Advanced Portfolio Construction

FIN-413: Financial applications of blockchains and distributed ledgers

Group 9

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2025

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1 Question 1

For any further plots (raw prices times series, log prices time series, histogram of linear/log returns, kernel density estimates of log returns) and specific parts of data exploration done, please refer to the Jupyter notebook "Question 1.ipynb" in the .zip file.

1.a

- **Linear returns:** $r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$

  Linear returns aggregate across assets, making it easy to calculate a portfolio's one-period return as the weighted sum of its constituent returns.

  Because of this property, risk and portfolio managers rely on linear returns for risk analysis, performance attribution, and portfolio optimization.

  By contrast, in the simple-return framework, extreme moves appear directly as massive outliers, which can bias estimates of statistical moments.

- **Logarithmic returns:** $r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$

  Log returns aggregate across time, so the compounded return over K periods equals the sum of the intermediate one-period log returns. The log transform compresses large positive jumps and produces a series that is closer to stationarity, even though its tails remain fat.

Meucci (2010) [7] notes that when securities are not too volatile and the time step of the return is short, linear and compounded returns have similar distributions. However, since we are working with cryptocurrencies, highly volatile assets, we choose to use log returns. Log returns are preferable because they are time-additive, simplifying cumulative return calculations, and better approximate a normal distribution despite the presence of heavy tails, which are typical in crypto markets. Given these characteristics, log returns are the most appropriate choice for our analysis. Moreover, Meucci (2010) point out that we may have issues when doing optimization with log returns and suggests: 1) Estimate our one-day invariants on log-returns 2) Project to horizon k by summing log returns 3) Map back to simple returns 4) Compute horizon means and covariances to put in our mean-variance optimizer

Everything that follows is based on return calculations after dropping prices on weekends. This choice will be discussed later in point 1.d.

1.b

There are many outliers as can be seen in table 1.

Note that all outlier computations and boxplots are based on the 1.5×IQR rule.

Table 1: Number of detected outliers per asset

Assets	**Outliers count**
ADA-USD/log	142
BCH-USD/log	175
BTC-USD/log	149
DOGE-USD/log	186
ETH-USD/log	125
LINK-USD/log	102
LTC-USD/log	135
MANA-USD/log	125
XLM-USD/log	152
XRP-USD/log	179
SPXT/log	127
XCMP/log	125
SOFR/log	119
VIX/log	104

We plot the boxplots to better visualize the distribution and extremity of log-return outliers:

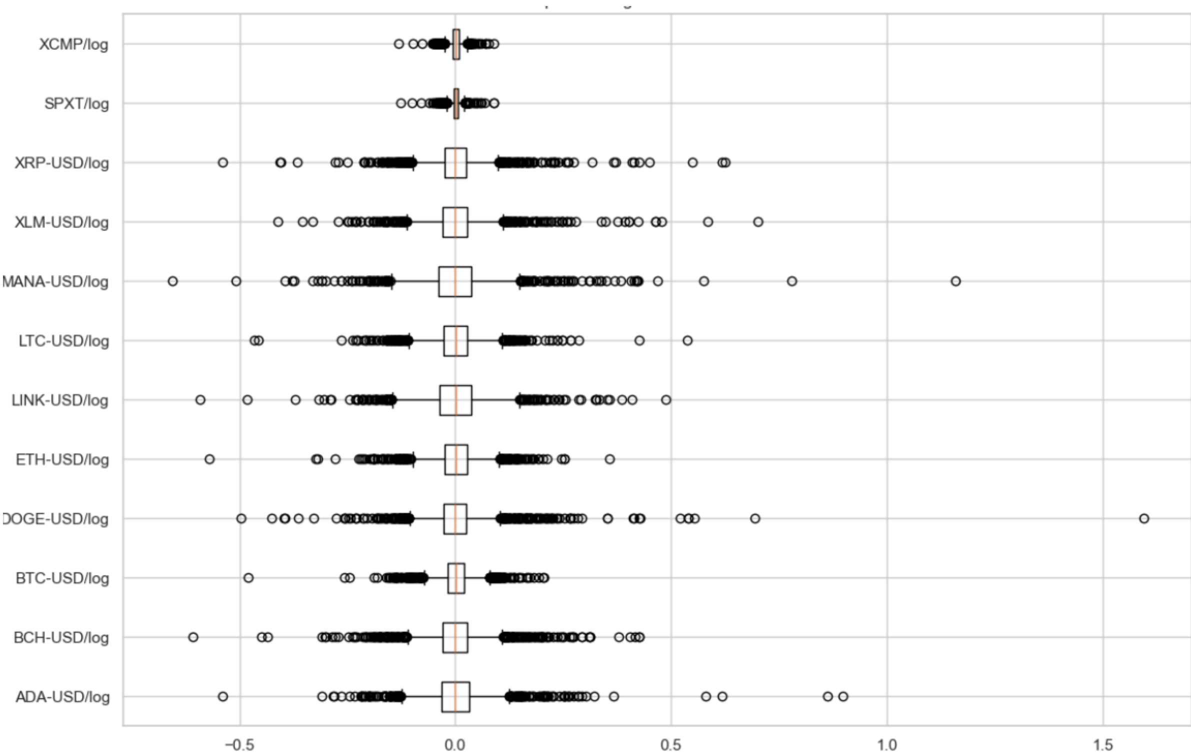


Figure 1: Boxplots of log returns

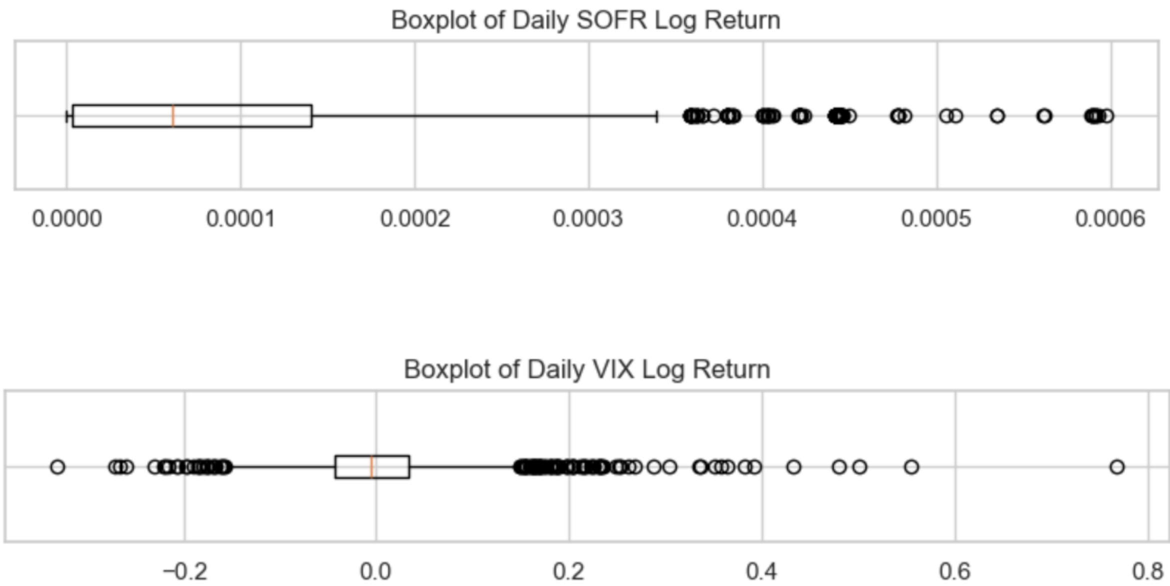


Figure 2: Boxplots of SOFR and VIX log returns

Cryptocurrencies exhibit significantly fatter tails compared to equities. The interquartile range (IQR) for cryptocurrencies is notably wider than that for equities, which appear more concentrated around zero. SOFR returns remain consistently close to zero, except during macroeconomic policy shifts, resulting in a right-skewed distribution. For the VIX, we observe extreme values on both sides, particularly pronounced on the positive side. The presence of numerous extreme outliers implies that these tails could disproportionately affect raw moment estimates.

To ensure robustness and avoid undue influence from extreme market events, while still preserving genuine asset movements, especially important for inherently volatile assets like cryptocurrencies, we apply winsorization. We treat all indices uniformly as tradable assets. Consequently, we winsorize returns symetrically at 1% on each tail for cryptocurrencies, equities, and the VIX, consistent with the methodology outlined in the course. However, given that SOFR log returns typically remain close to zero, we adopt a lighter winsorization of 0.5% on each tail to avoid excluding authentic, small fluctuations in rates.

Finally, we present boxplots of the returns after winsorization in figure 3, allowing visualization of the adjusted distributions.

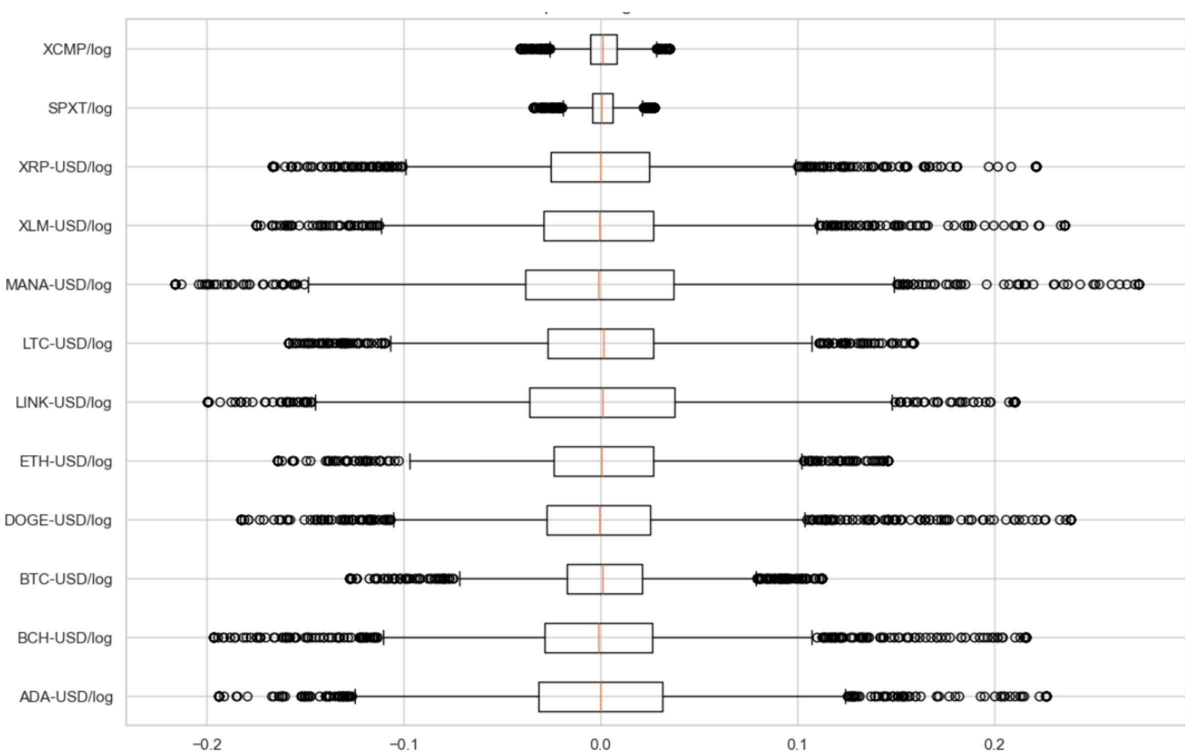


Figure 3: Boxplots of log returns after winsorization

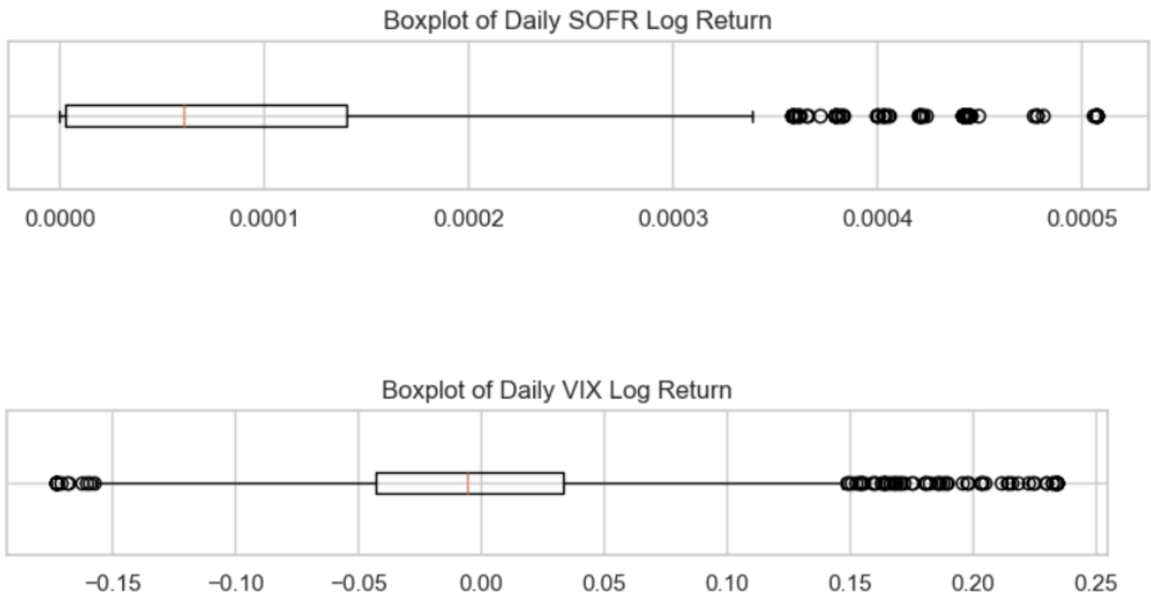


Figure 4: Boxplots of SOFR and VIX log returns after winsorization

After applying 1% winsorization to the log-returns, the most extreme values on both tails have been capped. This visibly reduces the influence of outliers, especially in high-volatility crypto assets such as DOGE, XRP, and BCH. The resulting distributions appear more symmetric and compact, particularly in the interquartile range, improving the robustness of moment estimates.

