

CS2800 Prelim 1 Review

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To prove...

$P \wedge Q$	show P and show Q
$P \vee Q$	show P or show Q
$\forall a \in A. P(a)$	Take an arbitrary a , show $P(a)$
$\exists a \in A. P(a)$	Pick any concrete a , show $P(a)$
$P \Leftrightarrow Q$	Show $P \Rightarrow Q$ and $Q \Rightarrow P$ separately

To prove $P \Rightarrow Q$

1. *Directly*: Assume P is true. Show that Q is true
2. *By Contrapositive*: Prove $\neg Q \Rightarrow \neg P$
3. *By Contradiction*: Assume P and $\neg Q$. Derive a contradiction

It may be helpful to break down your assumptions into different cases.

To disprove...

Prove the *negation*:

$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	show P is false or Q is false
$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	show P is false and show Q is false
$\neg(\forall a \in A. P(a))$	$\exists a \in A \neg P(a)$	Find/create an a which shows $\neg P(a)$
$\neg(\exists a \in A. P(a))$	$\forall a \in A \neg P(a)$	Take an arbitrary a and show $\neg P(a)$

Ex: Proving a Proposition

$$\forall g \in G. P(g) \Rightarrow (\exists g' \in G. Q(g')) \wedge Q(g)$$

1. Take an arbitrary g
2. Assume $P(g)$
 - 2.1 Pick any g' and show $Q(g')$
 - 2.2 Prove $Q(g)$

Ex: Proving the Negation

$$\exists g \in G. P(g) \wedge ((\forall g' \in G. \neg Q(g')) \vee \neg Q(g))$$

1. Pick a g and show $P(g)$ and $\neg Q(g)$ OR
2. Pick a g and show $P(g)$
 - 2.1 Take an arbitrary g'
 - 2.2 Show $\neg Q(g')$

How do we write a graph?

If you're asked to write a graph, you need to provide:

1. The **vertices** of the graph
2. The **edges** of the graph

Ex. *Undirected Graph*

Let $V = \{v, w, x\}$.

Let $E = \{\{v, w\}, \{w, x\}\}$.

Ex. *Directed Graph*

Let $V = \{0, 1, 2\}$.

Let $E = \{(0, 1), (2, 0)\}$.

What is a tree?

An *undirected* graph $G = (V, E)$ is a tree if it is **connected** and has **no** cycles.

Vertices with degree 1 in a tree are **leaves**.

Property: Every tree with at least one edge has at least one leaf.

Inducting on Vertices vs. Edges

Consider what information is actually relevant to the property that we're trying to prove. Is the number of edges or the number of vertices more important to know?

Try working out small examples!

Examples: Inductive Statement

1. Prove that for a graph $G = (V, E)$ without a cycle,
 $|E| \leq |V| - 1$.
2. Prove that every connected graph with n vertices has at least $n - 1$ edges.
3. Prove that every connected graph with n vertices and $n - 1$ edges is a tree.
4. Prove that every graph with n vertices and $n - 1$ edges is connected.

Inductive Step

In Homework 4 Q2, we saw the importance of starting at an arbitrary graph of $k + 1$ vertices rather than k vertices.

What's going on??? Aren't we proving $\mathbb{P}(k) \implies \mathbb{P}(k + 1)$?

The key is in the word arbitrary.

Inductive Step: Starting at $K + 1$

By starting at an **arbitrary** graph of $k + 1$ vertices, we already have this powerful modifier that we don't have to prove later on. Then, after removing a vertex to get a graph of k vertices, we use the inductive hypothesis and get some useful information.

Starting at an **arbitrary** graph of k vertices is way harder!! We must prove the graph we create by adding in a vertex is arbitrary.

Inductive Step: What to Remove?

Since you're the one doing the proof, you can decide what to remove! For example, in Homework 4, we removed a leaf in 2b) and a vertex in a cycle in 2c).

Since we already have that nice **arbitrary** modifier, we don't have to remove an arbitrary vertex in the graph. Wisely choosing what you remove can make the proof fall into place :)

2-Colorability Proof

Let 2-colorability be the property that using only two colors, we can color every vertex in such a way that no two adjacent vertices share the same color.

Prove, using induction, that every connected graph with n vertices and $n - 1$ edges is 2-colorable.

Example 2

Let $G = (V, E)$ be a finite (simple) undirected graph that has exactly two vertices $u, v \in V$ with odd degree. Use induction on $|E|$ to argue that there must be a path between u and v .

