A : y = g(x))$$

1.6.3 * Sketch of the proof of the Implicit Function Theorem

Not examinable.

1.6.4 The general from of the Implicit Function Theorem

Theorem 1.17. *For $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ both open, $f : \Omega \times \Omega' \to \mathbb{R}^m$ continuously differentiable everywhere, and $p \in \Omega \times \Omega'$ such that $f(p) = 0$ and the $m \times m$ matrix with entries $(D_{n+j} f^i(p))$ is invertible, there exist $A \subset \Omega$ and $B \subset \Omega'$ both open and $g : A \to B$ continuously differentiable such that*

$$\forall y \in B,\ (\exists x \in A : f(x, y) = 0) \iff (\exists x \in A : y = g(x))$$

1.6.5 * Equivalence of the two theorems

Not examinable.

2 Metric and topological spaces

2.1 Metric spaces

2.1.1 Motivation and definition

Definition 2.1. A **metric** on a set X is a function $\mathrm{d} : X \times X \to \mathbb{R}$ where for all $x, y, z \in X$

- $\mathrm{d}(x, y) \geq 0$
- $\mathrm{d}(x, y) = 0 \iff x = y$
- $\mathrm{d}(x, y) = \mathrm{d}(y, x)$
- $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y)$

Definition 2.2. A **metric space** is a pair (X, d) where d is a metric on X. The elements of X are **points** and d is the **distance** between two points.

2.1.2 Examples of metric spaces

Definition. A common metric on $\mathbb{R}$ is $\mathrm{d}_1(x, y) = |x - y|$.

Definition. The **Euclidean metric** on $\mathbb{R}^n$ is $\mathrm{d}_2(x, y) = ||x - y||$.

Lemma 2.1. *For $f : [a, b] \to \mathbb{R}$ continuous and non-negative, with $f(x)$ non-zero for at least one x, $\int_a^b f(t)\mathrm{d}t > 0$.*

Definition. For any set X, the **discrete metric** is

$$\mathrm{d}_{\mathrm{disc}}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Definition. The **British rail metric** on $\mathbb{R}$ is

$$\mathrm{d}_{\mathrm{rail}}(x, y) = \begin{cases} \|x - y\| & \text{if } x = ky, k \in \mathbb{R} \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

Definition. The **supremum** or **uniform metric** on $C([a,b])$ is

$$\mathrm{d}_\infty(f,g) = \max |f(t) - g(t)|$$

Definition 2.3. For a metric space (X, d) and a subset $Y \subset X$, the restriction of d to Y is the **induced metric** $\mathrm{d}\,|_Y$. $(Y, \mathrm{d}\,|_Y)$ is a **metric subspace** of X, and is also an MS.

Definition 2.4. For metric spaces (X, d_X) and (Y, d_Y) and a metric d on $X \times Y$ defined in terms of d_X and d_Y, $(X \times Y, \mathrm{d})$ is a **product metric space**. Important examples of such a d include:

- $\mathrm{d}((x_1, x_2), (y_1, y_2)) = \max \{\mathrm{d}_1(x_1, y_1), \mathrm{d}_2(x_2, y_2)\}$
- $\mathrm{d}((x_1, x_2), (y_1, y_2)) = \mathrm{d}_1(x_1, y_1) + \mathrm{d}_2(x_2, y_2)$

2.1.3 Normed vector spaces

Definition 2.5. A **norm** on a vector space V over $\mathbb{R}$ is a function $\|\cdot\| : V \to \mathbb{R}$ where for all $u, v \in V$ and $\lambda \in \mathbb{R}$

- $\|v\| \geq 0$
- $\|v\| = 0 \iff v = 0$
- $\|\lambda v\| = |\lambda| \|v\|$
- $\|u + v\| \leq \|u\| + \|v\|$

Definition. A **normed vector space** is a pair $(V, \|\cdot\|)$ where $\|\cdot\|$ is a norm on V.