For the SOFR and the VIX, the effect of winsorization is more subtle. SOFR returns remain tightly clustered around zero, and the light 0.5% trimming applied was sufficient

to remove only a few marginal observations without distorting the core distribution. In contrast, VIX shows a noticeable reduction in extreme values while preserving its natural asymmetry.
We compute moments, before and after winsorization:

	Log Return				Log Return Winsorised			
	mean	std	skew	kurtosis	mean	std	skew	kurtosis
ADA-USD/log	0.001705	0.075637	2.138372	25.123757	0.000754	0.064645	0.331076	2.175843
BCH-USD/log	-0.000118	0.071049	0.040709	9.246112	-0.000129	0.063278	0.197078	2.469498
BTC-USD/log	0.001515	0.043422	-0.817376	10.905432	0.001619	0.039635	-0.137368	1.556347
DOGE-USD/log	0.002609	0.082284	4.661481	81.051812	0.001471	0.062278	0.655612	3.247427
ETH-USD/log	0.000975	0.055481	-0.649550	9.651523	0.001135	0.050479	-0.130141	1.563508
LINK-USD/log	0.001891	0.075208	0.013394	6.459691	0.001765	0.068371	0.061370	1.140882
LTC-USD/log	0.000287	0.059894	0.094316	10.076927	0.000156	0.053331	-0.106731	1.390420
MANA-USD/log	0.001499	0.089623	1.676269	22.535994	0.000822	0.075871	0.418069	2.144416
XLM-USD/log	0.001578	0.071085	1.624939	14.436242	0.000827	0.061291	0.567050	2.863358
XRP-USD/log	0.001275	0.068511	1.247989	16.950890	0.000822	0.057632	0.482847	2.850345
SPXT/log	0.000479	0.011993	-0.828098	15.488848	0.000461	0.010288	-0.537011	1.636453
XCMP/log	0.000546	0.014470	-0.630714	7.473330	0.000544	0.013235	-0.427910	1.108111
VIX/log	0.000380	0.078721	1.550248	9.577290	-0.000571	0.070890	0.657213	1.376315
SOFR/log	0.000091	0.000110	2.070460	4.769815	0.000091	0.000108	1.963681	4.049580

Figure 5: Summary statistics of log returns before and after winsorization

Before winsorization, we can see that there are different orders of magnitude (in terms of volatility). The ten cryptocurrencies show daily volatility that are really higher compared to the traditional assets. Within cryptocurrencies, there are different levels of magnitude, with DOGE, MANA and LINK on top of the spectrum and BTC, ETH and LTC on the bottom. Every crypto shows huge kurtosis, highlighting extreme tail risk. Coins such as ADA, BCH, DOGE, MANA, XLM, XRP have a large positive skew, reflecting big up days. BTC and ETH are actually negatively skewed, indicating more frequent (or with higher magnitude) large down-moves than up-moves. SPXT and XCMP show large kurtosis and negative skew shock log-return. Those are also fat-tailed but less extreme than crypto. VIX log-returns exhibit high average day-to-day volatility. Fat tails with right skewed distribution indicating very large positive jumps in VIX. Given that the overnight rate hardly moves day-to-day, those log-returns are almost always close to zero, with few policy moves.
After winsorization, we can see that by cutting the most extreme values, our moments are now less impacted by some outliers that dominated the raw moments. The mean changed and the volatility dropped significantly. The distribution of each asset has become more Gaussian-like, with skewness approaching zero and kurtosis aligning more closely with the normal benchmark with respect to the unwinsorized data.

Now, let us inspect the correlation matrix and 60-days rolling correlations:

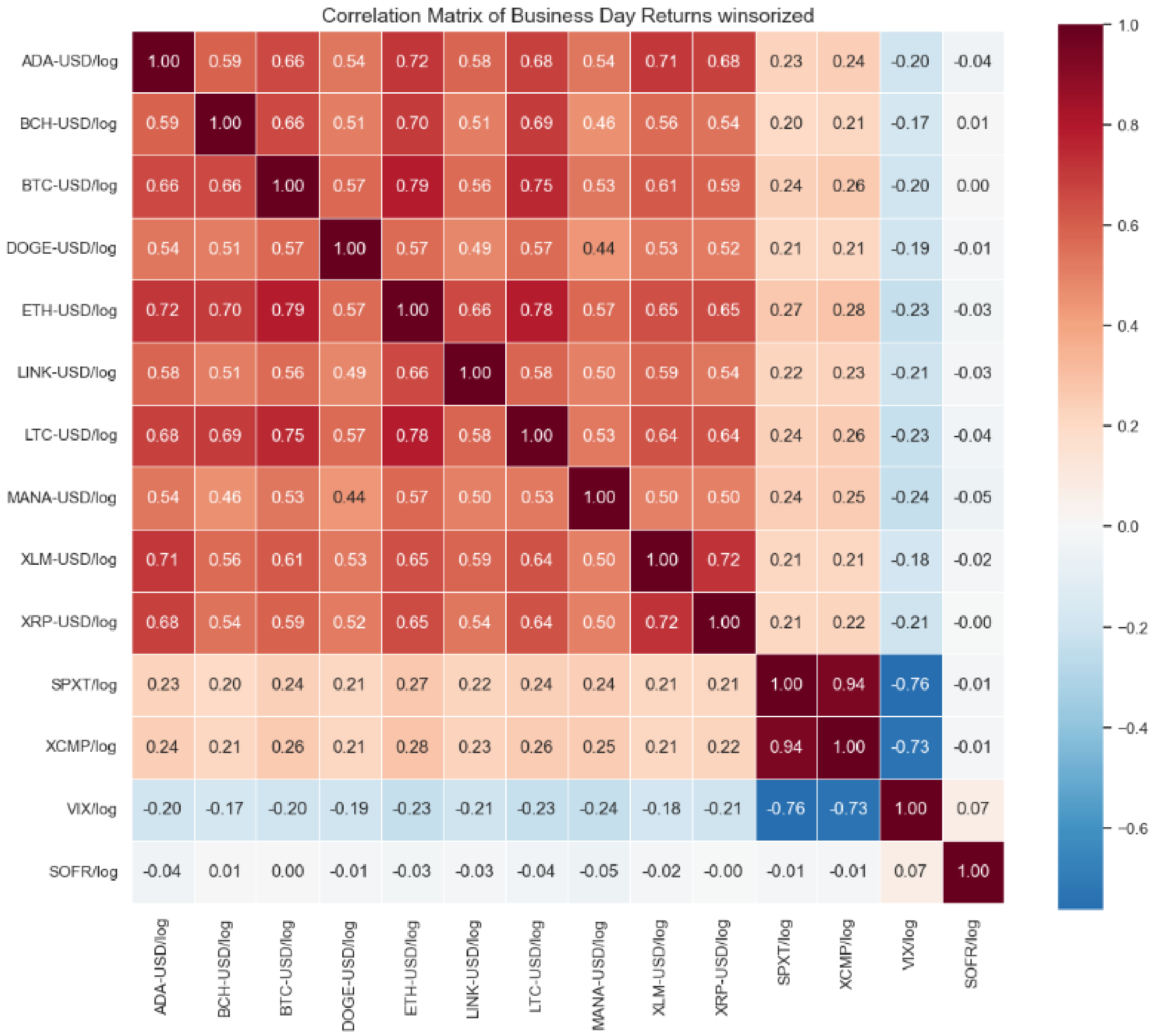


Figure 6: Correlation matrix of business day returns winsorized

We can see a really high correlation among the cryptocurrencies, suggesting that they often move together in response to market dynamics. Cryptos and equities show modest positive correlation. SPXT and XCMP show a correlation that is close to 1, which makes sense given that they both represent broad U.S. equity indices. They are almost interchangeable (in a correlation context) and both show a high negative correlation with the VIX. The VIX is a real-time market index that measures the expected volatility of the S&P 500 over the next 30 days, based on options prices. When the market panics (S&P 500 down), fear spikes (VIX up). When the market rallies, calm returns (VIX down). Even though the VIX is calculated from S&P 500 options, it's used as a proxy for market-wide fear, including the NASDAQ. This also explains the strong negative correlation between the VIX index and the NASDAQ index. SOFR appears essentially uncorrelated with other assets. We need to keep in mind that the correlation and the moments are computed over the entire time series, whereas in reality different trends can occur within that time frame.

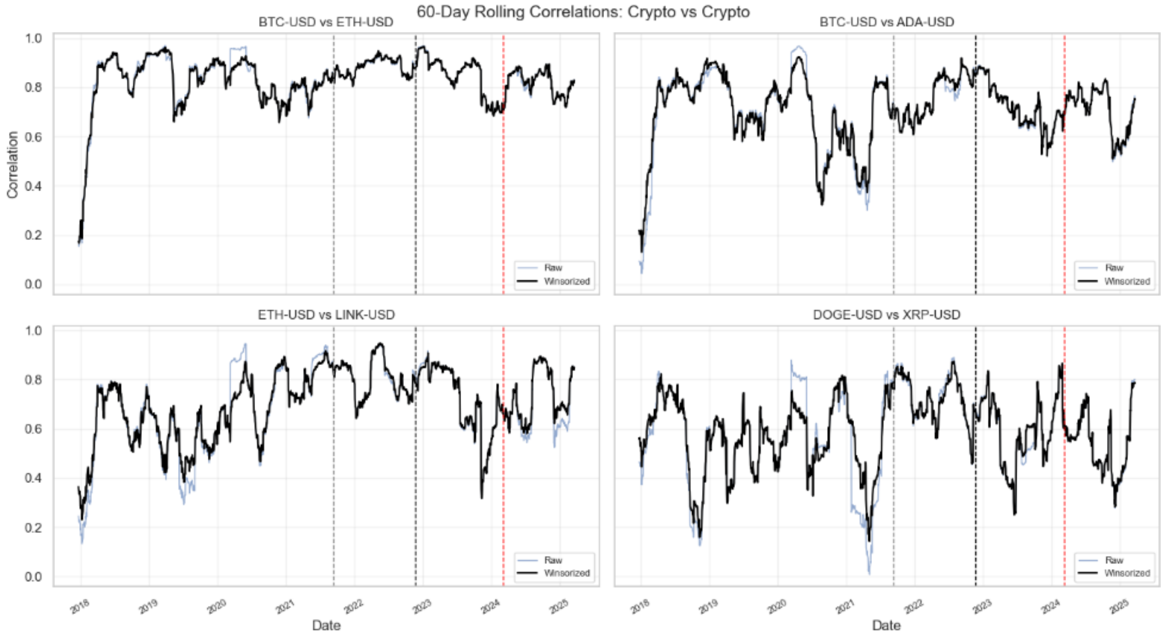


Figure 7: 60-day Rolling Correlations: Crypto Vs Crypto

The correlation is very high almost all the time, we get near to zero diversification benefit, (aside from meme coin) in a risk-parity or minimum-variance portfolio. Crisis and recoveries show up as dips and spikes, during the crash (between datePP to dateTr windows), correlation surge even higher. Around the recovery date (March 24), we see a slight decoupling with correlation dips back down at almost 0,7. During market stress, cryptocurrencies behave almost similarly (correlation close to 1). This correlation decreases during calm or recovery phases. In early 2018–19, cryptos correlations were low, around 0.2–0.4, meaning coins mostly moved on their own rather than in lockstep. This early stage reflects a still immature fragmented market. When we include that 2018–19 window in our full-sample correlation, those low 0.2–0.4 days pull mechanically the overall correlation down.

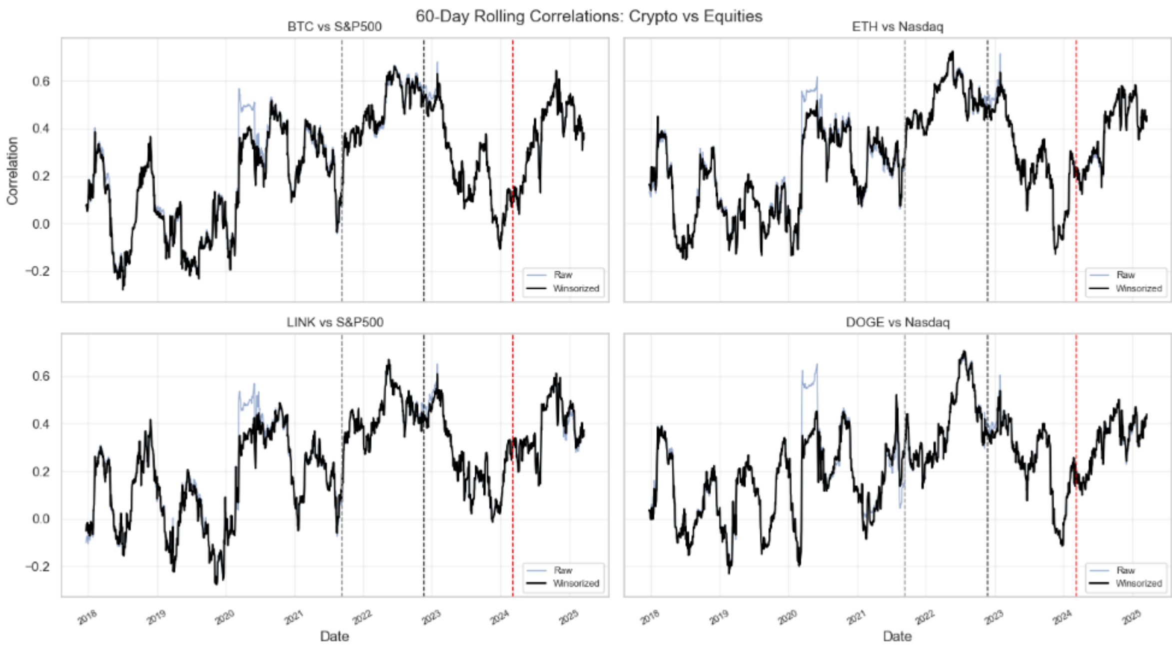


Figure 8: 60-day rolling correlations: Crypto vs Equities

For all four pairings, crypto–equity correlations generally sit between –0.2 and +0.6 and almost never break above 0.6—meaning Bitcoin, Ethereum, LINK, and DOGE usually move quite independently of US stocks.

