Expressiveness of Temporal Logic

Course in Automatic system Verification: Theory and Applications

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July 3, 2023

Abstract

In this work, I consider the expressive power of various temporal logics. First, I recall some basic results about expressiveness of first order logic. Then I consider the case of LTL and I show a theorem that can be used to prove that the concept of parity is not definable in this context. I discuss a counterexample that proves that the mentioned theorem doesn't directly apply to $LTL{+}P$ and I briefly highlight how a possible investigation may lead to a generalization of the theorem to the $LTL{+}P$ case. Next, I relate first order definable languages with LTL ones and I present an extension to LTL which allows us to increase the expressive power and capture regular languages without changing the complexity of the decision procedure. Finally, I move to the more interesting case of interval logic. I introduce the notion of bisimulation and its use in modal logic and, in particular, I show how to apply it to prove that the logic $A\overline{A}$ is strictly more expressive than its future fragment A.

Contents

Preliminaries

Model theory

Model theory offers a way to measure expressive power [Stu00]

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- Which kind of structures are definable in a logic $\mathcal{L}$?
- Which pairs of structures are distinguishable by means of formulae of logic $\mathcal{L}$?

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- Let $\mathcal{L}$ be a logic, let σ be a signature, and let C be a class of σ-structures

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Basic definitions:

- Let $\mathcal{L}$ be a logic, let σ be a signature, and let C be a class of σ-structures
- We say C is $\mathcal{L}$-**elementary** if there exists a possibly infinite set of formulae $\tau = \{\tau_1, \tau_2, \tau_3, \dots\}$ such that for each σ-structure s it holds $s \in C \iff s \models \tau$
- If τ is finite we say C is $\mathcal{L}$-**basic elementary**

Expressive power of FO Logic

- As seen in the first part of the course we have several tools to prove that properties are not expressible in first order logic:
  - Ehrenfeucht-Fraïssé games
  - 0/1 laws
  - Locality of first order formulas (Hanf/Gaiffman theorems)
  - Compactness theorem (just for the infinite setting)

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- Temporal logics ??

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Even in FO: the finite case

- The concept of *even* is not capturable by *FO* definable languages

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- The language $(aa)^*$ is regular but not star-free or equivalently not first order definable. Two ways to prove the result:
  - The minimum DFA has a counter
  - EF games: consider the set of structures $EVEN_{aa} = \{([2k+1], +1, <, Q_a) \ : \ k >= 0, Q_a = [2k+1]\}$

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  - EF games: consider the set of structures $EVEN_{aa} = \{([2k+1], +1, <, Q_a) \ : \ k >= 0, Q_a = [2k+1]\}$
- The language $(ab)^*$ is star-free. In fact:
  - The minimum DFA doesn't have a counter
  - We can build a generalized regular expression with star height 0 $(ab)^* = \overline{b\Sigma^* \cup \Sigma^* a \cup \Sigma^* aa\Sigma^* \cup \Sigma^* bb\Sigma^*} = \overline{b\overline{\emptyset} \cup \overline{\emptyset}a \cup \overline{\emptyset}aa\overline{\emptyset} \cup \overline{\emptyset}bb\overline{\emptyset}}$
- Notice that in $(ab)^*$,we are not counting the number of a's

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Even in FO: the infinite case

- Considering ω-words we need a different notion of parity, in fact we have $(pp)^\omega = (p)^\omega$
- $\Sigma = \{p, \neg p\}$
  $EVEN_{POSITIONS} = (p(p + \neg p))^\omega$
- As can be shown with similar arguments $EVEN_{POSITIONS}$ is not ω-star-free
- On the contrary $(p \cdot \neg p)^\omega$ is ω-star-free
  - $(p \ \neg p)^\omega = \emptyset \cdot ((p \cdot \neg p)^*)^\omega$
  - $\emptyset$ and $(p \cdot \neg p)^*$ are star-free
  - $(p \cdot \neg p)^* \cdot (p \cdot \neg p)^* \subseteq (p \cdot \neg p)^*$

