CS2800 Prelim 1 Review

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To prove...

$P \wedge Q$	show P and show Q
$P \vee Q$	show P or show Q
$\forall a \in A.\ P(a)$	Take an arbitrary a, show $P(a)$
$\exists a \in A.\ P(a)$	Pick any concrete a, show $P(a)$
$P \Leftrightarrow Q$	Show $P \Rightarrow Q$ and $Q \Rightarrow P$ separately

To prove $P \Rightarrow Q$

1. *Directly*: Assume P is true. Show that Q is true
2. *By Contrapositive*: Prove $\neg Q \Rightarrow \neg P$
3. *By Contradiction*: Assume P and $\neg Q$. Derive a contradiction

It may be helpful to break down your assumptions into different *cases*.

To disprove...

Prove the *negation*:

$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	show P is false or Q is false
$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	show P is false and show Q is false
$\neg(\forall a \in A.\ P(a))$	$\exists a \in A\ \neg P(a)$	Find/create an a which shows $\neg P(a)$
$\neg(\exists a \in A.\ P(a))$	$\forall a \in A\ \neg P(a)$	Take an arbitrary a and show $\neg P(a)$

Ex: Proving a Proposition

$$\forall g \in G.\ P(g) \Rightarrow (\exists g' \in G.\ Q(g')) \wedge Q(g)$$

1. Take an arbitrary g
2. Assume $P(g)$
   2.1 Pick any g' and show $Q(g')$
   2.2 Prove $Q(g)$

Ex: Proving the Negation

$$\exists g \in G.\ P(g) \land ((\forall g' \in G.\ \neg Q(g')) \lor \neg Q(g))$$

1. Pick a g and show $P(g)$ and $\neg Q(g)$ OR
2. Pick a g and show $P(g)$
   2.1 Take an arbitrary g'
   2.2 Show $\neg Q(g')$

How do we write a graph?

If you're asked to write a graph, you need to provide:

1. The vertices of the graph
2. The edges of the graph

Ex. *Undirected Graph*

Let $V = \{v, w, x\}$.
Let $E = \{\{v, w\}, \{w, x\}\}$.

Ex. *Directed Graph*

Let $V = \{0, 1, 2\}$.
Let $E = \{(0, 1), (2, 0)\}$.

What is a tree?

An *undirected* graph $G = (V, E)$ is a tree if it is **connected** and has **no** cycles.

Vertices with degree 1 in a tree are **leaves**.

Property: Every tree with at least one edge has at least one leaf.

Inducting on Vertices vs. Edges

Consider what information is actually relevant to the property that we're trying to prove. Is the number of edges or the number of vertices more important to know?

Sometimes inducting on vertices is stronger than inducting on edges since we remove the edge attached to the vertex in the process. We can't have a hanging edge.

Try working out small examples!

Examples: Inductive Statement

1. Prove that for a graph $G = (V, E)$ without a cycle, $|E| \leq |V| - 1$.
2. Prove that every connected graph with n vertices has at least $n - 1$ edges.
3. Prove that every connected graph with n vertices and $n - 1$ edges is a tree.
4. Prove that every acyclic graph with n vertices and $n - 1$ edges is connected.

Examples: Inductive Statement

1. Prove that for a graph $G = (V, E)$ without a cycle, $|E| \leq |V| - 1$. Vertices
2. Prove that every connected graph with n vertices has at least $n - 1$ edges. Vertices
3. Prove that every connected graph with n vertices and $n - 1$ edges is a tree. Either
4. Prove that every acyclic graph with n vertices and $n - 1$ edges is connected. Either

Inductive Step

In Homework 4 Q2, we saw the importance of starting at an arbitrary graph of $k+1$ vertices rather than k vertices.

What's going on??? Aren't we proving $\mathbb{P}(k) \implies \mathbb{P}(k+1)$?

The key is in the word arbitrary.

Inductive Step: Starting at $K + 1$

By starting at an **arbitrary** graph of $\mathbf{k+1}$ vertices, we already have this powerful modifier that we don't have to prove later on. Then, after removing a vertex to get a graph of k vertices, we use the inductive hypothesis and get some useful information.

Starting at an **arbitrary** graph of **k** vertices is way harder!! We must prove the graph we create by adding in a vertex is arbitrary.

Inductive Step: What to Remove?

Since you're the one doing the proof, you can decide what to remove! For example, in Homework 4, we removed a leaf in 2b) and a vertex in a cycle in 2c).