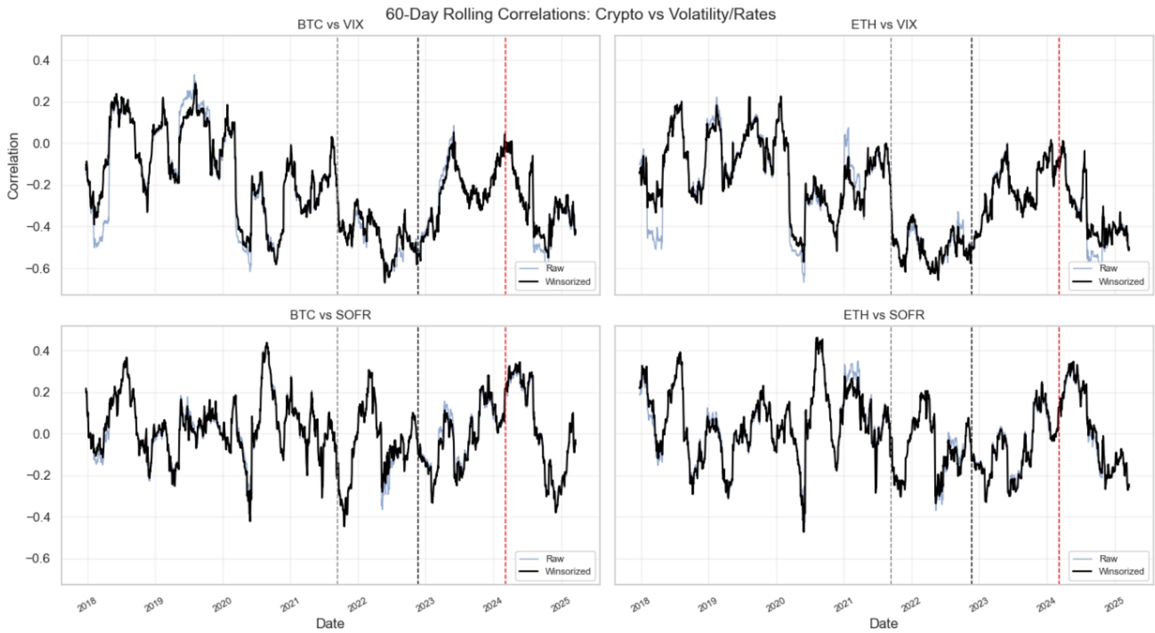


Figure 9: 60-day rolling correlations: Crypto vs Volatility / Rates

Bitcoin and Ether usually move in the opposite direction of stock market volatility (as measured by VIX), and this inverse relationship becomes even stronger during times of market panic or major downturns. Their correlation with SOFR oscillates around zero, turning mildly positive in crypto rallies and dipping negative during stress episodes.

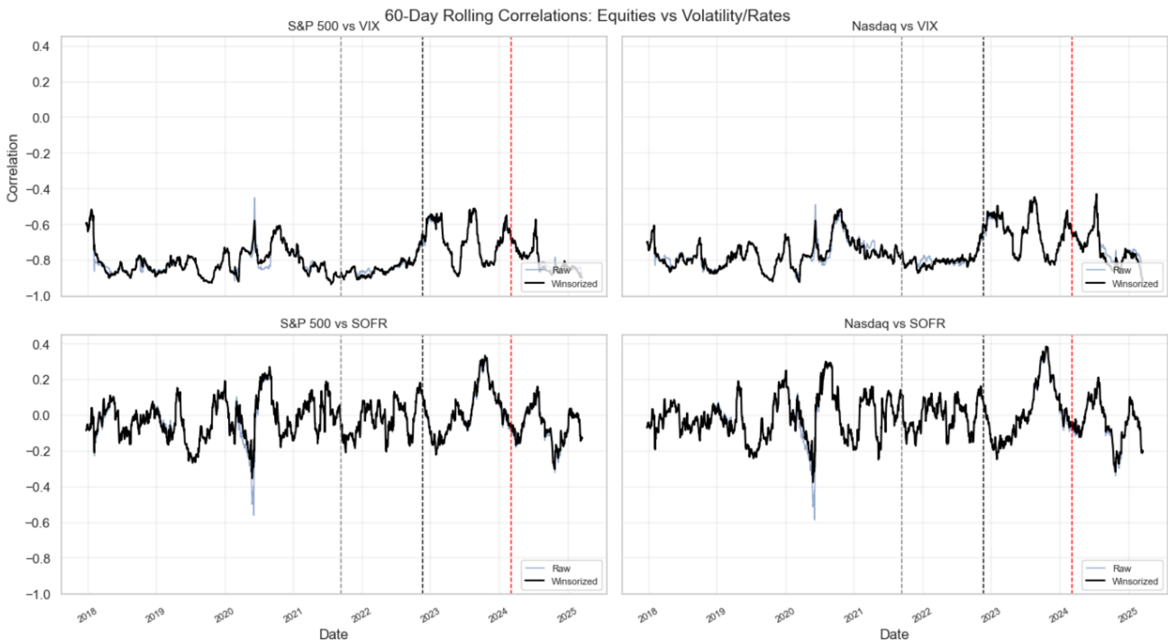


Figure 10: Equities vs Volatility / Rates

The S&P 500 and Nasdaq tend to decline when market volatility (VIX) increases, showing a strong and consistent inverse relationship. However, their relationship with

SOFR is generally weak (around zero), but becomes slightly positive during peak tightening cycles and slightly negative during market sell-offs.

1.c

1.c.i

Price indices only reflect changes in the level of the index (capital gains). However, total-return indices include dividends (or other distributions) reinvested back into the index, which give the actual wealth growth an investor would realize. Over long time horizons, the total return series will show higher compounded returns compared with the price index (which underestimates real growth), thanks to reinvested dividends (or other distributions). Moreover, the day-to-day volatility (around ex-dividend dates) of a total-return index may be lower than that of a price index, since dividends smooth out part of the capital-gain swings (they offset the drop in price on ex-dividend dates). However, over months or years, total return indices exhibit higher mean and higher variance because dividends are reinvested. So yes, this matters in general, especially if we care about reflecting investor wealth growth.

1.c.ii

Risk based portfolio optimization will optimize w.r.t. the variance-covariance matrix. In our crypto/equity setup, using total return indices ensures that cryptocurrency returns (already total return since no dividends) and equity returns are both measured on their full investor return (including distributions). This would not be true if we had used equity price-only series. Total-return indices smooth out the mechanical volatility spikes on ex-dividend dates and incorporate the extra variance from dividend compounding, resulting in a cleaner, economically meaningful variance–covariance matrix and risk–based allocations that reflect the true, tradeable risks of equities versus crypto.

1.d

SPXT, XCMP, SOFR, and VIX show zero returns over weekends, as traditional markets are closed on Saturdays and Sundays. Unlike cryptocurrencies that are traded 24/7. Weekend zero-returns in equities and rates artificially suppress their measured co-movements with cryptocurrencies. Cryptos move nonstop, but traditional assets sit at zero, pulling down correlations. Including non-trading days with zero returns biases the volatility estimate of traditional assets downward, as the standard deviation is diluted by flat returns. Finally, mixing weekend and weekday returns introduces a structural break in the return process, violating the i.i.d. assumption typically required for time-series models.
In order to treat this issue, we will drop the prices of all assets during the weekend and compute the return for Monday on the basis of the Friday price and the Monday price.

2 Question 2

For any further plots and specific parts of the code we used, please refer to the jupyter notebook "Questions 2 + 3 + 4.ipynb" in the .zip file.

2.a

An equally weighted (EW) portfolio gives every asset within it the same weight, regardless of its size or value. This means each asset in the portfolio contributes equally to its overall performance. In our case, $w \in \mathbb{R}^{14}$ and $w_i = \frac{1}{14}$.
To evaluate the performance of our equally weighted portfolio between the previous peak (PP) date (11-Sep-2021) and the trough (Tr) date (21-Nov-2022), we initialize the portfolio with a starting wealth of 1. We then observe how the portfolio's value evolves across this period.

- Portfolio value at the PP date: 1
- Portfolio value at the Tr date: 0.47

This reflects a significant decline in portfolio value over the period. Additionally, to assess the performance of each individual asset, we compute their percentage price change between the PP and Tr dates :

Table 2: Percentage changes of various assets over the period considered

Asset	**Percentage Change (%)**
ADA–USD	-88.46
BCH–USD	-83.78
BTC–USD	-65.11
DOGE–USD	-69.06
ETH–USD	-66.14
LINK–USD	-78.26
LTC–USD	-65.69
MANA–USD	-55.60
XLM–USD	-74.09
XRP–USD	-66.27
SPXT	-9.75
XCMP	-26.38
SOFR	1.24
VIX	6.73

The results indicate a sharp decline in the value of nearly all crypto assets, with losses ranging from around -55% to -88%.

2.b

For the selection of the appropriate data window, we consider Lopez de Prado (2016) [6], which suggests that, in general, we need at least $\frac{1}{2}N(N+1)$ independent and identically distributed (IID) observations in order to estimate a covariance matrix of size N that is

not singular. In our case $N = 14$, so we need at least 105 observations.
We choose to use a window of 252 business days rather than 365 calendar days because it aligns with the actual trading calendar and still captures a full cycle of market events. This avoids having to fill every Saturday and Sunday with artificial "0%" asset-index returns, which would underestimate asset-index volatility and attenuate crypto–asset-index covariances. By incorporating each weekend's actual crypto movements into Monday's return, we preserve the true risk dynamics in both traditional and 24/7 markets.

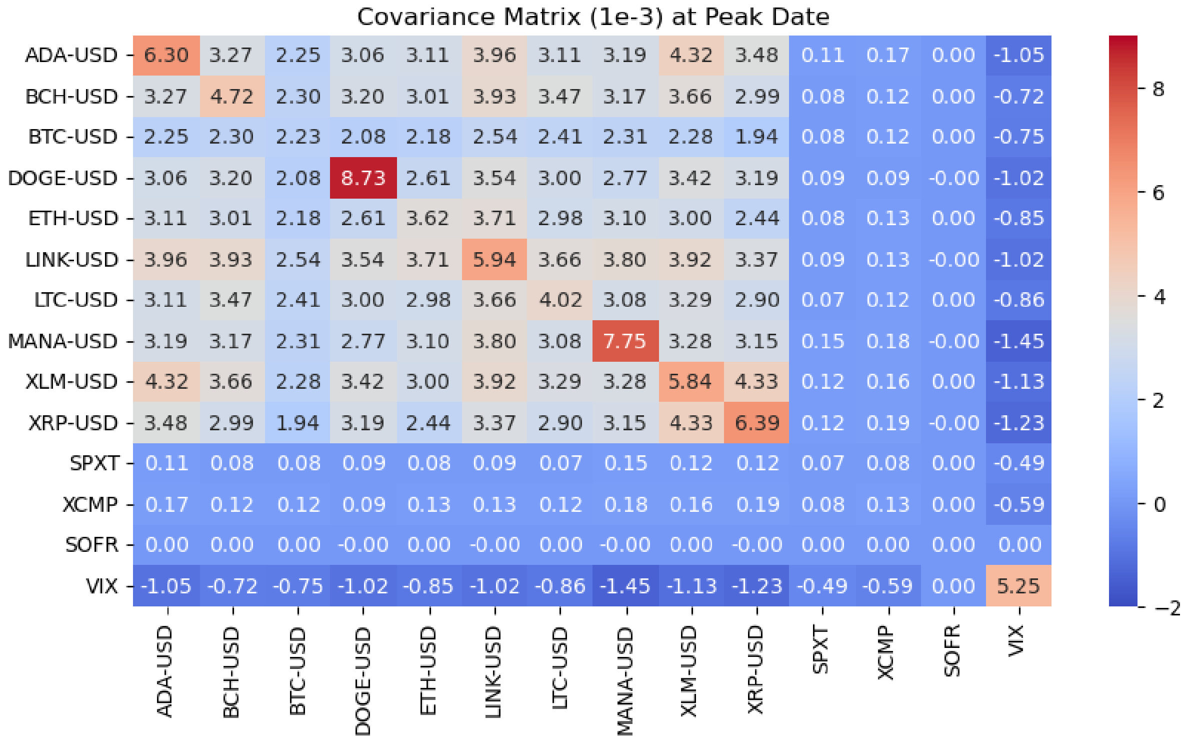


Figure 11: Covariance matrix at the previous peak date

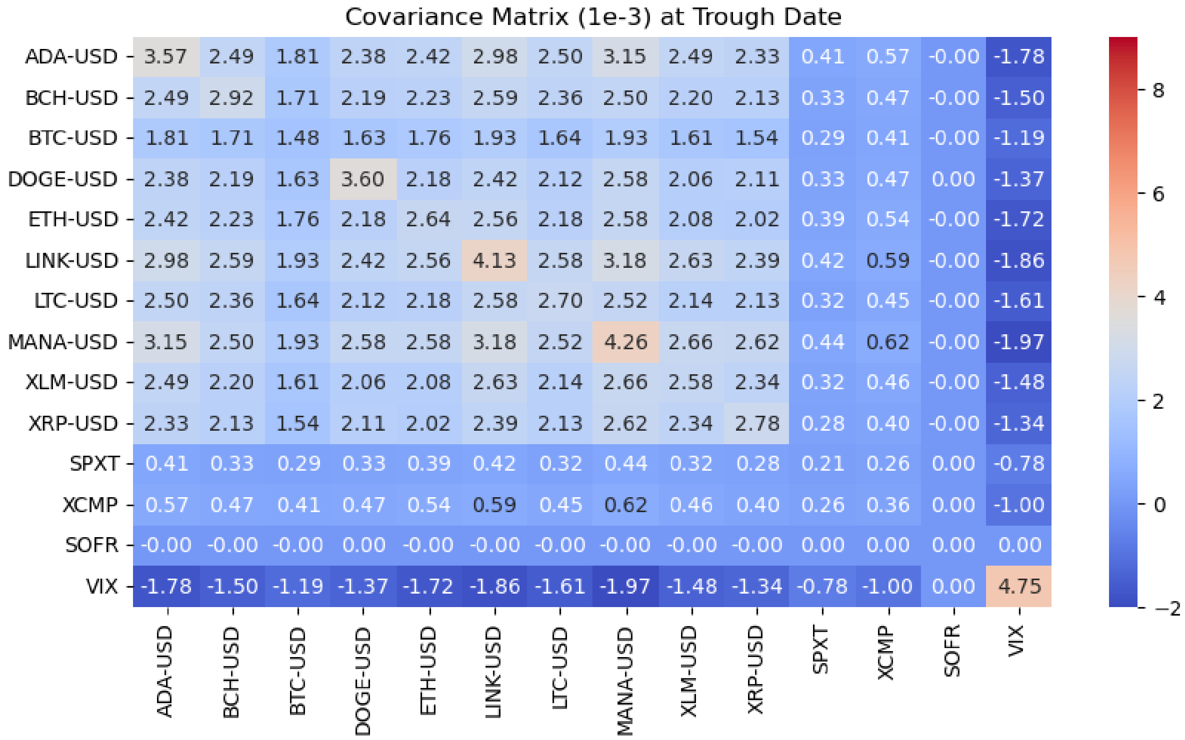


Figure 12: Covariance matrix at the trough date

2.c

The Eigenvalues clipping (Bouchaud & Potters 2016) [1]method keeps the $[N\alpha]$ top eigenvalues and shrink the others to a constant γ that preserves the trace, $\mathrm{Tr}(\Xi^{\mathrm{clip}}) = \mathrm{Tr}(E) = N$:

$$\Xi^{\text{clip.}} := \sum_{k=1}^{N} \xi_k^{\text{clip.}} \, u_k u_k^*, \quad \xi_k^{\text{clip.}} := \begin{cases} \lambda_k, & k \le \lceil N\alpha \rceil, \\ \gamma, & \text{otherwise.} \end{cases}$$

A simple procedure to choose α is to assume that all empirical eigenvalues beyond the Marčenko and Pastur upper edge can be deemed to contain some signal and are therefore kept without change. This is the procedure we have chosen to follow. The implementation is inspired by the github [5].

