CS2800 Prelim 2 Review

Robyn Burger, Clarence Dagins, Tuni Le, William Leung, Max Pace, Ambrose Yang

Review Session Feedback Form

General Tips:

- Have an example in the back of your mind for each counting rule
- Remember to simplify your final answer
- Show your work mathematically AND briefly explain your reasoning in words

Multiplication Rule

Count a set S by breaking down choices into multiple steps where each step is independent from the others.

You might apply the *multiplication rule* to a problem like this:

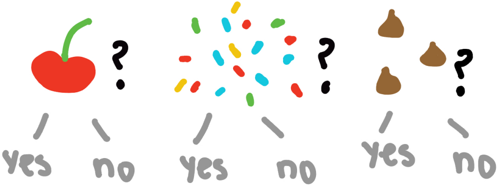
- n choices for step 1 ... n choices for step 5
- Then there are $n * n * n * n * n = n^5$ choices for the entire set

Ex. Ice Cream Cones - Multiplication Rule

Number of distinct ice cream cones with 3 toppings

Ex. Ice Cream Cones - Multiplication Rule

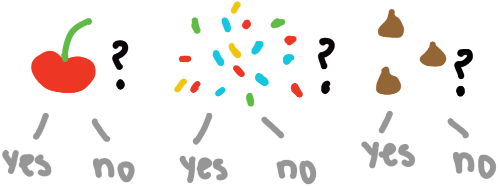
Number of distinct ice cream cones with 3 toppings



- Step 1: Cherries or no cherries = 2 choices
- Step 2: Sprinkles or no sprinkles? = 2 choices
- Step 3: Chocolate chips or no chocolate chips? = 2 choices

Ex. Ice Cream Cones - Multiplication Rule

Number of distinct ice cream cones with 3 toppings



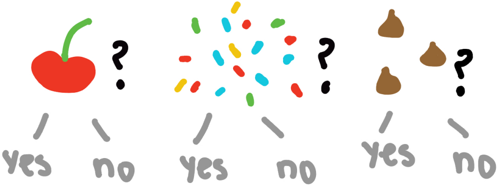
- Step 1: Cherries or no cherries = 2 choices
- Step 2: Sprinkles or no sprinkles? = 2 choices
- Step 3: Chocolate chips or no chocolate chips? = 2 choices

So there are three steps, each with 2 possibilities.

1. 2^3
2. 3^2

Ex. Ice Cream Cones - Multiplication Rule

Number of distinct ice cream cones with 3 toppings



- Step 1: Cherries or no cherries = 2 choices
- Step 2: Sprinkles or no sprinkles? = 2 choices
- Step 3: Chocolate chips or no chocolate chips? = 2 choices

So there are three steps, each with 2 possibilities.

$2 * 2 * 2 = 2^3$! So, there are 8 possible distinct ice cream cones.

Ex. License Plates - Multiplication Rule

Number of distinct license plates of length 7

H	X	M		7	1	2	8
A-Z	A-Z	A-Z		0-9	0-9	0-9	0-9

Ex. License Plates - Multiplication Rule

Number of distinct license plates of length 7

H	X	M	7	1	2	8
A-Z	A-Z	A-Z	0-9	0-9	0-9	0-9

- Steps 1-3 each have 26 choices (A-Z)

Ex. License Plates - Multiplication Rule

Number of distinct license plates of length 7

H	X	M		7	1	2	8
A-Z	A-Z	A-Z		0-9	0-9	0-9	0-9

- Steps 1-3 each have 26 choices (A-Z)
- Steps 4-7 each have 10 choices (0-9)

Ex. License Plates - Multiplication Rule

Number of distinct license plates of length 7

H	X	M		7	1	2	8
A-Z	A-Z	A-Z		0-9	0-9	0-9	0-9

- Steps 1-3 each have 26 choices (A-Z)
- Steps 4-7 each have 10 choices (0-9)

How many possibilities?