Since we already have that nice **arbitrary** modifier, we don't have to remove an arbitrary vertex in the graph. Wisely choosing what you remove can make the proof fall into place :)

2-Colorability Proof

Let 2-colorability be the property that using only two colors, we can color every vertex in such a way that no two adjacent vertices share the same color.

Prove, using induction, that every connected graph with n vertices and $n - 1$ edges is 2-colorable.

2-Colorability Proof

Inductive Statement: Let P(n) = "Every connected graph with n vertices and $n-1$ edges is 2-colorable."
Base Case: A graph with one vertex is trivially 2-colorable.
Inductive Hypothesis: Assume $P(k)$ is true for $k \geq 1$.

2-Colorability Proof

Inductive Statement: Let P(n) = "Every connected graph with n vertices and $n-1$ edges is 2-colorable."
Base Case: A graph with one vertex is trivially 2-colorable.
Inductive Hypothesis: Assume $P(k)$ is true for $k \geq 1$.
Inductive Step: Consider an arbitrary, connected graph G with $k+1$ vertices and k edges. Since G is connected and has $k+1$ vertices, it is acyclic and thus a tree. Therefore, it must contain a vertex v of degree 1.
Let G' be the graph obtained by removing vertex v and its incident edge from G. By the **inductive hypothesis**, G' is 2-colorable.

2-Colorability Proof

Inductive Step: Consider an arbitrary, connected graph G with $k+1$ vertices and k edges. Since G is connected and has $k+1$ vertices, it is acyclic and thus a tree. Therefore, it must contain a vertex v of degree 1.

Let G' be the graph obtained by removing vertex v and its incident edge from G. By the **inductive hypothesis**, G' is 2-colorable. Now, add vertex v and its incident edge back to G'. Color v differently from its only neighbor. Thus, since G' is 2-colorable, G is similarly 2-colorable.

Therefore, by induction, every connected graph with n vertices and $n-1$ edges is 2-colorable.

Example 2

Let $G = (V, E)$ be a finite (simple) undirected graph that has exactly two vertices $u, v \in V$ with odd degree. Use induction on $|E|$ to argue that there must be a path between u and v.

How do we write a function?

If you're asked to write a function, you need to provide:

1. The domain of the function
2. The codomain of the function
3. The function mappings

Ex. *Constructing Sets*

Let $X := \{0, 1\}$, $Y := \{a, b\}$.

$f : X \to Y \quad f(0) = a, f(1) = b$

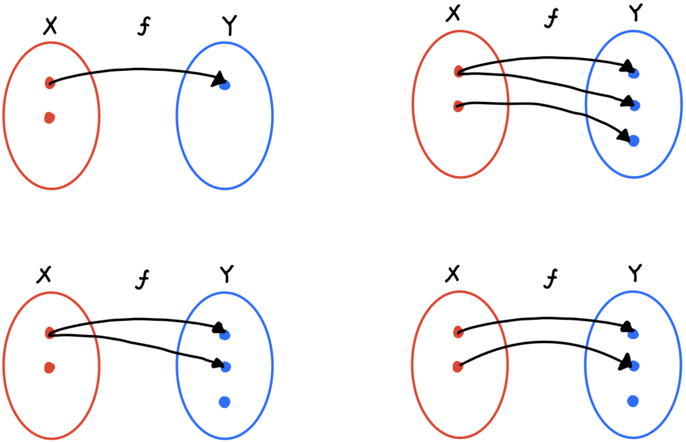
Ex. *Using Existing Sets*

$f : \mathbb{N} \to \mathbb{N} \quad f(x) = x^2$

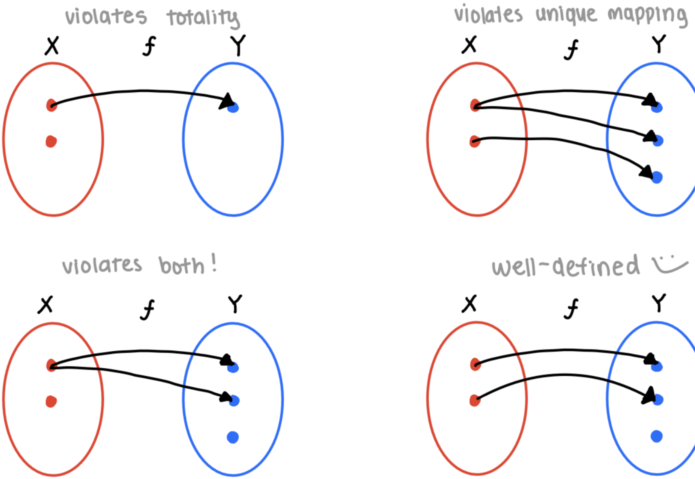
A function $f : X \to Y$ is *well-defined* if every input maps to a unique output.