Additionnaly, we choose to force our reconstructed correlation matrix to have all 1's on the diagonal before getting our covariance matrix. We do so because we want a "real" correlation matrix, which has all 1's on the diagonals (The correlation of a random variable with itself is always 1). We also tested without it, and we observed better results by forcing the diagonal to 1.

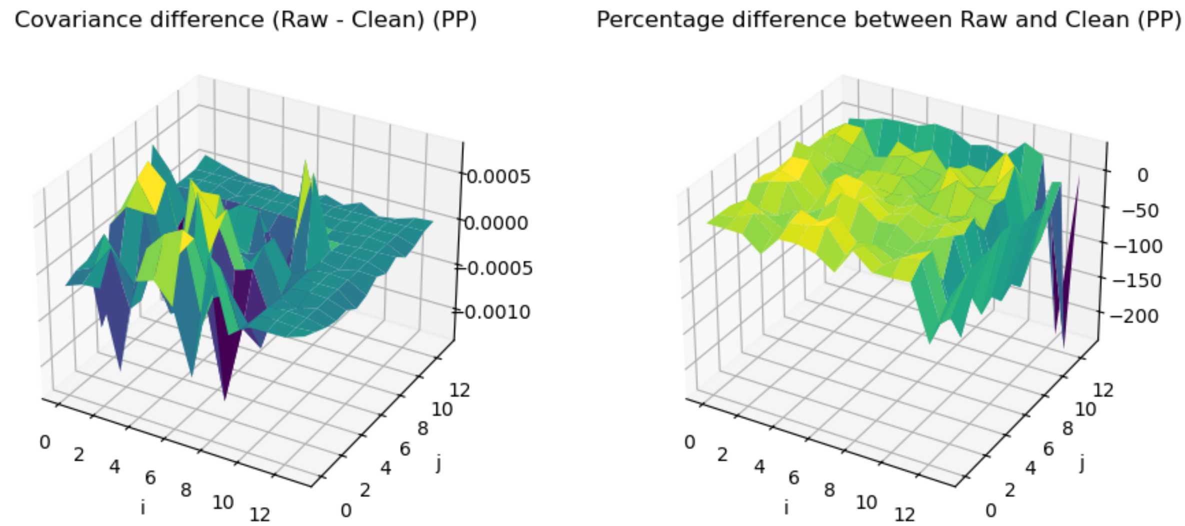


Figure 13: Comparison of raw and cleaned covariance matrices at the previous peak date

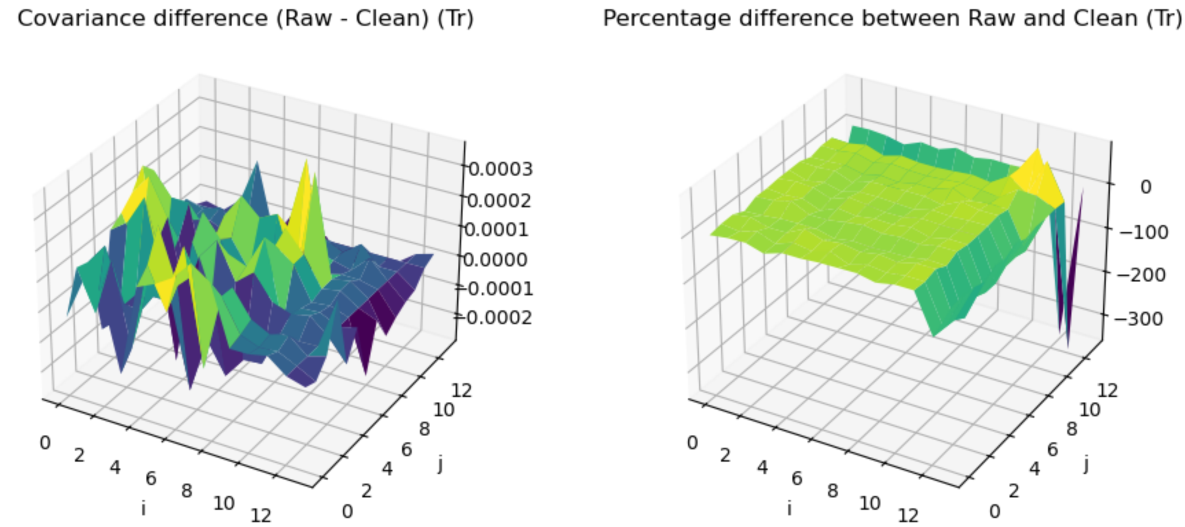
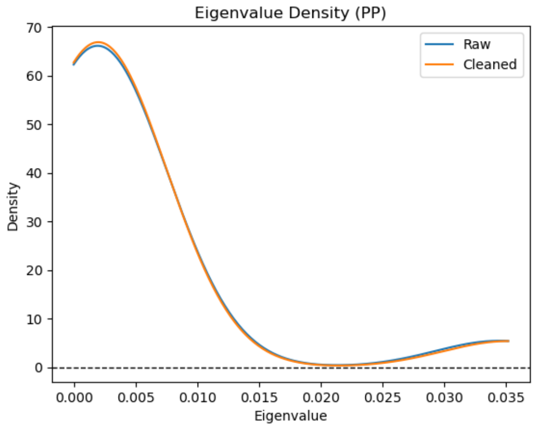


Figure 14: Comparison of raw and cleaned covariance matrices at the trough date

We clearly see that the covariance at the Tr date is much "cleaner" than the covariance at the PP date. We also see that the percentage change is very unstable around the SOFR. This is not surprising, as we clearly saw that the covariance line and columns corresponding to the SOFR are way smaller than the other values (0.00e-03). So this jump represents numerical instability more than error in our approach.



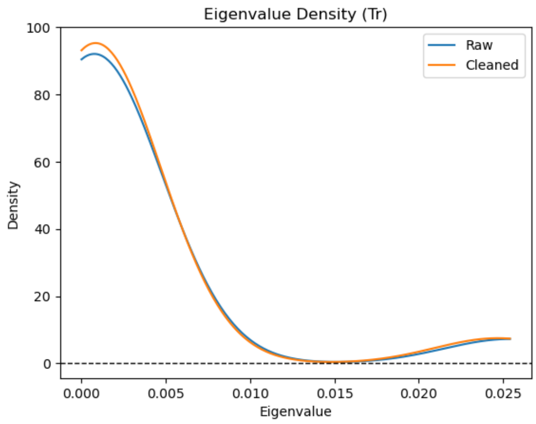


Figure 15: Eigenvalue densities raw and cleaned at both PP and Tr dates respectively

Regarding the eigenvalue density plot, we see the cleaned version is slightly shifted to the right, which shows the impact of putting all the small eigenvalues to a same constant. The tail on the right shows that we have kept the bigger eigenvalues, while in the smaller region, the orange line is slightly above the blue, which shows all the smaller eigenvalues which have been set to a constant. Remember that we are plotting the kernel density, which "smoothes" the actual distribution of the eigenvalues into a continuous curve.

- PP condition number $\rightarrow$ raw: 1.03e+10, cleaned: 4.99e+09
- Tr condition number $\rightarrow$ raw: 5.94e+06, cleaned: 3.94e+06

We also see that the condition number is very high initially for both dates, and that the clipping process reduces their magnitude by 2 and 1.5 respectively. This is what is expected, as with unstable random variables such as the covariance and correlation, the condition number is always very high. Nonetheless, the clipping has reduced it, making the matrices more prone to invertability.

2.d

To construct the Euler risk-contribution structures, we proceed as follows. Let $\mathbf{w}$ be the portfolio weights at the solution date and Σ the corresponding covariance matrix. First, we compute the total portfolio risk (standard deviation) $\sqrt{V} = \sigma = \sqrt{\mathbf{w}^\top \Sigma \, \mathbf{w}}$.
Next, we obtain the marginal risk contributions $\mathrm{MRC}_i = \frac{\partial \sigma}{\partial w_i} = \frac{(\Sigma \, \mathbf{w})_i}{\sqrt{\mathbf{w}^\top \Sigma \, \mathbf{w}}} = \frac{(\Sigma \, \mathbf{w})_i}{\sigma}$, and form the Euler risk contributions $\mathrm{RC}_i = w_i \, \mathrm{MRC}_i = w_i \, \frac{(\Sigma \, \mathbf{w})_i}{\sigma}$.
Finally, we normalize by dividing by σ, $\mathrm{NRC}_i = \frac{\mathrm{RC}i}{\sigma} \implies \sum_i \mathrm{NRC}_i = 1$, so that each NRC_i measures the fraction of total risk attributed to asset i.
By applying this recipe to both portfolios at their respective dates, and using each of the raw-sample and cleaned covariances, we obtain the four required Euler risk-contribution structures.
To summarize the dispersion of our normalized Euler risk contributions $\{NRC_i\}_{i=1}^N$ with a single metric, we compute the Herfindahl index $H = \sum_{i=1}^N (NRC_i)^2$. By squaring each contribution and summing, H captures concentration: in the extreme of perfect equality ($NRC_i = 1/N$ for all i), we get $H = 1/N$, indicating maximal diversification,

whereas if one asset bears nearly all the risk ($NRC_j \approx 1$), then $H \approx 1$, reflecting high concentration.

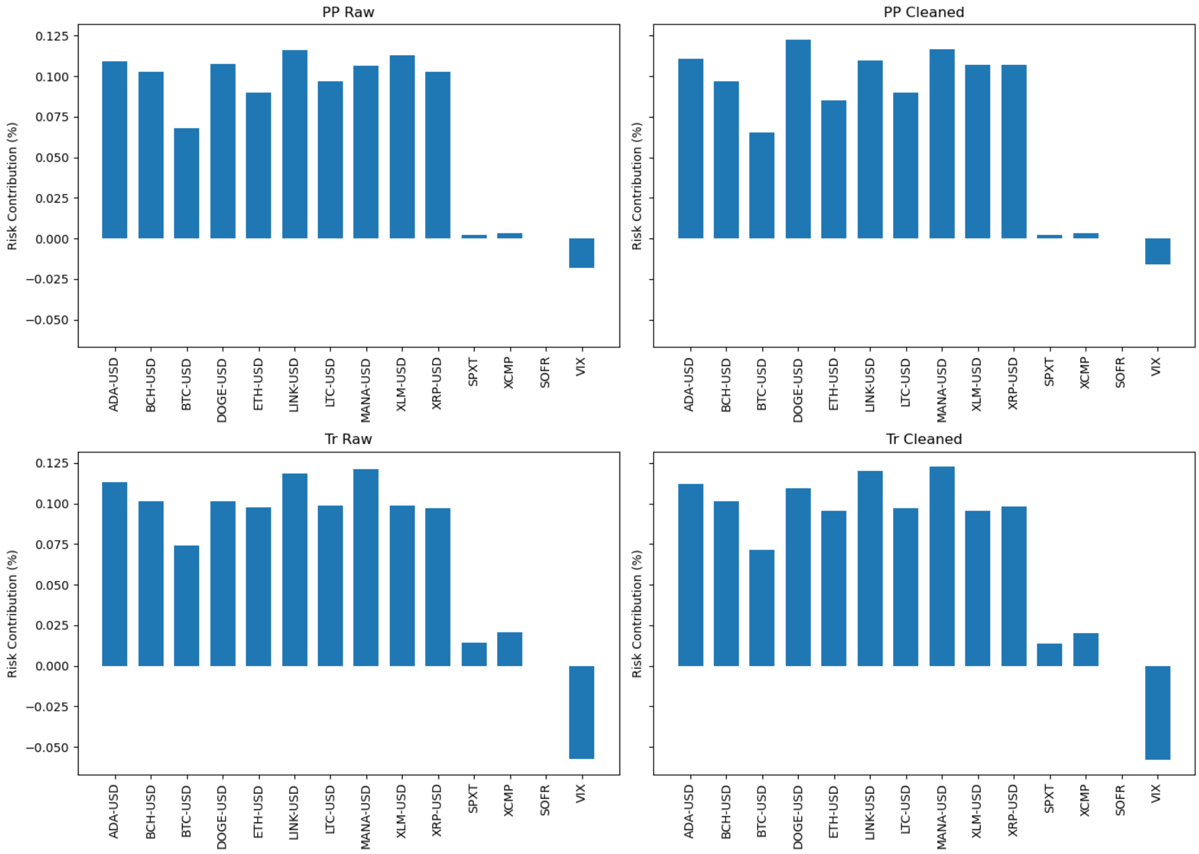


Figure 16: Risk contribution of assets in portfolios for PP and Tr dates, before and after covariance matrix cleaning.

We can see that the risk contribution is pretty evenly spread out across the 10 crypto assets, traditional assets remain, for the most part, virtually neutral in terms of risk contribution, with the notable exception of the VIX, which consistently exhibits a negative contribution. This changes slightly at the trough date, with the SPXT and XCMP taking up some of the risk. But the major risk bearers of the portfolios are in both cases the crypto assets. This is not surprising, as crypto assets are notoriously volatile. It is however interesting to note that while the crypto currencies take up most of the risk, BTC has the least amount of risk contribution compared to the other cryptos, reinforcing the idea that it behaves more and more like an equity asset (as we discussed in class).
We observe that cleaning the covariance matrix has only a marginal effect on the Herfindah index. However, the values for the Tr date are slightly higher than those for PP, indicating a somewhat greater concentration of risk in the portfolio during that time.

