One-dimensional Casimir force at short and long ranges between two parallel metal plates of finite thickness using the radiation pressure method

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Abstract.

Using the radiation pressure method and Drude's model to calculate the force and refractive index, we obtain an expression that models the attractive one-dimensional Casimir force between two identical, parallel plates of finite thickness and infinite length and width. When applied to aluminium at 1.5 Kelvin, the behavior of the calculated force differs significantly from the case of perfectly reflective plates outside a certain range of plate thickness and separation. This difference is attributed to the electron's inertia when exposed to the vacuum electromagnetic field, which is absent in the ideal case. For plates of any fixed thickness $a > 4 \times 10^{-7}$ m, the force is practically independent of the distance d between them up to at least 2 m, except when d is smaller than one plasma wavelength. For any fixed value of $4 \times 10^{-6}\,\mathrm{m} \leq d \leq 2\,\mathrm{m}$, the force is multiplied by a factor of approximately 4300 as a increases from 4×10^{-8} m to 1.6×10^{-7} m. From this last value of a, the attraction force increases very slowly up to $a = 4 \times 10^{-4}$ m. For $4\,\mathrm{m} < a < 2 \times 10^{15}$ m, the force is a function of a given by $\mathcal{F}(a) = -Ka$, where K is a constant whose value is approximately 3.76×10^{-10} kg/s^2.

1. Introduction

The Casimir effect can be interpreted as an attractive or repulsive force between uncharged conducting surfaces [1-5]. There are two ways to understand this effect. The first and most common explanation is by assuming that the universe is filled with a vacuum electromagnetic field that satisfies the boundary conditions set up by conducting surfaces. This viewpoint was originally applied by Casimir [6] to two perfectly reflecting parallel plates through a method based on the quantum vacuum electromagnetic energy. The second interpretation explains the effect resulting from interactions of the electromagnetic field originating from the material of one of the conducting surfaces with the material of the second one. This latter perspective was first adopted by Lifshitz [7], who applied it to characterize the Casimir force between two dielectric bodies filling the two half-spaces of the universe with parallel planes as boundaries. In the case of perfect conductivity, the Lifshitz result coincides with that of Casimir. Since the predictions based on either of these interpretations are identical, there is a consensus that they are equivalent [8].

Various calculational techniques have been developed to derive the Casimir force. Still, only a few of them can be applied, with limited successes, to real cases mostly because the substratum of the bodies involved is idealized in one way or another (see, e.g., [9-17]). The present paper uses the radiation pressure method, which was introduced in [18], to describe the Casimir force between two parallel and perfectly reflecting plates. Subsequently, the method was applied to parallel dielectric plates, first only for waves whose wave vector is normal to the plates [19] and afterward for all waves in three spatial dimensions [20]. In the present paper, we apply the radiation pressure method to get a mathematical model to explain the physics behind the Casimir force between two identical parallel metal plates when their thickness as well as the distance between them vary. Our model is simplified first as it assumes that the plates are infinite, second as the electrical conduction of the material making up the plates is assumed to fit the Drude model (hereafter, such a material will be called Drude-conductor), and third as it is limited to the one-dimensional configuration, that is we consider only vacuum electromagnetic modes whose wave vectors are normal to each plate surface.

The outline of the paper is as follows. By the method of radiation pressure, in Section 2 we deduce an expression for the one-dimensional Casimir force between two parallel perfectly conducting metallic plates. In Section 3, we find the radiation pressure on one Drude-conductor plate due to the quantum vacuum electromagnetic field in terms of the reflection and transmission coefficients and introduce a cutoff function to take care of the fact that the plate becomes more and more transparent as the electromagnetic wave frequency increases to reach its material plasma frequency. In Section 4, we determine the effective cutoff function for a beam of vacuum electromagnetic waves inside a cavity formed by two identical parallel Drude-conductor plates. We also use this cutoff function to express the one-dimensional Casimir force between two identical parallel Drude-conductor plates. These results are applied to aluminium plates in Section 5, where we show that the force of mutual attraction of the plates has the following particularities. First, it decreases with the distance d between the plates only when both d and their thickness a are small, but becomes independent of d if $d > 2 \times 10^{-6}\,\mathrm{m}$. Second, for any fixed $d > 4 \times 10^{-6}\,\mathrm{m}$, there is a steep increase of the force for $4 \times 10^{-8}\,\mathrm{m} < a < 1.6 \times 10^{-7}\,\mathrm{m}$, where it is multiplied by a factor of approximately 4300. Third, for $d > 4 \times 10^{-6}\,\mathrm{m}$, the force increases very little for $1.6 \times 10^{-7}\,\mathrm{m} < a < 4 \times 10^{-4}\,\mathrm{m}$, its value then being approximately equal to $-3.3 \times 10^{-11}\,\mathrm{N}$. Fourth, for $4\,\mathrm{m} < a < 2 \times 10^{15}\,\mathrm{m}$ and any fixed value of d between $10^{-8}\,\mathrm{m}$ and $2\,\mathrm{m}$, the force in newtons can be described by the function $\mathcal{F}(a) = -Ka$, where K is approximately equal to $3.76 \times 10^{-10}\,\mathrm{kg/s^2}$. In Section 6, we present our conclusion with a brief discussion and outlook.

2. Perfectly conducting metallic plates

Let us start by reviewing the radiation pressure method for perfectly conducting metallic plates [18]. The Casimir force between two parallel plates may be seen as a consequence of the exclusion of certain electromagnetic modes between the plates, this cavity allowing only discrete modes, whereas the modes outside the cavity form a continuum of frequencies. Due to interferences, the frequency spectrum of waves inside the cavity may be visualized as a Dirac comb where waves of allowed frequencies have infinite amplitudes while the amplitudes of waves with forbidden frequencies are null. Inside the cavity, the modes reflecting off the plates act to push the plates away from each other, whereas those outside the cavity act to push them together. As the distance between the plates decreases, more and more different modes disappear between the plates. To describe quantitatively this feature, we consider a three-dimensional Cartesian system of coordinates such that the x- and y-axes are parallel to the plates. The z-axis is, therefore, perpendicular to the plate surfaces, and we suppose that $z = 0$ is situated at mid-distance between the plates. Also, let L_x, L_y, be the length and width of the plates placed at a distance d apart, the values of L_x and L_y being much larger than d. We denote by V_{in} the volume of the cavity determined by the plates and by V_{out} the volume of each exterior side of the cavity, the exact nature of V_{out} will be specified shortly.