1. Totality: $\forall x \in X$, $\exists y \in Y$ such that $f(x) = y$,
   i.e it maps to **something**
2. Unique Mapping: $\forall x \in X$, $\forall y_1, y_2 \in Y$,
   if $f(x) = y_1 \wedge f(x) = y_2$ then $y_1 = y_2$,
   i.e. it's mapping is **unique**!

Ex. Well-Defined?



Ex. Well-Defined?



Notice: you only need to look at the left (domain) blob to tell if the function is well-defined!

Injectivity

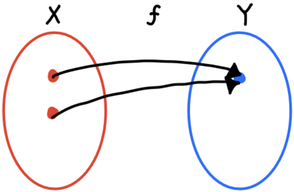
A function $f : X \to Y$ is *injective* if each output is mapped to by at most one input.

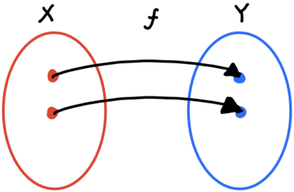
Two equivalent definitions: $\forall x_1, x_2 \in X \ldots$

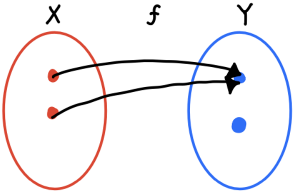
- $(f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)$
  - (*direct*) if two inputs map to the same output, the two inputs are equal
- $(x_1 \neq x_2) \Rightarrow f(x_1) \neq f(x_2)$
  - (*contrapositive*) if two inputs are not equal, their outputs are not equal

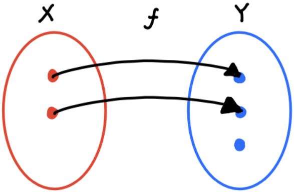
* *Why are these equivalent? The first is $P \Rightarrow Q$, the second is $\neg Q \Rightarrow \neg P$.*

Ex. Injective?

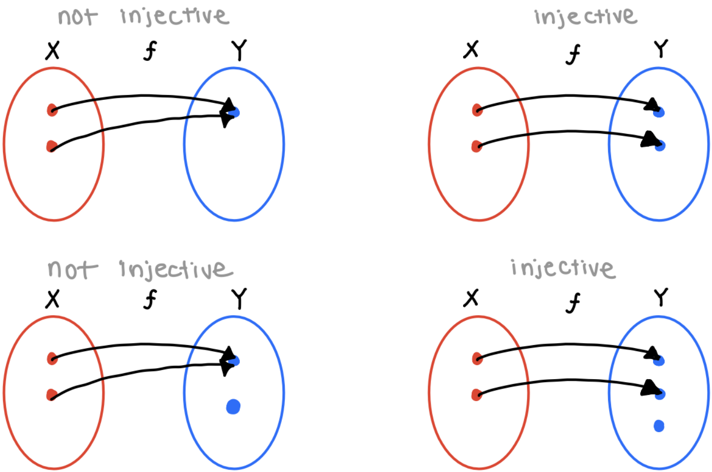








Ex. Injective?



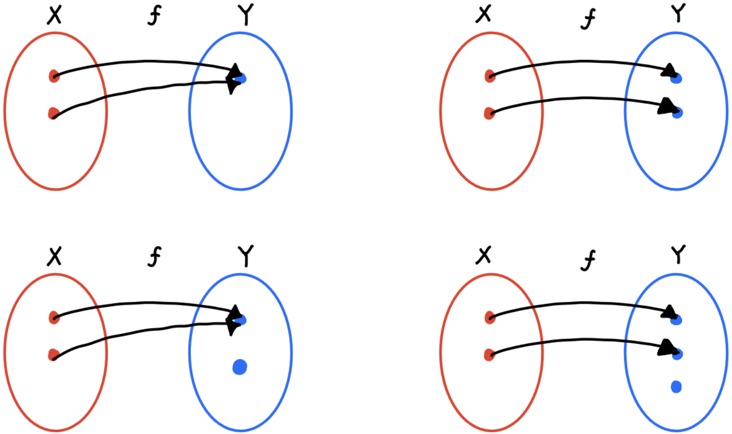
Notice: you only need to look at the *right* (codomain) blob to see if a function is injective!