Table 3: Herfindahl Index values

	Herfindahl Index
Minimum value ($1/N$)	0.0714
PP Raw	0.1047
PP Cleaned	0.1050
Tr Raw	0.1101
Tr Cleaned	0.1108

2.e

Following the method in Meucci (2009) [8], a generic portfolio can be seen either as a combination of the original securites with weights $\mathbf{w}$, or as a combination of the uncorrelated principal portfolios with weights $\tilde{\mathbf{w}} \equiv \mathbf{E}^{-1}\mathbf{w}$, where the matrix $\mathbf{E} \equiv (\mathbf{e_1}, ..., \mathbf{e_N})$ contains the eigenvectors of Σ, the returns covariance. In terms of the latter, we can introduce the variance concentration curve

$$v_n \equiv \tilde{w}_n^2 \lambda_n, \quad n = 1, ..., N.$$

The generic entry v_n of this concentration curve represents the variance due to the n-th principal portfolio. The total portfolio variance is the sum of these terms :

$$\mathrm{Var}\{R_w\} = \sum_{n=1}^{N} v_n$$

Finally, we can compute the diversification distribution :

$$p_n \equiv \frac{\tilde{w}_n^2 \lambda_n}{\mathrm{Var}\{R_w\}}$$

Portfolio diversification can then be summarized by the entropy of $\{p_n\}$, or equivalently by its exponential, the Effective Number of Bets:

$$\mathcal{N}_{\mathrm{Ent}} \equiv \exp\Big(-\sum_{n=1}^{N} p_n \ln p_n\Big).$$

Table 4: Effective number of bets and effective number of assets

	NEnt	**Eff. number of assets**
PP raw	1.13	9.55
PP cleaned	1.13	9.52
Tr raw	1.18	9.08
Tr cleaned	1.19	9.02

The Effective Number of Bets, defined as the exponential of the entropy of the diversification distribution, remains very low (around 1.1) in both peak and trough periods, indicating that risk is effectively concentrated in a single dominant factor, as can be seen in figure 17. In contrast, the effective number of assets (the reciprocal of the Herfindahl index) is around 9, showing that many assets contribute to total variance but in a correlated way. Cleaning the covariance matrix has virtually no impact on NEnt and only minor effects on the asset-based index, slightly reducing concentration after cleaning. This highlights that, despite a large asset pool, the true number of independent "bets" is very limited, driven by a few principal risk factors.

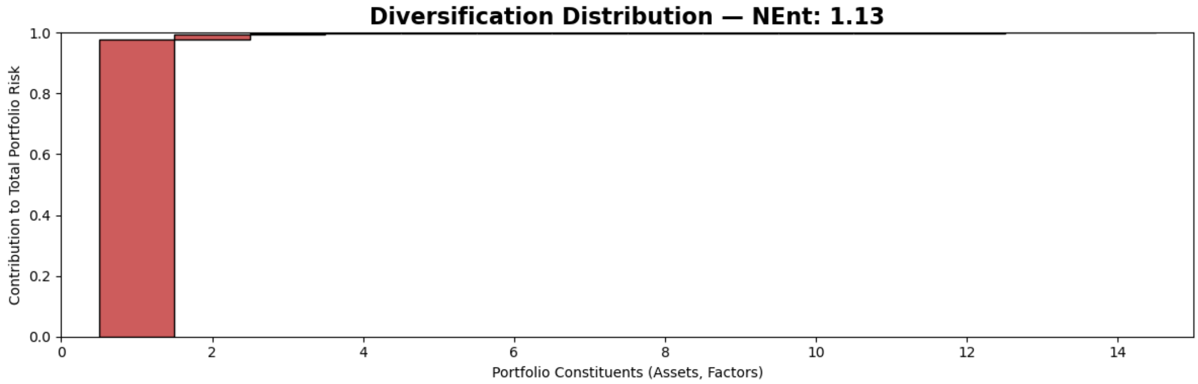


Figure 17: Diversification distribution of the EW portfolio at the PP date

2.f

The scatter plot of the rank of the risk contribution versus the rank of % loss on the trough date exhibits a clear positive association, with Kendall's $\tau = 0.65$ (p = 0.001), indicating a statistically significant moderate-to-strong concordance between our ex-ante risk contributions and actual losses realized. Assets that the cleaned covariance model flagged as more "risky" tended indeed to suffer larger drawdowns.
The plot also reveals a few notable deviations. For example, ADA-USD, while ranked 3rd in risk, suffered the largest loss (rank 1), and XLM-USD, ranked 8th in risk, experienced the 4th largest drawdown. MANA-USD, despite having the highest risk contribution (rank 1), recorded a small drawdown (rank 10). Such outliers remind us that even a well-specified covariance model cannot capture all idiosyncratic shocks or sudden regime shifts. Overall, the correspondence ($\tau = 0.65$) demonstrates that Euler risk contributions provide a meaningful, though not perfect, forecast of stress losses, validating the cleaned-matrix approach for identifying the portfolio's most vulnerable positions.

Table 5: Loss percentages, risk contributions, and corresponding rankings for selected assets

Asset	Loss %	Rank Loss	Risk Contrib. %	Rank Risk
ADA-USD	−88.456	1	11.2319	3
BCH-USD	−83.780	2	10.1471	5
LINK-USD	−78.263	3	11.9985	2
XLM-USD	−74.086	4	9.5740	8
DOGE-USD	−69.059	5	10.9533	4
XRP-USD	−66.265	6	9.7986	6
ETH-USD	−66.141	7	9.5644	9
LTC-USD	−65.686	8	9.7254	7
BTC-USD	−65.106	9	7.1437	10
MANA-USD	−55.604	10	12.2921	1
XCMP	−26.385	11	1.9815	11
SPXT	−9.746	12	1.3761	12
SOFR	1.236	13	−0.0040	13
VIX	6.730	14	−5.7825	14

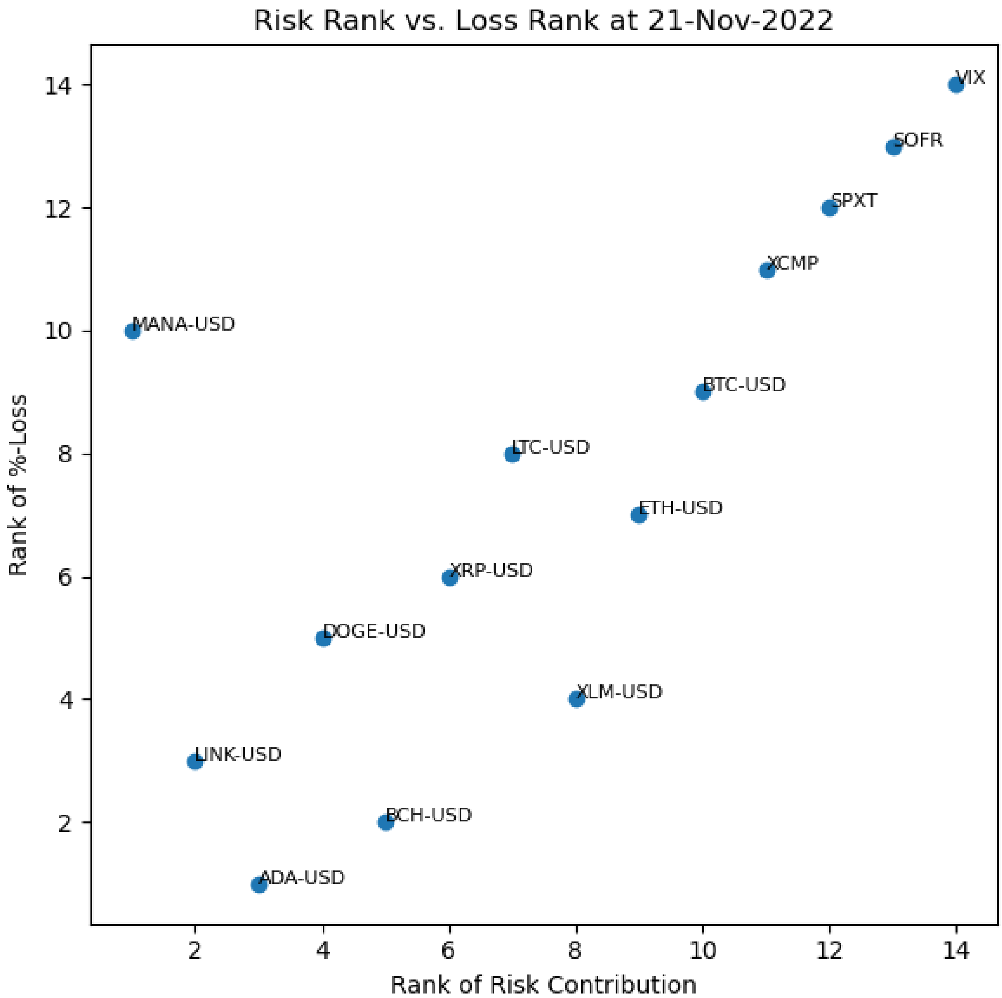


Figure 18: Scatter plot of risk contribution rank vs. loss rank at the trough date

3 Question 3

For any further plots and specific parts of the code we used, please refer to the jupyter notebook "Questions 2 + 3 + 4.ipynb" in the .zip file.

3.a

To construct the following portfolios, we used the cleaned covariance matrix obtained in Question 2, evaluated separately at the previous peak date and the trough date.

3.a.i

The minimum variance portfolio is obtained by solving the following quadratic program, where w is the vector of asset weights and Σ is the covariance matrix:

$$\begin{aligned} &\min_{w\in\mathbb{R}^N} \; w^\top \Sigma\, w \\ \text{s.t.} \quad & w_i \;\geq 0, \quad i = 1, \dots, N, \\ & \sum_{i=1}^{N} w_i \;=\; 1 \end{aligned}$$

This optimization problem is solved separately using the covariance matrix evaluated at the PP date and the Tr date.

3.a.ii

The equal risk contribution (ERC) portfolio aims to equalize the individual asset contributions to total portfolio risk. Let w be the vector of asset weight, Σ the covariance matrix and $\sigma_p = \sqrt{w^\top \Sigma w}$ the total portfolio volatility. The marginal risk contribution (MRC) of asset i is

$$\mathrm{MRC}_i = \frac{\partial \sigma_p}{\partial w_i} = \frac{(\Sigma w)_i}{\sigma_p}$$

and the total risk contribution (RC) of asset i is

$$\mathrm{RC}_i = w_i \cdot \mathrm{MRC}_i = \frac{w_i (\Sigma w)_i}{\sigma_p}$$

The ERC portfolio solves the following:

$$\begin{aligned} &\min_{w\in\mathbb{R}^N} \sum_{i=1}^{N}\sum_{j=1}^{N} (\mathrm{RC}_i - \mathrm{RC}_j)^2 \\ \text{s.t.} \quad & w_i \;\geq 0, \quad i = 1, \dots, N, \\ & \sum_{i=1}^{N} w_i \;=\; 1 \end{aligned}$$

3.a.iii

The maximum effective number of bets portfolio aims to maximize diversification by maximizing the effective number of uncorrelated risk factors. Basically, the argument made is that two separate investments into two different but highly correlated stocks are essentially making the same "bet", so we should avoid that.
Let w be the vector of asset weight and Σ the covariance matrix. The minimum effective number of bets portfolio solves the following program: (cf. 2.e)

$$\begin{aligned} &\max_{w \in \mathbb{R}^N} \exp\Big(-\sum_{n=1}^{N} p_n \ln p_n\Big) \\ \text{s.t.} \quad & w_i \geq 0, \quad i = 1, \ldots, N, \\ & \sum_{i=1}^{N} w_i = 1 \end{aligned}$$

3.a.iv

The hierarchical risk parity (HRP) portfolio offers a more robust alternative to traditional risk-based allocations by avoiding the inversion of noisy covariance matrices. It uses hierarchical clustering to structure portfolio weights, enhancing stability in the presence of estimation errors and outliers.
The portfolio construction can be implemented through the following steps:

1. **Hierarchical clustering**
   Compute a distance matrix from the asset correlation matrix using $d_{ij} = \sqrt{0.5(1 - \rho_{ij})}$, and perform hierarchical clustering.

2. **Quasi-diagonalization**
   Reorder the rows and columns of the covariance matrix based on the hierarchical clustering output. This places similar assets adjacent to one another, resulting in a block-like structure in the matrix.

3. **Recursive bisection**
   Allocate portfolio weights by recursively splitting the ordered list of assets into clusters. At each split, weights are assigned in inverse proportion to the variance of each cluster, ensuring risk-balanced allocation across the hierarchy.

3.b

Our results are summarized in the below figures 19 20 22 23.

