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Position-Force Synthesis of Closed-Loop Linkages

The dimensional synthesis of spatial closed-loop linkages to match both force and position specifications is investigated. An efficient means of formulating force equations is introduced through the application of linear algebra to screw theory. The synthesis of spatail four-bar linkages is discussed in detail; it is shown that the maximum number of allowable design positions is not decreased after force constraints are imposed on classical position synthesis problems.

1 Introduction

The purpose of the dimensional synthesis process of linkage design is to determine the dimensions of linkages so that they can perform prescribed tasks. Normally the kinematic synthesis of linkages deals with designing linkages to match only position or motion specifications. In many applications, however, force specifications are essential because of the interactions between a linkage and its environment. Without force considerations during the dimensional synthesis process, large and costly actuators or costly control efforts may be needed to satisfy force specifications. Our research seeks to avoid this by considering force specifications in the dimensional synthesis process of linkage design. We call this process force synthesis. Although the idea of determining dimensions based on force considerations has been known for a long time, the concept of force synthesis as a formal synthesis theory was introduced only recently. It was originated by Roth (1989) and was extended by Raghavan (1989). These concepts were used by Huang and Roth (1990) to study the force synthesis of planar four-bar linkages.

In this paper, we consider the design of spatial closed-loop linkages to match both force and position specifications. We first discuss the synthesis of passive linkages and then describe how we utilize the results for the synthesis of passive linkages to design active linkages. Passive linkages are defined as linkages without actuators and without auxiliary force devices (such as springs or dashpots) attached to them. In passive linkages, specified external forces are structurally equilibrated by the linkages. The central conceptual tool in this research is the use of reciprocal screw systems. We apply linear algebra to screw systems and develop an efficient approach to formulate force equations for closed-loop linkages. To illustrate the concept of position-force synthesis, we present a detailed study of the synthesis of spatial four-bar linkages.

2 Theoretical Background

2.1 Screws and Screw Systems. This section briefly reviews the theory of screws. A screw **S** is defined by a straight line (screw axis) in space with an associated pitch. It is useful to represent a screw by screw coordinates, $\mathbf{S} = (\mathbf{s}, \mathbf{s}_o)$, where $\mathbf{s}$ and $\mathbf{s}_o$ are Cartesian 3-vectors, and **S** is thus a 6-vector. Since a screw depends only on five parameters, the 6-vector can be thought of as the homogeneous coordinates of a point of 5-dimensional projective space. The pitch of the screw is defined as

$$p = \frac{\mathbf{s}\cdot\mathbf{s}_o}{\mathbf{s}\cdot\mathbf{s}}.$$

There are two common ways to represent a screw using a 6-vector: one uses the so-called radial coordinate representation $(\mathbf{s}, \mathbf{s}_o)$, the other uses the reverse order, the so-called axial coordinates $(\mathbf{s}_o, \mathbf{s})$. Two screws $\mathbf{S}_1 = (\mathbf{s}_1, \mathbf{s}_{o1})$ and $\mathbf{S}_2 = (\mathbf{s}_2, \mathbf{s}_{o2})$ are said to be reciprocal if they satisfy the condition

$$\mathbf{s}_1\cdot\mathbf{s}_{o2} + \mathbf{s}_2\cdot\mathbf{s}_{o1} = 0.$$

If one of the two screws is represented by axial coordinates and the other by radial coordinates, then the reciprocal condition can be thought of as requiring that the inner product of the two screws equals zero, i.e,

$$\mathbf{S}_1\cdot\mathbf{S}_2 = 0. \tag{1}$$

All the screws constructed by the linear combinations of n independent screws form a screw system of the nth order, which is usually called an n-system. A reciprocal screw system is a system of screws in which every screw is reciprocal to all the screws in the original screw system, and the reciprocal screw system of an n-system is a $(6-n)$-system. For more rigorous developments of screw theory, see Ball (1900), Hunt (1978), and Roth (1984).

2.2 Vector Spaces and Screw Systems. Since a screw can be represented by a 6-vector, it is obvious that screws have vector properties in 6-space, denoted by $\mathbf{V}_6$. In the context of vector spaces, a screw system of the nth order can be represented by an n-dimensional subspace of $\mathbf{V}_6$, and a reciprocal screw system can be represented by the nullspace of the subspace representing the original screw system. Note that when we discuss screw systems in the context of linear algebra, a reciprocal system is always represented in a coordinate type (axial or radial) different from that for the original screw system, so that we can use Eq. (1) as the reciprocal condition of the two screws. In this paper, the terms screw systems and

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vector spaces are used interchangeably. For more discussions of screw systems within the context of linear algebra, see Davies and Primrose (1971) and Sugimoto and Duffy (1982).