Surjectivity

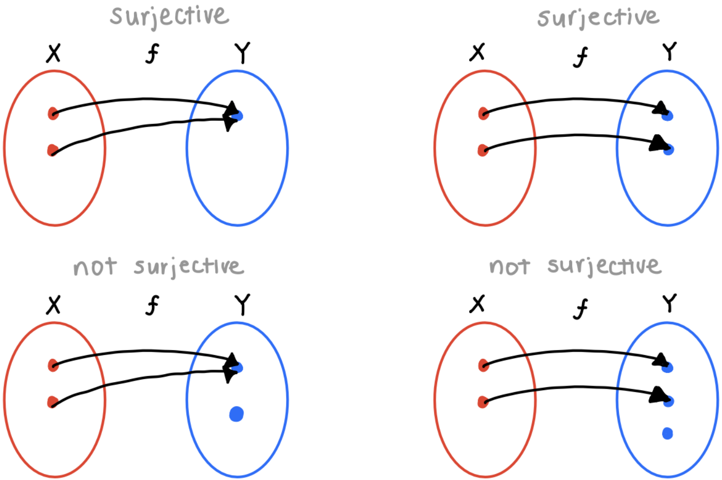
A function f is *surjective* if all outputs are mapped to by some input.

$\forall {\color{blue}Y} \in {\color{blue}\mathcal{Y}}, \exists {\color{red}X} \in {\color{red}\mathcal{X}}$ such that $f({\color{red}X}) = {\color{blue}Y}$

Ex. Surjective?



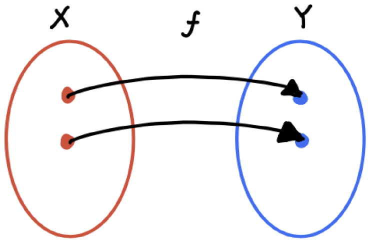
Ex. Surjective?



Notice: you only need to look at the *right* (codomain) blob to see if a function is surjective!

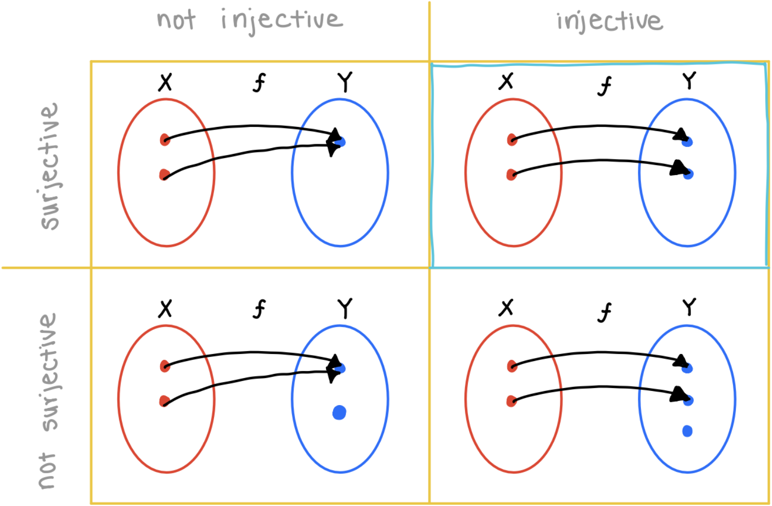
Bijective

A function $f : X \to Y$ is *bijective* if it is both *injective* and *surjective*.



If you're asked to prove if a function if bijective, you must seperately argue that it is injective and surjective.

Comparing 'Jectivities



Notice: you only need to look at the *right* (codomain) blob to see if a function is 'jective!

Well-Defined Vs. 'Jectivitive

Essentially, *unique mapping* and *injectivity* show the same property (no two arrows on one dot) but *unique mapping* describes dots in the domain and *injectivity* describes dots in the codomain.

Likewise, *totality* and *surjectivity* show the same property (no unmapped dots), but *totality* describes dots in the domain and *surjectivity* describes dots in the codomain.

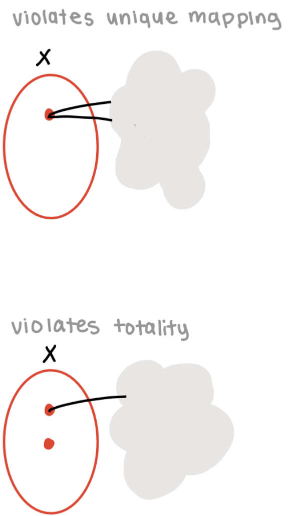
Key Takeaway:

Well-defined? Look at domain (left blob)

'Jective? Look at codomain (right blob)

* *And by dots, I of course mean elements!*

Cloudy Comparison



General Injectivity Proof Stategies

- There are two definitions, if you have trouble using one, try the other!
- You are proving an implication $P \Rightarrow Q$, so you should assume P and try to show Q is true.
- Write out the definition somewhere you can reference to keep yourself on track.

