Expressivness of Temporal Logic

Course in Automatic system Verification: Theory and Applications

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Abstract

In this presentetion I consider the expressive power of temporal logic

Contents

Background

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- If τ is finite we say C is **base elementary**

Expressive power in classical logic

- As seen in the first part of the course we have several tools to prove that properties are not expressible in first order logic:
  - Ehrenfeucht-Fraïssé games
  - 0/1 laws
  - Locality of first order formulas (Hanf/Gaiffman theorems)
  - Compactness theorem (just for the infinite setting)

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- Temporal logic ??

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  - $Gn \in EVEN_{GRAPHS}$ and $H_n \notin EVEN_{GRAPHS}$

$\Longrightarrow$ We do not have a first order formula which defines $EVEN_{GRAPHS}$

Expressive power of LTL

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- As a matter of fact we can use infinite formulas of finite length
  $\tau = \{p, XXp, XXXXp, \dots\}$

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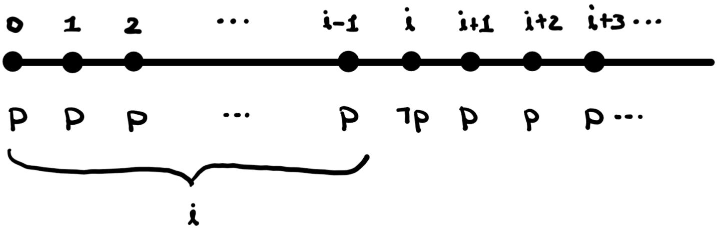

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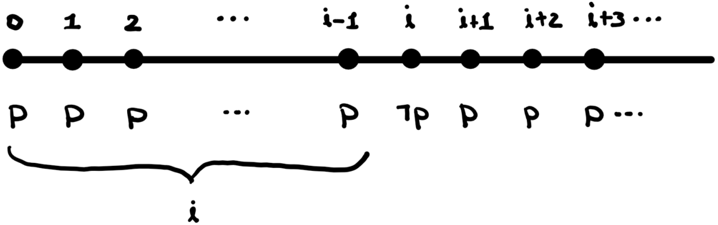

Theorem

Let $f(p)$ be an $(LTL + P)$ formula,
Let n denote the number of X operators in f
Every sequence $p^i(\neg p)p^\omega$ where $i > n$ has the same truth value on f:
$p^{(n+1)}(\neg p)p^\omega \models f(p) \iff p^{(n+2)}(\neg p)p^\omega \models f(p) \iff \dots$

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- Case $f(p) = Xf$. We have $eval_j(Xf) = eval_{j-1}(f)$. f contains $n-1$ X opertors so we have $j-1 > n-1$ and hence by the inductive hypothesis the value of $eval_{j-1}(f)$ is indipendent of j.

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  we have $eval_j(f_1 \ U \ f_2) = eval_j(f_2) \vee (eval_j(f_1) \wedge eval_{j-1}(f_1 \ U \ f_2))$

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  we have $eval_j(f_1 \; U \; f_2) = eval_j(f_2) \vee (eval_j(f_1) \wedge eval_{j-1}(f_1 \; U \; f_2))$ and unfolding n times:
  $eval_j(f_1 \; U \; f_2) = eval_j(f_2) \vee (eval_j(f_1) \wedge eval_{j-1}(f_2) \vee (eval_{j-1}(f_1) \wedge \cdots \wedge (eval_{n+1}(f_2) \vee (eval_{n+1}(f_1) \wedge eval_n(f_1 \; U \; f_2))) \ldots))$

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  By the inductive hypothesis
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- Case $f(p) = f_1 \ U \ f_2$. From
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but $eval_{km+1}(f(p)) = eval_{km}(f(p))$ □

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- It turns out that operators corresponding to any property definable by a right-linear grammar (or in other words regular properties) can be added to LTL while not increasing the complexity of its decision procedure
- The logic obtained in this way is called ETL (Extended Temporal Logic)

Conclusion

Concluding remarks

- Often we observe that more expressive power comes at a cost

Bibliography I

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