Consider a body with m degrees of freedom. Its instantaneous screw system (ISS) is of order m and is spanned by m screws, $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$ (expressed in radial coordinates). To determine the reciprocal wrench system, $\mathbf{W}$ (expressed in axial coordinates), to the body's motion, we have the following linear equations:

$$\mathbf{S}_1 \cdot \mathbf{W} = 0,\ \mathbf{S}_2 \cdot \mathbf{W} = 0;\ \ldots;\ \mathbf{S}_m \cdot \mathbf{W} = 0. \tag{2}$$

It is obvious that $\mathbf{W}$ lies in the nullspace of the space spanned by $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$. To determine the screw system that $\mathbf{W}$ belongs to, we simply need to find a basis for the nullspace. Let $\mathbf{W} = (\omega_1, \omega_2, \ldots, \omega_6)$. We select any $6-m$ entries of the unknown wrench $\mathbf{W}$, say $\omega_1, \omega_2, \ldots, \omega_{6-m}$, and then solve the system of equations [given by Eqs. (2)] for the other m entries in terms of the selected ones. The solution for $\mathbf{W}$ takes the form:

$$\mathbf{W} = \omega_1 \mathbf{N}_1 + \omega_2 \mathbf{N}_2 + \ \ldots\ + \omega_{6-m} \mathbf{N}_{6-m},$$

where $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_{6-m}$ span the nullspace and serve as a basis for it. The nullspace of a space $\mathbf{S}$ is the space of all vectors orthogonal to $\mathbf{S}$; it is denotes by $\overline{\mathbf{S}}$, and called the orthogonal complement (Strang, 1980) of $\mathbf{S}$.

In what follows, we first quote some fundamental definitions and theorems on vector spaces (Shephard, 1966), then introduce two theorems that are especially useful in performing instantaneous kinematic and static analyses for closed-loop linkages.

- **Definition (Sum of subspaces)**: Let $\mathbf{U}$ and $\mathbf{W}$ be two subspaces of a vector space $\mathbf{V}$. The sum of $\mathbf{U}$ and $\mathbf{W}$ (denoted by $\mathbf{U}+\mathbf{W}$) is the set of all elements of $\mathbf{V}$ that can be written in the form $\mathbf{u}+\mathbf{w}$, where $\mathbf{u} \in \mathbf{U}$ and $\mathbf{w} \in \mathbf{W}$.
- **Definition (Intersection of subspaces)**: Let $\mathbf{U}$ and $\mathbf{W}$ be two subspaces of a vector space $\mathbf{V}$. The intersection of $\mathbf{U}$ and $\mathbf{W}$ (denoted by $\mathbf{U} \cap \mathbf{W}$) is the set of all elements that belong to both $\mathbf{U}$ and $\mathbf{W}$.
- **Definition (Direct sum of subspaces):** If $\mathbf{U}$ and $\mathbf{W}$ are independent subspaces of a vector space $\mathbf{V}$, i.e., $\mathbf{U} \cap \mathbf{W} = 0$, then the subspace $\mathbf{U}+\mathbf{W}$ is called the direct sum of the subspaces $\mathbf{U}$ and $\mathbf{W}$ and is denoted by $\mathbf{U} \oplus \mathbf{W}$.
- **Theorem (Orthogonal complement of intersection of subspaces):** The orthogonal complement of $\mathbf{U} \cap \mathbf{W}$ (denoted by $\overline{\mathbf{U} \cap \mathbf{W}}$) equals the sum of the orthogonal complements of $\mathbf{U}$ and $\mathbf{W}$, i.e., $\overline{\mathbf{U} \cap \mathbf{W}} = \overline{\mathbf{U}} + \overline{\mathbf{W}}$.
- **Theorem (Dimension of sum of subspaces)**:

  $$\mathrm{Dim}(\mathbf{U}+\mathbf{W}) = \mathrm{Dim}(\mathbf{U}) + \mathrm{Dim}(\mathbf{W}) - \mathrm{Dim}(\mathbf{U} \cap \mathbf{W})$$
- **Theorem (Dimension of direct sum of subspaces):**

  $$\mathrm{Dim}(\mathbf{U} \oplus \mathbf{W}) = \mathrm{Dim}(\mathbf{U}) + \mathrm{Dim}(\mathbf{W})$$

Consider a single closed-loop linkage, as shown in Fig. 1. Let $\mathbf{S}$ be the instantaneous screw system (ISS) of the coupler, C, and assume we separate the coupler into two parts, C_1 and C_2, so that the closed-loop linkage is broken down into two open-loop linkages. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the instantaneous screw systems of C_1 and C_2, respectively, relative to ground. In a general configuration of the linkage, the following relations are true:

$$\mathbf{V}_6 = \mathbf{S}_1 + \mathbf{S}_2,$$

$$\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_2.$$

And we have the following theorem:

Theorem 1 Let $\overline{\mathbf{S}}$, $\overline{\mathbf{S}}_1$, and $\overline{\mathbf{S}}_2$ be the reciprocal systems of $\mathbf{S}$, $\mathbf{S}_1$, and $\mathbf{S}_2$, respectively. Then $\overline{\mathbf{S}} = \overline{\mathbf{S}}_1 \oplus \overline{\mathbf{S}}_2$, i.e., $\overline{\mathbf{S}}_1 \cap \overline{\mathbf{S}}_2 = 0$ ($\overline{\mathbf{S}}_1$ and $\overline{\mathbf{S}}_2$ are linearly independent).