How do we write a function?

If you're asked to write a function, you need to provide:

1. The **domain** of the function
2. The **codomain** of the function
3. The function mappings

Ex. *Constructing Sets*

Let $X := \{0, 1\}$, $Y := \{a, b\}$.

$$f : X \rightarrow Y \quad f(0) = a, f(1) = b$$

Ex. *Using Existing Sets*

$$f : \mathbb{N} \rightarrow \mathbb{N} \quad f(x) = x^2$$

Well Defined Function

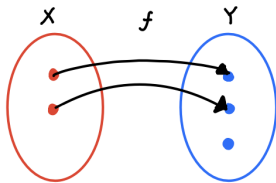
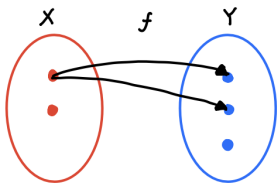
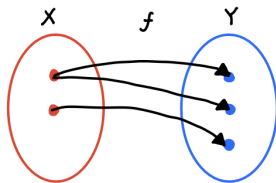
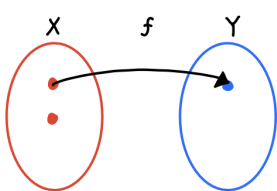
Functions & 'Jectivity

A function $f : X \rightarrow Y$ is *well-defined* if every input maps to a unique output.

1. Totality: $\forall x \in X, \exists y \in Y$ such that $f(x) = y$,
i.e it maps to **something**
2. Unique Mapping: $\forall x \in X, \forall y_1, y_2 \in Y$,
if $f(x) = y_1 \wedge f(x) = y_2$ then $y_1 = y_2$,
i.e. it's mapping is **unique!**

Ex. Well-Defined?

For each function, determine if it is well-defined. If it is not, state which property it violates (totality, unique mapping, or both)



Injectivity

A function $f : X \rightarrow Y$ is *injective* if each output is mapped to by at most one input.

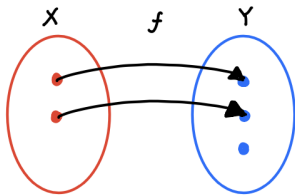
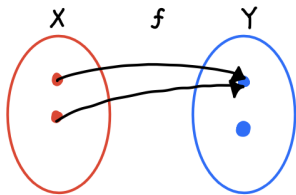
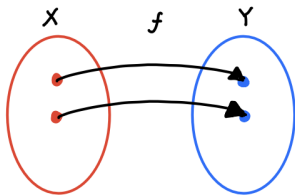
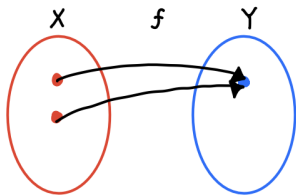
Two equivalent definitions: $\forall x_1, x_2 \in X \dots$

- $(f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)$
 - (*direct*) if two inputs map to the same output, the two inputs are equal
- $(x_1 \neq x_2) \Rightarrow f(x_1) \neq f(x_2)$
 - (*contrapositive*) if two inputs are not equal, their outputs are not equal

* Why are these equivalent? The first is $P \Rightarrow Q$, the second is $\neg Q \Rightarrow \neg P$.

Ex. Injective?

For each function, determine if it is injective.

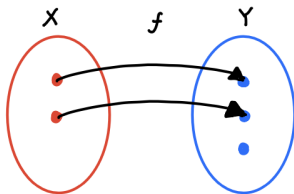
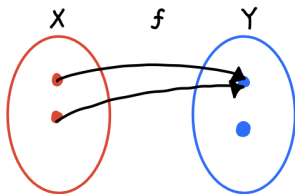
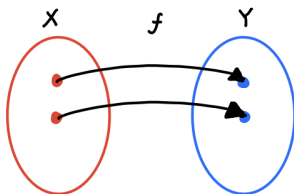
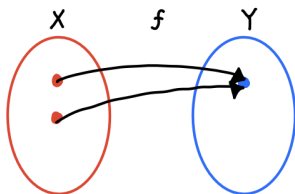


Surjectivity

A function f is *surjective* if all outputs are mapped to by some input.

