CSC 481: First Order Logic

2- Semantics: interpretation, denotation and logical models

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The basics

A language is composed of:

- **Syntax:** what symbols may be used and what combinations of symbols are well-formed
  - "I drove to work today" is a well-formed English sentence
  - "I work to today drove" is not
- **Semantics:** what each sentence means
  - Sentences are used to convey that the world is one way and not another
- **Pragmatics**: what the language is used for
  - The sentence "fire!" suggests a different response in a crowded theater or in a shooting range

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Last class

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This class

Interpretations - Propositional Logic

Assumptions - Propositional Logic

- There are propositions
- Each proposition has a truth value of true or false
- No other aspects of the world matter
  - This means that propositions have no “internal structure”. If I call P the proposition “Toto is a dog” and Q the proposition “Rex is a dog”, there’s no way to tell that these two propositions are about the same topic.

Interpretation in Propositional Logic

- An interpretation consists on an assignment of a truth value to all propositions in a domain

- In other words, to create an interpretation, we simply choose which propositions are true and which propositions are false

- As we'll see, the value of an expression like $(P \vee Q) \rightarrow \neg(X \wedge (Y \vee Z))$ can be computed based on the truth assignments within a given interpretation

- In practice, we typically don't know all the facts, so we don't care about a specific interpretation, but about *statements that are true in **every** interpretation* where the facts we **do** know hold true.

Interpretations - First Order Logic

Assumptions - First order Logic

- The world consists of objects and relations (predicates)
- For a predicate $P(o_1, ..., o_n)$ of arity n, some n-tuples of objects satisfy P, and some do not.
- Every function is total: $F(o_1, ..., o_n)$ of arity n returns some object of the world for every possible n-tuple of objects.
    - Even if it doesn't make much sense: in a domain containing as objects both people and cities, we would have to define what bestFriend("SanLuisObispo") means.
- No other aspects of the world matter

Semantics - Interpretation

Intuition

- Without further information, a sentence like dog(“Toto”) or bestPlaceToLive(“California”) = “San Luis Obispo” or “It will rain tomorrow” is neither true nor false
- We need to decide for ourselves what “Toto” means and whether it is a dog or not, and which city in California is the best to live, and wether or not it will actually rain tomorrow
- Semantics is always relative to an interpretation, which defines:
  - What objects exist in the domain
  - Which objects have which properties
    - That is, which predicates evaluate to true or false given every possible combination of objects as arguments
  - What objects are returned by each function
- An interpretation is sometimes called a “possible world”

Examples:

- An interpretation might define that the predicate **dog** is satisfied by the object denoted by **‘Toto’** but not by the objects denoted by **‘Garfield’**or **‘San Luis Obispo’**
    - But it might define otherwise: <u>beware of reading too much into names of identifiers</u>

Examples:

- An interpretation might define that the tuple denoted by **<‘Ryan’,’Blake’>** satisfies the predicate **married**
  - In other words, **married(‘Ryan’,’Blake’)** returns True

Examples:

- An interpretation might define that the best friend of the object denoted by **‘Bob’** is the object denoted by **‘Jon’**
  - That is, bestFriend(‘Bob’) should return ‘Jon’
  - We might need to define that **bestFriend(‘San Luis Obispo’)** returns some dummy object (every function is total)

Formalizing Interpretation

- An interpretation $\mathfrak{I}$ is a pair $< D, I >$ where:
- D is called the domain and can be any non-empty set
- I is called the interpretation mapping and assigns meaning to predicate symbols and function symbols

Formalizing Interpretation - Domain

- D is called the domain and can be any non-empty set
  - Including non-mathematical objects such as people, places, numbers, fictional characters…

Formalizing Interpretation - Mapping

- I is called the interpretation mapping and assigns meaning to predicate symbols and function symbols:
  - Given a predicate symbol P of arity n:
    - $I[P]$ is a subset of D^n
      - $I[P] \subseteq D^n$
    - Alternatively, $I[P]$ can be understood as a function mapping n-tuples from D to {True, False}
      - $I[P] : D^n \mapsto \{true, false\}$

Formalizing Interpretation - Mapping

- I is called the interpretation mapping and assigns meaning to predicate symbols and function symbols:
  - Given a function symbol f of arity n:
    - $I[f]$ maps n-tuples of D to objects of D
      - $I[f] : D^n \mapsto D$

Denotation

- We need next to specify which elements of D are “denoted” (represented) by any variable-free term of FOL.
- For terms containing variables, we need a variable assignment u from variables in FOL to the domain D
- Denotation is defined recursively:
    - Any variable denotes the object from D it maps to via u
    - Any function term $f(t_1, \ldots, t_n)$ denotes the object from D returned from $I[f]$ with the objects denoted by $< t_1, \ldots, t_n >$ as parameters
        - Remember that constants like “Bob” can be seen as simply functions of arity 0
- The book uses $\|t\|_{\mathfrak{I},\mu}$ to represent the object denoted by term t under interpretation $\mathfrak{I}$ and assignment u

Satisfaction and Models

Assume that $t_1, \ldots, t_n$ are terms, P is a predicate of arity n, α and β are formulas, and x is a variable.

1. $\Im, \mu \models P(t_1, \ldots, t_n)$ iff $\langle d_1, \ldots, d_n \rangle \in \mathcal{P}$, where $\mathcal{P} = \mathcal{I}[P]$, and $d_i = \|t_i\|_{\Im,\mu}$;
2. $\Im, \mu \models t_1 = t_2$ iff $\|t_1\|_{\Im,\mu}$ and $\|t_2\|_{\Im,\mu}$ are the same element of $\mathcal{D}$;
3. $\Im, \mu \models \neg\alpha$ iff it is not the case that $\Im, \mu \models \alpha$;
4. $\Im, \mu \models (\alpha \wedge \beta)$ iff $\Im, \mu \models \alpha$ and $\Im, \mu \models \beta$;
5. $\Im, \mu \models (\alpha \vee \beta)$ iff $\Im, \mu \models \alpha$ or $\Im, \mu \models \beta$ (or both);
6. $\Im, \mu \models \exists x.\alpha$ iff $\Im, \mu' \models \alpha$, for some variable assignment μ' that differs from μ on at most x;
7. $\Im, \mu \models \forall x.\alpha$ iff $\Im, \mu' \models \alpha$, for every variable assignment μ' that differs from μ on at most x.

1. married('Ryan','Blake')
2. spouse('Ryan') = 'Blake'
3. ¬ dog('Garfield')
4. dog('Toto') ^ married('Ryan','Blake')
5. dog('Garfield') v dog('Toto')
6. $\exists x$ married(x, 'Blake')
7. $\forall x\, dog(x) \rightarrow mammal(x)$

We say $\Im \models \alpha$ or α is satisfied in $\Im$ or (conversely) $\Im$ satisfies α

Satisfaction and Models - Notes

- If α and β are *sentences* (no free variables), then satisfaction does not depend on u
- If $\mathfrak{I} \models \alpha$, we say α is true under $\mathfrak{I}$, otherwise it is false
- If S is a set of sentences and $\mathfrak{I} \models$ S, we say $\mathfrak{I}$ is a **logical model** of S

Next class

- We will look at the pragmatics of FOL: putting everything together to see how a KB in First-Order Logic can be used to help an agent answer interesting questions