We now consider the radiation pressure exerted by one electromagnetic wave from a vacuum, which is incident normally on one of the plates. Due to the perfect conductivity, and thus to the perfect reflectivity of the plate, this pressure is equal to twice the energy per unit of volume of the incident field. The energy of every quantum vacuum electromagnetic wave with frequency ω being $\hbar\omega/2$, each such wave in this one-dimensional model decomposes into two components, one moving to the left (toward $z < 0$), the other one moving to the right (toward $z > 0$), each one with energy $\hbar\omega/4$. In the cavity between the plates, the density of energy due to this standing wave is then $\hbar\omega/2V_{\text{in}}$, whereas in each exterior side of the cavity, the density of energy due to the same standing wave is $\hbar\omega/2V_{\text{out}}$. Note that to determine the pressure on the perfectly reflecting plates forming the cavity, one must consider first only the waves moving rightward on the left side of the cavity, second only those moving leftward on its right side, and third waves moving rightward or leftward between the two plates.

If a plate is perfectly reflecting for waves of frequency ω, then its reflection coefficient is given by $R(\omega) = 1$. In fact, this occurs only for ω up to a very high but finite value ω_p, which is the plasma frequency of the material making up the plate, that is, the frequency from which the material becomes transparent to electromagnetic waves. For $\omega > \omega_p$, the reflection coefficient tends toward 0. To describe how much of the momentum of each incident wave is transformed into a pressure, we introduce the cutoff function of each plate for a wave of frequency ω, denoted by $\mathcal{F}_1(\omega)$, which is defined similarly to the function R, that is the values of $\mathcal{F}_1(\omega)$ tend to 1 for $\omega \ll \omega_p$ and to 0 for $\omega \gg \omega_p$.

Let $k_0 = \omega/c$ be the wavenumber of a quantum vacuum electromagnetic wave of frequency ω, where c stands for the speed of light in vacuum. Inside the cavity, the total pressure on one plate is the sum of the pressures due to all quantum vacuum electromagnetic waves that can exist in the cavity, that is, those whose wavenumbers are $k_{0,n} = \pi n/d$ for $n \in \mathbf{N}^*$. Therefore, the total pressure on the interior face of this plate is

$$P_{\text{in}}(d) = \frac{\hbar c}{V_{\text{in}}} \sum_{n=0}^{\infty} k_{0,n} \mathcal{F}_1(k_{0,n}), \tag{1}$$

where $V_{\text{in}} = L_x L_y d$ and the results are multiplied by two to take account of the two polarizations of light. Another expression for (1) is

$$P_{\text{in}}(d) = \frac{\hbar c \pi}{L_x L_y d^2} \sum_{n=0}^{\infty} n \mathcal{F}_2(n), \tag{2}$$

where $\mathcal{F}_2$ is the function $\mathcal{F}_1$ expressed in terms of n instead of $k_{0,n}$.

The total pressure on the other side of the same plate results from all quantum vacuum electromagnetic waves with allowed modes in $V_{\text{out}} = L_x L_y L_z$, where L_z is half the size of the universe, which is here assumed to be a very large but finite real number. The wavenumbers of these modes are $k_{0,n} = \pi n/L_z$ for $n \in \mathbf{N}^*$. Since L_z is very large, we can consider these wavenumbers as forming a continuum. Thus, the total pressure on one plate due to the waves outside the cavity is

$$P_{\text{out}}(d) = \frac{\hbar c}{V_{\text{out}}} \sum_{n=0}^{\infty} k_{0,n} \mathcal{F}_1(k_{0,n}) \Delta n, \tag{3}$$

where $\Delta n = 1$ is inserted to transform (3) into an integral. Again, we have multiplied by two the right-hand side of (3) to take care of the two polarizations of light. Since $\Delta k_{0,n} = \pi \Delta n/L_z$, letting L_z tend to ∞ turns (3) into

$$P_{\text{out}}(d) = \frac{\hbar c}{L_x L_y \pi} \int_0^{\infty} k_0 \mathcal{F}_1(k_0) \mathrm{d}k_0. \tag{4}$$

Introducing the variable $u = k_0 d/\pi$, then (4) becomes

$$P_{\text{out}}(d) = \frac{\hbar c \pi}{L_x L_y d^2} \int_0^\infty u \mathcal{F}_2(u) \mathrm{d}u, \tag{5}$$

where as for (2) $\mathcal{F}_2$ is the function $\mathcal{F}_1$ where u replaces k_0 and becomes n when u takes positive integer values. Despite appearances, (5) is independent of d because the integration with respect to u gives an expression having d^2 as a factor. Therefore, from (2) and (5) it follows that the net pressure on one of the plates forming the cavity is

$$P_{\text{net}}(d) = P_{\text{in}}(d) - P_{\text{out}}(d) = \frac{\hbar c \pi}{L_x L_y d^2} \left(\sum_{n=0}^{\infty} n \mathcal{F}_2(n) - \int_0^\infty u \mathcal{F}_2(u) \mathrm{d}u \right). \tag{6}$$

Using the Euler-Maclaurin formula to regularize (6), that is, to calculate its right-hand side (see e.g. [21]), it is straightforward to obtain

$$P_{\text{net}}(d) = -\frac{\hbar c \pi}{12 L_x L_y d^2},$$

from which results the following expression for the one-dimensional Casimir force generated by two parallel plates perfectly reflecting up to the plasma frequency of their material [19, 20]:

$$F(d) = -\frac{\hbar c \pi}{12 d^2}. \tag{7}$$

3. Vacuum electromagnetic field pressure on one Drude-conductor plate

The mathematical model developed in the preceding section, or in [20] for the corresponding mathematical model in three spatial dimensions, does not hold for cavities formed by Drude-conductor plates because it disregards most of their physical properties. For instance, the thickness and the index of refraction of the plates should act upon the value of the Casimir force. Consequently, with Drude-conductor plates, the Casimir forces should depend upon more than just the distance between the plates. We shall now describe how much a Drude-conductor plate allows the transmission of vacuum electromagnetic waves by introducing into the regularization procedure a cutoff function $\mathcal{F}$ that will take into account important physical properties of the plates. The new expression of $\mathcal{F}$ will act upon the reflection and transmission coefficients, $R(\omega)$ and $T(\omega)$, of the plates for vacuum electromagnetic waves of frequency ω.