Contrapositive Injectivity Proof

1. **Define:** *"W.T.S.* $\forall x_1, x_2 \in X$*"*, $(x_1 \neq x_2) \Rightarrow f(x_1) \neq f(x_2)$*."*
   - Helps you know your goal - refer back to this if you get confused/lost
   - Shows us you know the def.
2. **Assume:** *"Assume* $x_1, x_2 \in X$ *where* $(x_1 \neq x_2)$*."*
   - Notice: we assumed LHS of def. is true.
   - Common mistake: *"I'll just skip this step, and use the* x_1, x_2 *from the definition."* No! Please don't! Those values are hypothetical! You need to assume you actually have such values before you can use them.
3. **Prove:** Use assumption & def. of f to show RHS is true.
   - Apply f to x_1 and x_2,
   - Use function def. to show $f(x_1) \neq f(x_2)$.
4. **Conclude:** *"Thus,* f *is injective."*
   - Generalizes above for any $x_1, x_2 \in X$.
   - If you skipped writing the definition above, you should write it here.

Ex. Contrapositive Injectivity Proof

Show $f : \mathbb{Z}^{\text{even}} \to \mathbb{Z}^{\text{odd}}$, $f(x) = 2x + 3$ is injective.

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Define: We want to show $\forall x_1, x_2 \in \mathbb{Z}^{\text{even}}$ such that $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

Ex. Show $f : \mathbb{Z}^{\text{even}} \to \mathbb{Z}^{\text{odd}}$, $f(x) = 2x + 3$ is injective.

Define: We want to show $\forall x_1, x_2 \in \mathbb{Z}^{\text{even}}$ such that $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

Assume: Assume we have some arbitrary $x_1, x_2 \in \mathbb{Z}^{\text{even}}$ such that $x_1 \neq x_2$. By the fact they are even, we can rewrite $x_1 = 2k_1$ and $x_2 = 2k_2$ for $k_1, k_2 \in \mathbb{Z}$. Since $x_1 \neq x_2$, $k_1 \neq k_2$.

Ex. Show $f : \mathbb{Z}^{\text{even}} \to \mathbb{Z}^{\text{odd}}$, $f(x) = 2x + 3$ is injective.

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Prove: We apply f to both x_1 and x_2.

$f(x_1) = f(2k_1) = 2(2k_1) + 3 = 4k_1 + 3$

$f(x_2) = f(2k_2) = 2(2k_2) + 3 = 4k_2 + 3$

For the sake of contradiction, assume $4k_1 + 3 = 4k_2 + 3$. Subtracting 3 we get $4k_1 = 4k_2$, then dividing by 4 we get $k_1 = k_2$. This is a contradiction, so our assumption is false, so $4k_1 + 3 \neq 4k_2 + 3$

Ex. Show $f : \mathbb{Z}^{\text{even}} \to \mathbb{Z}^{\text{odd}}$, $f(x) = 2x + 3$ is injective.

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Assume: Assume we have some arbitrary $x_1, x_2 \in \mathbb{Z}^{\text{even}}$ such that $x_1 \neq x_2$. By the fact they are even, we can rewrite $x_1 = 2k_1$ and $x_2 = 2k_2$ for $k_1, k_2 \in \mathbb{Z}$. Since $x_1 \neq x_2$, $k_1 \neq k_2$.

Prove: We apply f to both x_1 and x_2.

$f(x_1) = f(2k_1) = 2(2k_1) + 3 = 4k_1 + 3$

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For the sake of contradiction, assume $4k_1 + 3 = 4k_2 + 3$. Subtracting 3 we get $4k_1 = 4k_2$, then dividing by 4 we get $k_1 = k_2$. This is a contradiction, so our assumption is false, so $4k_1 + 3 \neq 4k_2 + 3$

Conclude: Since x_1, x_2 are arbirary, this applies to all of $\mathbb{Z}^{even}$. So, f is injective.