An ω-language L is ω-star-free if it can be written as $L = \cup_{i \in I} U_i \cdot V_i^\omega$ with U, V star-free and $V \cdot V \subseteq V$

Expressive power of LTL

P. Wolper. "Temporal logic can be more expressive". In: *Information and Control* 56.1 (1983), pp. 72–99. ISSN: 0019-9958

Even in LTL

- $EVEN_{COMPUTATIONS}(p) =$
  $\{\sigma : \sigma$ is a computation on which p is true on all even states$\}$
- $EVEN_{COMPUTATIONS}(p)$ is not expressible in LTL

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- There are infinite computations in $EVEN_{COMPUTATIONS}(p)$ which are not model ψ

- As a matter of fact we can use infinite formulas of finite length
  $\tau = \{p, XXp, XXXXp, \dots\}$

Even in LTL

To prove the previous result we consider the sequence $p^i(\neg p)p^\omega$

0	1	2	$\cdots$	$i-1$	i	$i+1$	$i+2$	$\cdots$
p	p	p	$\dots$	p	$\neg p$	p	p	$\dots$

Theorem

Let $f(p)$ be an LTL formula,
Let n denote the number of X operators in f
Every sequence $p^i(\neg p)p^\omega$ where $i > n$ has the same truth value on f:
$p^{(n+1)}(\neg p)p^\omega \models f(p) \iff p^{(n+2)}(\neg p)p^\omega \models f(p) \iff \dots$

Even in LTL

Proof.

Denote by $eval_i(f)$ the truth value of f on the sequence $p^i(\neg p)p^\omega$, we want to prove that $eval_{n+1}(f(p)) = eval_{n+2}(f(p)) = eval_{n+3}(f(p)) = \ldots$ In other words we want to prove that $eval_i(f(p))$ is independent of $i > n$.

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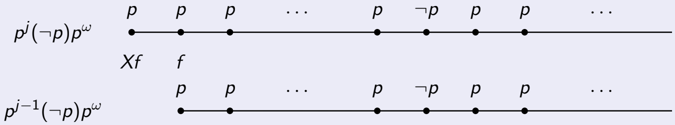
- Propositional cases are straightforward.
- Case $f(p) = Xf$. We have $eval_j(Xf) = eval_{j-1}(f)$. f contains $n-1$ X operators so we have $j-1 > n-1$ and hence by the inductive hypothesis the value of $eval_{j-1}(f)$ is independent of j.



Even in LTL

Proof.

- Case $f(p) = f_1 \ U \ f_2$. From
  - $(f_1 \ U \ f_2) = f_2 \vee (f_1 \wedge X(f_1 \ U \ f_2))$
  - $eval_j(Xf) = eval_{j-1}(f)$

  we have $eval_j(f_1 \ U \ f_2) = eval_j(f_2) \vee (eval_j(f_1) \wedge eval_{j-1}(f_1 \ U \ f_2))$

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  we have $eval_j(f_1 \ U \ f_2) = eval_j(f_2) \vee (eval_j(f_1) \wedge eval_{j-1}(f_1 \ U \ f_2))$ and unfolding n times:
  $eval_j(f_1 \ U \ f_2) = eval_j(f_2) \vee (eval_j(f_1) \wedge eval_{j-1}(f_2) \vee (eval_{j-1}(f_1) \wedge \cdots \wedge (eval_{n+1}(f_2) \vee (eval_{n+1}(f_1) \wedge eval_n(f_1 \ U \ f_2))) \ldots))$
  By the inductive hypothesis
  $eval_j(f_k) = eval_{j-1}(f_k) = \cdots = eval_{n+1}(f_k)$ for $k = 1, 2$

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  By the inductive hypothesis
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  so we have:
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  $eval_{n+1}(f_1 \; U \; f_2) = eval_{n+2}(f_1 \; U \; f_2) = eval_{n+3}(f_1 \; U \; f_2) = \ldots$

$\square$

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Corollary

For each $m \geq 2$ *the set* $m_{COMPUTATIONS}(p) =$
$\{\sigma.\ \sigma$ *is a computation on which* p *is true at each state multiple of* $m\}$ *is not expressible in LTL*

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For each $m \geq 2$ the set $m_{COMPUTATIONS}(p) = \{\sigma.\ \sigma$ is a computation on which p is true at each state multiple of $m\}$ is not expressible in LTL

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Suppose that the formula $f(p)$ with n X operators captures $m_{COMPUTATIONS}(p)$ for a given m.