Let us first determine $R(\omega)$ and $T(\omega)$ for one plate in a vacuum using the Drude model [22]. We denote the vacuum and the plate material by the indices 0 and 1, respectively, the index of refraction of the latter being denoted by $n(\omega)$. The coefficients $R(\omega)$ and $T(\omega)$ associated with a quantum vacuum electromagnetic wave of frequency ω and incident normally on one face of the plate can be obtained from Fresnel's reflection $r_{01}(\omega)$ and transmission $t_{01}(\omega)$ indices ($r_{10}(\omega)$ and $t_{10}(\omega)$ when the wave goes from inside to outside the material of the plate), which are given by

$$r_{10}(\omega) = -r_{01}(\omega) = \frac{n(\omega) - 1}{n(\omega) + 1}, \qquad t_{10}(\omega) = n(\omega)\, t_{01}(\omega) = \frac{2n(\omega)}{n(\omega) + 1},$$

where $n(\omega) = c[\alpha(\omega) + i\beta(\omega)]/\omega$ for (see [23])

$$\alpha(\omega) = \left[\frac{\omega^2 \mu_1}{2}\left(\epsilon_1 - \frac{\sigma_0 \tau}{1 + \omega^2 \tau^2} + \sqrt{\left(\epsilon_1 - \frac{\sigma_0 \tau}{1 + \omega^2 \tau^2}\right)^2 + \left(\frac{\sigma_0}{\omega(1 + \omega^2 \tau^2)}\right)^2}\right)\right]^{\frac{1}{2}}$$

and

$$\beta(\omega) = \left[\frac{\omega^2 \mu_1}{2}\left(\frac{\sigma_0 \tau}{1 + \omega^2 \tau^2} - \epsilon_1 + \sqrt{\left(\epsilon_1 - \frac{\sigma_0 \tau}{1 + \omega^2 \tau^2}\right)^2 + \left(\frac{\sigma_0}{\omega(1 + \omega^2 \tau^2)}\right)^2}\right)\right]^{\frac{1}{2}},$$

whereas ϵ_1 and μ_1 designate the permittivity and permeability of the material making up the plates, σ_0 is its conductivity when $\omega \to 0$, and τ is the mean free path time of their conduction electrons. Also, let a be the thickness of the plate and $\gamma_1(\omega)$ be the wave phase shift after its round displacement between the two faces of the plate. Then, it is straightforward to show that

$$\gamma_1(\omega) = 2a[\alpha(\omega) + i\beta(\omega)].$$

Taking account of the successive reflections between its faces, we get $R(\omega) = r(\omega)(r(\omega))^*$ and $T(\omega) = t(\omega)(t(\omega))^*$, where [23]

$$r(\omega) = \frac{r_{01}(\omega)(1 - e^{-i\gamma_1(\omega)})}{1 - (r_{01}(\omega))^2 e^{-i\gamma_1(\omega)}}, \qquad t(\omega) = \frac{t_{01}(\omega) t_{10}(\omega) e^{-i\gamma_1(\omega)/2}}{1 - (r_{10}(\omega))^2 e^{-i\gamma_1(\omega)}},$$

and $*$ designates the complex conjugate. The plasma frequency of the plate material is given by $\omega_p = \sqrt{N q_e^2 / m\epsilon_1}$, where N is the number of conduction electrons per unit of volume, q_e is the electric charge of the electron and m is its mass [22]. The coefficients $R(\omega)$ and $T(\omega)$ of one plate will show oscillations due to the interference of waves between the faces of that plate.

The radiation pressure exerted on the plate surfaces by a quantum vacuum electromagnetic wave of frequency ω is the same as that of a classical electromagnetic wave with energy $\hbar\omega/4$ and amplitude $\sqrt{\hbar\omega/4\epsilon_0 V}$ [23], where ϵ_0 is the vacuum permittivity and V is the finite volume of the cavity between the plates, or outside this cavity, still assumed to be finite. We shall now use this equivalence to determine the pressure exerted by the former on one Drude-conductor plate.

Suppose a quantum vacuum electromagnetic wave of frequency ω that is incident normally on such a plate and whose reflection and transmission coefficients are respectively $R(\omega)$ and $T(\omega)$. The pressure this wave exerts on the plate results from the momentum variation when the wave meets the plate surface. It is straightforward to show that this pressure $P(\omega)$ is given by

$$P(\omega) = \mathcal{E}(\omega)(1 + R(\omega) - T(\omega)),$$

where $\mathcal{E}(\omega)$ designates the energy density of the quantum vacuum electromagnetic wave. The value of $\mathcal{E}(\omega)$ is averaged over a period of time t to get the mean pressure of the wave on the plate. Using $< >_t$ to denote this mean value, we have

$$< P(\omega) >_t = < \mathcal{E}(\omega) >_t (1 + R(\omega) - T(\omega)). \tag{8}$$

The total average pressure then follows by integrating (8) over ω from 0 to ∞, still taking account of the two polarizations of the quantum vacuum electromagnetic waves. Also, we define the cutoff function of the plate by

$$\mathcal{F}(\omega) = (1 + R(\omega) - T(\omega))/2, \tag{9}$$

where the factor $1/2$ is necessary in this definition in order to get $\mathcal{F}(\omega) = 1$ when $R(\omega) = 1$.

4. Effective cutoff function and Casimir force for Drude-conductor plates

According to Heisenberg's uncertainty principle, when a quantum electromagnetic wave of frequency ω, with $\Delta\omega = 0$, appears in a vacuum, it does it everywhere with the same frequency. For the cavity formed by two parallel Drude-conductor plates, this means that the wave is everywhere outside the cavity, as well as between the plates and inside each plate. In each of the plates and in the cavity, the wave is constrained by the boundary conditions and by the type of material of the plates.

To describe the one-dimensional Casimir force, the electromagnetic wave corresponding to a wave of the quantum vacuum electromagnetic field can be considered as formed of two components, one moving to the right, the other one moving to the left. These two components act on each other through interference. The energy of each of these components being $\hbar\omega/4$, its momentum is given by $\hbar\omega/4c$.

The interference of the above electromagnetic waves will act upon the expression of the cutoff function applying to the Drude-conductor plates. For each vacuum quantum fluctuation, the component of the wave moving initially to the right keeps non-random phase relations between each of its parts, even when these parts travel to the right or to the left between or inside the plates. The same behavior applies to the component of the wave moving initially to the left (see Fig. 1). However, we consider that these two components are in a random phase relation for each vacuum quantum fluctuation. Then, on average, over a period of time, the interference of these two components cancels, and the total pressure on the plates depends only on the pressure produced independently by each component of the vacuum quantum fluctuation.