Direct Injectivity Proof

1. **Define:** *"W.T.S.* $\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \Rightarrow (x_1 = x_2)$*."*
   - Helps you know your goal - refer back to this if you get confused/lost.
   - Shows you understand the definition.
2. **Assume:** *"Assume we have arbitrary* $x_1, x_2 \in X$ *such that* $f(x_1) \neq f(x_2)$ *"*
   - Notice: we assumed LHS of definition is true.
   - Please do this step
3. **Prove:** *Use assumption & def. of* f *to show RHS is true.*
   - Replace $f(x_1) = f(x_2)$ with the function definitions of $f(x_1)$ and $f(x_2)$.
   - Often, this is where you do a bit of algebra! Simplify until you reach the conclusion of $x_1 = x_2$
4. **Conclude:** *"Thus,* f *is injective."*
   - Generalizes above for any $x_1, x_2 \in X$.
   - If you skipped writing the definition above, you should write it here.

Direct Injectivity Proof

Show $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 6x^2$ is injective.

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Define: We want to show f is injective, meaning $\forall x_1, x_2 \in \mathbb{N}$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Show $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 6x^2$ is injective.

Define: We want to show f is injective, meaning $\forall x_1, x_2 \in \mathbb{N}$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Assume: Assume we have arbitrary x_1, $x_2 \in X$ such that $f(x_1) = f(x_2)$.

Show $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 6x^2$ is injective.

Define: We want to show f is injective, meaning $\forall x_1, x_2 \in \mathbb{N}$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Assume: Assume we have arbitrary x_1, $x_2 \in X$ such that $f(x_1) = f(x_2)$.

Prove: By the definiton of f, we know

$$\begin{aligned} f(x_1) = f(x_2) &\Leftrightarrow 6(x_1)^2 = 6(x_2)^2 \text{def. of f} \\ &\Leftrightarrow (x_1)^2 = (x_2)^2 \text{ dividing both sides by 6} \\ &\Leftrightarrow \pm x_1 = x_2 \text{ square root of both sides} \\ &\Leftrightarrow x_1 = x_2 \text{ since } x_1, x_2 \in \mathbb{N} \end{aligned}$$

Show $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 6x^2$ is injective.

Define: We want to show f is injective, meaning $\forall x_1, x_2 \in \mathbb{N}$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Assume: Assume we have arbitrary x_1, $x_2 \in X$ such that $f(x_1) = f(x_2)$.

Prove: By the definiton of f, we know

$$
\begin{aligned}
f(x_1) = f(x_2) &\Leftrightarrow 6(x_1)^2 = 6(x_2)^2 \text{def. of f} \\
&\Leftrightarrow (x_1)^2 = (x_2)^2 \text{ dividing both sides by 6} \\
&\Leftrightarrow \pm x_1 = x_2 \text{ square root of both sides} \\
&\Leftrightarrow x_1 = x_2 \text{ since } x_1, x_2 \in \mathbb{N}
\end{aligned}
$$

Conclude: Therefore, f is injective.

Ex. Not Injective Proof

Prove $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 6x^2$ is not injective.

Ex. Not Injective Proof

Prove $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 6x^2$ is not injective.

To prove a property doesn't hold, it suffices to find just one counterexample!

Ex. Not Injective Proof

Consider $x_1 = 1$, $x_2 = -1$.

$f(x_1) = f(1) = 6(1^2) = 6$ $f(x_2) = f(-1) = 6(-1)^2 = 6$

So $f(x_1) = f(x_2) = 6$. However, $x_1 \neq x_2$. So, f is not injective.

General Surjectivity Proof Strategies

- Remember your goal is to prove the existence of an x
- Don't preemptively use x before you prove it exists
- Show that your choice of x actually satisfies $f(x) = y$

Surjective Proof

1. **Define**: "We want to show $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$"
   - This helps you remember what you're trying to prove, so I encourage you to write this, but if you are short on time, this is okay to skip.
2. **Assume**: Assume $\exists y \in Y$. We want to show that $\exists x \in X$ such that $f(x) = y$.
   - Writing "We want to show..." is another way to keep your goals straight and clearly show us what your proof is trying to show.
3. **Find x**: "I claim that $x = \ldots$ satisfies this because..."
   - Give a specific value of x that satisfies $f(x) = y$
   - Often, this value is in terms of y
4. **Show f(x) = y**: Use the function definition to evaluate $f(x)$, simplifying until you arrive at y.
5. **Conclude:** "... So, f is surjective"
   - If you skipped the definition at the beginning, you should state it here, e.g. "We have found such a $x \in X$ such that $f(x) = y$. Since y was arbitrary, this applies $\forall y \in Y$."

Ex. Surjective Proof

Show $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 - 1, \frac{x_2}{7})$ is surjective.