1. $26^3 * 10^4$
2. $3^{2}6*10^{4}$

Ex. License Plates - Multiplication Rule

Number of distinct license plates of length 7

H	**X**	**M**	**7**	**1**	**2**	**8**
A-Z	A-Z	A-Z	0-9	0-9	0-9	0-9

- Steps 1-3 each have 26 choices (A-Z)
- Steps 4-7 each have 10 choices (0-9)

Since each choice is independent, there are
$26 * 26 * 26 * 10 * 10 * 10 * 10 = 26^3 * 10^4$
$= 175,760,000$ possible license plates

Multiplication Rule

- $\star$ Used when previous choices do not affect future choices
- $\star$ Guiding examples: License plate, ice cream cones

Addition Rule

- Current choices DO affect later choices
- Break up possibilities into distinct cases, add them

Addition Rule

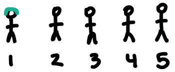
- Current choices DO affect later choices
- Break up possibilities into distinct cases, add them

Might look like:

- 5 possible distinct cases
- Case 1: In this case, there are n choices .... Case 5: In this case there are n choices
- So, in total there are $n + n + n + n + n = 5n$ possibilities

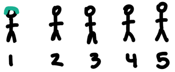
Ex. Addition Rule - Line Order

Anke, Martha, Tuni, Clarkson, and Obama are standing in line. If Tuni wants to stand in an odd position in the line, how many possible ways can they do this?



Ex. Addition Rule - Line Order

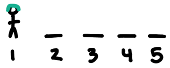
Anke, Martha, Tuni, Clarkson, and Obama are standing in line. If Tuni wants to stand in an odd position in the line, how many possible ways can they do this?



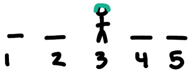
- Case 1: Tuni stands in position 1
- Case 2: Tuni stands in position 3
- Case 3: Tuni stands in position 5

Ex. Addition Rule - Line Order

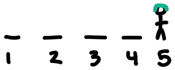
- Case 1: Tuni stands in position 1



- Case 2: Tuni stands in position 3

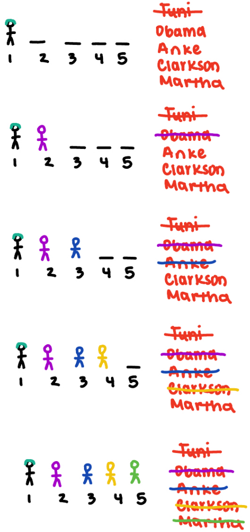


- Case 3: Tuni stands in position 5



For each of these we fix Tuni's position – there is one choice for that position.

Ex. Addition Rule - Line Order



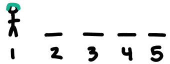
Ex. Addition Rule - Line Order

To recap: 1 choice for position 1 * 4 choices for position 2 * 3 choices for position 3 * 2 choices for position 4 * 1 choice for position 5

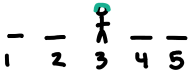
$1 * 4 * 3 * 2 * 1 = 4! = 24$ ways to arrange Tuni at position 1.

Ex. Addition Rule - Line Order

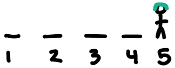
- Case 1: Tuni stands in position 1 **24 choices**



- Case 2: Tuni stands in position 3 **24 choices**



- Case 3: Tuni stands in position 5 **24 choices**



$24 + 24 + 24 = 72$ arrangements w/ Tuni at odd position

Subtraction Rule

If $A \subseteq S$, then $|A| = |S| - |S \backslash A|$.