For Drude-conductor plates, we have $R(\omega) < 1$, which implies that the amplitude of allowed modes between the plates is described by a continuous function of the wavenumber $k_0 = k_0(\omega)$ instead of Dirac-delta peaks as in the case of perfectly reflecting plates. To calculate the net pressure on the plates, the coefficients of reflection and transmission must be adjusted to take account of the multiple reflections, first between the plates and second inside each plate. The new coefficients will be said effective and will be designated by $\mathcal{R}(\omega)$ and $\mathcal{T}(\omega)$.

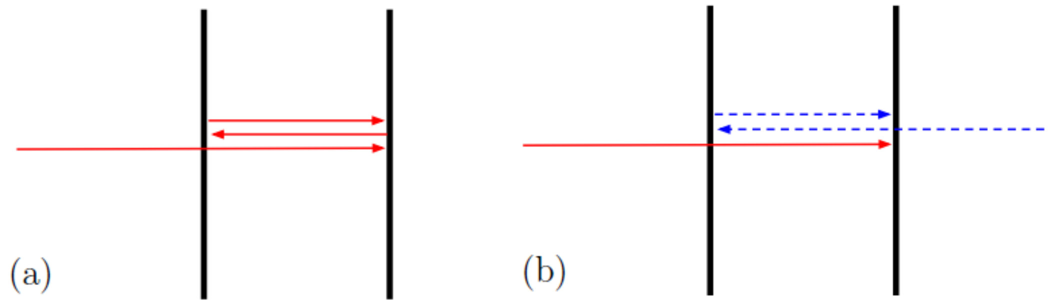

Figure 1: The diagram on the left (a) shows two beams between the plates, heading to the right and originating from the same source. These beams will interfere in both classical and quantum electromagnetic settings. In the diagram on the right (b), the two beams between the plates pointing to the right are not from the same source. These beams will interfere only in the classical electromagnetic setting.

To deduce expressions for $\mathcal{R}(\omega)$, $\mathcal{T}(\omega)$, and for the ensuing effective cutoff function $\mathcal{F}^{\mathrm{eff}}(\omega)$, let a be the thickness of each plate and d the distance between them. Expressions for $\mathcal{R}(\omega)$ and $\mathcal{T}(\omega)$ will be obtained by considering separately the vacuum electromagnetic waves with $k_0 > 0$ and those with $k_0 < 0$, and then by summing the resulting expressions. Here, we follow the method of [19]. Starting with $k_0 > 0$, we have that each wave initially moves rightward, and its reflection on any plate surface gives a wave moving leftward, which in turn generates interferences after further reflections. To simplify the notation, let $k = k(\omega) = \sqrt{\mu_1\omega(\epsilon_1\omega - i\sigma(\omega))}$, where $\sigma(\omega)$ designates the plate material conductivity. Taking into account the continuity of the functions describing these waves and the continuity of their spatial derivatives at each surface of each plate, it is straightforward to show that the amplitudes of the waves with $k_0 > 0$ in each of the five regions determined by the plates (see Fig. 2) satisfy the equations

$$
\begin{aligned}
A_1(\omega)e^{ik_0(\frac{d}{2}+a)} + A_2(\omega)e^{-ik_0(\frac{d}{2}+a)} &= B_1(\omega)e^{ik(\frac{d}{2}+a)} + B_2(\omega)e^{-ik(\frac{d}{2}+a)} \\
A_1(\omega)e^{ik_0(\frac{d}{2}+a)} - A_2(\omega)e^{-ik_0(\frac{d}{2}+a)} &= n(\omega)\left(B_1(\omega)e^{ik(\frac{d}{2}+a)} - B_2(\omega)e^{-ik(\frac{d}{2}+a)}\right) \\
B_1(\omega)e^{ik\frac{d}{2}} + B_2(\omega)e^{-ik\frac{d}{2}} &= C_1(\omega)e^{ik_0\frac{d}{2}} + C_2(\omega)e^{-ik_0\frac{d}{2}} \\
n(\omega)\left(B_1(\omega)e^{ik\frac{d}{2}} - B_2(\omega)e^{-ik\frac{d}{2}}\right) &= C_1(\omega)e^{ik_0\frac{d}{2}} - C_2(\omega)e^{-ik_0\frac{d}{2}} \\
C_1(\omega)e^{-ik_0\frac{d}{2}} + C_2(\omega)e^{ik_0\frac{d}{2}} &= D_1(\omega)e^{-ik\frac{d}{2}} + D_2(\omega)e^{ik\frac{d}{2}}
\end{aligned}
$$

$$
\begin{aligned}
C_1(\omega)e^{-ik_0\frac{d}{2}} - C_2(\omega)e^{ik_0\frac{d}{2}} &= n(\omega)\left(D_1(\omega)e^{-ik\frac{d}{2}} - D_2(\omega)e^{ik\frac{d}{2}}\right) \\
D_1(\omega)e^{-ik(\frac{d}{2}+a)} + D_2(\omega)e^{ik(\frac{d}{2}+a)} &= G(\omega)e^{-ik_0(\frac{d}{2}+a)} \\
n(\omega)\left(D_1(\omega)e^{-ik(\frac{d}{2}+a)} - D_2(\omega)e^{ik(\frac{d}{2}+a)}\right) &= G(\omega)e^{-ik_0(\frac{d}{2}+a)}
\end{aligned}
$$

where the meaning of the functions $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$, and G describing these amplitudes is clear from Fig. 2, and $n(\omega)$ denotes the complex refraction index of the material of each plate. Since the electric field amplitude of the wave of frequency ω originally propagating to the right is given by $E_0(\omega) = \sqrt{\hbar\omega/4\epsilon_0 V_{\rm in}}$ and its phase shift for a round-displacement between the plates by $\gamma_0(\omega) = 2k_0(\omega)d$, it is then direct to show that the solutions of the above equations are [23]