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Suppose that the formula $f(p)$ with n X operators captures $m_{COMPUTATIONS}(p)$ for a given m. Choose k such that $km > n$. Then we have

- $p^{km+1}(\neg p)p^{\omega} \in m_{COMPUTATIONS}(p)$
- $p^{km}(\neg p)p^{\omega} \notin m_{COMPUTATIONS}(p)$

but $eval_{km+1}(f(p)) = eval_{km}(f(p))$ □

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Express Even in LTL

To express EVEN in LTL, [Wol83] proposed 2 solutions

1. Add quantifiers to *LTL* and use the the formula
   $\psi(p) = \exists q(q \wedge G(q \rightarrow X\neg q) \wedge G(\neg q \rightarrow Xq) \wedge G(q \rightarrow p))$
   The logic obtained in this way is called *QLTL* and its satisfiability problem is non-elementary [KPV01]
2. Add the operator $even(p)$ directly in the syntax of LTL

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Regarding point 2, which kind of operators can we add to LTL to increase its expressive power while keeping its decision problem PSPACE-complete?

$\rightarrow$ We can add operators corresponding to right-linear grammars

The logic obtained in this way is called **ETL** (Extended Temporal Logic)

ETL: Extended Temporal Logic

- Grammar $G = (V, T, P, S)$ with $n = |V|$ and $m = |T|$
  Variables: $V = \{V_1, \ldots, V_n\}$
  Terminals: $T = \{t_1, \ldots, t_m\}$
  Productions of the form $V_i \rightarrow t_{ij}$ or $V_i \rightarrow t_{ij} V_{ij}$

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- To each variable V_i we associate an m-ary operator $\mathcal{G}_i(f_1, \ldots, f_m)$

- **ETL syntax** $\phi ::= AP \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid X\phi \mid \phi U \phi \mid \mathcal{G}_i(\phi, \ldots, \phi)$

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- **ETL syntax** $\phi ::= AP \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid X\phi \mid \phi U \phi \mid \mathcal{G}_i(\phi, \dots, \phi)$
- **ETL semantics** Given an LTL structure $M = (S, x, L)$ we say

$$M, x \models \mathcal{G}_i(f_1, \dots, f_m)$$
$$\Longleftrightarrow$$
$$\exists w \in L(G[S = V_i]) \text{ of the form } w = t_{w_0} t_{w_1} t_{w_2} \dots (1 \leq w_j \leq m) \text{ s.t. }$$
$$\forall j \geq 0 \;\; M, x^j \models f_{w_j}$$

where the j-th argument of $\mathcal{G}_i$, f_j, is the temporal formula related to the terminal t_j

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$$
\begin{aligned}
M, x \models & \mathcal{G}_S(p, \neg p \vee p) \\
& \iff \forall j \; ((j \text{ is even} \to M, x^j \models f_a) \wedge (j \text{ is odd} \to M, x^j \models f_b)) \\
& \iff \forall j \; ((j \text{ is even} \to M, x^j \models p) \wedge (j \text{ is odd} \to M, x^j \models \neg p \vee p)) \\
& \iff \forall j \; (j \text{ is even} \to M, x^j \models p)
\end{aligned}
$$

ETL: Complexity

Theorem

The satisfiability problem for ETL is PSPACE-hard

The proof is a reduction from *finite automation inequivalence*. Given two DFA A and B we construct an ETL formula which is satisfiable iff $\mathcal{L}(A) \neq \mathcal{L}(B)$

If we allowed Context-free Grammar operators ETL would be undecidable.