Lemma 2.2. *For a normed vector space $(V, \|\cdot\|)$, $\mathrm{d}_{\|\ \|}(u,v) = \|u - v\|$ is a metric on V.*

2.1.4 Open sets in metric spaces

Definition 2.6. The **ball** of radius ϵ about x (or **ϵ-neighbourhood** of x) is

$$B_\epsilon(x) = \{y \in X \mid d(x,y) < \epsilon\}$$

Definition 2.7. $U \subseteq X$ is **open** $\iff \forall u \in U,\ \exists \delta > 0 : B_\delta(u) \subseteq U$

Lemma 2.3. *All balls are open.*

Lemma 2.4. *In any (X, d), X and $\emptyset$ are open.*

Lemma 2.5. *A union of open sets is open.*

Lemma 2.6. *An intersection of finitely many open sets is open.*

Definition 2.8. Two metrics d_1 and d_2 on X are **topologically equivalent** iff

$$\forall U \subseteq X,\ (U \text{ open in } (X, \mathrm{d}_1) \iff U \text{ open in } (X, \mathrm{d}_2))$$

2.1.5 Convergence in metric spaces

Definition 2.9. A sequence (x_n) of elements of X **converges** in $(X, \mathrm{d}) \iff$ there exists a **limit** $x \in X$ such that

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N,\ \mathrm{d}(x_n, x) < \epsilon$$

We use the usual notation of $x_n \to x$.

Definition. A sequence (x_n) is **eventually constant** iff

$$\exists N \in \mathbb{N} : \forall n \geq N,\ x_n = x_N$$

Proposition. *Under the discrete metric, a sequence converges iff it is eventually constant.*

Lemma 2.7. *If the limit of a sequence exists, it is unique.*

Corollary 2.8. *If* d_1 *and* d_2 *are topologically equivalent, convergence under* d_1 *is equivalent to convergence under* d_2.

2.1.6 Closed sets in metric spaces

Definition 2.10. $V \subseteq X$ is **closed** in $(X, \mathrm{d}) \iff$ all convergent sequences whose elements are all in V have their limit also in V.

Proposition. *Under the discrete metric, all sets are closed.*

Theorem 2.9. V *is closed* $\iff X \setminus V$ *is open.*

Lemma 2.10. *An intersection of closed sets is closed. A union of finitely many closed sets is closed.*

2.1.7 Interior, isolated, limit, and boundary points in metric spaces

Definition 2.11. For $V \subset X$, $x \in X$ (not necessarily $x \in V$):

- x is an **interior** or **inner point** of $V \iff \exists \delta > 0 : B_\delta(x) \subset V$
- x is an **isolated point** of $V \iff \exists \delta > 0 : V \cap B_\delta(x) = \{x\}$
- x is a **limit** or **accumulation point** of $V \iff \forall \delta > 0,\ (B_\delta(x) \cap V) \setminus \{x\} \neq \emptyset$
- x is a **boundary point** of $V \iff \forall \delta > 0,\ B_\delta(x) \cap V \neq \emptyset$ and $B_\delta(x) \setminus V \neq \emptyset$

Definition 2.12. For (X, d) and $V \subset X$:

- The **interior** of V is V°, the set of its interior points.
- The **closure** of V is $\overline{V}$, the union of V and the set of its limit points.
- The **boundary** of V is ∂V, the set of its boundary points.

Proposition. V *is open* $\iff V = V^\circ$.

Proposition. V *is closed* $\iff V = \overline{V}$.

Proposition. $V \subseteq W \implies V^\circ \subseteq W^\circ$

Proposition. $V \subseteq W \implies \overline{V} \subseteq \overline{W}$

Proposition. $\overline{V \cup W} = \overline{V} \cup \overline{W}$

Lemma 2.11. *x is a limit point of $V \iff$ there is a sequence of points in $V \setminus \{x\}$ converging to x.*

Definition 2.13. $V \subseteq X$ is **dense** in $X \iff \overline{V} = X$. (X, d) is **separable** $\iff$ there is a countable dense subset.

2.1.8 Continuous maps of metric spaces

Notation. From now on, let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a map.