$$
\begin{aligned}
A_1(\omega) &= E_0(\omega)\, e^{i\omega t} \\
A_2(\omega) &= \left(r_{01}(\omega) + \frac{t(\omega)e^{-ika}(r_{10}(\omega) + r(\omega)e^{-i\gamma_0(\omega)})}{1-(r(\omega))^2 e^{-i\gamma_0(\omega)}}\right) E_0(\omega)\, e^{ik_0(d+2a)}e^{i\omega t} \\
B_1(\omega) &= \frac{t(\omega)E_0(\omega)(1+r_{10}(\omega)r(\omega)e^{-i\gamma_0(\omega)})e^{i\omega t}}{t_{10}(\omega)(1-(r(\omega))^2e^{-i\gamma_0(\omega)})}\, e^{ik_0(\frac{d}{2}+a)}e^{-ik\frac{d}{2}} \\
B_2(\omega) &= \frac{t(\omega)E_0(\omega)(r_{10}(\omega)+r(\omega)e^{-i\gamma_0(\omega)})e^{i\omega t}}{t_{10}(\omega)(1-(r(\omega))^2e^{-i\gamma_0(\omega)})}\, e^{ik_0(\frac{d}{2}+a)}e^{ik\frac{d}{2}} \\
C_1(\omega) &= \frac{t(\omega)E_0(\omega)e^{i\omega t}}{1-(r(\omega))^2e^{-i\gamma_0(\omega)}}\, e^{ik_0 a} \\
C_2(\omega) &= \frac{r(\omega)t(\omega)E_0(\omega)e^{i\omega t}}{1-(r(\omega))^2e^{-i\gamma_0(\omega)}}\, e^{-ik_0(d-a)} \\
D_1(\omega) &= \frac{(t(\omega))^2E_0(\omega)e^{i\omega t}}{t_{10}(\omega)(1-(r(\omega))^2e^{-i\gamma_0(\omega)})}\, e^{-ik_0(\frac{d}{2}-a)}e^{ik(\frac{d}{2}+a)} \\
D_2(\omega) &= \frac{(t(\omega))^2r_{10}(\omega)E_0(\omega)e^{i\omega t}}{t_{10}(\omega)(1-(r(\omega))^2e^{-i\gamma_0(\omega)})}\, e^{-ik_0(\frac{d}{2}-a)}e^{-ik(\frac{d}{2}+a)} \\
G(\omega) &= \frac{(t(\omega))^2E_0(\omega)e^{i\omega t}}{1-(r(\omega))^2e^{-i\gamma_0(\omega)}}\, e^{2ik_0 a}
\end{aligned}
$$

Inside the region between the plates, the intensity of an electromagnetic wave incident on a plate surface depends upon the distance between the plates because of the interferences they sustain. One easily shows [23] that this intensity is given by

$$
\mathcal{I}(\omega) = \frac{|C_1(\omega)|^2}{(t(\omega))^2|E_0(\omega)|^2e^{2ik_0a}} = \frac{1}{1-2\,\mathrm{Re}\{(r(\omega))^2e^{-i\gamma_0(\omega)}\} + (R(\omega))^2}.
$$

The effective coefficients of reflection and transmission applying to an electromagnetic

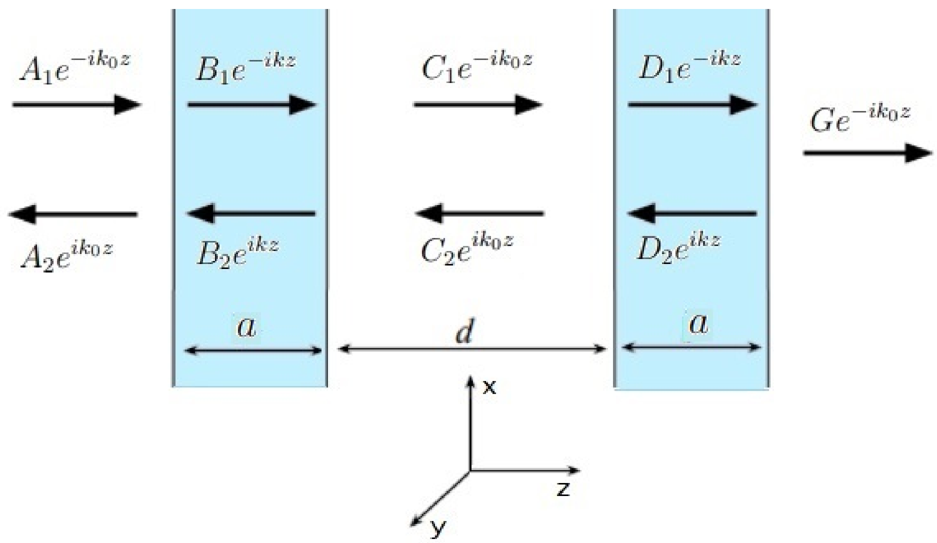

Figure 2: The system of two plates (shaded areas) and three regions of quantum vacuum. The diagram represents electromagnetic waves initially propagating to the right in each sector.

wave in the cavity and incident on a plate are then respectively given by $\mathcal{R}(\omega) = R(\omega)\mathcal{I}(\omega)$ and $\mathcal{T}(\omega) = T(\omega)\mathcal{I}(\omega)$.

To determine the effective cutoff function, we consider a beam with initial intensity I_0 formed by the quantum vacuum electromagnetic waves of frequency ω moving rightward toward the left-hand plate of the cavity. The intensity of this beam following its entry into the cavity is $I_1(\omega) = T(\omega)I_0(\omega)$. After multiple reflections between the plates, the Poynting vector modules of the incident, reflected, and transmitted beams on the right-hand plate are proportional to $I_1(\omega)$, as well as to $\mathcal{I}(\omega)$, $\mathcal{R}(\omega)$ and $\mathcal{T}(\omega)$, respectively.

The above results are for quantum vacuum electromagnetic waves with $k_0 > 0$. For a wave with wavenumber $k_0 < 0$, the phase at $z = \frac{d}{2} + a$ is not necessarily equal to that of the corresponding wave with $k_0 > 0$ at $z = -\frac{d}{2} - a$. To see how this influences the total electromagnetic wave intensity inside the cavity and within the plates, let us consider the electric field resulting from the multiple reflections of an electromagnetic wave initially moving leftward outside the cavity. We suppose that the phase difference between a wave initially moving leftward and one initially moving rightward is $\phi = \phi(\omega)$ at $z = 0$ when the two plates are temporarily removed. Taking account of the multiple reflections between the plates of the initially moving rightward electromagnetic wave with frequency ω, the amplitude of the electric field on the interior face of the right-hand plate is

$$E_1(\omega) = C_1(\omega) e^{-i\gamma_0(\omega)/2}.$$

The amplitude of the initially moving leftward electromagnetic wave with frequency ω on the same face is

$$E_2(\omega) = \frac{E_1(\omega) r(\omega) e^{i(\omega t + \phi)} e^{-i3k_0 d/2}}{1 - (r(\omega))^2 e^{-i\gamma_0(\omega)}}.$$