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A decision procedure can be obtained by extending the tableau method proposed by Manna and Pnueli (1981).

Theorem

The satisfiability problem for ETL is in PSPACE

Expressive power of PNL

D. Della Monica, A. Montanari, and P. Sala. "The Importance of the Past in Interval Temporal Logics: The Case of Propositional Neighborhood Logic". In: *Logic Programs, Norms and Action: Essays in Honor of Marek J. Sergot on the Occasion of His 60th Birthday*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 79–102. ISBN: 978-3-642-29414-3

Interval temporal logic

- Linearly-ordered domain: $\mathbb{D} = (\mathcal{D}, <)$
- The basic time-unit are strict intervals: $[i, j], i, j \in \mathcal{D}, i < j$
- Interval structure: $\mathbb{I}(\mathbb{D})$ the set of intervals over $\mathbb{D}$
- Interval model: $M = (\mathbb{I}(\mathbb{D}), V)$ where V is a labeling function $V : AP \to 2^{\mathbb{I}(\mathbb{D})}$

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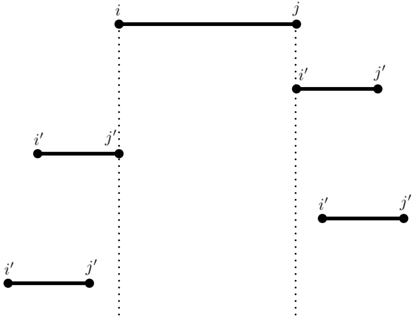
- Interval model: $M = (\mathbb{I}(\mathbb{D}), V)$ where V is a labeling function $V : AP \rightarrow 2^{\mathbb{I}(\mathbb{D})}$
- There are 13 $(6 \times 2 + 1)$ relations between pairs of intervals (Allen's interval relations)
- These relations are treated as accessibility relation in the Kripke structure
- To each relation R_X we associate a unary modal operator $\langle X \rangle$

Some of the Allen's relations

Among all 13 relations (which are mutually exclusive and jointly exhaustive), consider *after*, *meets* and their inverses

- $\langle A \rangle$: $[i, j] \ R_A \ [i', j'] \iff j = i'$
- $\langle \overline{A} \rangle$: $[i', j'] \ R_{\overline{A}} \ [i, j] \iff j' = i$
- $\langle L \rangle$: $[i, j] \ R_L \ [i', j'] \iff j < i'$
- $\langle \overline{L} \rangle$: $[i', j'] \ R_{\overline{L}} \ [i, j] \iff j' < i$



The logic $\mathcal{HS}$ and related fragments

The logic $\mathcal{HS}$:

- **syntax** $\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \langle X \rangle \phi$
  where $\langle X \rangle \phi$ is any modal operator related to the Allen's relation R_X

- **semantics**
  $M, [i,j] \models \langle X \rangle \phi \iff \exists [i',j']$ s.t. $[i,j] \; R_X \; [i',j']$ and $M, [i',j'] \models \phi$
  where M is an interval model

The logic $\mathcal{HS}$ and related fragments

The logic $\mathcal{HS}$:

- **syntax** $\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \langle X \rangle \phi$
  where $\langle X \rangle \phi$ is any modal operator related to the Allen's relation R_X
- **semantics**
  $M, [i,j] \models \langle X \rangle \phi \iff \exists [i',j']$ s.t. $[i,j]\ R_X\ [i',j']$ and $M, [i',j'] \models \phi$
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There are 4096 fragments of $\mathcal{HS}$ and about 90% of them are undecidable [Del11]

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We consider the following fragments:

- The logic RPNL A: $\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \langle A \rangle \phi$
- The logic PNL $A\overline{A}$: $\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \langle A \rangle \phi \mid \langle \overline{A} \rangle \phi$
- The logic $A\overline{L}$: $\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \langle A \rangle \phi \mid \langle \overline{L} \rangle \phi$