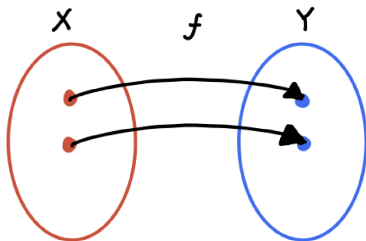
$$\forall Y \in Y, \exists X \in X \text{ such that } f(X) = Y$$

Ex. Surjective?



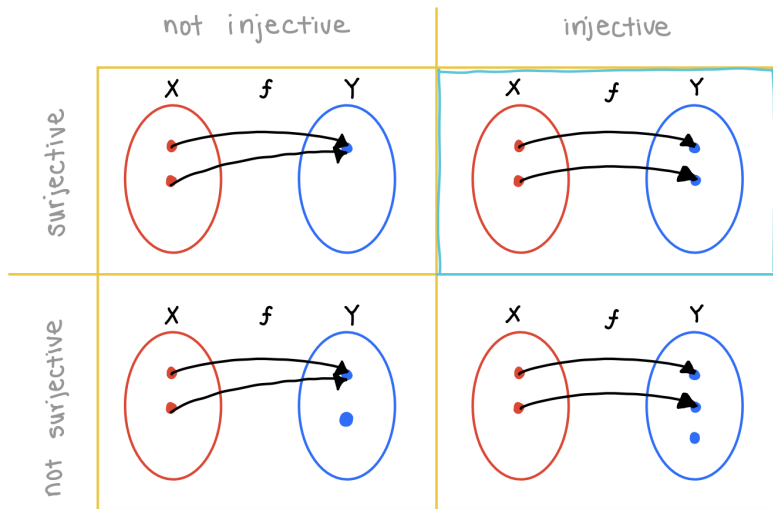
Bijjective

A function $f : X \rightarrow Y$ is *bijjective* if it is both *injective* and *surjective*.



If you're asked to prove if a function is bijective, you must separately argue that it is injective and surjective.

Comparing 'Jectivities



Notice: you only need to look at the *right* (codomain) blob to see if a function is 'jective!

Well-Defined Vs. 'Jectivity

Essentially, *unique mapping* and *injectivity* show the same property (no two arrows on one dot) but *unique mapping* describes dots in the **domain** and *injectivity* describes dots in the **codomain**.

Likewise, *totality* and *surjectivity* show the same property (no unmapped dots), but *totality* describes dots in the **domain** and *surjectivity* describes dots in the **codomain**.

Key Takeaway:

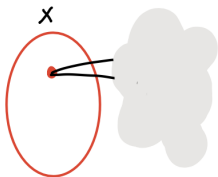
Well-defined? Look at **domain** (left blob)

'Jective? Look at **codomain** (right blob)

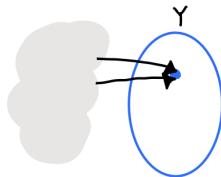
** And by dots, I of course mean elements!*

Cloudy Comparison

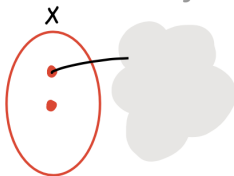
violates unique mapping



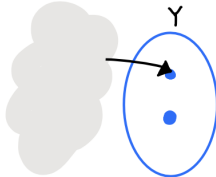
not injective



violates totality



not surjective



General Injectivity Proof Strategies

- There are two definitions, if you have trouble using one, try the other!
- You are proving an implication $P \Rightarrow Q$, so you should assume P and try to show Q is true.
- Write out the definition somewhere you can reference to keep yourself on track.

Contrapositive Injectivity Proof

1. **Define:** “W.T.S. $\forall x_1, x_2 \in X$,
 $(x_1 \neq x_2) \Rightarrow f(x_1) \neq f(x_2)$.”

- Helps you know your goal - refer back to this if you get confused/lost
- Shows us you know the def.

2. **Assume:** “Assume $x_1, x_2 \in X$ where $(x_1 \neq x_2)$.”

- Notice: we assumed LHS of def. is true.
- Common mistake: “I’ll just skip this step, and use the x_1, x_2 from the definition.” No! Please don’t! Those values are hypothetical! You need to assume you actually have such values before you can use them.

3. **Prove:** Use assumption & def. of f to show RHS is true.

- Apply f to x_1 and x_2 ,
- Use function def. to show $f(x_1) \neq f(x_2)$.

4. **Conclude:** “Thus, f is injective.”

- Generalizes above for any $x_1, x_2 \in X$.
- If you skipped writing the definition above, you should write it here.

