Discussion 11: Combinatorics

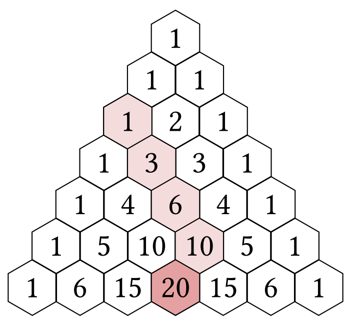
Problem 1. In this problem, we'll give a combinatorial proof of another identity involving the binomial coefficients. For all $m, n \in \mathbb{N}$:

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n}$$

This identity is often referred to as the *Hockey Stick Identity* because of how the involved binomial coefficients are arranged in Pascal's Triangle. For example, if we let $m = 2$ and $n = 3$, the identity says that

$$\binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \binom{5}{3} = \binom{6}{3}.$$

We highlight the coefficients on the left side in light pink and the coefficient on the right side in dark pink in the following figure.



To write a combinatorial proof, we must identify a set S that we can count with both sides of the identity. In this case, suppose that we have a cup with n red and $m+1$ blue marbles. Then, let S be the set of all distinct arrangements of these marbles into a line. In other words,

$$S = \left\{ w \in \{R, B\}^* : |w|_R = n \text{ and } |w|_B = m + 1 \right\}$$

(a) Explain how the right side of the identity counts the set S.

To reason about the left side of the identity, we need to figure out the meaning of the variable k. Suppose that k represents the number of red marbles to the left of the rightmost blue marble.

(b) What are the values of k that are associated with the following arrangements?

B R B B R   B B B B R R R   R B R B R B R B R   R R B R B B R R B

(c) What are all possible values of k?

(d) Explain why are there exactly $\binom{m+k}{k}$ arrangements associated with each value of k.

(e) Put together the ideas from parts (d) and (e) to explain how the left side of the identity counts the set S.

For some extra practice (outside of discussion), try to give an algebraic proof of the *Hockey Stick Identity*. One approach is to let $m \in \mathbb{N}$ be arbitrary and then perform induction on n.

Problem 2. Alfred returned home with a box of donuts with exactly 2 donuts each of 6 different flavors. In how many different ways can Alfred select 7 of these donuts to bring to his coworkers the following morning?

(Here, two selections as "different" if they disagree on the number of at least one flavor of donuts. The order in which Alfred selected the donuts and the way they are arranged do not matter.)

(a) Consider the following counting process:

We can use "stars and stripes", treating the 7 *selections as indistinguishable balls and the* 6 *flavors as distinguishable bins. Thus, there are* $\binom{7+6-1}{6-1} = \binom{12}{5} = 792$ *possible selections.*

Explain the mistake here. Have we over or under-counted the possible selections?

In order to correct this mistake, we will use a counting approach that combines "stars and stripes" with the (bad event) Inclusion-Exclusion Rule.

(b) Let S be the set of selections of 7 donuts in a hypothetical world where there are 7 donuts of each flavor. What is $|S|$?

(c) Next, let's define the bad events given your findings in part (a). What are six things that cannot happen for the six flavors in order to ensure that an element of S corresponds to a selection that Alfred can make?

We'll call the bad events $S_1, \ldots S_6$.

(d) What is $|S_1|$?

Hint: Think about how we can generate a selection that we are guaranteed is in S_1. We saw something similar when we did the surjective "stars and strings" proof in the lecture.

Notice that this is the same as $|S_i|$ for each $i \in [6]$.

(e) What is $|S_1 \cap S_2|$? A similar approach to part (b) should work.

Again, this should be the same as $|S_i \cap S_{i'}|$ for all $i \neq i' \in [6]$.

(f) What is the cardinality of any j-way intersection of the bad events for $j \geq 3$? How do you know this?

(g) Recall that the bad event form of the Inclusion-Exclusion Rule says that the cardinality of our desired set is

$$|S| + \sum_{j=1}^{n} (-1)^j \sum_{\substack{I \subseteq [n] \\ |I| = j}} \left| \bigcap_{i \in I} S_i \right|.$$

Plug your answers from parts (d), (e), and (f) into this formula to calculate the number of possible selections.