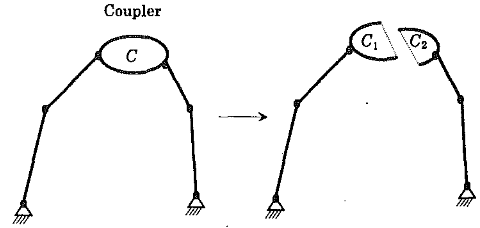

Fig. 1 A single closed-loop linkage

(Proof)

Since reciprocal screw systems are orthogonal complements of their original screw systems,

$$\mathbf{V}_6 = \mathbf{S}_1 \oplus \overline{\mathbf{S}}_1 = \mathbf{S}_2 \oplus \overline{\mathbf{S}}_2 = \mathbf{S} \oplus \overline{\mathbf{S}}. \tag{3}$$

Let $\mathrm{Dim}(\mathbf{S}) = n$, $\mathrm{Dim}(\mathbf{S}_1) = p$, and $\mathrm{Dim}(\mathbf{S}_2) = q$.
Since $\mathbf{V}_6 = \mathbf{S}_1 + \mathbf{S}_2$ and $\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_2$,

$$\mathrm{Dim}(\mathbf{V}_6) = 6 = \mathrm{Dim}(\mathbf{S}_1) + \mathrm{Dim}(\mathbf{S}_2) - \mathrm{Dim}(\mathbf{S}) = p + q - n.$$

From the above equation, we have

$$n = p + q - 6. \tag{4}$$

Since $\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_2$,

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}_1 + \overline{\mathbf{S}}_2.$$

From Eqs. (3) and (4), we have

$$\begin{aligned} \mathrm{Dim}(\overline{\mathbf{S}}) &= 6 - \mathrm{Dim}(\mathbf{S}) \\ &= 6 - (p + q - 6) \\ &= (6 - p) + (6 - q) \\ &= \mathrm{Dim}(\overline{\mathbf{S}}_1) + \mathrm{Dim}(\overline{\mathbf{S}}_2). \end{aligned}$$

Therefore,

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}_1 \oplus \overline{\mathbf{S}}_2.$$

q.e.d.

The above theorem indicates that the reciprocal screw system of the ISS of the coupler can be decomposed into the direct sum of the reciprocal screw systems of the ISS's of the two chains. It follows that a set of basis vectors of $\overline{\mathbf{S}}$ can be obtained by the combination of any set of basis vectors of $\overline{\mathbf{S}}_1$ and any set of basis vectors of $\overline{\mathbf{S}}_2$.

A similar theorem exists for an n-chain multiple-loop linkage. Let $\mathbf{S}$ be the instantaneous screw system of the coupler of the linkage, and $\mathbf{S}_1, \mathbf{S}_2, \ldots$, and $\mathbf{S}_n$ be the ISS's of the n chains that connect the platform to the ground. In a general configuration of the linkage, the following relations are true:

$$\mathbf{V}_6 = \mathbf{S}_i + \mathbf{S}_j,\ i,j = 1, 2, \ldots, n,\ i \neq j; \tag{5}$$

$$\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_2 \cap \ldots \cap \mathbf{S}_n, \tag{6}$$

and we have the following theorem:

Theorem 2 Let $\overline{\mathbf{S}}$ and $\overline{\mathbf{S}}_i$, $i = 1, \ldots, n$, be the reciprocal screw systems of $\mathbf{S}$ and $\mathbf{S}_i$, respectively. Then

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}_1 \oplus \overline{\mathbf{S}}_2 \oplus \ \ldots\ \oplus \overline{\mathbf{S}}_n,$$

i.e., $\overline{\mathbf{S}}_1 \cap \overline{\mathbf{S}}_2 \cap \ \ldots\ \cap \overline{\mathbf{S}}_n = 0$ (they are linearly independent).

(Proof)

From Eq. (5), we have

$$(\mathbf{S}_j + \mathbf{S}_{j+1}) \cap (\mathbf{S}_j + \mathbf{S}_{j+2} \cap \ldots \cap)(\mathbf{S}_j + \mathbf{S}_n) = \mathbf{V}_6, j = 1, 2, \ldots, n.$$

The above equation can be rewritten as

$$\mathbf{S}_{i-1} + (\mathbf{S}_i \cap \mathbf{S}_{i+1} \cap \ \ldots\ \cap \mathbf{S}_n) = \mathbf{V}_6, i = 2, \ldots, n.$$

Let

$$\mathbf{S}_{in} = \mathbf{S}_i \cap \mathbf{S}_{i+1} \cap \ldots \cap \mathbf{S}_n.$$

Equation (6) can be rewritten as

$$\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_{2n}.$$

Since $\mathbf{S}_1 + \mathbf{S}_{2n} = \mathbf{V}_6$, according to Theorem 1,

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}_1 \oplus \overline{\mathbf{S}}_{2n}. \tag{7}$$

Since $\mathbf{S}_{2n} = \mathbf{S}_2 \cap \mathbf{S}_{3n}$ and $\mathbf{S}_2 \cap \mathbf{S}_{3n} = \mathbf{V}_6$, according to Theorem 1,

$$\overline{\mathbf{S}}_{2n} = \overline{\mathbf{S}}_2 \oplus \overline{\mathbf{S}}_{3n}.$$

Substituting the above equation into Eq. (7) yields

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}_1 \oplus \overline{\mathbf{S}}_2 \oplus \overline{\mathbf{S}}_{3n}.$$

By mathematical induction, it can be shown that

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}_1 \oplus \overline{\mathbf{S}}_2 \oplus \ldots \oplus \overline{\mathbf{S}}_n.$$

q.e.d.