Therefore, the total amplitude of the electric field on the interior face of the right-hand plate is

$$E_{\mathrm{tot}}(\omega) = \frac{E_1(\omega) e^{i\omega t} e^{-ik_0 d/2}(1 + r(\omega) e^{i\phi} e^{ik_0 d})}{1 - (r(\omega))^2 e^{-i\gamma_0(\omega)}}.$$

Since the intensity of this incident electric field, denoted by $I_{\mathrm{i\,tot}}$, is proportional to E_{tot} multiplied by its complex conjugate, then [23]

$$\frac{I_{\mathrm{i\,tot}}(\omega)}{I_1(\omega)} = \frac{|E_{\mathrm{i\,tot}}(\omega)|^2}{|E_1(\omega)|^2} = \frac{1 + 2\,\mathrm{Re}\{r(\omega) e^{i\phi'}\} + R(\omega)}{1 - 2\,\mathrm{Re}\{(r(\omega))^2 e^{-i\gamma_0(\omega)}\} + (R(\omega))^2}. \tag{10}$$

where $\phi' = \phi - k_0 d$ is the phase difference between the rightward and leftward moving waves when both arrive on the interior face of the right-hand plate.

If a beam of intensity given by (10) is incident on one of the plates, then the intensity of the reflected beam is

$$I_{\mathrm{r\,tot}}(\omega) = \frac{I_1(\omega) R(\omega)(1 + 2\,\mathrm{Re}\{r(\omega) e^{i\phi'}\} + R(\omega))}{1 - 2\,\mathrm{Re}\{(r(\omega))^2 e^{-i\gamma_0(\omega)}\} + (R(\omega))^2}$$

and that of the transmitted beam is

$$I_{\mathrm{t\,tot}}(\omega) = \frac{I_1(\omega) T(\omega)(1 + 2\,\mathrm{Re}\{r(\omega) e^{i\phi'}\} + R(\omega))}{1 - 2\,\mathrm{Re}\{(r(\omega))^2 e^{-i\gamma_0(\omega)}\} + (R(\omega))^2}.$$

Thus, the pressure on the interior face of the right-hand plate (or the interior face of the left-hand plate) is

$$< P_{\mathrm{in}}(\omega) >_t = \frac{< \mathcal{E}_1(\omega) >_t}{I_1(\omega)} (I_{\mathrm{i\,tot}}(\omega) + I_{\mathrm{r\,tot}}(\omega) - I_{\mathrm{t\,tot}}(\omega)),$$

where $\mathcal{E}_1(\omega)$ is the density of energy of a wave having $I_1(\omega)$ as initial intensity inside the cavity. Therefore

$$< P_{\text{in}}(\omega) >_t = < \mathcal{E}_0(\omega) >_t \Lambda(\omega)(1 + 2\,\text{Re}\{r(\omega)e^{i\phi'}\} + R(\omega))(1 + R(\omega) - T(\omega)), \qquad (11)$$

where $\mathcal{E}_0(\omega)$ is the energy density of a wave with $I_0(\omega)$ as initial intensity outside the cavity and

$$\Lambda(\omega) = T(\omega)/(1 - 2\,\text{Re}\{(r(\omega))^2 e^{-i\gamma_0(\omega)}\} + (R(\omega))^2).$$

Taking (9) into account, the comparison of (1) and (11) shows that inside the cavity, the effective cutoff function for each plate is

$$\mathcal{F}^{\text{eff}}(k_0) = \frac{1}{2}\Lambda(k_0)(1 + 2\,\text{Re}\{r(k_0)e^{i\phi'}\} + R(k_0))(1 + R(k_0) - T(k_0)), \qquad (12)$$

where we have used the same symbols to represent the functions r, γ_0, Λ, R and T even if they are now expressed in terms of k_0 instead of ω.

When considering many measures of the Casimir force for a given distance between the plates, the beam moving rightward and the one moving leftward have relative random phases. For the average value of this force over time, the term $2\,\text{Re}\{r(k_0)e^{i\phi'}\}$ thus vanishes in (12). Consequently

$$\mathcal{F}^{\text{eff}}(k_0) = \frac{1}{2}\Lambda(k_0)(1 + R(k_0))(1 + R(k_0) - T(k_0)).$$

The expression of the force on the exterior side of one plate of the cavity formed by Drude-conductor plates can be obtained from (4) with the cutoff function of Section 3. In the case of waves with $k_0 > 0$, that is the plate on the left-hand side of the cavity, this force is given by

$$F_{\text{out}}(a, d) = \frac{\hbar c}{2\pi} \int_0^\infty k_0(1 + R(k_0) - T(k_0))\text{d}k_0.$$

Using (11), we also obtain that the force on the interior side of the same plate is

$$F_{\text{in}}(a, d) = \frac{\hbar c}{2\pi} \int_0^\infty k_0\Lambda(k_0)(1 + R(k_0))(1 + R(k_0) - T(k_0))\text{d}k_0.$$

Therefore, the net force on each of these Drude-conductor plates is given by

$$\begin{aligned} F(a,d) &= F_{\text{in}}(a,d) - F_{\text{out}}(a,d) \\ &= \frac{\hbar c}{2\pi}\int_0^\infty k_0[\Lambda(k_0)(1+R(k_0))-1](1+R(k_0)-T(k_0))\mathrm{d}k_0. \end{aligned} \tag{13}$$

5. The Casimir force for Drude-aluminium plates

Let us apply (13) to two identical plates of aluminium kept at 1.5 Kelvin, which is just a bit higher than their superconductivity temperature [24, 25]. The other relevant physical characteristics of aluminium are : $\sqrt{\epsilon_1\mu_1} = 1$, $N = 1.806 \times 10^{29}\,\mathrm{m}^{-3}$, $\sigma_0 = 10^{12}\,\mathrm{S/m}$ [26] and $\tau = 1.965 \times 10^{-10}\,\mathrm{s}$, so that $\omega_p = 2.397 \times 10^{16}\,\mathrm{Hz}$ and $k_p = \omega_p/c = 7.99 \times 10^7\,\mathrm{m}^{-1}$.