Definition 2.14. f is **continuous at** $x \in X$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 : \forall x' \in X,\ (\mathrm{d}_X(x, x') < \delta \implies \mathrm{d}_Y(f(x), f(x')) < \epsilon)$$

f is **continuous** $\iff$ f is continuous at every $x \in X$. f is **uniformly continuous** $\iff$ f is continuous with δ independent of x (but still dependent on ϵ).

Theorem 2.12. *f is continuous $\iff \forall U \subseteq Y$ (U is open $\implies f^{-1}(U)$ is open).*

Proposition. *f is continuous $\iff \forall U \subseteq Y$ (U is closed $\implies f^{-1}(U)$ is closed)*

Theorem 2.13. *f is continuous at $x \in X \iff$ for all sequences $(x)_n$, $x_n \to x \implies f(x_n) \to f(x)$.*

Definition 2.15. f is a **homeomorphism** $\iff$ f is bijective and continuous and f^{-1} is continuous. Two metric spaces are **homeomorphic** $\iff$ there is a homeomorphism between them.

Definition 2.16. f is **Lipschitz** iff

$$\exists M > 0 : \forall x_1, x_2 \in X,\ \mathrm{d}_Y(f(x_1), f(x_2)) \leq M \mathrm{d}_X(x_1, x_2)$$

f is **bi-Lipschitz** iff

$$\exists M_1, M_2 > 0 : \forall x_1, x_2 \in X,\ M_2 \mathrm{d}_X(x_1, x_2) \leq \mathrm{d}_Y(f(x_1), f(x_2)) \leq M_1 \mathrm{d}_X(x_1, x_2)$$

f is an **isometry** or **distance preserving** iff

$$\forall x_1, x_2 \in X,\ \mathrm{d}_Y(f(x_1), f(x_2)) = \mathrm{d}_X(x_1, x_2)$$

Proposition. *f is surjective and bi-Lipschitz $\implies$ f is a homeomorphism.*

2.2 Topological spaces

2.2.1 Motivation

Nothing important in this section.

2.2.2 Topology on a set

Definition 2.17. A **topology** on a set A is τ, a collection of subsets of A where

- $\emptyset \in \tau$
- $A \in \tau$
- Any union of elements of τ is in τ
- Any intersection of finitely many elements of τ is in τ

A **topological space** is a pair (A, τ) where τ is a topology on A. The elements of A are **points**. The elements of τ are **open sets**. An open set containing $a \in A$ is a **neighbourhood** of a.

Definition. The **coarse topology** on any A is $\{\emptyset, A\}$.

Definition. The **discrete topology** on any A is the set of all its subsets.

Definition. The **Sierpinski topological space** is $(\{a, b\}, \{\emptyset, \{a, b\}, \{b\}\})$

Definition. The **order topology** on $\mathbb{R}$ is the collection of all sets of the form $(a, +\infty)$ with $a \in \mathbb{R} \cup \{-\infty, +\infty\}$, where $(+\infty, +\infty) = \emptyset$.

Definition. The **co-finite topology** on any A is the collection of subsets whose complements are finite, and also the empty set.

Definition. The **induced topology** on any X from a metric d on X is the collection of open sets under d. Two topologically equivalent metrics induce the same topology.

Definition. The **Euclidean topology** is the topology induced from the Euclidean metric.

Definition. (A, τ) is **metrisable** if there is a metric on A which induces τ.

Definition. For (A, τ) and $Y \subset A$, the **induced** or **subspace topology** on Y is $\tau_Y = \{U \cap Y \mid U \in \tau\}$

Definition. For (A, τ_A) and (B, τ_B), the **product topology** on $A \times B$ is

$$\tau \star \mu = \{\Omega \subseteq A \times B \mid \forall (a, b) \in \Omega,\ \exists U \in \tau, V \in \mu : a \in U \wedge b \in V \wedge U \times V \in \Omega\}$$

Definition 2.18. A topology τ is **stronger** or **finer** than another topology μ on the same set $\iff \mu \subset \tau$.