3 Force Synthesis of Passive Linkages

3.1 Statement of the Problem. In a linkage, rigid bodies (links) are constrained through kinematic joints that connect one link to another. When performing force analysis on a link, two kinds of forces are considered. One is the constraint forces transmitted through the kinematic joints connecting the link to the other links of the linkage. The other is the forces exerted on the link by the environment. If, in a certain configuration, a linkage is designed to support the external forces exerted on one of its links, the link under consideration can be thought of as a rigid body carried by the linkage. In our analysis, the position of that link can be assumed to be completely or partially specified. To take into account the external forces, several linearly independent wrenches, $\mathbf{W}_i$, $i = 1, 2, \ldots, d$, are specified. Our goal here is to determine the dimensions of the linkage so that any linear combination of the specified independent wrenches exerted on the externally loaded link can be equilibrated by the constraint forces exerted on that link by the joints. Usually more than one design position is specified, and the linkage has to guide the body to pass through all the specified positions and to support the specified wrenches.

3.2 Force Specifications at a Single Position. In this section, we investigate both complete and incomplete problems. We call a problem complete or incomplete depending on whether or not it can be converted to a completely specified kinematic problem. We also illustrate the correspondence between force specifications and kinematic constraints.

Based on the concept of reciprocal screws, a body with n degrees of freedom can maintain static equilibrium under the action of a wrench belonging to a $(6-n)$-system that is reciprocal to the motion of the body. On the other hand, if the body is constrained under the action of $6-n$ independent wrenches, we say that the specified wrench system is complete in the sense that the first-order motion of the body is completely constrained. Take for example the planar four-bar linkage shown in Fig. 2, and assume it is under static equilibrium without any actuator. For a certain configuration, if two independent (nonparallel, in this case) pure forces, $\mathbf{f}_1$ and $\mathbf{f}_2$, are exerted on its coupler, the instantaneous center I, of the coupler relative to ground, is readily determined because the instantaneous center is simply the intersection of the lines of action of the two forces. Thus the force specifications have been converted to kinematic constraints. Similarly, if five independent wrenches are exerted on the coupler of a single-degree-of-freedom spatial four-bar linkage, the instantaneous screw, of the coupler's motion relative to ground, is simply the reciprocal screw of the system spanned by the five specified wrenches.

For the case in which the number of prescribed independent

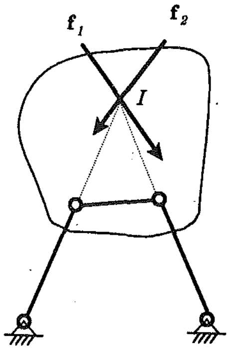

Fig. 2 Planar four-bar linkage with complete force specifications

wrenches is less than 6-n, the first-order motion of the body cannot be uniquely determined from the external loading. We call this an incompletely specified problem. In incompletely specified problems, we need to specify some motion parameters in addition to the specified external wrenches to determine the first order motion. However, in many applications, it is desirable to have incomplete specifications at many positions. In that case we do not specify any motion parameter in order to obtain the complete first order motion; instead, we accumulate force equations obtained from the static analyses of all the specified positions and solve them simultaneously with the position constraint equations. Unlike completely specified problems, incompletely specified problems cannot be converted to classical kinematic synthesis problems.

3.3 Force Equations for Closed-Loop Linkages. The force equations for closed-loop linkages can be formulated in several ways, such as by free-body analysis and by the use of the principle of virtual work. Here we introduce an efficient way to formulate the equations by using the reciprocal relation between two screw systems.

Consider an n-degree-of-freedom, closed single-loop linkage formed by two chains connected by the coupler as their common outermost body. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the instantaneous screw systems of the outermost bodies of the two chains. Let $\mathbf{S}_1$ be a p-systems, and $\mathbf{S}_2$ be a q-system. We first find a base of their orthogonal complements, $\overline{\mathbf{S}}_1$ and $\overline{\mathbf{S}}_2$, by using the previously described method. Let the basis of $\overline{\mathbf{S}}_1$ be denoted by $(\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_p)$ and that of $\overline{\mathbf{S}}_2$ by $(\mathbf{B}_{p+1}, \mathbf{B}_{p+2}, \ldots, \mathbf{B}_{p+q})$. Any wrench $\mathbf{W}$ applied on the coupler must be equilibrated by the structural forces supported by the two chains, that is, $\mathbf{W}$ must belong to the sum of the two screw systems $\overline{\mathbf{S}}_1$ and $\overline{\mathbf{S}}_2$, and therefore

$$\mathbf{W} = \alpha_1 \mathbf{B}_1 + \alpha_2 \mathbf{B}_2 + \ldots + \alpha_p \mathbf{B}_p + \ldots + \alpha_{p+q} \mathbf{B}_{p+q}.$$

From Theorem 1, we know that $\mathbf{B}_1$, $\mathbf{B}_2$, $\ldots$, and $\mathbf{B}_{p+q}$ are always linearly independent. The equilibrium equation becomes the condition for $\alpha_1, \alpha_2, \ldots, \alpha_{p+q}$ to have a nontrivial solution. In the case of a one-degree-of-freedom linkage, the condition is that the determinant of the 6×6 matrix $[\mathbf{W}, \mathbf{B}_1, \ldots, \mathbf{B}_{p+q}]$ must be zero. For multiple-loop linkages, we can, according to Theorem 2, take a similar approach to formulate the force equations.