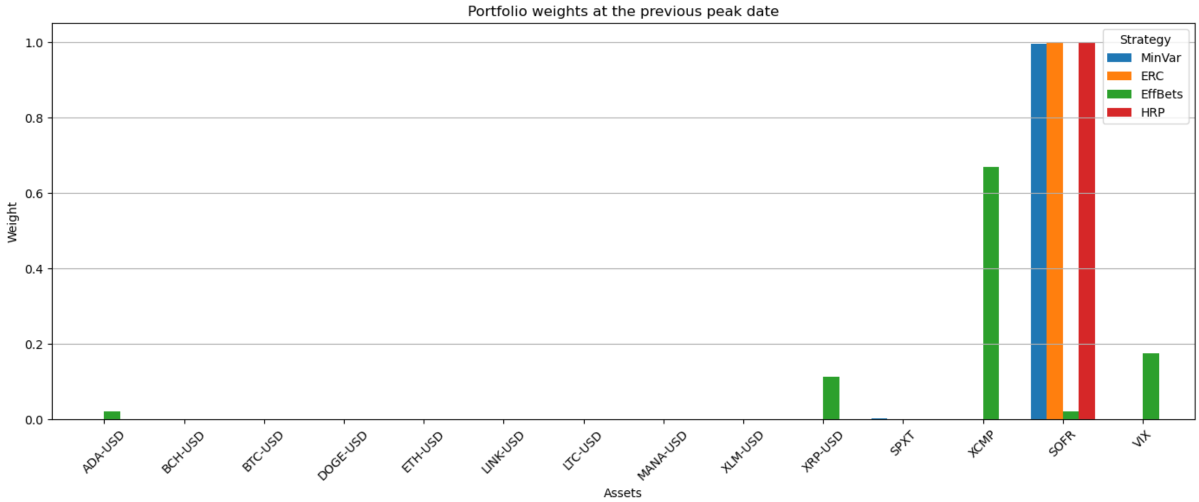


Figure 19: Portfolio weights at the previous peak date

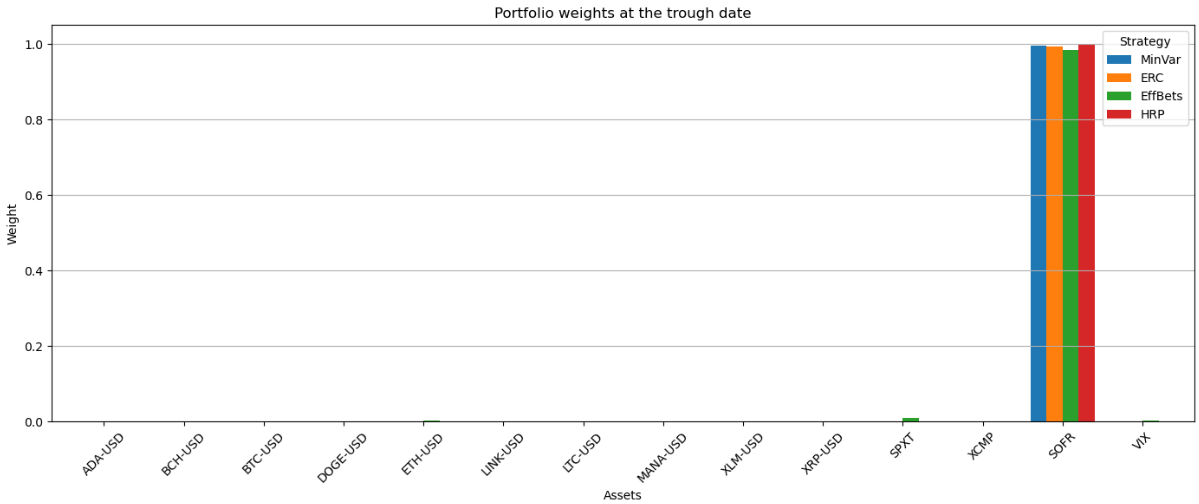


Figure 20: Portfolio weights at the trough date

We clearly see that 3 out of the 4 strategies allocate all of our wealth to the SOFR at the PP date. At the Tr date, all four strategies allocate all the wealth to SOFR. This does make sense mathematically, let us break down why:

- **MinVar**: Seeks to minimize portfolio variance. As a result, nearly all of the allocation ends up in SOFR, an asset with near-zero volatility and very low correlation to the rest, since that placement drives total variance down to its minimum.
- **ERC**: Allocates so that each asset contributes equally to total portfolio risk. Because SOFR has almost zero volatility and thus zero marginal risk, the optimizer must assign it a disproportionately high weight (approaching 100 %) in order for its risk contribution to match that of the other more volatile assets.
- **EffBets**: Seeks to construct a portfolio exposed to the largest possible number of distinct "bets" (i.e. uncorrelated risk factors). Beyond a certain point, adding more assets that don't introduce new independent risk factors only increases exposure to risks already in your portfolio, without raising the effective number of bets.

We found that the effective number of bets was 6.19 at the PP date (see figure 21) and 7.45 at the Tr date.

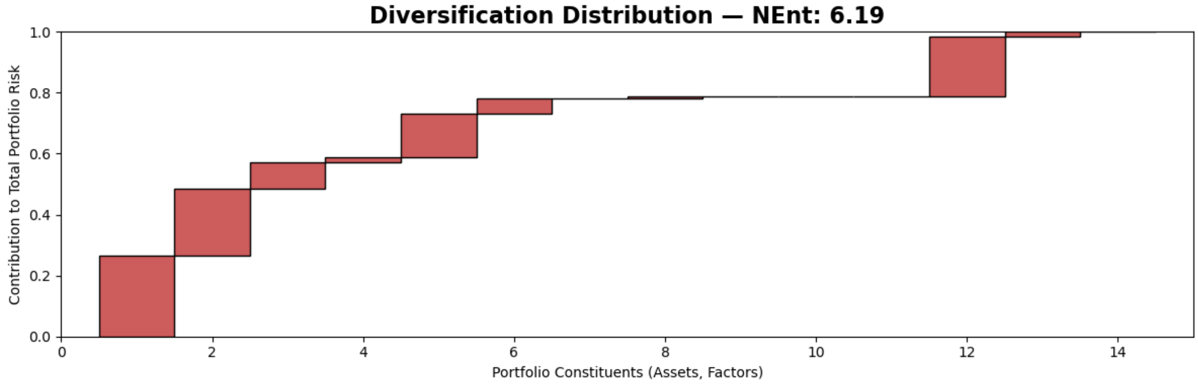


Figure 21: Diversification distribution of the maximum effective number of bets portfolio at the PP date

- **HRP**: Seeks to cluster assets that are highly correlated together, and then takes a risk parity approach on each of them, attributing more weight to each class that has lower risk. We can clearly see that the SOFR has very low volatility, and is not strongly correlated to the other assets (c.f. covariance matrices in figures 11 and 12), so it is not surprising to see it have a high weight.

However, this does not make sense intuitively. No investor looking for profit would invest only in the risk less asset. In our investments class, and generally every time we have learned about portfolio construction, we have treated the risky assets separately from the risk less asset. We only recombine them at the end, and the risk less asset becomes a tool for the investors to expose themselves to more/less risk through borrowing/lending. That is why we have chosen to do the analysis again, but this time without including the SOFR, which we here treat like the risk less asset. We justify this assumption by looking at the results, and also looking at the variance matrices (figures 11 and 12). We clearly see that the line and column associated to the SOFR are 0. This means that the covariance matrix is almost certainly not of full rank, i.e. uninvertible (or at least, it will be numerically very close to not full rank). It also means that we introduce a lot of numerical instability, as small changes in some values may have big impacts on the results. The findings are shown in figures 22 and 23.

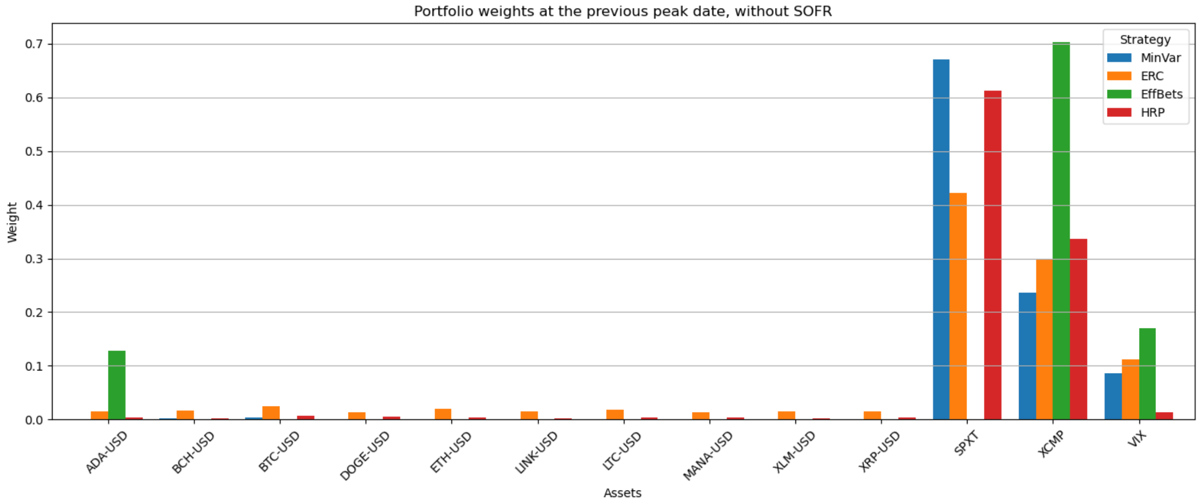


Figure 22: Portfolio weights at the previous peak date, without the SOFR

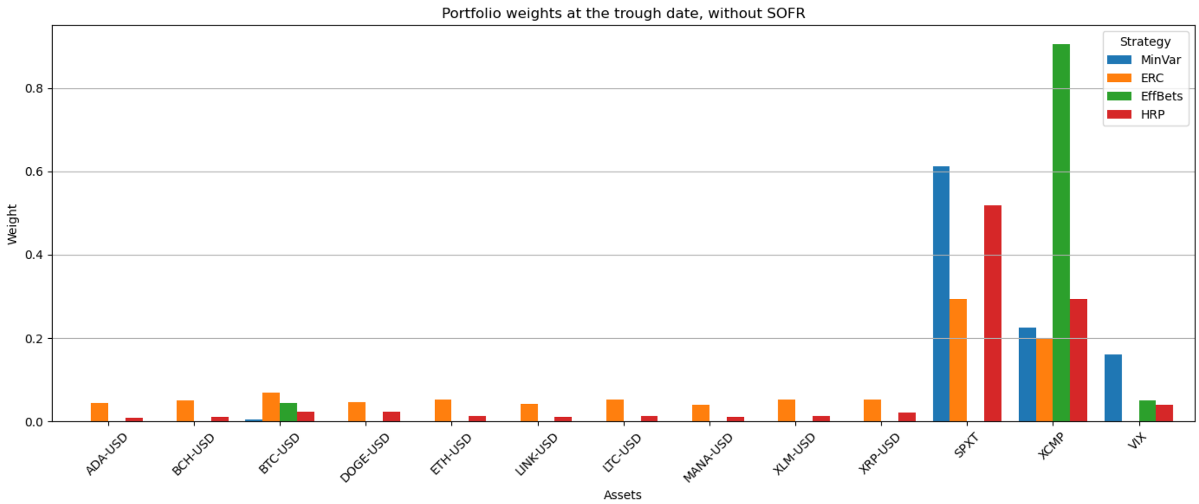


Figure 23: Portfolio weights at the trough date, without the SOFR

Now, we can already see the results look more interesting! Clearly our optimization has done something, and not simply allocated all the wealth to one asset. Let us break it down method by method.

- **MinVar**: At the PP date, the minimum variance is achieved when we are invested approximately 67% in the SPXT, 24% in the XCMP and 9% in the VIX. This seems to make sense, as the SPXT and XCMP are both indexes, known to be less volatile than crypto assets. Aditionally, it seems some diversification is possible through the VIX (Volatility index) which makes sense, since the VIX has negative covariance with the other assets. At the Tr date, we see that the 3 main assets are the same, but the distribution has changes slightly. We are now invested approximately 61% in SPXT, 22% in XCMP and 16% in VIX. The reasoning is the same as at date PP.

- **ERC**: At the PP date, the equal risk contribution is achieved when we are invested approximately 42% in the SPXT, 30% in the XCMP, 11% VIX and some small investments (between 1% and 3%) in the crypto assets. At the Tr date, the weight is shifted more towards the cryptos, with 29% in SPXT, 20% XCMP, nothing in the VIX, and some more realistic approximately equal positions in the crypto assets (now between 4% and 7%). When we look at our weight distribution and the risk contribution distribution of the equally weighted portfolio computed in 2.d, we see that the positively correlated assets seem to exhibit an inverse dependence between their risk contribution from 2.d and their weights in the optimal equal risk contribution portfolio. The higher the assets risk contribution, the smaller its weight in our portfolio, and vice-versa. This is however not the case for the VIX, as it has negative covariance with all other assets, and has one of the highest variances across all assets. This means that its high variance multiplied by its high weight at PP will balance out the negative contributions from the other assets, making its risk contribution equal with the other assets. At the Tr date, its weight is 0, indicating that the actual theoretical weight computed was negative, but it was clipped by our constraint. Running the experiment without restricting the weights to be positive confirmed our hypothesis.

- **EffBets**: At the PP date, the maximum number of effective bets is achieved while being invested 70% in XCMP, 17% in VIX and 13% in ADA. At the Tr date, we are now mostly invested in the XCMP at 90%, and at 5% in both BTC and the VIX. We found that the effective number of bets was 5.82 at the PP date (see figure 24) and 3.53 at the Tr date.

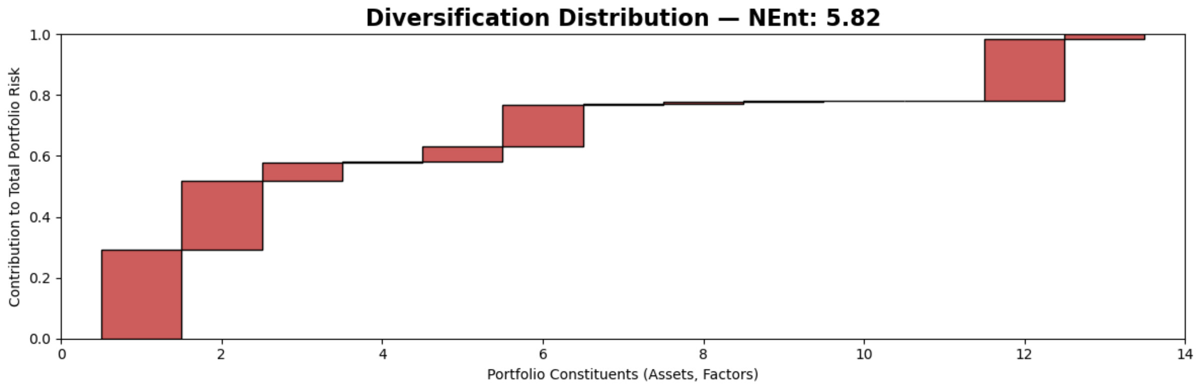


Figure 24: Diversification distribution of the maximum effective number of bets portfolio at the PP date

- **HRP**: At the PP date, the hierarchical risk parity portfolio is achieved by investing 61% in the SPXT and 34% in the XCMP, as well some very minor positions in the crypto assets and VIX (less than 2 %). At the Tr date, we are now slightly less invested in the traditional assets SPXT and XCMP (52% and 30% respectively), and a little bit more in the cryptos (between 1% and 5%). This follows from the fact that we follow a risk parity approach, so we will invest more into assets with lower volatility. So again we invest more heavily in the low volatile traditional index assets, and less into the more volatile cryptos. We can clearly see the change between PP and Tr, as the crypto assets became less volatile (they take up more wealth of our portfolio), and the traditional assets whose variance increased (they take up less wealth) (see covariance matrices in figures 11 and 12 for variance changes).