Proposition. *The coarse topology is weaker than all others. The discrete topology is stronger than all others.*

Lemma 2.14. $G \in \tau \iff \forall x \in G,\ \exists U \in \tau : x \in U \wedge U \subseteq G.$

Definition 2.19. For $\Omega \subseteq A$, $x \in \Omega$ is an **interior point** of $\Omega \iff \exists U \in \tau : z \in U \wedge U \subseteq \Omega$. The **interior** of Ω is Ω°, the set of all interior points.

2.2.3 Convergence, and the Hausdorff property

Definition 2.20. A sequence (x_n) of elements of A **converges** $\iff$ there is a **limit** x such that

$$\forall G \in \tau \text{ where } x \in \tau,\ \exists N \in \mathbb{N} : \forall n \geq N,\ x_n \in G$$

. Limits are not necessarily unique.

Proposition. *Under the coarse topology, all sequences are convergent. Under the discrete topology, (x_n) converges $\iff$ (x_n) is eventually constant.*

Definition. U and V **separate** x and y $\iff$ $x \in U \wedge y \in V \wedge U \cap V = \emptyset$.

Definition 2.21. A topological space is **Hausdorff** $\iff$ any pair of distinct elements can be separated.

Theorem 2.15. *In Hausdorff topological spaces, limits are unique.*

2.2.4 Closed sets in topological spaces

Definition 2.22. $V \subseteq A$ is **closed** in (A, τ) $\iff$ $A \setminus V$ is open in (A, τ).

Theorem 2.16. *The lecturer messed up the numbering.*

Lemma 2.17. *$\emptyset$ and A are closed. An intersection of closed sets is closed. A union of finitely many closed sets is closed.*

Lemma 2.18. *For (A, τ) Hausdorff and $a \in A$, $\{a\}$ is closed.*

Definition 2.23. For $\Omega \subseteq A$, $x \in \Omega$ is a **limit** or **accumulation point** of Ω iff

$$\forall U \in \tau,\ x \in U \implies (X \cup U) \setminus \{x\} \neq \emptyset$$

The **closure** of Ω is $\overline{\Omega}$, the union of Ω and the set of its limit points.

Lemma 2.19. *$S \subset T \implies \overline{S} \subset \overline{T}$. S is closed $\iff$ $S = \overline{S}$.*

2.2.5 Continuous maps on topological spaces

Notation. In this section, let (A, τ_A) and (B, τ_B) be topological spaces and let $f : A \to B$ be a map.

Definition 2.24. f is **continuous** $\iff$ $\forall U \in \tau_B,\ f^{-1}(U) \in \tau_A$.

Proposition. *τ_A is the discrete topology $\implies$ f is continuous. τ_B is the coarse topology $\implies$ f is continuous.*

Theorem 2.20. *f is continuous $\iff$ $\forall U \in \tau_B,\ U$ is closed $\implies$ $f^{-1}(U)$ is closed.*

Theorem 2.21. *A composition of continuous maps is continuous.*

Lemma 2.22. *The constant map is continuous.*

Definition 2.25. f is a **homeomorphism** $\iff$ f is bijective and continuous and f^{-1} is continuous. Two topological spaces are **topologically equivalent** or **homeomorphic** $\iff$ there is a homeomorphism between them.

Proposition. *If (A, τ_A) and (B, τ_B) are homeomorphic, (A, τ_A) is Hausdorff $\iff$ (B, τ_B) is Hausdorff.*

2.3 Connectedness

2.3.1 Connected sets

Notation. From now on (X, d), (X, d_X) and (Y, d_Y).

Definition 2.26. $T \subseteq X$ is **disconnected** if there are open sets $U, V \subseteq X$ such that

- $U \cap V = \emptyset$
- $T \subseteq U \cup V$
- $T \cap U \neq \emptyset$ and $T \cap V \neq \emptyset$

Definition 2.27. T is **connected** $\iff$ T is not disconnected.