Ex. Contrapositive Injectivity Proof

Show $f : \mathbb{Z}^{\text{even}} \rightarrow \mathbb{Z}^{\text{odd}}$, $f(x) = 2x + 3$ is injective.

Direct Injectivity Proof

1. **Define:** “W.T.S. $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow (x_1 = x_2).$ ”
 - Helps you know your goal - refer back to this if you get confused/lost.
 - Shows you understand the definition.
2. **Assume:** “Assume we have arbitrary $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ ”
 - Notice: we assumed LHS of definition is true.
 - Please do this step
3. **Prove:** Use assumption & def. of f to show RHS is true.
 - Replace $f(x_1) = f(x_2)$ with the function definitions of $f(x_1)$ and $f(x_2)$.
 - Often, this is where you do a bit of algebra! Simplify until you reach the conclusion of $x_1 = x_2$
4. **Conclude:** “Thus, f is injective.”
 - Generalizes above for any $x_1, x_2 \in X$.
 - If you skipped writing the definition above, you should write it here.

Ex. Injectivity Proof

Show $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(x) = 6x^2$ is injective.

Ex. Not Injective Proof

Prove $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = 6x^2$ is not injective.

General Surjectivity Proof Strategies

- Remember your goal is to prove the existence of an x
- Don't preemptively use x before you prove it exists
- Show that your choice of x actually satisfies $f(x) = y$

Surjective Proof

1. **Define:** "We want to show $\forall y \in Y, \exists x \in X$ such that $f(x) = y$ "
 - This helps you remember what you're trying to prove, so I encourage you to write this, but if you are short on time, this is okay to skip.
2. **Assume:** Assume $\exists y \in Y$. We want to show that $\exists x \in X$ such that $f(x) = y$.
 - Writing "We want to show..." is another way to keep your goals straight and clearly show us what your proof is trying to show.
3. **Find x:** "I claim that $x = \dots$ satisfies this because..."
 - Give a specific value of x that satisfies $f(x) = y$
 - Often, this value is in terms of y
4. **Show $f(x) = y$:** Use the function definition to evaluate $f(x)$, simplifying until you arrive at y .
5. **Conclude:** "... So, f is surjective"
 - If you skipped the definition at the beginning, you should state it here, e.g. "We have found such a $x \in X$ such that $f(x) = y$. Since y was arbitrary, this applies $\forall y \in Y$."

Ex. Surjective Proof

Show $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x_1, x_2) = (x_1 - 1, \frac{x_2}{7})$ is surjective.

A binary relation R on a set A is a subset of $A \times A$. When an ordered pair (x, y) satisfies $(x, y) \in R$, we say that xRy (“ x is related to y ”).

Properties of Relations

A relation is:

- *reflexive* if $\forall a \in A. aRa$
- *symmetric* if $\forall a, b \in A. aRb \implies bRa$
- *transitive* if $\forall a, b, c \in A. aRb \text{ and } bRc \implies aRc$
- *anti-symmetric* if $\forall a, b \in A. aRb \text{ and } bRa \implies a = b$

Example

Let \square be a relation on S which is reflexive, symmetric, and transitive. Let \triangle be a relation on S which is reflexive, anti-symmetric, and transitive.

Define \approx to be a relation on $S \times S$ such that:

$$(a_1, b_1) \approx (a_2, b_2) \iff a_1 \square a_2 \text{ and } b_1 \triangle b_2$$

Determine, with proof, whether \approx is reflexive, symmetric, transitive, or anti-symmetric.

Partial Orders

A binary relation R on the set A is a **partial order** if R is *reflexive*, *anti-symmetric*, and *transitive*.

A relation R' on the set A is **total** if every pair of elements are “comparable” according to the relation. That is:

$$\forall a, b \in A. (a, b) \in R' \vee (b, a) \in R'$$

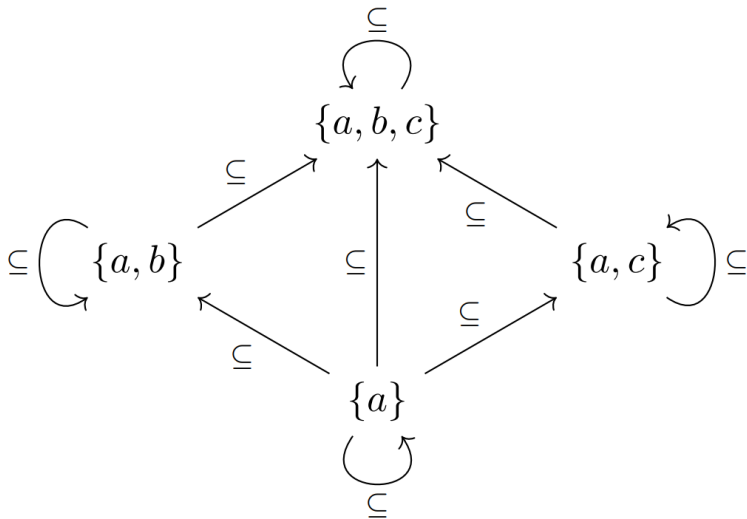
Ex. Subset

Given a set B , The subset relation (\subseteq) on $\mathcal{P}(B)$ is a partial order. We can observe that:

1. \subseteq is reflexive: $\forall a \in \mathcal{P}(B). a \subseteq a$
2. \subseteq is anti-symmetric: $\forall a, b \in \mathcal{P}(B). a \subseteq b \wedge b \subseteq a \Rightarrow a = b$
3. \subseteq is transitive: $\forall a, b, c \in \mathcal{P}(B). a \subseteq b \wedge b \subseteq c \Rightarrow a \subseteq c$

\subseteq is not total. For example, when $B = \{a, b, c, d\}$,
 $\{a, b\} \not\subseteq \{c, d\}$ and $\{c, d\} \not\subseteq \{a, b\}$

Ex. Subset



**Empty set not depicted, but it is a subset of all of these*

Equivalence Relation

A relation R on a set A is an **equivalence relation** if R is *reflexive*, *transitive*, and *symmetric*.

Ex. $=$ is an equivalence relation

1. An element $a \in A$, $a = a$
2. For elements $a, b, c \in A$, if $a = b$ and $b = c$, then $a = c$.
3. For elements $a, b \in A$, if $a = b$, then $b = a$

Equivalence Class

Given an equivalence relation R on the set A , and an element $x \in A$, the **equivalence class** of x , denoted $[x]_R$, is the set of all elements that are equivalent to x .

$$[x]_R = \{y \in A \mid x R y\}$$

Equivalence Class

Equivalence classes **partition** the set. That is, any two equivalence classes are either the same or disjoint.

$$\forall x, y \in A. [x]_R = [y]_R \vee [x]_R \cap [y]_R = \emptyset$$

In other words, for any two elements in the set, they are either equivalent (in the same equivalence class) or not equivalent.

All the elements inside an equivalence class are equivalent to each other, and all of the elements outside the equivalence class are not equivalent to any element in the equivalence class.

Ex. \mathbb{Z} modulo 8

$\forall a, b \in \mathbb{Z}, a \equiv b \pmod{8}$ iff $\exists k \in \mathbb{Z}. b - a = 8k$. Show if equivalence mod 8 is:

1. *reflexive*:
2. *transitive*:
3. *symmetric*:

The **quotient** of an equivalence relation R on the set A , denoted A/R , is the set of all *equivalence classes*.

In our previous example

$$\mathbb{Z}/\equiv_8 =$$

To Prove...

1. A relation is a partial order: show it is reflexive, anti-symmetric, and transitive
2. A relation is an equivalence relation: show it is reflexive, symmetric, and transitive
3. A set is an equivalence class: show all elements within the set are equivalent, and any element outside the set is not equivalent to any element inside the set

Excercise

Let S be a set with a partial order \sqsubseteq . For an arbitrary $n \in \mathbb{N}$, let \leq be a relation on S^n so that $s = (s_1, \dots, s_n) \leq w = (w_1, \dots, w_n)$ if either

1. $\exists i \in \mathbb{N}^+ \ i \leq n. \forall k \in \mathbb{N}^+ \ k < i. s_k = w_k \wedge s_i \sqsubseteq w_i \wedge s_i \neq w_i$
2. $s = w$

Example: if S is the set of letters a to z, and \sqsubseteq is the alphabetical order of characters. Then \leq on S^3 is “alphabetical order” of 3 letter strings. So $(c, a, t) \leq (d, o, g)$ and $(m, a, p) \leq (m, a, t)$.

Prove \leq is a partial order on S^n

References



Lecture Notes. Matthew Eichorn.



Lecture Notes. Anke Van Zuylen.

Thank you for listening!

You're gonna do great!!