None of our optimization methods use any other constraints than the ones given in the problem set (namely $w \geq 0 \quad \text{and} \quad w^T 1 = 1$).
Now lets take a look at the pros and cons of holding each portfolio, namely discussing risk contributions and general comfortability in holding said portfolios.
Minimum variance:

- Pros: Minimizes the risk, and is supposed to be the "safest bet".
- Cons: Since it is not very diversified among assets, and heavily invests in one or two securities, the risk contributions are very unequal. Additionally, being savvy investors, we know we can get a portfolio with better return and the same risk as the minimum variance portfolio by finding the tangent portfolio (with highest Sharpe ratio) and combine it with the risk free asset. However one pitfall of the tangent portfolio is that it is based on expected return, which can be difficult to compute.

Equal risk contribution:

- Pros: The risk contribution for every position is equal across the whole portfolio. It is intuitive, as we invest more in less risky assets, and less into more risky assets.
- Cons: It has non-zero positions in every asset, and has an overlapping 'pie-charting' of risk, as was discussed in the guest lecture by Mister Du Plessis.

Maximum number of effective bets:

- Pros: Maximizes exposure to uncorrelated risk components (eigenvectors), and reduces exposure to dominant market factors. It does true diversification in terms of risk direction.
- Cons: It is not as intuitive, as it uses PCA and eigenvalue decomposition. Assets do not necessarily have an equal risk decomposition. Some of the positions also seem unrealistic (too small).

Hierarchical risk parity:

- Pros: Since we don't use any matrix inversion, it is robust to estimation errors. Risk contributions are more or less balanced across groups of similar assets.
- Cons: There is no guarantee it is optimal in any classical sense, since it is not based on any optimization, but is a rather heuristic approach.

4 Question 4

4.a

The objective here is to enrich the HRP optimization model of Lopez De Prado by using a new distance measure not based on the correlation in order to avoid the pitfalls mentionned in Embrechts, McNeil, Straumann (1999) [4] in Correlation : Pitfalls and Alternatives.
Indeed, correlation is very usefull in some cases. But the risks should have a joint multivariate normal distribution. That is the case in elliptical distributions. If we are not convinced that our risks respect this condition, we cannot use correlation freely and there are a lot of pitfalls (A correlation of zero doesn't mean independance of risks...). As mentionned in Embrechts, McNeil, Straumann (1999) in Correlation : Pitfalls and Alternatives, using rank correlation with the kendall definitions solves a lot of problems. Therefore, we enrich our HRP by 2 ways and we will compare them. Firstly, by computing the distance using the rank correlation with the Kendall's definition, secondly by using the first distance metric defined on page 17 of Embrechts et Al., [3] Correlations and dependency in risk management , properties and pitfalls (based on the work of Schweizer and Wolf 1981).

1/ Kendall's Tau rank correlation

The Kendall's tau coefficient between two variables X and Y is defined as (found in Embrechts et Al., Correlationsand dependency in risk management, properties and pitfalls):

$$\tau(X,Y) = \mathbb{P}((X_1 - X_2)(Y_1 - Y_2) > 0) - \mathbb{P}((X_1 - X_2)(Y_1 - Y_2) < 0)$$

We can estimate this tau by using the following formula

$$\tau = \frac{\#\text{ concordant pairs} - \#\text{ discordant pairs}}{\frac{1}{2}n(n-1)}$$

Where:

- A **concordant pair** $(x_i, y_i), (x_j, y_j)$ satisfies $(x_i - x_j)(y_i - y_j) > 0$,
- A **discordant pair** satisfies $(x_i - x_j)(y_i - y_j) < 0$,
- n is the number of observations.

Distance metric based on Kendall's Tau

To construct a distance matrix from Kendall's tau for use in HRP, we apply the following transformation:

$$d_{ij}^{(\tau)} = \sqrt{\frac{1}{2}\left(1 - \tau_{ij}\right)}$$

This yields a proper distance metric:

- $d_{ij} = 0$ when $\tau_{ij} = 1$ (perfect concordance),

- $d_{ij} = 1$ when $\tau_{ij} = -1$ (perfect discordance),
- $d_{ij} \in [0, 1]$, satisfying the properties of a metric.

2/ Using Schweizer and Wolf distance metric

The distance metric is defined as follows:

$$\delta(X, Y) = 12 \int_0^1 \int_0^1 |C(u, v) - uv)| du dv$$

C is the copula, which is simply the joint density function of X and Y, which we can estimate with our data (see Jupyter notebook 'Questions 2 + 3 + 4.ipynb' to see how we proceeded). We will also estimate the integral numerically (Again, see Jupyter notebook).

4.b

Let's begin with the enriched HRP that uses Kendall's tau:

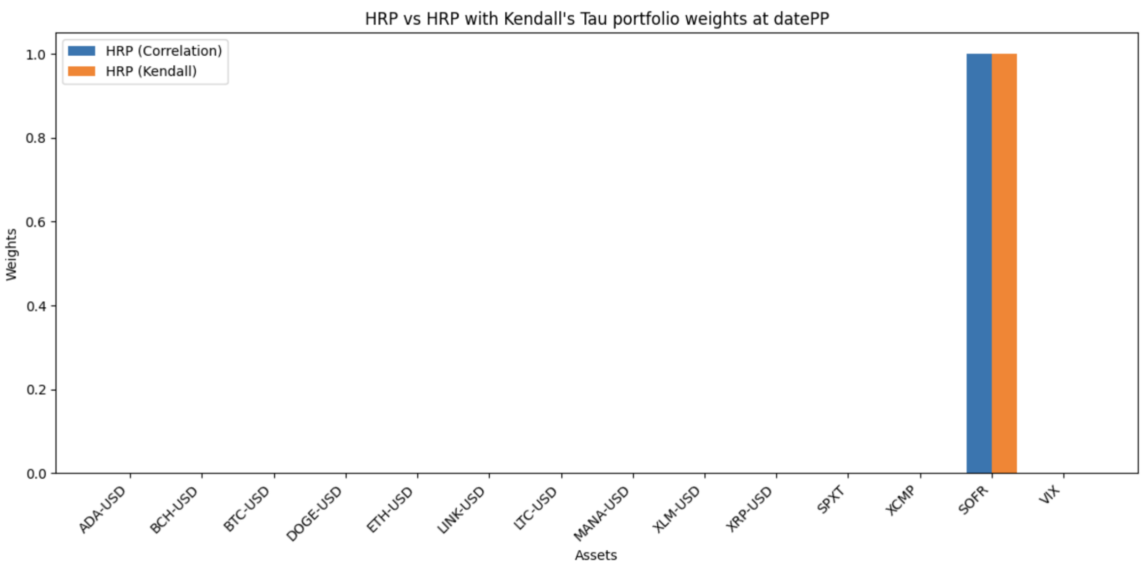


Figure 25: Portfolio weights at the PP date computed with HRP and 'revisited' HRP with rank correlation with Kendall definition.

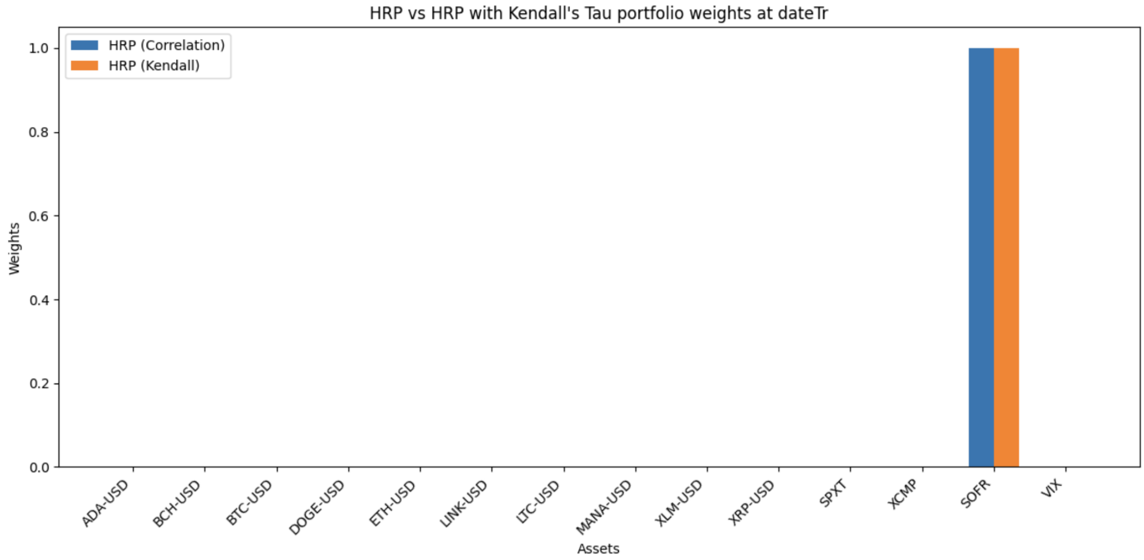


Figure 26: Portfolio weights at the Tr date computed with HRP and 'revisited' HRP with rank correlation with Kendall definition.

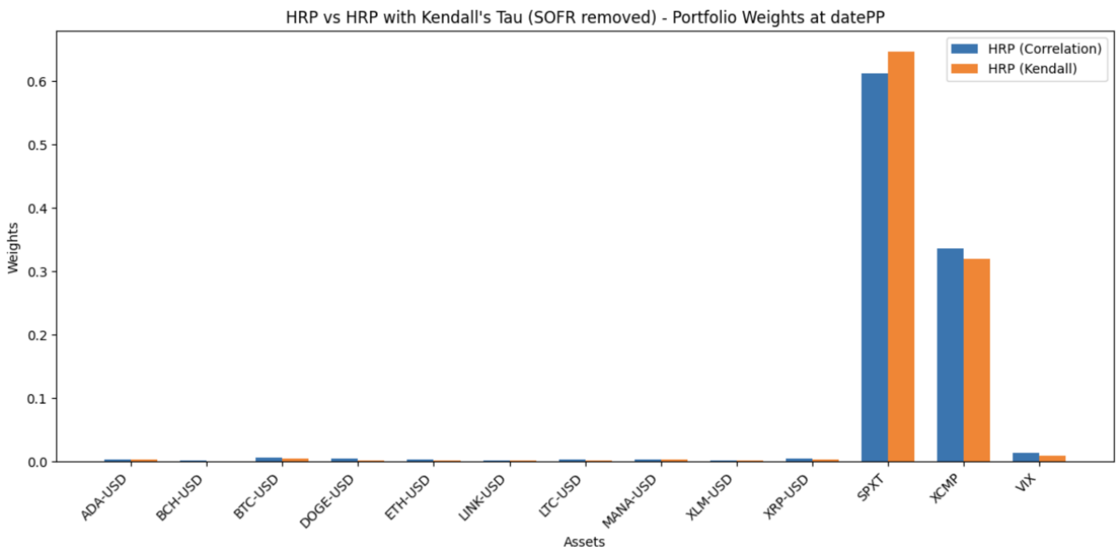


Figure 27: Portfolio weights at the PP date computed with HRP and 'revisited' HRP with rank correlation with Kendall definition, without SOFR.

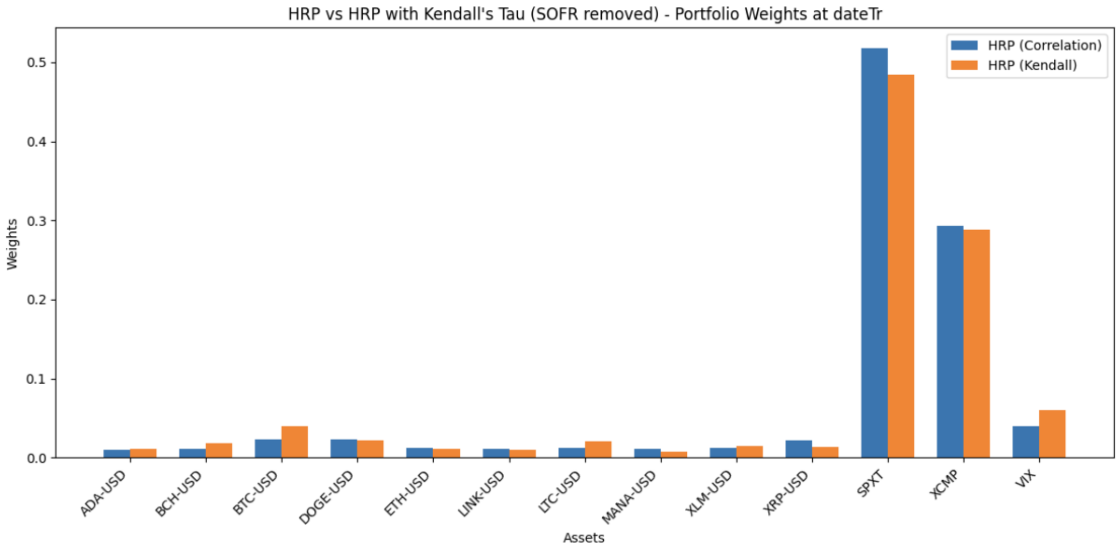


Figure 28: Portfolio weights at the Tr date computed with HRP and 'revisited' HRP with rank correlation with Kendall definition, without SOFR.