Evaluating (13) numerically requires skillful handling because the graph of its integrand may have a large number of very high and narrow peaks with barely predictable positions for $k_0 < k_p$ and $a < 8\times 10^{-7}\,\mathrm{m}$. Furthermore, a simple numerical analysis of the integrand seen as a function of k_0 shows that for $k_0 \gg k_p$ it tends to 0 faster than $1/k_0^2$, so that the integral converges. A very good approximation of the force values can then be obtained through the numerical integration of this function for k_0 between 0 and $2\,k_p$. For $a > 8 \times 10^{-7}\,\mathrm{m}$, due to scattering, these peaks rapidly disappear, and the integrand still decreases to 0 faster than $1/k_0^2$. In these cases, we have done the numerical integration up to values of k_0 where the variation of the integral becomes insignificant. These computations were performed for a large set of values for $5\times 10^{-9}\,\mathrm{m} \le a \le 2\times 10^{15}\,\mathrm{m}$ and $10^{-8}\,\mathrm{m} \le d \le 2\,\mathrm{m}$ (see [27]). A geometrical representation of the results is given in Fig. 3, where $F(a,d)$ is expressed in newtons (N), via a log scale, while a and d are expressed in $d_p = c\pi/\omega_p = 3.93 \times 10^{-8}\,\mathrm{m}$, also via log scales. A two-dimensional perspective for $5 \times 10^{-9}\,\mathrm{m} \le a \le 2 \times 10^{-5}\,\mathrm{m}$ is also shown in Fig. 4 to highlight the force variations differently.

The Casimir force between the two plates of aluminium differs significantly from the force generated by two perfectly reflecting plates within a certain range, which, according to (7), tends to 0 as d^{-2}. Similarly, the force between two thin dielectric plates shows comparable behavior to the case of two aluminium plates when their thickness-to-separation ratio increases, as noted in reference [19]. Hence, our calculated value of the Casimir force for Drude-aluminium plates for $d = 2 \times 10^{-5}\,\mathrm{m}$ and $a = 1.2 \times 10^{-7}\,\mathrm{m}$ is about 10^6 times that for perfectly reflecting plates. More generally, one can observe that $F(a, d)$ becomes independent of d for $d > 10^{-5}\,\mathrm{m}$ if $a > 4 \times 10^{-8}\,\mathrm{m}$, and for $d > 2 \times 10^{-6}\,\mathrm{m}$ if $a > 4 \times 10^{-7}\,\mathrm{m}$. Another notable difference is that for any fixed $d > 4 \times 10^{-6}\,\mathrm{m}$, there is a cliff for $4 \times 10^{-8}\,\mathrm{m} < a < 1.6 \times 10^{-7}\,\mathrm{m}$, where the force is multiplied by a factor of approximately 4300. Also for $d > 4 \times 10^{-6}\,\mathrm{m}$, the force stays practically constant and equal to $-3.3 \times 10^{-11}\,\mathrm{N}$ for $1.6 \times 10^{-7}\,\mathrm{m} < a < 4 \times 10^{-4}\,\mathrm{m}$. For $4\,\mathrm{m} < a < 2 \times 10^{15}\,\mathrm{m}$ and any fixed value of d between $10^{-8}\,\mathrm{m}$ and $2\,\mathrm{m}$, the force in newtons can be described by the function $\mathcal{F}(a) = -Ka$, where K is approximately equal to $3.76 \times 10^{-10}\,\mathrm{kg/s^2}$.

The aforementioned differences from perfectly reflecting plates, and more generally, the general appearance of the surface in Fig. 3, can be explained using the following mathematical and physical arguments. First, observe that the quantum vacuum electromagnetic field outside the system of two parallel plates is independent of the distance between these plates, as well as of their thickness. Thus, for every pair of values (a, d), the force between the plates is determined by what happens only in the regions between or inside the plates. The values of this force are obtained by integrating a function whose graph may include many high and narrow peaks, which correspond to resonances of the electromagnetic waves between the plates or inside them. These resonances are at the origin of a mutual repulsion force between the plates. A close examination of such a graph shows that these peaks are essentially similar to one another. To understand the physics underlying the total force between the plates, let us consider a typical peak of this graph and its integration when the values of d or a vary.

If a is very small, then the quantum vacuum electromagnetic waves have almost no interaction with the conduction electrons in the plate material, and, consequently, they are almost not scattered by these, the level of this scattering being here described by τ. For such very small values of a, the analysis of a typical peak shows that, as the value of a increases, its height also increases while its width, at one-tenth of its height,

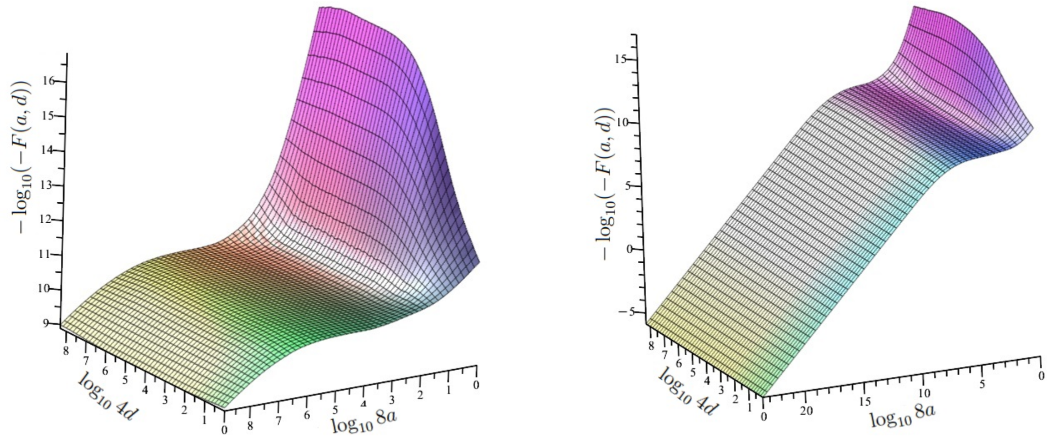

Figure 3: The function $-\log_{10}(-F(a,d))$ for two parallel aluminum plates at 1.5 Kelvin, whose thickness a expressed by $\log_{10} 8a$ is between $d_p/8$ and $5 \times 10^{22}\, d_p$, i.e. between 5×10^{-9} m and 2×10^{15} m, and their separation distance d expressed by $\log_{10} 4d$ is between $d_p/4$ and $5 \times 10^7\, d_p$, i.e. between 10^{-8} m and 2 m. The surface on the left is a close-up of that on the right for a between $d_p/8$ and $10^8\, d_p$, i.e. between 5×10^{-9} m and 4 m.

decreases. Moreover, the integral of this peak tends to a constant beyond a relatively small value of a, so that the increase in the height of the peak balances the reduction in its width. This observation applies regardless of the value of d, except for very small values of d such that no resonance can take place. Therefore, the mutual repulsion force of the plates caused by any single peak is roughly independent of d and a, beyond some relatively small value of a. But as d increases, the region between the plates contains more and more different vibrational modes, and thus, there are more and more resonant peaks between the plates. Hence, the repulsive force due to all these peaks increases with d. However, the frequencies of waves outside the cavity always belong to a larger set than the one of those inside. Consequently, the scattering by the plates of waves outside the cavity will exert on them a net residual attractive force independent of d. For instance, if $a = 5 \times 10^{-9}$ m and $d > 4 \times 10^{-2}$ m, then the force between the plates is practically constant and equal to -3.68×10^{-17} N.