$\mathcal{F}$-Bisimulation: definition

Let $\mathcal{F}$ be a logic, $M = (\mathbb{I}(\mathbb{D}), V)$ and $M' = (\mathbb{I}(\mathbb{D}'), V')$ two models
The binary relation $B \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')s$ is an $\mathcal{F}$-**bisimulation** if

1. $(I_1, I_2) \in B \implies$ the intervals I_1, I_2 satisfy the same propositional letters

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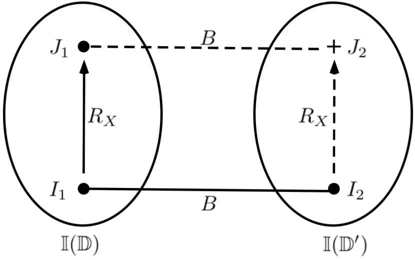
2. $(I_1, I_2) \in B \wedge (I_1, J_1) \in R_X \implies \exists J_2\ (I_2, J_2) \in R_X \wedge (J_1, J_2) \in B$



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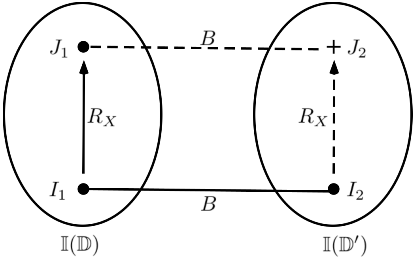

3. $(I_1, I_2) \in B \wedge (I_2, J_2) \in R_X \implies \exists J_1\ (I_1, J_1) \in R_X \wedge (J_1, J_2) \in B$

where I_1, I_2, J_1, J_2 are intervals and $\langle \overline{X} \rangle$ is an operator in the logic $\mathcal{F}$

$\mathcal{F}$-Bisimulation: invariance result

Theorem

Let M and M' be two interval models, $I \in \mathbb{I}(\mathbb{D})$ and $J \in \mathbb{J}(\mathbb{D}')$ two intervals. **If** there exists a $\mathcal{F}$-bisimulation B between M and M' with $(I, J) \in B$ **then** $M, I \equiv M', J$

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- In other words, any $\mathcal{F}$-Bisimulation preserves the truth of all formulas in $\mathcal{F}$
- Sketch proof. Consider the B such that $(I, J) \in B$. Suppose that $M, I \models \phi$, show that $M', J \models \phi$ by structural induction on ϕ. Propositional cases are straightforward. Modal cases are dealt by using the forward condition of bisimulation
- The theorem is an instantiation of a general result on modal logics [BRV01]
- In general the converse of the theorem does not hold

Expressiveness results in $\mathcal{HS}$ fragments

$\mathcal{F}_1$ is *strictly less expressive than* $\mathcal{F}_2$ ($\mathcal{F}_1 \prec \mathcal{F}_2$) if all operators $\langle X \rangle$ in $\mathcal{F}_1$ are definable in $\mathcal{F}_2$ but not viceversa

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To prove $\langle X \rangle$ is not definable in $\mathcal{F}$

- Find models M, M' and a $\mathcal{F}$-bisimulation B
- Find $\mathcal{F}$-bisimilar intervals I and J and...
- Show that $M, I \models \langle X \rangle p$ and $M', J \not\models \langle X \rangle p$

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Notice the similarity between the use of EF Games in FO and LTL theorem seen before

Past operators in A: the $\langle \overline{L} \rangle$ modality

Theorem

The modality $\langle \overline{L} \rangle$ is not definable in A over $\mathbb{N}$

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Proof.

- Consider $M = (\mathbb{I}(\mathbb{N}), V)$ and $M' = (\mathbb{I}(\mathbb{N}), V)$
- $AP = \{p\}$ and $V(p) = \{[i, i+1] : i \geq 0\}$

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- $AP = \{p\}$ and $V(p) = \{[i, i+1] : i \geq 0\}$
- A-Bisimulation
  $B = \{([i,j],[i,j]) : 0 \leq i < j\} \cup \{([i,j],[i+1,j+1]) : 0 \leq i < j\}$
- Intervals $[1,2]$ and $[2,3]$ are A-bisimilar but $M,[1,2] \not\models \langle \overline{L} \rangle p$ while $M',[2,3] \models \langle \overline{L} \rangle p$

$\square$

Past operators in A: the $\langle \bar{L} \rangle$ modality

Corollary

A is strictly less expressive than $A\bar{L}$. $\square$

Past operators in A: the $\langle \overline{L} \rangle$ modality

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Proof.