3.4 Multiple-Position Force Synthesis. It is usually desirable to design a linkage with multiple-position specifications. Since the number of force equations is proportional to the degrees of freedom of a system, the concept of multiple-position force synthesis is particularly useful when dealing with low-degree-of-freedom systems. Usually the external forces and moments are specified as a resultant wrench. In that case the number of force equations at each position is equal to the degrees of freedom of the system. For one-degree-of-freedom systems we get only one force equation at each position and the system of equations remains treatable. Hence we apply the concept to the synthesis of one-degree-of-freedom closed-loop linkages whose coupler members are required to satisfy motion and force specifications. The design equations are obtained from the motion synthesis of each chain that connects the coupler to ground, and from the force synthesis of the whole closed-loop system. In incompletely specified problems, the force specification can be arranged in many ways since, at each position, forces may or may not be specified. If specified, the number of independent forces may vary. In most cases, we want to be able to specify as many design positions as possible.

The total number of design equations, N, in the multiple-position and force synthesis problem is

$$N = N_1 + N_2 + \ldots + N_l + D, \tag{8}$$

where N_i, $i = 1, \ldots, l$, is the number of equations obtained from the position synthesis of the ith chain, and D is the number of force equations obtained from the force synthesis of the closed linkage. Note that the equations obtained from the position synthesis of a chain involve only the variables related to that chain, while the D force equations involve variables of the whole closed-loop linkage.

From Eq. (8), we may expect the maximum number of allowable positions for the position synthesis to decrease when we add force constraints. Fortunately, the maximum number of positions is usually limited by some of the chains (we call them the dominant chains of the linkage), and there are usually plenty of free parameters left after the other chains are synthesized. As a result, those free parameters can be used to satisfy force constraints without decreasing the maximum number of allowable positions for the position-synthesis problem. The main difficulty in the force synthesis of closed-loop mechanisms is that the force equations are usually complicated. This difficulty can be resolved by separating the synthesis into two stages, namely, position synthesis of the dominant chains and position-force synthesis of the other chains.

The first stage of the synthesis is to recognize the dominant chains and to use the corresponding position constraints to determine the dimensions of these chains. Once the size parameters of dominant chains are determined, we then proceed to the second stage, namely, to synthesize the other chains using both their position constraints and the force constraints (as applied on the whole linkage). Note that the force equations are simplified because some of the design parameters have already been determined in the first stage.

Table 1 Position synthesis of dyads

Dyad	DOF	Reciprocal Force System	Max. # of of Positions	# of Free Parameters	# of Free Parameters for 3-P
RR	2	4	3	0	0
RC	3	3	3	2	2
SS	6	1	7	0	4
RS	4	2	4	1	3
CC	4	2	5	0	4
CS	5	1	8	0	5
PS	4	2	3	1	1
PC	3	3	2	3	N/A
PR	2	4	2	2	N/A
PH	2	4	2	3	N/A
PP	2	4	1	4	N/A
HH	2	4	3	2	2
HC	3	3	4	0	3
HR	2	4	3	1	1
HS	4	2	5	0	4

In this table the column headings have the following meanings:

- Dyad: the first moving link is jointed to ground by the first joint-type and to the second moving link by the second joint-type.
- DOF: degree of freedom of the motion of the second moving link relative to ground.
- Reciprocal Force System: order of the screw system reciprocal to the motion of the second moving link relative to ground.
- Max. # of Positions: maximum number of possible design positions for the dyad.
- # of Free Parameters: number of free parameters remaining after the maximum number of design positions are specified in a synthesis.
- # of Free Parameters for 3-P: number of free parameters remaining after a three-position synthesis.

Table 2 One-degree-of-freedom spatial four-bar linkages

4-Bar Linkage	DOF	Max. # of Positions	# of Free Parameters	Dominant Dyad	# of Free Parameters for the Other Dyad	# of Force Specifications
RRSS	1	3	4	RR	4	4
RRSC	1	3	5	RR	5	5
RCCC	1	3	6	RC	4	4
RCSR	1	3	5	RC	3	3
RCSP	1	3	3	N/A	N/A	3
RCSH	1	3	6	RC	3	3
HCCC	1	3	7	HC	4	4
HCSR	1	3	6	HC/RS	3	3
HCSP	1	3	4	PS	3	3
HCSH	1	3	7	HC	4	4
HRSS	1	3	5	HR	3	3
HRSC	1	3	5	HR	3	3
HHSS	1	3	5	HH	3	3
HHSC	1	3	5	HH	3	3

In this table the column headings have the following meanings:

- DOF: degree of freedom of the motion of the coupler of the four-bar linkage (relative to ground).
- Max. # of Positions: maximum number of possible design positions for position synthesis.
- # of Free Parameters: number of free parameters when maximum number of design positions are specified for position synthesis.
- Dominant Dyad: the dyad that limits the maximum number of design positions.
- # of Free Parameters for the Other Dyad: number of free parameters when synthesizing the nondominant dyad with the number of design positions limited by the dominant dyad.
- # of Force Specifications: maximum number of possible force specifications, i.e., the maximum number of wrenches.