Proposition. *X is connected $\iff$ the only subsets which are both open and closed are X and $\emptyset$.*

Lemma 2.23. *T is disconnected $\iff$ there is a continuous map $f : T \to \mathbb{R}$ with $f(T) = \{0, 1\}$.*

Lemma 2.24. *$S \subseteq \mathbb{R}$ is an interval $\iff$ $\forall x, y \in S, z \in \mathbb{R},\ x < z < y \implies z \in S$.*

Theorem 2.25. *In $(\mathbb{R}, \mathrm{d}_1)$, all connected sets are intervals.*

Theorem 2.26. *In $(\mathbb{R}, \mathrm{d}_1)$, all closed intervals (of the form $[a, b]$) are connected.*

Proposition. *In fact, all intervals in $(\mathbb{R}, \mathrm{d}_1)$ are connected.*

2.3.2 Continuous maps and connected sets

Theorem 2.27. *If $f : X \to Y$ is continuous and $S \subseteq X$ is connected, $f(S)$ is connected.*

Corollary 2.28. *If (X, d_X) and (Y, d_Y) are homeomorphic, X is connected $\iff$ Y is connected.*

Theorem 2.29. *If X is connected, $f : X \to \mathbb{R}$ is continuous, and $a, b \in X$*

$$f(a) < 0 \wedge f(b) > 0 \implies \exists c \in X : f(c) = 0$$

Corollary 2.30. *If $f : [a, b] \to \mathbb{R}$ is continuous and $x, y \in [a, b]$*

$$f(x) < 0 \wedge f(y) > 0 \implies \exists z \in [a, b] : f(z) = 0$$

2.3.3 Path connected sets

Definition 2.28. A **path** from a to b is a continuous map $f : [0, 1] \to X$ where $f(0) = a$ and $f(1) = b$.

Definition 2.29. (X, d) is **path-connected** if every pair of points is joined by a path.

Theorem 2.31. *(X, d) is path connected $\implies$ (X, d) is connected.*

2.4 Compactness

2.4.1 Compactness by covers

Definition 2.30. An **open cover** for $Y \subseteq X$ is a collection $\mathcal{R}$ of open subsets of X where $Y \subseteq \bigcup_{U \in \mathcal{R}} U$. A **sub-cover** of $\mathcal{R}$ for Y is $\mathcal{C} \subseteq \mathcal{R}$ such that $\mathcal{C}$ is also an open cover for Y. A **finite cover** is an open cover with finitely many elements.

Definition 2.31. Y is **compact** $\iff$ every open cover for Y has a finite sub-cover.

Proposition 2.32. *In* $(\mathbb{R}, \mathrm{d}_1)$*, closed intervals are compact.*

Proposition 2.33. *X is compact and $Y \subseteq X$ is closed $\implies$ Y is compact.*

Theorem 2.34. *Y is compact $\implies$ Y is closed.*

Theorem 2.35. *For both of the examples of product metric spaces in Definition 2.4, X and Y are compact $\implies$ $X \times Y$ is compact.*

Corollary 2.36. *A Cartesian product of closed intervals is compact in $\mathbb{R}^n$.*

Definition 2.32. $Z \subseteq X$ is **bounded** iff

$$\exists M \in \mathbb{R} : \forall x, y \in Z,\ \mathrm{d}(x, y) \leq M$$

For an arbitrary set S, $f : S \to X$ is **bounded** $\iff$ $f(S)$ is bounded.

Proposition. *A union of finitely many bounded sets is bounded.*

Lemma 2.37. (X, d) *is compact* $\implies$ *X is bounded.*

Theorem 2.38. *(Heine Borel Theorem) In* $(\mathbb{R}^n, \mathrm{d}_2)$*, a subset is compact $\iff$ it is closed and bounded.*

2.4.2 Sequential compactness

Definition 2.33. (X, d) is **sequentially compact** $\iff$ every sequence has a convergent subsequence.

