ISyE 4133 – Advanced Optimization – Spring 2022

Problem Set 2, Due: February 20, 2022

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Problem Set Rules:

1. Each student should hand in an individual problem set on Canvas.
2. You may interact with fellow students when preparing your homework solutions. However, at the end, you must write up solutions on your own. Duplicating a solution that someone else has written (verbatim or edited) is not permissible.
3. Late assignments will *not* be accepted. However, each student has one "flex day" during the semester (use it wisely) that grants a 24-hour extension for a problem set. It must be requested to one of the TAs before the original due date of the assignment. A flex day cannot be split into multiple fractional flex days!
4. The solutions and Python codes should be submitted electronically on Canvas prior to the deadline. Any handwritten solutions should be scanned so that they are easily read by the TAs, and then submitted electronically.
5. For questions involving computations, the Python code as well as the answers will be graded.
6. Show your work. Answers requiring a justification will not receive full grades if the justification is not provided. Similarly, answers requiring intermediate computations will not receive full grades if those computations are not shown in your solutions.

Problem 1

Radiation Therapy (25 points)

Cancer is the second leading cause of death in the United States, with an estimated 600,000 deaths in 2017. Over 1.8 million new cases of cancer were diagnosed in the United States alone in 2018. About half of these patients are treated with radiation therapy, which uses beams of high energy photons to cells.

In this problem, we will optimize treatment decisions for intensity-modulated radiation therapy (IMRT), which was invented in the early 1980s. Since radiation must pass through healthy tissue to reach the tumor, radiation therapy damages both healthy and cancerous tissue. IMRT was invented so that the radiation could "fit" the tumor more closely and reduce the dose to healthy tissue.

This is done by making the intensity profile of each beam non-uniform. Figure 1 shows a simple example of how traditional radiation therapy works. There are three beams, which are each aimed at the target, or the cancerous tumor. Unfortunately, the target is surrounded by healthy tissue and some critical structures (these could be a kidney, liver, etc.). So while the positioning of the beams gives a high dose of radiation to the target, it also gives a high dose of radiation to the critical structures.

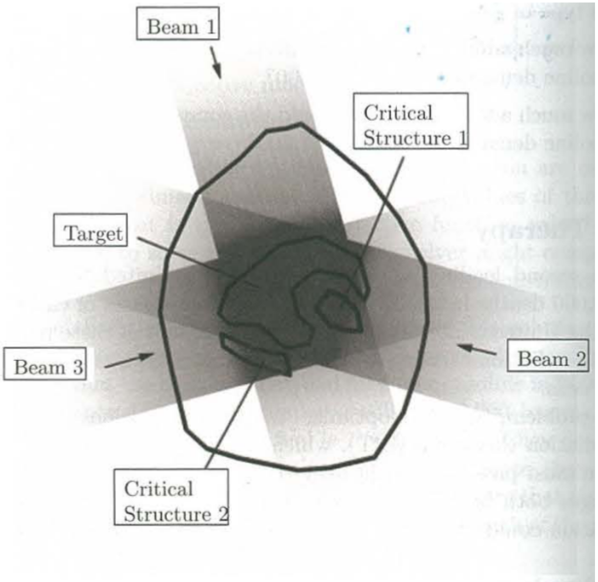

Figure 1: An example treatment plan using traditional radiation therapy.

Figure 2 shows how the treatment can be improved with IMRT. Each beam is divided into beamlets, so the intensity of each beamlet can be determined independently. The beams are in the same position, but by altering the intensity of the beamlets, we are able to give a high dose of radiation to the target while giving a low dose of radiation to the critical structures.

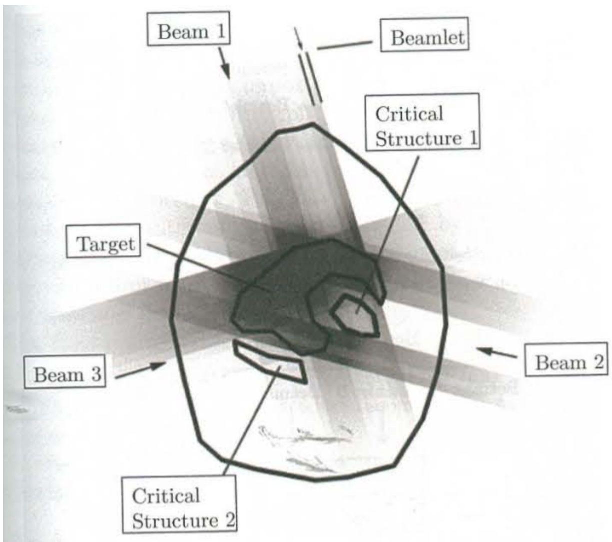

Figure 2: An example treatment plan using intensity-modulated radiation therapy (IMRT).

So the fundamental problem is deciding how the beamlet intensities should be selected to deliver a high dose of radiation to the tumor, while minimizing damage to healthy tissue. Oncologists solve this problem by first taking a CT scan of the patient, and dividing the area with the tumor into a grid of *voxels*. In this problem, we will be looking at a patient case with

9 voxels and 6 beamlets. (In practice, there are typically over 100,000 voxels and hundreds of beamlets.) This is shown in Figure 3. Beam 1 is divided into three beamlets, which all come in from the right and each give a dose to three voxels. For example, beamlet 1 hits voxel 3, then voxel 2, and then voxel l. Beam 2 is also divided into three beamlets which proceed in straight lines. The white voxels (voxel 2, voxel 6, voxel 7, and voxel 8) are tumor voxels, the dark blue voxel (voxel 5) is a spinal cord voxel, and the light blue voxels (voxel 1, voxel 3, voxel 4, and voxel 9) are other healthy tissue voxels.