Note that the $\langle \overline{L} \rangle$ modality is definable in $A\overline{A}$.
It holds that $\langle \overline{L} \rangle \phi = \langle \overline{A} \rangle \langle \overline{A} \rangle \phi$ □

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Which type of relation between $A\overline{A}$ and $A\overline{L}$??

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Past operators in $A\overline{L}$: the $\langle\overline{A}\rangle$ modality

Theorem

The modality $\langle\overline{A}\rangle$ is not definable in $A\overline{L}$ over $\mathbb{N}$

Proof.

- Consider $M = (\mathbb{I}(\mathbb{N}), V)$ and $M' = (\mathbb{I}(\mathbb{N}), V)$
- $AP = \{p\}$ and $V(p) = \mathbb{I}(\mathbb{N})$
- $A\overline{L}$-Bisimulation $B = \{([i,j],[i,j]) : 0 \leq i < j\} \cup \{([0,2],[1,2])\}$
  $\rightarrow$ 2 types of forward and backward conditions need to be checked
- Intervals $[0,2]$ and $[1,2]$ are $A\overline{L}$-bisimilar but $M,[0,2] \not\models \langle\overline{A}\rangle$ *true* while $M',[1,2] \models \langle\overline{A}\rangle$ *true*

$\square$

Past operators in $A\overline{L}$: the $\langle \overline{A} \rangle$ modality

Corollary

$A \prec A\overline{L} \prec A\overline{A}$ $\square$

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- In this setting past modalities increase the expressive power (in contrast to LTL)
- As an example $A\overline{A}$ is able to separate $\mathbb{R}$ and $\mathbb{Q}$ while A is not
- The previous proofs can be generalized to work with $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and finite prefixes of $\mathbb{N}$

Conclusions

Expressiveness in a picture

$$LTL \equiv LTL + P \equiv FO[<] \equiv \text{Counter-free automata}$$

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$$A \prec A\overline{L} \prec A\overline{A}$$

Concluding remarks

- The concept of *even* is not *FO*-definable
- We can use bisimulation dealing with modal logic
- Adding past modalities may or may not increase the expressive power (but other properties like succinctness or complexity may change)
- Not always the increase in expressive power comes at the cost of an increase in complexity for the decision problem
  - *LTL* and *ETL* are both *PSPACE*-complete
  - $A, A\overline{L}$ and $A\overline{A}$ are *NEXPTIME*-complete

Future investigations

- The considered *LTL* theorem (slide 9) doesn't directly apply to $LTL{+}P$
  Consider $\psi_k = pU(\neg p \wedge XGp) \wedge F(\neg p \wedge Y^k true \wedge Z^{k+1} false)$
  - For a computation σ, $\sigma \models pU(\neg p \wedge XGp)$ iff $\sigma = p^i(\neg p)p^\omega$ for some i
  - $p^i(\neg p)p^\omega \models \psi_k$ iff $i = k$

  $\rightarrow$ We have $p^k(\neg p)p^\omega \models \psi_k$ but $p^{k+1}(\neg p)p^\omega \not\models \psi_k$ contradicting the LTL theorem

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  $\rightarrow$ We have $p^k(\neg p)p^\omega \models \psi_k$ but $p^{k+1}(\neg p)p^\omega \not\models \psi_k$ contradicting the LTL theorem
- How can we extend the *LTL* theorem to *LTL + P*?
  - If we take an algorithm to translate *LTL* formula into *LTL+P*, can we say something about the number of next operators?
  - Forgetting about translation algorithms, can we directly generalize the theorem taking into account the *LTL+P* semantics and some other properties, like the number of *yesterday* operators?

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