4 Summary of Design of Spatial Four-Bar Linkages

Four-bar linkage synthesis has been widely studied in connection with the classical rigid-body guidance problem (Roth, 1968; Suh and Radcliffe, 1978). In this section, we demonstrate the concept of combining force synthesis with position synthesis of spatial four-bar linkages. We first investigate the properties of the component chains (dyads) and then investigate the design of various four-bar linkages made up of these components. The basic properties of all the dyads containing R, C, P, H, S joints are tabulated in Table 1.

Various combinations of dyads make up the one-degree-of-freedom mechanisms listed in Table 2. Inversions of mechanisms, such as CRCC and RCCC, are considered members of the same family of mechanisms, and only one of them is listed. Note that we can impose force specifications for all the linkages listed without reducing the number of maximum allowable design positions in the rigid-body guidance problem. The reason is that when we design a spatial four-bar linkage to match position specifications, the maximum number of design positions is usually limited by one of the dyads, therefore an abundance of free parameters remain for the position synthesis of the other dyad. These free parameters can be used to match force specifications. It is because of this that our proposed position-force synthesis is a highly effective design methodology.

5 Force Synthesis of Active Linkages

In the force synthesis of active linkages, a closed loop is again considered to be composed of a combination of chains from ground to the coupler, with the coupler being their common outermost link. The active elements are the actuators, springs, and other devices that are attached to the mechanism and apply forces on it. In a nonredundant, active, closed linkage, the number of active elements equals the degrees of freedom of the mechanism. When the number of active elements is greater than the degrees of freedom of the linkage, the linkage is considered to have force redundancy. The procedures for the position-force synthesis of active chains are the same as those for passive linkages, except that force equations need to be formulated differently.

5.1 Force Equations for Active Closed-Loop Linkages. In an active closed-loop linkage, the number of force elements equals that of the degrees of freedom of the mechanism. To perform the static analysis, we first treat the joints that are driven by force elements as locked. Then the mechanism can be treated as a passive linkage, and the force systems reciprocal to the "open" chains, which connect the coupler to ground, are formulated as usual. As in the analysis of passive mechanisms, we should be able to prove that the forces supported by different chains belong to linearly independent force sys-

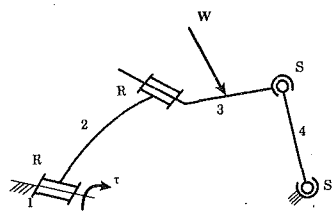

Fig. 3 An active RRSS mechanism

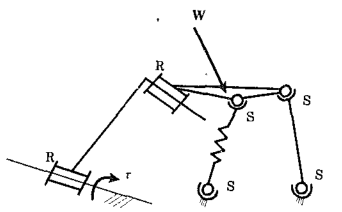

Fig. 4 A redundant closed-loop linkage

tems. However, unlike the synthesis of a passive mechanism, the force synthesis of an active mechanism results in a system of equations of the following form:

$$\mathbf{W}_{6\times 1} = \mathbf{B}_{6\times 6}\alpha_{6\times 1}, \tag{9}$$

where $\mathbf{B}$ is a full-rank 6×6 matrix obtained by adding the force systems of all the chains, $\mathbf{W}$ is the external force acting on the coupler of the closed linkage, and α is the 6×1 vector to be determined. Solving Eq. (9) yields a unique solution for α, from which the force supported by each chain can be found. Note that the force analysis of the closed-loop linkage has been broken down into the analyses of open-loop linkages. Finally, the equilibrium equations are obtained from the equilibrium analysis of all the chains that contain force elements.

As an example, we analyze a spatial four-bar mechanism to illustrate the concept. It should be noted that this approach is also readily applicable to multiple-loop mechanisms. Figure 3 shows an RRSS four-bar mechanism with the ground R-joint driven by an unknown torque τ. If the ground R-joint is treated as locked, the force system reciprocal to the motion of the RR dyad is simply the system reciprocal to the motion of the moving R-joint. The force system is a five system, i.e., the force supported by the RR dyad with the ground R-joint locked can be expressed as

$$\mathbf{W}_{RR} = \alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \ \ldots \ + \alpha_5\mathbf{B}_5. \tag{10}$$

On the other hand, the force supported by the SS dyad can be expressed as

$$\mathbf{W}_{ss} = \alpha_6\mathbf{B}_6. \tag{11}$$

From Theorem 1, we know that $\mathbf{W}_{RR}$ and $\mathbf{W}_{SS}$ belong to linearly independent force systems. Thus the external force $\mathbf{W}$ can be expressed as

$$\mathbf{W} = \mathbf{W}_{RR} + \mathbf{W}_{SS} = \alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \ \ldots \ + \alpha_6\mathbf{B}_6. \tag{12}$$

Since $\mathbf{B}_i$, $i = 1, 2, \ldots, 6$, are linearly independent wrenches, we can uniquely determine α_i by solving Eq. (12) for the α_i. Once the α_i are determined, $\mathbf{W}_{RR}$ can be found. The equilibrium equation is obtained from the equilibrium analysis of the RR dyad, i.e., from

$$\mathbf{W}_{RR} \cdot \hat{\mathbf{S}}_R = \tau, \tag{13}$$

where $\hat{\mathbf{S}}_R$ is the unit instantaneous screw of the ground R-joint.