So, another way to do the line order problem would be:

$|S|$ = Number of Arrangements
$|A|$ = Number of Arrangements with Tuni at odd position
$|S \backslash A|$ = Number of Arrangements with Tuni at even position

Subtraction Rule - Line Order

This problem goes similarly

- fix Tuni at position 2, there are 24 arrangements
- fix Tuni at position 4, there are 24 arrangements
- So, by addition rule: $|S \backslash A| = 24 + 24 = 48$
- There are $5! = 120$ total arrangements

Subtraction Rule

$|S| = \#$ Arrangements $= 120$
$|A| = \#$ of Arrangements with Tuni at odd position $= ?$
$|S \backslash A| = \#$ of Arrangements with Tuni at even position $= 48$

$|A| = |S| - |S \backslash A|$.

$|A| = 120 - 48$
$|A| = 72$

Addition/Subtraction Rule

- Break up possibilities into distinct cases, add them
- Consider finding the complement of the set you want (using the Subtraction rule)

Bijection Rule

If there is a bijection from $S \to T$, then $|S| = |T|$.

- Lets you reuse previous calculations
- Allows you to calculate the size of an easier set

Bijection Rule

Guiding Examples

- From Last HW: number of cone arrangements where all three had whipped cream is equal to the number of cone arrangements where none of the three have whipped cream
- Number of ways to order [5] set where 3 is to the right of 2

$$[\,1\,,\,4\,,\,2\,,\,3\,,\,5\,] \quad \mapsto \quad [\,1\,,\,4\,,\,23\,,\,5\,]$$

Ex. Division Rule

Suppose you have sets S and T, where each element in S maps to exactly k elements of $|T|$. Then, $|S| = \frac{|T|}{k}$.

- Suppose you want to count the number of cows in a field, given the number of cow legs.
- There is a 4-to-1 correspondence between number of cow legs $|L|$ and number of cows $|C|$. So, if there are $|L|$ legs, then there are $|C| = \frac{|L|}{4}$ cows

Binomial Coefficient

Definition 25.1 — Binomial Coefficient.

Given $k \leq n \in \mathbb{N}$, the *binomial coefficient* $\binom{n}{k}$ is the number of k-combinations of a set of n elements. That is,

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

You are creating the next big 5-person K-Pop group. You hold auditions, and 80 people show up. You want to test all their chemistry to make sure you get the optimal group, so you want to test every possible group of 5. How many groups do you have to test?

How many groups do you have to test?

$\binom{80}{5}$

$= \frac{80!}{5!(80-5)!}$

$= \frac{80!}{5!(75)!}$

$= \frac{80*79*78*77*76*75!}{5!75!}$

$= \frac{80*79*77*78*76}{5!}$

$= \frac{2884801920}{120}$

$= 2,4040,016$ groups!

Balls and Bins!

These are the only ones you need to worry about!

n Balls	m Bins	Arrangements	Injective Arrangements	Surjective Arrangements
Distinguishable	Distinguishable	m^n	$n! \cdot \binom{m}{n}$	
Indistinguishable	Distinguishable	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$

Distinguishable Balls Distinguishable Bins

Number of arrangements:

- For each of the n balls, we have m choices for which bin to put them into
- by the multiplication rule, we have m^n arrangements

Distinguishable Balls Distinguishable Bins

Number of injective arrangements:

- If our function is $\{balls\} \rightarrow \{bins\}$, we need to make sure each ball maps to a distinct bin, i.e. at most 1 ball per bin
- Doesn't work if n $>$ m
- For the first ball, we choose which of the m bins to place it in. For the next ball, we choose which of the remaining $m - 1$ bins to place it in. We continue until we exhaust all bins – giving us the $n!$ coefficient.
- For each ball, there are $\binom{m}{n}$ choices.
- So, we have $n!$ * $\binom{m}{n}$

Indistinguishable Balls Distinguishable Bins

Number of arrangements:

- Stars and stripes!
- Consider a string made up of n stars * and $m-1$ stripes |
- The $m-1$ stripes represent the boundaries between bins – so there is one after each bin except for the last
- In total our string has $n+m-1$ characters (number of stars + number of stripes)
- We now want to place our $m-1$ stripes in the $n+m-1$ positions. There are $\binom{n+m-1}{m-1}$ ways to do this
- We then place indistinguishable stars in the remaining positions

Indistinguishable Balls Distinguishable Bins

Number of injective arrangements

- Each bin has at most 1 ball
- We choose n of the m bins to place a ball into
- $\binom{m}{n}$

Combinatorial Proof

(10 points) Write a combinatorial proof showing that $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$. (Recall that a combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.) *Hint: Imagine you have $n+1$ red balls and k blue balls that you are trying to place in a line.*

Don't freak out!