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Define: We want to show $\forall (y_1, y_2) \in \mathbb{R}^2$, $\exists (x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

Show $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 - 1, \frac{x_2}{7})$ is surjective.

Define: We want to show $\forall (y_1, y_2) \in \mathbb{R}^2$, $\exists (x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

Assume: Assume we have an arbitrary $(y_1, y_2) \in \mathbb{R}^2$. We want to find some $(x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

Show $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 - 1, \frac{x_2}{7})$ is surjective.

Define: We want to show $\forall (y_1, y_2) \in \mathbb{R}^2$, $\exists (x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

Assume: Assume we have an arbitrary $(y_1, y_2) \in \mathbb{R}^2$. We want to find some $(x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

Find x: Consider $(x_1, x_2) = (y_1 + 1, 7y_2)$.

Show $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 - 1, \frac{x_2}{7})$ is surjective.

Define: We want to show $\forall (y_1, y_2) \in \mathbb{R}^2$, $\exists (x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

Assume: Assume we have an arbitrary $(y_1, y_2) \in \mathbb{R}^2$. We want to find some $(x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

Find x: Consider $(x_1, x_2) = (y_1 + 1, 7y_2)$.

Show: Apply f, we then have

$$
\begin{aligned}
& f(x_1, x_2) \\
& = f(y_1 + 1, 7y_2) \\
& = (y_1 + 1 - 1, \frac{7 \cdot y_2}{7}) \\
& = (y_1, y_2)
\end{aligned}
$$

Show $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 - 1, \frac{x_2}{7})$ is surjective.

Define: We want to show $\forall (y_1, y_2) \in \mathbb{R}^2$, $\exists (x_1, x_2) \in \mathbb{R}^2$ such that $f(x_1, x_2) = (y_1, y_2)$.

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Find x: Consider $(x_1, x_2) = (y_1 + 1, 7y_2)$.

Show: Apply f, we then have

$$\begin{aligned}
&f(x_1, x_2) \\
&= f(y_1 + 1, 7y_2) \\
&= (y_1 + 1 - 1, \frac{7 \cdot y_2}{7}) \\
&= (y_1, y_2)
\end{aligned}$$

Conclude: Since (y_1, y_2) was arbitrary, this applies to all pairs from Y. So, f is surjective.

Relations

A binary relation R on a set A is a subset of $A \times A$. When an ordered pair (x, y) satisfies $(x, y) \in R$, we say that xRy ("x is related to y").

Properties of Relations

A relation is:

- *reflexive* if $\forall a \in A.\ aRa$
- *symmetric* if $\forall a, b, \in A.\ aRb \implies bRa$
- *transitive* if $\forall a, b, c \in A.\ aRb$ and $bRc \implies aRc$
- *anti-symmetric* if $\forall a, b \in A.\ aRb$ and $bRa \implies a = b$

Example

Let $\Box$ be a relation on S which is reflexive, symmetric, and transitive. Let $\triangle$ be a relation on S which is reflexive, anti-symmetric, and transitive.
Define $\approx$ to be a relation on $S \times S$ such that:

$$(a_1, b_1) \approx (a_2, b_2) \iff a_1 \Box a_2 \text{ and } b_1 \triangle b_2$$

Determine, with proof, whether $\approx$ is reflexive, symmetric, transitive, or anti-symmetric.

Partial Orders

A binary relation R on the set A is a **partial order** if R is *reflexive*, *anti-symmetric*, and *transitive*.

A relation R' on the set A is **total** if every pair of elements are "comparable" according to the relation. That is:

$$\forall a, b \in A.\ (a, b) \in R' \vee (b, a) \in R'$$

Ex. Subset

Given a set B, The subset relation ($\subseteq$) on $\mathcal{P}(B)$ is a partial order. We can observe that:

1. $\subseteq$ is reflexive: $\forall a \in \mathcal{P}(B).\ a \subseteq a$
2. $\subseteq$ is anti-symmetric: $\forall a, b \in \mathcal{P}(B).\ a \subseteq b \wedge b \subseteq a \Rightarrow a = b$
3. $\subseteq$ is transitive: $\forall a, b, c \in \mathcal{P}(B).\ a \subseteq b \wedge b \subseteq c \Rightarrow a \subseteq c$

$\subseteq$ is not total. For example, when $B = \{a, b, c, d\}$, $\{a, b\} \not\subseteq \{c, d\}$ and $\{c, d\} \not\subseteq \{a, b\}$