5.2 Force Equations for Redundant Closed-Loop Linkages. Redundant closed-loop linkages are linkages that have more force elements than the degrees of freedom of their couplers. For example, Fig. 4 shows an RRSS four-bar mechanism driven by an actuator, and a spring element is attached to its coupler. An external force, $\mathbf{W}$, is applied to the coupler. The system has one degree of freedom; however, it has two independent force producing elements with which to equilibrate $\mathbf{W}$: the driving actuator with torque τ and the spring. Note that from a design viewpoint the spring is an active element because the magnitude of the spring force $\mathbf{f}_K$ varies according to the designer's choice of spring.

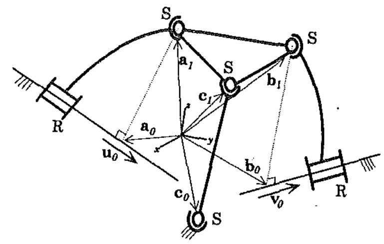

Fig. 5 An RSSRSS linkage

The force analysis is similar to that for nonredundant linkages. For example shown in Fig. 4, we have, in addition to Eqs. (10) and (11),

$$\mathbf{W}_{SPR} = \alpha_7\mathbf{B}_7, \tag{14}$$

where $\mathbf{W}_{SPR}$ is the force supported by the spring element. The external force is then expressed as

$$\mathbf{W} = \alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \ \ldots \ + \alpha_6\mathbf{B}_6 + \alpha_7\mathbf{B}_7. \tag{15}$$

The force analysis results in systems of equations similar to those obtained for the analysis of the multifingered grasping problem (Kerr and Roth, 1986). Generally, the systems of equations are of the following form:

$$\mathbf{W}_{6\times 1} = \mathbf{B}_{6\times n}\alpha_{n\times 1}, \tag{16}$$

where $n > 6$. There is no unique solution to the above system of equations. Any solution for α can be broken into two parts α_p and α_h, where α_h corresponds to the internal grasp forces (Kerr and Roth, 1986) that do not result in a net force to balance the external force. For mechanical fingers, internal grasp forces contribute to the stability of the contacts. However, for closed linkages, since the coupler and the chains are connected by closed joints, internal forces are not essential for the force equilibrium of closed linkages. In fact, internal forces are usually unwanted loads on the members, so the optimal solution occurs when $\alpha_h = 0$,

$$\alpha = \alpha_p = \mathbf{B}_r^+\mathbf{W}, \tag{17}$$

where $\mathbf{B}_r^+$ denotes the right generalized inverse of $\mathbf{B}$.

Once α is determined, the force supported by each chain can be found. We then proceed with the equilibrium analyses of all the chains that contain force elements and obtain the equilibrium equations. Note that the number of equilibrium equations equals the number of active elements. For the example shown in Fig. 4, we have, in addition to Eq. (13),

$$f_{SPR} = f_K, \tag{18}$$

where f_{SPR} and f_K are the magnitudes of $\mathbf{W}_{SPR}$ and $\mathbf{f}_K$, respectively.

6 Numerical Examples

In this section, we give two examples of synthesizing a spatial RSSRSS linkage to match three position-force specifications. In the first, the linkage is passive, in the second, one of the R-joints is actuated with known torque. Figure 5 shows a skeleton diagram of the RSSRSS linkage. The coupler of the linkage is required to pass through three positions and to support a specified force (and moment) at each position. Data for position synthesis have been adapted from Roth (1968). The screw displacements in Roth (1968) have been converted into equivalent homogeneous transformation matrices as follows:

$$
\mathbf{D}_{12} = \begin{bmatrix} -0.4662298 & -0.6179617 & 0.6331190 & 2.466281 \\ 0.7350041 & -0.6688602 & -0.1115741 & 0.265085 \\ 0.4924280 & 0.4132833 & 0.7661579 & 0.507641 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}
$$

and

$$
\mathbf{D}_{13} = \begin{bmatrix} 0.3460161 & -0.9363812 & 0.0594521 & 0.654105 \\ 0.9287878 & 0.3328481 & -0.1632059 & 1.071321 \\ 0.1330325 & 0.1116829 & 0.9848526 & 1.867013 \\ 0 & 0 & 0 & 1.0 \end{bmatrix},
$$

where $\mathbf{D}_{12}$ is the transformation matrix for the displacement from the first to the second position, and $\mathbf{D}_{13}$ is the matrix for the displacement from the first to the third position.