- Which side looks easier?
- What's a known set you can count in this (easier) way?

Explain how you can count it in the harder way

- Remember your counting rules, balls and bins, etc.
- When in doubt – use your words! Your explanation doesn't have to be pretty to be correct!

Combinatorial Proof

(10 points) Write a combinatorial proof showing that $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$. (Recall that a combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.) *Hint: Imagine you have $n+1$ red balls and k blue balls that you are trying to place in a line.*

Right side looks easier! What set does right side count?

- n+k+1 items
- want to make a group of k

Combinatorial Proof

(10 points) Write a combinatorial proof showing that $\sum_{j=0}^{k}\binom{n+j}{j}=\binom{n+k+1}{k}$. (Recall that a combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.) *Hint: Imagine you have $n+1$ red balls and k blue balls that you are trying to place in a line.*

We create our set based on the RHS:

Claim: Both sides count the number of ways to order $n+1$ indistinguishable red balls and k indistinguishable blue balls.

Combinatorial Proof

(10 points) Write a combinatorial proof showing that $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$. (Recall that a combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.) *Hint: Imagine you have $n+1$ red balls and k blue balls that you are trying to place in a line.*

We explain how the RHS counts this set:

RHS: There are $n+k+1$ spots, and we select k of the spots to place the blue balls, the rest red. Thus there are $\binom{n+k+1}{k}$ ways to order the red and blue balls.

Combinatorial Proof

(10 points) Write a combinatorial proof showing that $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$. (Recall that a combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.) *Hint: Imagine you have $n+1$ red balls and k blue balls that you are trying to place in a line.*

Now, we explain you we can count it in the harder way:

LHS: We partition the set of possible orderings by the number of blue balls at the beginning of the ordering. Suppose there are exactly b blue balls at the beginning of the ordering. Then the sequence must begin with exactly b blue balls, followed by a red ball. There are $j = k - b$ blue balls remaining, and we have that $0 \leq j \leq k$. After the first red ball there are also n red balls remaining. Thus among the $n + j$ spots that are left we must choose j of them to have blue balls, so for fixed j, there are $\binom{n+j}{j}$ ways to order the remaining red and blue balls after starting with exactly b blue balls. Summing over all possible j, there are $\sum_{j=0}^{k} \binom{n+j}{j}$ ways to order the red and blue balls.

Probability

General tips

- Generally pretty similar to combinatorics — justify your answers with both math and words, and keep in mind what events you're actually representing
- Can be helpful sometimes to think about Venn diagrams or probability trees

Probability Spaces

A probability space $(\Omega, \mathcal{F}, Pr)$ consists of...

- a sample space Ω of all possible outcomes,
- a set of events $\mathcal{F}$ which they can fulfill, and
- a probability function Pr that says how likely those events are to occur.

Probability Spaces

A probability space $(\Omega, \mathcal{F}, Pr)$ consists of...

- a sample space Ω of all possible outcomes,
- a set of events $\mathcal{F}$ which they can fulfill, and
- a probability function Pr that says how likely those events are to occur.

Complementary events: For the outcome of a fair coin flip, let A be the event that it's heads and A' be the event that it's not heads (i.e., tails)

- A and A' are considered "complements" of each other
- Complements are mutually exclusive: $Pr(A \cap A') = 0$
- Complements are exhaustive: $Pr(A \cup A') = 1$
- So for any outcome, either one happens or the other

Ex. Probability - Strawberry Picking

Clarence bought a box of 36 strawberries, but unfortunately 2 of them are rotten! Robyn picks 5 strawberries at random.