Beam 2

Beamlet 4 ↓	Beamlet 5 ↓	Beamlet 6 ↓	Beam 1
Voxel 1	Voxel 2	Voxel 3	← Beamlet 1
Voxel 4	Voxel 5	Voxel 6	← Beamlet 2
Voxel 7	Voxel 8	Voxel 9	← Beamlet 3

Figure 3: An example patient case with nine voxels and two beams. The tumor voxels are shown in white, the spinal cord voxel is shown in violet, and other healthy tissue voxels are shown in light blue.

When each beamlet is turned on at unit intensity (beamlet intensity = 1), it gives a different radiation dose to each voxel. This is due to the amount of tissue the beamlet has to go through to reach the voxel, as well as the type of tissue. Figure 4 shows the radiation doses to each voxel from the beamlets in Beam 1, and Figure 5 shows the radiation doses to each voxel from the beamlets in Beam 2. So if Beamlet 1 is turned on with intensity = 1, it will give a dose of 2 to Voxel 3, 2 to Voxel 2, and 1 to Voxel l. If it is instead turned on with intensity = 2, it will give a dose of 4 to Voxel 3, 4 to Voxel 2, and 2 to Voxel 1 (the doses are doubled).

			Beam 1
1.0	2.0	2.0	← Beamlet 1
1.0	2.0	2.5	← Beamlet 2
1.5	1.5	2.5	← Beamlet 3

Figure 4: The dose given to each voxel when Beamlets 1, 2, and 3 are turned on with intensity equal to 1.

Beam 2

Beamlet 4 ↓	Beamlet 5 ↓	Beamlet 6 ↓
1.0	2.0	2.0
1.0	2.0	2.5
1.5	1.5	2.5

Figure 5: The dose given to each voxel when Beamlets 4, 5, and 6 are turned on with intensity equal to 1.

The problem we are trying to solve here is to determine what the beamlet intensities should be. Our objective is to minimize the total dose to all healthy tissue (the spinal cord plus other healthy tissue). We also need to make sure that the tumor voxels each get a total dose of at least 9 and the spinal cord voxel gets a total dose of at most 5. Lastly, the intensities should be non-negative.

a) (10 points) Formulate this problem as a linear optimization problem. Be sure to clearly indicate what your decision variables, objective, and constraints are.

b) (7 points) Solve this problem in a linear optimization software. What is the optimal solution? Intuitively, does this make sense?

c) (8 points) Suppose we would like to reduce the dose to the spinal cord as much as possible. We can do this by reducing the maximum dose that the spinal cord can receive. Adjust your spinal cord constraint to make sure that the spinal cord receives a total dose of no more than 4. How does the optimal solution change? Continue reducing the maximum total dose to the spinal cord by trying the values 3, 2, 1, and 0. What happens to the optimal solution? How would you describe this trade off to the oncologist?

Problem 2

Basic (Feasible) Solutions (15 points)

Consider the following polyhedron in $\mathbb{R}^7$:

$$\begin{pmatrix} 2 & 2 & 0 & 4 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 0 & -2 & 1 \\ 1 & 2 & 1 & 5 & 4 & 3 & 3 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$
$$\boldsymbol{x} \geq \mathbf{0}$$

and the following vectors

1. $(0, 0, 0, 0, 0, 1, 1)^\top$
2. $(-1, 0, 0, 2, 0, -1, 0)^\top$
3. $(1, 0, 0, 0, 2, -1, 0)^\top$
4. $(0, 1, 1, 0, 0, 1, 0)^\top$
5. $(1, 0, 1, 0, 1, 0, 0)^\top$

Which of these vectors are basic solutions? Also, which ones are basic feasible solutions? You need to justify your answers.

Problem 3

Simplex Algorithm (30 points)

a) (15 points) Solve the following LP using the simplex algorithm, starting from the basis $B = (1, 2)$ (there is no need to run Phase I). Show your work.

$$\min\ (-2, 1, 1, 0, 3)\boldsymbol{x}$$
$$\begin{pmatrix} 1 & -2 & 1 & -1 & 0 \\ 2 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

b) (15 points) Using the simplex algorithm, determine whether or not the following LP is feasible.

$$\min\ (-2, 1, 1)\boldsymbol{x}$$
$$\begin{pmatrix} 1 & -2 & 1 \\ 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Problem 5

Duality (30 points)

Consider the following primal problem:

$$\begin{aligned} \max \quad & x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 \leq 0 \\ & 2x_1 - 3x_2 \leq 0 \\ & x_1 - x_2 \leq 3 \\ & -x_1 + 2x_2 \leq 2 \\ & -2x_1 + x_2 \leq 0 \end{aligned}$$

a) (7 points) Find the optimal value and the optimal solution of this LP using the graphical method. Please show your work.

b) (8 points) Write down the dual problem.

c) (15 points) The goal of the remaining questions is to prove that the optimal solution you found using the graphical method in part a) is indeed optimal for the primal problem. We denote that solution $\boldsymbol{x}^*$.

   i) (3 points) Show that $\boldsymbol{x}^*$ is feasible for the primal problem.

ii) (8 points) Let $\boldsymbol{y}^*$ be an optimal dual solution and let us assume that $\boldsymbol{x}^*$ is optimal. Using complementary slackness, determine $\boldsymbol{y}^*$.

iii) (4 points) Conclude that $\boldsymbol{x}^*$ is indeed an optimal primal solution.