The force specified at the ith position is represented as a wrench $\mathbf{W}_i$:

$$\mathbf{W}_1 = [\,4\ 5\ 6\ 1\ 2\ 3];$$

$$\mathbf{W}_2 = [7\ 2\ 1\ 3\ 4\ 5];$$

$$\mathbf{W}_3 = [3\ 4\ 5\ 7\ 8\ 9].$$

From Table 2, we know that the dominant chains of the RSSRSS linkage are the two RS dyads. The configuration of one of the two RS dyads in the first position is determined using the corresponding position constraint equations (see Roth, 1968, and Suh and Radcliffe, 1978). The result is as follows:

$$\mathbf{a}_0 = [1.0\ 1.0\ 1.0],$$

$$\mathbf{u}_0 = [0.3948\ \ -0.0699\ \ -0.9161],$$

$$\mathbf{a}_1 = \ [7.7933\ \ -11.1583\ 4.8556],$$

where $\mathbf{u}_0$ is the direction-cosine vector of the axis of the R-joint; $\mathbf{a}_0$ gives the coordinates of the intersection of the R-axis and the normal to it from the S-joint; $\mathbf{a}_1$ gives the coordinates of the center of the S-joint. The configuration of the other RS dyad is as follows:

$$\mathbf{b}_0 = [-2.0\ 3.0\ 4.0],$$

$$\mathbf{v}_0 = [0.5564\ 0.7036\ \ -0.4421],$$

$$\mathbf{b}_1 = [10.8469\ 1.3995\ 17.6216],$$

where $\mathbf{v}_0$ is the direction-cosine vector of the axis of the R-joint; $\mathbf{b}_0$ gives the coordinates of the intersection of the R-axis and the normal to it from the S-joint; $\mathbf{b}_1$ gives the coordinates of the center of the S-joint.

Then we proceed with the second stage of the synthesis, namely the design of the SS dyad based on its position constraints and three force constraint equations. Let the position vectors of the center of the moving and ground S-joints be $\mathbf{c}_1$ and $\mathbf{c}_0$, respectively. In order for the last link of the SS dyad (the coupler of the RSSRSS linkage) to pass through the three prescribed positions, the following position constraints must be satisfied:

$$|\mathbf{c}_j - \mathbf{c}_0| = |\mathbf{c}_1 - \mathbf{c}_0|, j \ = \ 2,\ 3, \tag{19}$$

where

$$\mathbf{c}_j = \mathbf{D}_{1j}\mathbf{c}_1.$$

We obtain two equations in terms of six unknowns, the components of $\mathbf{c}_1$, (t_1, t_2, t_3), and the components of $\mathbf{c}_0$, (s_1, s_2, s_3).

For the passive linkage design, we follow the procedures described in Section 3 to derive the three force equations. We have a total of five, second degree, polynomial equations in the six unknowns: (t_1, t_2, t_3) and (s_1, s_2, s_3).

Table 3 Result of the passive RSSRSS linkage design

	s_2	s_3	t_1	t_2	t_3
Solution 1	0.8031	9.7347	4.3347	-1.6292	8.7301
Solution 2	0.1104	-10.6297	7.7952	-11.1817	4.8673
Solution 3	2.2513	4.2336	3.2898	0.1561	1.0078
Solution 4	0.8031	9.7347	4.3348	-1.6292	8.7301
Solution 5	40.2404	61.1912	-11.0595	-6.6762	-13.1533

Table 4 Result for the active RSSRSS linkage design

	s_2	s_3	t_1	t_2	t_3
Solution 1	13.0605	-2.2053	10.8452	1.3957	17.6669
Solution 2	-16.6119	-37.3597	8.4535	4.8326	41.6514
Solution 3	-6.9461	7.5097	0.5242	-1.2904	8.3281
Solution 4	0.0754	-10.6567	7.7888	-11.2652	4.9036

For the active linkage design, the R-joint, with direction cosines $\mathbf{u}_0$, is actuated at each position with the following torque

$$\tau_1 = 10; \tau_2 = 12; \tau_3 = 14.$$

We follow the procedures described in Section 5 to derive the three force equations. We also have a total of five, second degree, polynomial equations in the six unknowns: (t_1, t_2, t_3) and (s_1, s_2, s_3).

With the selection of $s_1 = 6$, we use the continuation method (Wampler et al., 1990) to solve each of the above two systems of equations for the rest of the unknowns. All the real solutions are listed in Tables 3 and Table 4, respectively. Here we have used the continuation method to solve the set of five equations. However, it is also possible to analytically reduce this set to two 4th degree equations in two unknowns or even a single polynomial in one unknown. This alternative is detailed in Huang (1992), where the explicit form of the five equations is also presented.

Regardless of which method we use to obtain the solutions listed in Table 3 and 4, we then select one of the listed solutions according to some additional criteria, such as space constraints. If we did not use position-force synthesis methods, but instead used only a position synthesis to determine (t_1, t_2, t_3) and (s_1, s_2, s_3), we would have to provide whatever torque τ_1, τ_2, τ_3 the structure required. Thereby we would lose control of this aspect of the design and could be forced into providing an unduly large torque motor or actuator.

7 Conclusion

This paper illustrated the concept of designing closed-loop linkages to match position and force specifications. First, we defined force synthesis problems and discussed the correspondence between them and classical kinematic synthesis problems. We showed that complete force specifications can be converted to first order motion constraints, while incompletely specified problems give force equations in terms of the dimensions of the linkages. Second, we looked into the design of passive linkages to match multiple position-force specifications. We showed that the maximum number of allowable design positions need not be decreased after the force synthesis is imposed on classical position synthesis problems. Third, we discussed the applications of linear algebra to screw systems and introduced two theorems that are useful in formulating the force equations for closed-loop linkages. Finally, we extended the concept of the synthesis of passive linkages to include synthesis of active linkages, and we discussed both nonredundant and redundant linkages.

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