When a reaches values large enough for the scattering of electromagnetic waves by conduction electrons in the plates to become important, a new phenomenon occurs. The analysis of a typical peak indeed shows that, as for very thin plates, the amplitude of

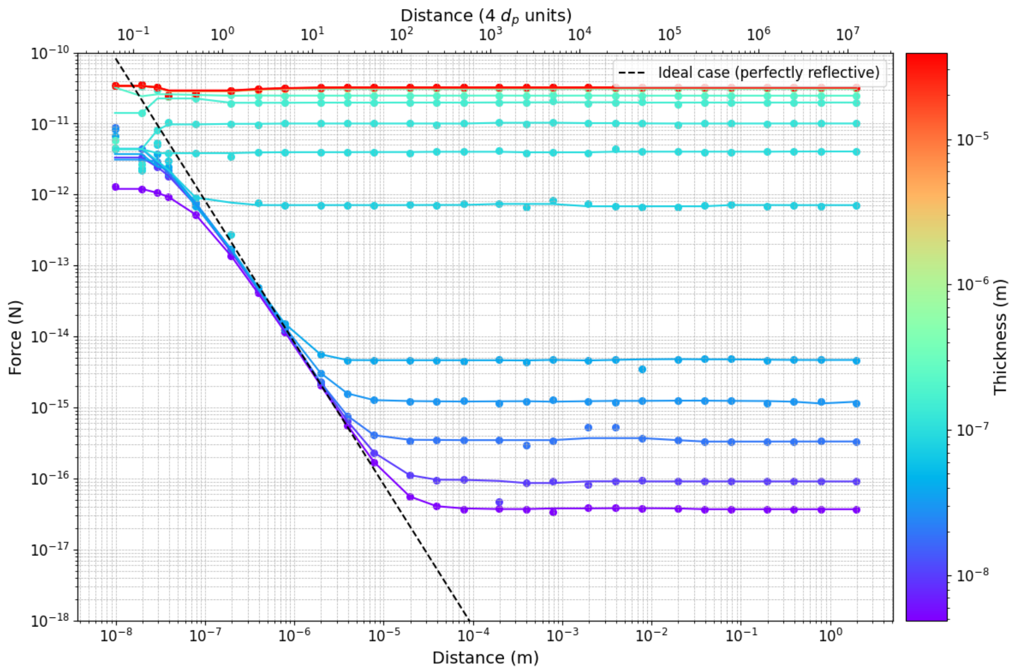

Figure 4: Force versus the distance at 1.5 Kelvin between two parallel aluminium plates for thicknesses between 5×10^{-9} m and 2×10^{-5} m. The ideal case of perfectly reflective plates is shown by the dashed line, while the solid lines show the processed data after applying a median filter with a window size of five.

the peak increases, and its width gradually decreases until a reaches a certain threshold. From this point on, the level of scattering by the plates causes a decrease in the amplitude of the peak. The integral of this peak, expressed as a function of a, thus determines a curve that reaches a maximum and then decreases with a very steep slope. This happens for any fixed value of d. This abrupt drop in force happens when all resonances are gradually annihilated due to the scattering of the vacuum electromagnetic fluctuations by the conduction electrons in the plates, which occurs when $a \geq 1.6 \times 10^{-7}$ m. The whole is what we call the cliff. Physically, this means that the repulsive force between the plates decreases substantially over a small interval of a, resulting in a large increase in their mutual attraction. Following this, the attraction force between the plates increases very slowly for $1.6 \times 10^{-7}\,\mathrm{m} < a < 4 \times 10^{-4}\,\mathrm{m}$. Geometrically, this is the relative flatness of the surface almost horizontal in the graphs of Fig. 3. Finally, from this last value of a, and more clearly for $4\,\mathrm{m} < a < 2 \times 10^{15}\,\mathrm{m}$, the force can be described by the function

$\mathcal{F}(a) = -Ka$ due to the absence of any significant repulsion force between the plates.

6. Conclusion and outlook

We have used the radiation pressure method to obtain a mathematical model of the one-dimensional Casimir force between two infinite parallel plates of Drude-conductor, which takes into account a number of physical characteristics of the material forming the plates. When applied to aluminium plates at 1.5 Kelvin and for a specific range of thickness and plate separation, the calculated force differs significantly from that acting on two perfectly reflective plates. The results of this paper also differ from those of [19], which are limited to the cases where $r_{01} \ll 1$ or $a \to \infty$ without taking into account the scattering of the plates. First, $F(a, d)$ is independent of d for $d > 10^{-5}$ m if $a > 4 \times 10^{-8}$ m, and for $d > 2 \times 10^{-6}$ m if $a > 4 \times 10^{-7}$ m. Second, the absolute value of this force increases abruptly as the thickness of the plates passes through a narrow threshold, for which the resonance of electromagnetic waves between and in the plates is almost completely eliminated due to scattering. For aluminium in the above particular conditions, this threshold is for a between 4×10^{-8} m and 1.6×10^{-7} m. Third, for $4\,\mathrm{m} < a < 2 \times 10^{15}$ m, the force is proportional to a.

The mathematical model of this paper is an attempt to describe the Casimir force between two parallel plates of the same finite thickness. Placed in the continuation of [18-20], the present paper can be seen as another step along the way toward a realistic mathematical model to describe the Casimir force through the radiation pressure method. The results here apply to the one-dimensional Casimir force when Drude's model is used to express the behavior of conduction electrons inside each plate. However, we think that these will likely continue to apply in a more elaborate model that would take into account all vacuum electromagnetic waves incident on the plates, not only those normal to the surfaces. Other improvements to the model could be to replace Drude's model by that of Drude-Lorentz, and to take account of the temperature [28-30] when formulating the reflection and the transmission coefficients. We hope to explore these questions in a future work.

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