What is the probability of Robyn having at least 1 rotten strawberry among the 5?

Ex. Probability - Strawberry Picking

Clarence bought a box of 36 strawberries, but unfortunately 2 of them are rotten! Robyn picks 5 strawberries at random.

What is the probability of Robyn having at least 1 rotten strawberry among the 5?

- What is our sample space?
- What is the event that we are counting?

Ex. Probability - Strawberry Picking

Clarence bought a box of 36 strawberries, but unfortunately 2 of them are rotten! Robyn picks 5 strawberries at random.

What is the probability of Robyn having at least 1 rotten strawberry among the 5?

- Sample space is all possibility to choose 5 strawberries out of 36.
- Event is all possibility where Robyn will pick at least 1 rotten strawberry.

Ex. Probability - Strawberry Picking

Clarence bought a box of 36 strawberries, but unfortunately 2 of them are rotten! Robyn picks 5 strawberries at random.

What is the probability of Robyn having at least 1 rotten strawberry among the 5?

- $|\Omega| = \binom{36}{5}$
- $|A| = \binom{36}{5} - \binom{34}{5}$

So the probability of interest will be:

$$Pr(A) = \frac{|A|}{|\Omega|} = \frac{\binom{36}{5} - \binom{34}{5}}{\binom{36}{5}} = \frac{11}{42}$$

Conditioning and Independence

Conditioning: How likely is A to happen given that B happens?

- $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B)$

Two events A and B are independent if...

- $\Pr(A \mid B) = \Pr(A)$
- (Equivalently, $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$)

Ex. Independence - Playing cards

Tuni thoroughly shuffles a standard deck of 52 playing cards, then draws a card from the top of the deck.

- Let R = the event that the card drawn is a red card.
- Let K = the event that the card drawn is a King.

Are R and K independent?

Ex. Independence - Playing cards

Tuni thoroughly shuffles a standard deck of 52 playing cards, then draws a card from the top of the deck.

- Let R = the event that the card drawn is a red card.
- Let K = the event that the card drawn is a King.

Are R and K independent?

- If R and K are independent, then $\Pr(R \cap K) = \Pr(R) \cdot \Pr(K)$
- There are 26 red cards and 4 kings in a standard deck, so $\Pr(R) = 26/52 = 1/2$ and $\Pr(K) = 4/52 = 1/13$
- There are 2 red kings, so $\Pr(R \cap K) = 2/52 = 1/26 = \Pr(R) \cdot \Pr(K)$

Yes, R and K are independent.

Independence on more than 2 events

A set of events A_1, ..., A_n are...

- pairwise independent if any two events are independent from one another (e.g., $\Pr(A_1 \cap A_2) = \Pr(A_1) \cdot \Pr(A_2)$)
- mutually independent if any intersection of any number of these events are independent from one another (e.g., $\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1) \cdot \Pr(A_2)) \cdot \Pr(A_3))$

Bayes's Rule and Law of Total Probability

Bayes's Rule: $\Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)}$

Law of Total Probability: Given a partition set of events A_1, ..., A_n of a sample space, and an event B...

- $Pr(B) = \sum_{i=1}^{n} Pr(B|A_i) \cdot Pr(A_i)$
- Likelihood of event B can be represented as adding up the possibilities within all of the A_i's

Bayesian Inference

Bayesian Inference: updating event knowledge with more info

- $\Pr(A)$ = "prior probability" = initial belief
- $\Pr(A|B)$ = "posterior probability" = new belief after observing new evidence B
- $\frac{Pr(B|A)}{Pr(B)}$ = whether B is more (> 1) or less (< 1) likely to occur during A

You can also consider the effect of multiple observations; e.g., considering B_1 after having already considered B_2:

- Prior probability becomes $\Pr(A|B_1)$
- Posterior probability becomes $\Pr(A|B_1 \cap B_2)$
- $\frac{Pr(B_2|A\cap B_1)}{Pr(B_2|B_1)}$ = whether B_2 is more or less likely to occur during A, given that B_1 also occurs
- Note that $\Pr(B_2|B_1) = \Pr(A|B_1) \cdot \Pr(B_2|A \cap B_1) + \Pr(\overline{A}|B_1) \cdot \Pr(B_2|\overline{A} \cap B_1)$ by law of total probability

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal (one head, one tail). They can only be distinguished by looking at them. All the subsequent parts must be read in sequence, i.e. the information in the previous steps is assumed known.

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal. They can only be distinguished by looking at them.

- Victoria shuts her eyes, chooses a coin at random, and toss it. What is the probability that the lower face of the coin is a head?

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Let us denote:

- DH = The coin is double-headed
- DL = The coin is double-tailed
- N = The coin is normal
- L_H = The lower face is head
- U_H = The upper face is head

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal. They can only be distinguished by looking at them.

- Victoria shuts her eyes, chooses a coin at random, and toss it. What is the probability that the lower face of the coin is a head?

$$
\begin{aligned}
Pr(L_H) &= Pr(L_H \mid DH)Pr(DH) + Pr(L_H \mid DT)Pr(DT) \\
&\quad + Pr(L_H \mid N)Pr(N) \\
&= 1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}
\end{aligned}
$$

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal. They can only be distinguished by looking at them.

- Now she opens her eyes and sees that the upper face is head. What is the probability that the lower face is a head?

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal. They can only be distinguished by looking at them.

- Now she opens her eyes and sees that the upper face is head. What is the probability that the lower face is a head?

We are interested in computing $Pr(L_H \mid U_H)$!

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal. They can only be distinguished by looking at them.

- Now she opens her eyes and sees that the upper face is head. What is the probability that the lower face is a head?

$$
\begin{aligned}
Pr(L_H \mid U_H) &= \frac{Pr(L_H \cap U_H)}{Pr(U_H)} \\
&= \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}
\end{aligned}
$$

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Tuni's Clarification: Intuition can be misleading when it comes to probability! It is possible for one to think that, since we are conditioning on U_H, we know that there can only be 4 coins that has head(s), so why can't we compute $Pr(L_H \mid U_H)$ as $\frac{\text{2 double-head}}{\text{4 coins}}$?

This is because, the computation $\frac{2}{4}$ is only true if the 4 coins are equally likely to be on our hand. But this is not true! Informally, if a head that is facing up, and there is a same number of double-head and normal coins, then double-head coins must be **more likely** to be in our hand than normal coins! Thus, the 4 coins are not equally likely to be on our hand, so we cannot just take 2 divides by 4 for our probability.

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal. They can only be distinguished by looking at them.

- She shuts her eyes again, picks up the coin, and tosses it again. What is the probability that the lower face is a head?

Ex. Bayes's Rule & Law of Total Probability - Coin Toss

Victoria has 5 fair coins. 2 are double-headed, 1 is double-tailed, and 2 are normal. They can only be distinguished by looking at them.

- She shuts her eyes again, picks up the coin, and tosses it again. What is the probability that the lower face is a head?

$$
\begin{aligned}
Pr(L_{2,H} \mid U_{1,H}) &= Pr(L_{2,H} \mid DH, U_{1,H}) Pr(DH \mid U_{1,H}) \\
&\quad + Pr(L_{2,H} \mid N, U_{1,H}) Pr(N \mid U_{1,H}) \\
&= 1 \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{6}
\end{aligned}
$$

References

- Lecture Notes. Matthew Eichorn.
- Lecture Notes. Anke Van Zuylen.

Thank you for listening!

You're gonna do great!!

Review Session Feedback Form