ISyE 4133 – Advanced Optimization – Spring 2022

Problem Set 1, Due: February 4, 2022

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Problem Set Rules:

1. Each student should hand in an individual problem set on Canvas.
2. You may interact with fellow students when preparing your homework solutions. However, at the end, you must write up solutions on your own. Duplicating a solution that someone else has written (verbatim or edited) is not permissible.
3. Late assignments will *not* be accepted. However, each student has one "flex day" during the semester (use it wisely) that grants a 24-hour extension for a problem set. It must be requested to one of the TAs before the original due date of the assignment. A flex day cannot be split into multiple fractional flex days!
4. The solutions and Python codes should be submitted electronically on Canvas prior to the deadline. Any handwritten solutions should be scanned so that they are easily read by the TAs, and then submitted electronically.
5. For questions involving computations, the Python code as well as the answers will be graded.
6. Show your work. Answers requiring a justification will not receive full grades if the justification is not provided. Similarly, answers requiring intermediate computations will not receive full grades if those computations are not shown in your solutions.

Problem 1

Equivalent formulations (20 points)

a) (10 points) Transform the following linear programs into standard form.

   i) (5 points)

$$\begin{aligned} \text{maximize } & x_1 + x_2 \\ \text{subject to } & x_1 - x_2 \leq 5 \\ & x_1 + 6x_2 = 10 \\ & x_1 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$

ii) (5 points)

$$\begin{aligned} \text{minimize } & x_1 - 3x_3 \\ \text{subject to } & 2x_1 + x_2 \geq 20 \\ & x_2 + 7x_3 \leq 40 \\ & x_1 \geq -10 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \end{aligned}$$

b) (10 points) Show that every linear program can be put in the following equivalent form

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

Problem 2

Geometry (30 points total)
Consider the following linear program:

$$\left.\begin{array}{lr} \text{min} & -x + y \\ \text{s.t.:} & \\ (1): & x - y \leq 3 \\ (2): & 2x - 3y \leq 4 \\ (3): & x \leq 5 \\ (4): & x, y \geq 0. \end{array}\right\}$$

a) (5 points) Graph the feasible region of the LP. Is the feasible region unbounded? Does the feasible region have any basic feasible solutions? If so, what are they?

b) (4 points) Using the graphical method (Lectures 4-5, Slide 6), determine the optimal solution for the LP.

c) (3 points) Is there more than one optimal solution?

d) (8 points) For this question, we will demonstrate (without using the graphical method, nor the simplex algorithm) that the solution found in part b) is indeed an optimal solution for the LP.

   i. (3 points) To this end, first show that the solution found in part b) is indeed a feasible solution for the LP. What is its objective value?

   ii. (5 points) Then, by looking at the constraints of the LP, argue that no feasible solution can have an objective value strictly better than that of the solution found in part b).

e) (5 points) If the objective were changed from minimization to maximization, what would the new optimal solution be?

f) (5 points) Are any of the above constraints redundant? If so, indicate which one(s). (Redundancy will play an important role in the formulation of integer programs.) As a reminder, a constraint is redundant if removing it does not change the feasible region.

Problem 3

Reformulations (25 points total)

a) (7 points) Convert the following mathematical program into an equivalent linear program. Please show your work.

$$\left.\begin{array}{lr} \min & (\max\{x-y, z\}) \\ \text{s.t.:} & \\ (1): & x+y=16 \\ (2): & x+y+z \le 22 \\ (3): & |x-y| \le 19 \\ (4): & z \ge 2 \\ (5): & x, y \ge 0. \end{array}\right\}$$

b) (3 points) If we replace $\min(\max\{x-y, z\})$ with $\max(\min\{x-y, z\})$ in the original problem, can it still be converted to a linear program? If so convert it, if not explain why not

c) (7 points) Are the optimal values in parts a) and b) identical? Justify your answer. (We are asking whether the minimax is equal to the maximin in this problem). You **cannot** code or solve the mathematical programs using Excel. *Hint: Solve part a) by hand (i.e., find/guess an optimal solution, and prove that it is indeed optimal). Then, show that the optimal value of part b) is higher. Remember, the optimal value of a maximization problem is as large as the objective value of any feasible solution.*

d) (8 points) Can the original problem be converted into an equivalent linear program if we add separately each of the following constraint:

   i. (4 points) $\dfrac{x-z}{21-x-y-z} \ge 10$

   ii. (4 points) $\dfrac{y-z}{20-y+x} \ge 5$

   If so convert the new constraint, if not explain why not.

Problem 4

Simple LP and Algorithms (25 points)

This problem is concerned with one of the simplest types of linear programs in which there is a single linear constraint and there are lower and upper bounds on variables:

a) (5 points) For every value b with $0 \le b \le 10$, $F(b)$ denotes the following problem:

$$\begin{array}{ll} \text{maximize} & 6x_1 + 4x_2 + 2x_3 + x_4 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 \le b \\ & 0 \le x_1 \le 2,\ 0 \le x_2 \le 3,\ 0 \le x_3 \le 1,\ 0 \le x_4 \le 4 \end{array}$$

   i) (3 points) Code the LP using Gurobi, and solve it for $b \in [\![0, 10]\!] (= \{0, 1, 2, 3, \ldots, 10\})$. (Remember to upload every code you write with your problem set.)

ii) (2 points) Let $f(\cdot)$ be the optimal value function. That is, $f(b)$ is the optimal value when the right-hand side is b. Graph $f(b)$ for $0 \leq b \leq 10$ using Python or Excel. *Hint: use a Scatter Plot with Straight Lines and Markers*

b) (5 points) Give a simple algorithm for finding the optimal solution and optimal value of $F(b)$. Your algorithm should compute $f(b)$ correctly for $0 \leq b \leq 10$. (You can describe the algorithm using Python or any other language. Or you can simply use pseudo-code. All we really want is for you to be able to describe how to find the optimal solution in a way that is understandable. Accordingly, we will be lenient in grading this part.)

c) (4 points) Write $f(b)$ as a piecewise linear function:

$$f(b) = \begin{cases} 6b & \text{if } 0 \leq b \leq 2 \\ 4b + 4 & \text{if } 2 \leq b \leq 5 \\ a_1 b + a_2 & \text{if } 5 \leq b \leq 6 \\ a_3 b + a_4 & \text{if } 6 \leq b \leq 10 \end{cases}$$

That is, determine a_1, a_2, a_3, a_4.

d) (5 points) Consider the problem $G(b)$. For every value b with $0 \leq b \leq 16$, $G(b)$ denotes the following problem:

$$\begin{aligned} \text{maximize} \quad & 6x_1 + 4x_2 + 15x_3 + 3x_4 \\ \text{subject to} \quad & 3x_1 + 1x_2 + 3x_3 + 1x_4 \leq b \\ & 0 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 3,\ 0 \leq x_3 \leq 1,\ 0 \leq x_4 \leq 4 \end{aligned}$$

i) (3 points) Code the LP using Gurobi, and solve it for $b \in [\![0, 16]\!]$. *(A for loop in your code might make your life easier...)*

ii) (2 points) Let $g(\cdot)$ be the optimal value function. Graph $g(b)$ for $0 \leq b \leq 16$ using Python or Excel.

e) (6 points) Give a simple algorithm for finding the optimal solution and optimal value of $G(b)$. Your algorithm should compute $g(b)$ correctly for $0 \leq b \leq 16$. (You can describe the algorithm in a similar way to the algorithm you wrote for part b).)