ISyE 4133 – Advanced Optimization – Spring 2022

Problem Set 3, Due: March 29, 2022

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Problem Set Rules:

1. Each student should hand in an individual problem set on Canvas.
2. You may interact with fellow students when preparing your homework solutions. However, at the end, you must write up solutions on your own. Duplicating a solution that someone else has written (verbatim or edited) is not permissible.
3. Late assignments will *not* be accepted. However, each student has one "flex day" during the semester (use it wisely) that grants a 24-hour extension for a problem set. It must be requested to one of the TAs before the original due date of the assignment. A flex day cannot be split into multiple fractional flex days!
4. The solutions and Python codes should be submitted electronically on Canvas prior to the deadline. Any handwritten solutions should be scanned so that they are easily read by the TAs, and then submitted electronically.
5. For questions involving computations, the Python code as well as the answers will be graded.
6. Show your work. Answers requiring a justification will not receive full grades if the justification is not provided. Similarly, answers requiring intermediate computations will not receive full grades if those computations are not shown in your solutions.

Problem 1

Dantzig-Wolfe Decomposition (30 points in total)

Consider the following optimization problem:

$$\begin{array}{lll}(\mathcal{O}) & \min & -2x_1 - x_2 \\ & \text{s.t.} & -x_1 + 3x_2 = 2 \\ & & |x_1 - 1| + |x_2 - 1| \leq 1 \\ & & x_1, x_2 \geq 0\end{array}$$

Let $P \coloneqq \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } |x_1 - 1| + |x_2 - 1| \leq 1,\ (x_1, x_2) \geq 0\}$ be the set of "easy" constraints.

a) (8 points) Draw the polyhedron P. What are the extreme points of P?

b) (22 points) Solve $(\mathcal{O})$ using the Dantzig-Wolfe decomposition. You will initiate the column generation algorithm with $\boldsymbol{x}^1$ being the left extreme point of P, and $\boldsymbol{x}^2$ being the right extreme point of P. You may not use any software to solve any optimization problem you formulate. Show your work. What is the optimal solution of $(\mathcal{O})$?

   You can solve the pricing subproblem using your drawing of P

Problem 2

Logic Problems (35 points in total)

a) In each of the following, suppose that x, y, and z are binary variables. Write down each statement using equivalent linear constraint(s).

i) (3 points) "$x = 1$ or $y = 0$"

ii) (3 points) "$x = 0$ or $y = 1$ but not both"

iii) (3 points) "$x = 0$ if and only if $y = 0$"

iv) (4 points) "If $x + y \geq 1$, then $z = 0$"

v) (5 points) "($x = 0$) or ($y = 0$ and $z = 1$)"

vi) (7 points) Model $x = y \times z$ using linear constraints. *Hint: This is equivalent to: "$x = 1$ if and only if $y = 1$ and $z = 1$"*

b) In the following problem, x and y are continuous variables such that $0 \leq x \leq 10$ and $0 \leq y \leq 10$.

i) (5 points) Transform the following statement into linear constraints, by introducing a single binary variable w and two nonnegative constants M_1 and M_2: "$2x - 3y \geq 5$ or $3x + y \leq 18$

ii) (5 points) What are the smallest values of M_1 and M_2 for which your answer is still correct? Justify your answer.

Problem 3

School Bus Routing in Atlanta (35 points in total)

The Atlanta board of education is taking bids on the city's four school bus routes. Four companies have made the bids in Table 1.

	Bids ($1000)			
	Route	Route	Route	Route
Company	1	2	3	4
1	8	7	-	-
2	8	9	-	5
3	3	-	2	-
4	-	-	4	4

Table 1: Bids made by companies

a) (15 points) Formulate an IP that assigns each route to one bidder (and each bidder must be assigned to only one route).

b) (5 points) Solve this problem using Python. What is the optimal assignment, and corresponding cost?

c) (5 points) In this question, we relax the integrality constraints of the IP formulated in part a). Specifically, replace all binary variables by continuous variables bounded between 0 and 1. Solve the resulting LP in Python. What property do you observe?

d) (10 points) In this question, we will determine a feasible solution of the IP formulated in part a) by iteratively (and myopically) assigning companies to routes. The algorithm is as follows: Select the (company,route) pair with lowest bid. If that company AND that route have not been previously assigned (e.g., to another route/company), then assign that company to that route. Repeat this process for every bid until every company is assigned to one route. What is the resulting assignment and corresponding total cost? What can you conclude regarding this algorithm?

Problem 4

Class Assignments (35 points in total)

The Salanter Akiba Riverdale (SAR) Academy is a coeducational, private Modern Orthodox Jewish day school located in New York City. Every summer, the SAR Academy must create class assignments for their elementary school students. Each grade of 80-100 students must be divided into different classes. Requests for assignments are made by parents, teachers, and school therapists. These requests include pairs of students that should be placed together, pairs of students that should not be placed together, and requests for students to be placed in classes that better suit their unique or special academic needs. These requests often conflict with each other, and it falls on the administration to prioritize which requests should be fulfilled over others.

In this exercise, we will solve a simplified version of the problem faced by the SAR Academy with 40 students and two classes. (The full optimization problem is currently being used to assist administrators at the SAR Academy.) The parents or guardians of each of the 40 students are asked to submit preferences for class 1 or class 2. These preferences often depend on the teaching style of the teachers, the teachers older siblings have had in the past, and characteristics of the class (one class is called an "inclusion class," which is better for students with academic needs). The parents give a ranking of 1 to the class they prefer (their first choice), and a ranking of 2 to their second choice. This data, as well as the gender of each of the students, is given in the spreadsheet *ClassAssignments.csv.*

a) (13 points) The problem faced by the SAR Academy is to decide which students should be assigned to which classes, to satisfy as many of the parent preferences as possible. Each student must be assigned to exactly one class, and there should be exactly 20 students in each class.

   i) (7 points) Formulate this problem as an integer optimization problem. Describe the decision variables, objective function, and constraints.

   ii) (3 points) Now solve this optimization problem. What is the optimal solution? Give the optimal assignment to classes, as well as the optimal objective function value.

iii) (3 points) How many students received their first choice class, according to the parent preferences? (*Hint:* You should be able to answer this by just looking at the optimal objective function value.)

b) (7 points) After looking at the optimal solution, the SAR academy decided that they would like to adjust the formulation to better boy/ girl ratio in the classes. They would like to limit the boys in each class to no more than 12.

i) (3 points) What constraint(s) do you need to add to your model to incorporate this adjustment?

ii) (2 points) Add the necessary constraints and resolve your model. What is the optimal solution now? How does it compare to the previous solution?

iii) (2 points) How many students received their first choice class in this new solution, according to the parent preferences?

c) (15 points) Now, we will add some logical constraints to capture additional preferences of parents, teachers, and school therapists.

i) (3 points) Students 10 and 11 are twins, and the school has a policy that twins must be placed in different classes. What constraint(s) needs to be added to the model to implement this policy?

ii) (3 points) Students 4, 9, 15, 25, 30, and 36 are all from the same neighborhood. The school would like to put at least 2 students from this neighborhood in each class. What constraint(s) needs to be added to the model to implement this policy?

iii) (4 points) The school therapist strongly recommends that students 20 and 21 be placed in the same classroom, that student 1 be placed in classroom 2, and that student 40 be placed in classroom 2. What constraint(s) needs to be added to the model to implement this policy?

iv) (5 points) Add all of these constraints to your model, and solve it again. What is the optimal solution now? How does the objective value compare to the one obtained in part b)? What does this tell us?