Ex. Subset

$$
\begin{tikzcd}
 & \{a,b,c\} \arrow[loop above, "\subseteq"] & \\
\{a,b\} \arrow[loop left, "\subseteq"] \arrow[ur, "\subseteq"] & & \{a,c\} \arrow[loop right, "\subseteq"] \arrow[ul, "\subseteq"'] \\
 & \{a\} \arrow[loop below, "\subseteq"] \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] \arrow[uu, "\subseteq"] &
\end{tikzcd}
$$

**Empty set not depicted, but it is a subset of all of these*

Equivalence Relation

A relation R on a set A is an **equivalence relation** if R is *reflexive*, *transitive*, and *symmetric*.

Ex. $=$ is an equivalence relation

1. An element $a \in A$, $a = a$
2. For elements $a, b, c \in A$, if $a = b$ and $b = c$, then $a = c$.
3. For elements $a, b \in A$, if $a = b$, then $b = a$

Equivalence Class

Given an equivalence relation R on the set A, and an element $x \in A$, the **equivalence class** of x, denoted $[x]_R$, is the set of all elements that are equivalent to x.

$$[x]_R = \{y \in A \mid x \, R \, y\}$$

Equivalence Class

Equivalence classes **partition** the set. That is, any two equivalence classes are either the same or disjoint.

$$\forall x, y \in A.\ [x]_R = [y]_R \vee [x]_R \cap [y]_R = \emptyset$$

In other words, for any two elements in the set, they are either equivalent (in the same equivalence class) or not equivalent.

All the elements inside an equivalence class are equivalent to each other, and all of the elements outside the equivalence class are not equivalent to any element in the equivalence class.

Ex. $\mathbb{Z}$ modulo 8

$\forall a, b \in \mathbb{Z}$, $a \equiv b \pmod{8}$ iff $\exists k \in \mathbb{Z}.\ b - a = 8k$

1. *reflexive:* $\forall a \in \mathbb{Z}.\ a - a = 8(0)$
2. *transitive:* Suppose $b - a = 8k$ and $c - b = 8k'$. So $b = c - 8k'$ and $c - a = 8(k + k')$.
3. *symmetric:* If $b - a = 8k$ then $a - b = 8(-k)$

Ex: $\mathbb{Z}$ modulo 8

$$\vdots$$

-16	-15	-14	-13	-12	-11	-10	-9
-8	-7	-6	-5	-4	-3	-2	-1
0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39

$$\vdots$$

Ex: $\mathbb{Z}$ modulo 8

$$\vdots$$

-16	-15	-14	-13	-12	-11	-10	-9
-8	-7	-6	-5	-4	-3	-2	-1
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39

$$\vdots$$

Quotient

The **quotient** of an equivalence relation R on the set A, denoted A/R, is the set of all *equivalence classes*.

In our previous example:

$$
\begin{aligned}
\mathbb{Z}/\equiv_8 &= \\
&= \{\{\ldots, -8, 0, 8, \ldots\}, \{\ldots, -7, 1, 7, \ldots\}, \{\ldots, -6, 2, 6, \ldots\}, \ldots\} \\
&= \{[0], [1], [2], [3], [4], [5], [6], [7]\}
\end{aligned}
$$

To Prove...

1. A relation is a partial order: show it is reflexive, anti-symmetric, and transitive
2. A relation is an equivalence relation: show it is reflexive, symmetric, and transitive
3. A set is an equivalence class: show all elements within the set are equivalent, and any element outside the set is not equivalent to any element inside the set

Excercise

Let S be a set with a partial order $\sqsubseteq$. For an arbitrary $n \in \mathbb{N}$, let $\leq$ be a relation on S^n so that $s = (s_1, ..., s_n) \leq w = (w_1, ..., w_n)$ if either

1. $\exists i \in \mathbb{N}^+ \; i \leq n. \, \forall k \in \mathbb{N}^+ \; k < i. \; s_k = w_k \wedge s_i \sqsubseteq w_i \wedge s_i \neq w_i$
2. $s = w$

Example: if S is the set of letters a to z, and $\sqsubseteq$ is the alphabetical order of characters. Then $\leq$ on S^3 is "alphabetical order" of 3 letter strings. So $(c, a, t) \leq (d, o, g)$ and $(m, a, p) \leq (m, a, t)$.

Prove $\leq$ is a partial order on S^n

References

Lecture Notes. Matthew Eichorn.

Lecture Notes. Anke Van Zuylen.

Thank you for listening!

You're gonna do great!!