Lemma 2.39. (x_n) *has a convergent subsequence* $\iff$ $\exists x \in X : \forall \epsilon > 0$*, there are infinitely many i where $x_i \in B_\epsilon(x)$.*

Proposition. (X, d) *is sequentially compact* $\implies$ *X is bounded.*

Theorem 2.40. (X, d) *is compact* $\implies$ (X, d) *is sequentially compact.*

Theorem 2.41. *(Bolzano-Weierstrass Theorem) In $\mathbb{R}^n$, every bounded sequence has a convergent subsequence.*

Theorem 2.42. (X, d) *is sequentially compact* $\implies$ (X, d) *is compact.*

2.4.3 Continuous maps and compact sets

Theorem 2.43. *$Z \subseteq X$ is compact and $f : X \to Y$ is continuous $\implies$ $f(Z)$ is compact.*

Corollary 2.44. *If (X, d_X) and (Y, d_Y) are homeomorphic, X is compact $\iff$ Y is compact.*

Theorem 2.45. *X is compact and $f : X \to Y$ is continuous $\implies$ f is uniformly continuous.*

Corollary 2.46. *$f : [a, b] \to \mathbb{R}$ is continuous $\implies$ f is uniformly continuous.*

Theorem 2.47. *If X is compact and $f : X \to \mathbb{R}$ is continuous, f is bounded from above and below and attains its upper and lower bounds.*

Theorem 2.48. *For $f : \mathbb{R} \to \mathbb{R}$ continuous and $[a, b] \subseteq \mathbb{R}$, $f([a, b])$ is a closed interval.*

2.5 Completeness

2.5.1 Complete metric spaces and Banach space

Definition 2.34. (x_n) is **Cauchy** iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n, m > N,\ \mathrm{d}(x_n, x_m) < \epsilon$$

Proposition. *(x_n) is convergent $\implies$ (x_n) is Cauchy.*

Proposition. *If a Cauchy sequence has a subsequence converging to some x, the whole sequence convergence to x.*

Definition 2.35. (X, d) is **complete** $\iff$ all Cauchy sequences in X converge. $(V, \|\cdot\|)$ is a **Banach space** $\iff$ $(V, \mathrm{d}_{\|\ \|})$ is complete.

Lemma 2.49. *For all n, $(\mathbb{R}^n, \mathrm{d}_2)$ is complete.*

Proposition 2.50. *The metric space $(C([a, b]), \mathrm{d}_2)$, where*

$$\mathrm{d}_2(f, g) = \sqrt{\int_a^b |f(t) - g(t)|^2 dt}$$

is not complete.

Theorem 2.51. *If (f_n) is a sequence of continuous functions which converges uniformly, its limit is also continuous.*

Theorem 2.52. *The metric space $(C([a, b]), \mathrm{d}_\infty)$, where*

$$\mathrm{d}_\infty(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)|$$

is complete.

Theorem 2.53. *(X, d) is compact $\implies$ (X, d) is complete.*

2.5.2 Arzelà-Ascoli

Definition 2.36. A collection $\mathcal{C}$ of functions $f : [a, b] \to \mathbb{R}$ is **uniformly bounded** iff

$$\exists M > 0 : \forall f \in \mathcal{C}, x \in [a, b],\ |f(x)| < M$$

$\mathcal{C}$ is **uniformly equi-continuous** iff

$$\forall \epsilon > 0,\ \exists \delta > 0 : \forall f \in \mathcal{C}, x, y \in [a, b],\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem 2.54. *(Arzelà-Ascoli Theorem) If $\mathcal{C}$ is a uniformly bounded and equi-continuous collection of continuous functions, every sequence in $\mathcal{C}$ has a convergent subsequence in $(C([a, b]), \mathrm{d}_\infty)$ with d_∞ defined as in Theorem 2.52.*

2.5.3 Fixed point theorem

Definition 2.37. $f : X \to Y$ is **contracting** iff

$$\exists K \in (0, 1) : \forall a, b \in X,\ \mathrm{d}_Y(f(a), f(b)) \leq K\mathrm{d}_X(a, b)$$

Theorem 2.55. *(Banach Fixed Point Theorem) If (X, d) is non-empty and complete and $f : X \to X$ is contracting, f has a unique fixed point.*