The HRP weights have been described in Question 3, here we want to compare them with new weights computed with the enriched HRP using rank correlation in the distance calculation (by using Kendall's definition).
Firstly, SOFR having such a little variance, both methods predict approximately a weight of 1 for SOFR on all dates. As discussed in Question 3, this makes sense but is not necessarily very interesting. Therefore we will remove the SOFR and consider it as the risk less asset.
Then, we can observe that the weights are pretty similar for the two models, the interpretations of the models aren't totally different and this is true for both dates (PP and Tr). However, we can see that the modifications on large weights (SPXT and XCMP) are less than 10% but the differences on small weights are more significant (for instance, the weight on BTC computed by HRP Kendall is approximately 40% higher than the one predicted by HRP). As mentioned in Q3, weights at the through date are more spread than weights at the previous peak, which means that smaller weights tend to be larger at the through date, leading to greater variation between the weights predicted by the two models.
As explained in Embrechts, McNeil, Straumann (1999) in Correlation : Pitfalls and Alternatives, the use of rank correlation is useful only if the assumption that the risks have a joint multivariate normal distribution is not satisfied. We noticed that the difference between weights predicted by the two models is higher at the Tr date. This means that at the PP, the risks have an almost absolutely joint multivariate normal distribution but at Tr this is not true anymore.
We observe an example where the use of correlation is a trap (even if it wasn't in the past), illustrating the importance of being aware of correlation pitfalls.
Now let's do the same with the second model, using the second distance metric based on Schweizer and Wolf:

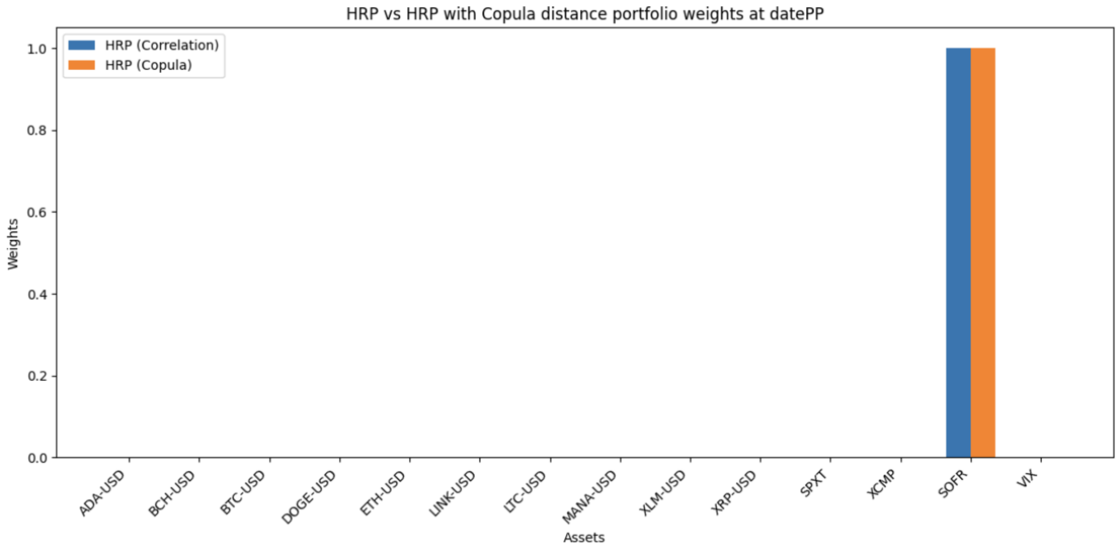


Figure 29: Portfolio weights at the PP date computed with HRP and 'revisited' HRP with copula distance.

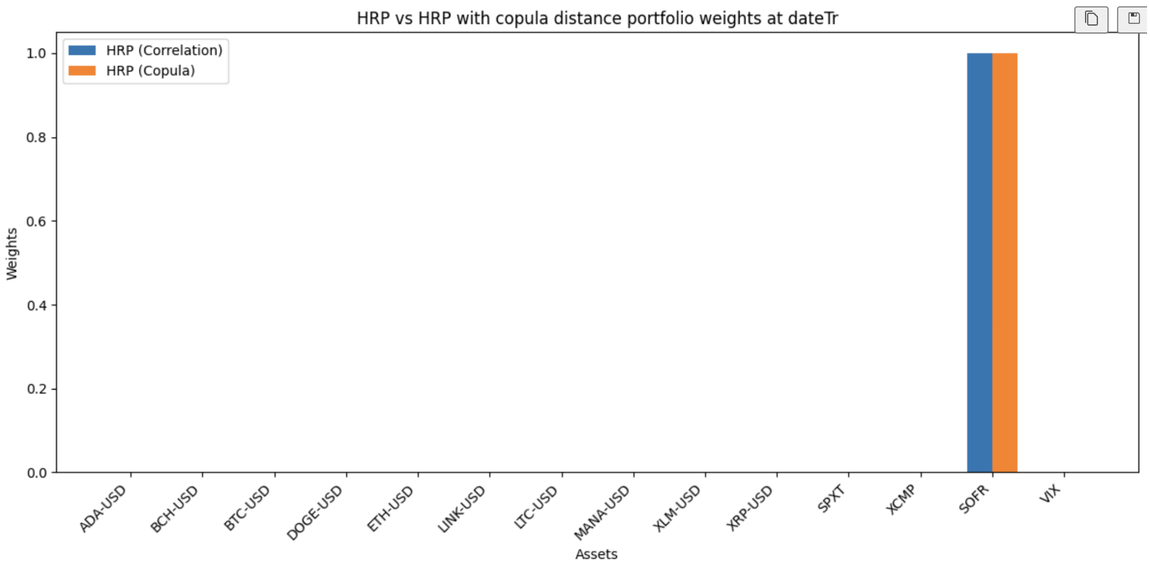


Figure 30: Portfolio weights at the Tr date computed with HRP and 'revisited' HRP with copula distance.

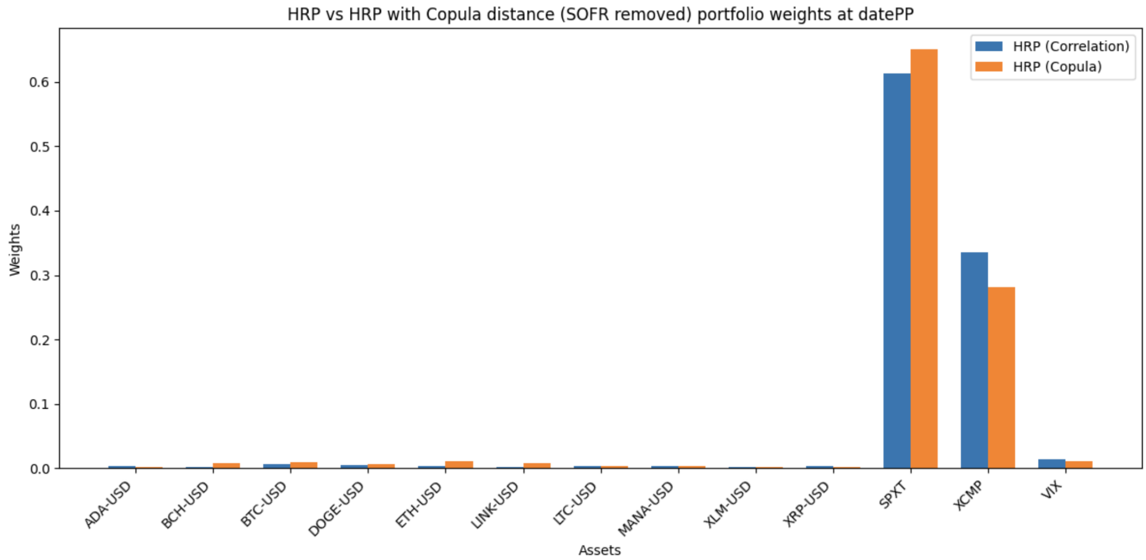


Figure 31: Portfolio weights at the PP date computed with HRP and 'revisited' HRP with copula distance, without SOFR.

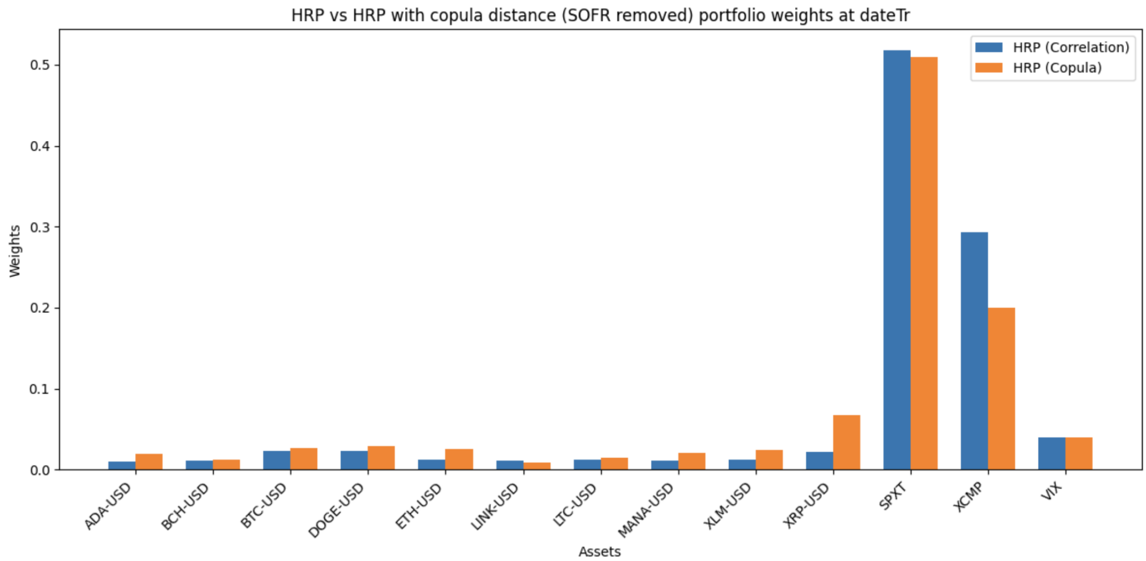


Figure 32: Portfolio weights at the Tr date computed with HRP and 'revisited' HRP with copula distance, without SOFR.

The interpretations are exactly the same as the ones we have done for the first enriched HRP (note that the two models have similar predictions on the weights, but there are still some differences, for example, the weight of XCMP at dateTr when the SOFR is removed). This seems logical, as our two ways to enrich the HRP are valid and explained in the project litterature. The similarity of the results suggests that our implementations of the methods are correct.

4.c

Time-Series Momentum (TSMOM), as presented in Moskowitz, Ooi, Pedersen (2010) [9], is a strategy that predicts an asset's future return based on its own past return trends.

If an asset has shown positive returns over a recent period, it is expected to continue to perform well in the near future, and vice versa for negative returns. Unlike cross-sectional momentum, TSMOM looks at each single asset in isolation. We form a signal for each asset whether it's trending up or down. If the asset had positive return over the past k days go long, otherwise go short. The signal is simply

$$s_{i,t}^{k} = \sum_{j=t-k}^{t} r_{i,j}$$

Since we are using log returns, this is simply the log return over the lookback period k. We generate binary trading signals for each asset based on whether its cumulative log return over the lookback period is positive or negative:

- +1 if the return is positive (go long),
- -1 if the return is negative (go short),
- 0 if the return is zero (neutral).

Then we implement the strategy where we combine the predictive power of time-series momentum and diversification discipline of rank correlation hierarchical risk parity. At fixed intervals (rebalance frequency) we select only the assets with active signals (long or short), estimate a robust covariance matrix using the clipped covariance matrix from Question 2 and apply HRPe (Copula) to allocate portfolio weights among the selected assets. The direction of each position is then adjusted based on the momentum signal, resulting in a long/short portfolio that is normalized. Finally, we compute daily portfolio returns over each holding period and aggregate these into a portfolio value series.

Note that the plotted graphs represent the value of the portfolio, given the value was 1 when we start

Figure 33: Portfolio value (starting at 1) calculated with time-series momentum and Copula-based Hierarchical Risk Parity diversification strategy

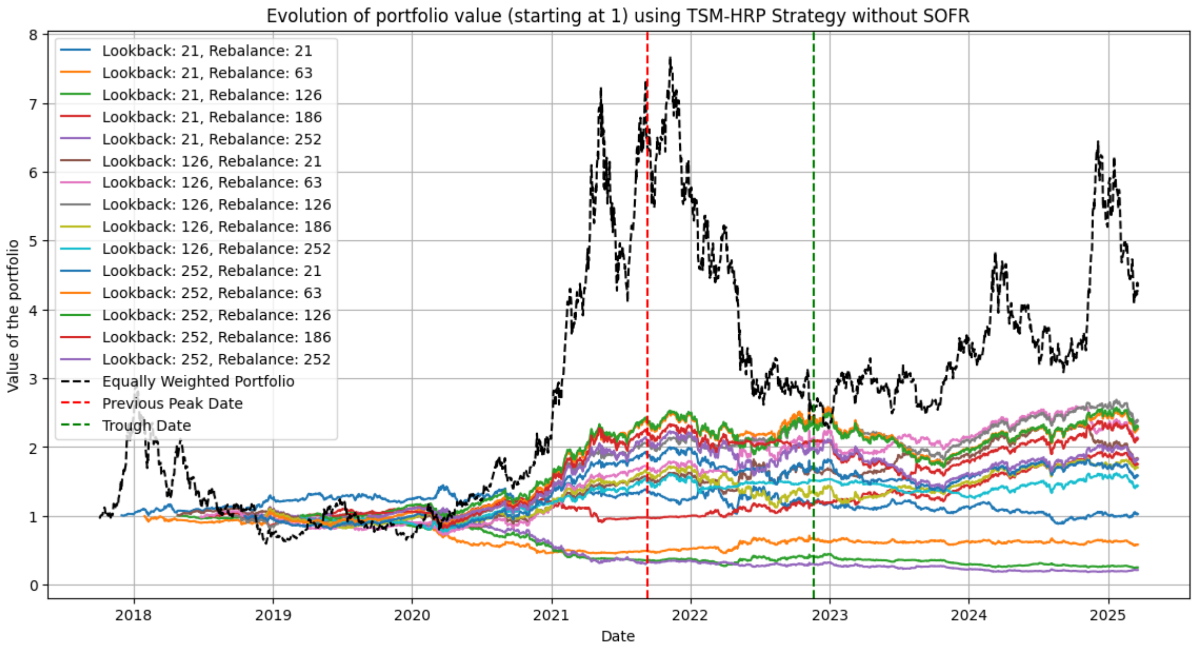


Figure 34: Portfolio value (starting at 1) calculated with time-series momentum and Copula-based Hierarchical Risk Parity diversification strategy, without SOFR

As seen before, including SOFR leads to dominant allocations in SOFR, due to its low volatility. This suggests the model becomes overly risk-averse when SOFR is included, which aligns with [2] observation that HRP can lead to conservative portfolios if nearly risk-free assets dominate. The figure shows extremely flat Portfolio values across all parameter configurations of the TSM-HRP strategy. In contrast, the equally weighted portfolio (black dashed line) shows high portfolio value and volatility.
On the other hand, figure 34 shows that the strategy now allocates across active, volatile assets, demonstrating the true performance potential of TSMOM when nearly riskless assets (SOFR) are removed. Removing SOFR restores meaningful momentum differentials across assets, which enhanced HRP can effectively exploit.

We have compared different lookback windows and rebalancing frequencies to calculate momentum and rebalance our strategy respectively. We observe that shorter windows respond more quickly to recent price movements and can better capture emerging trends. However, this increased reactivity may lead to overfitting short-term noise, reducing signal robustness. In contrast, longer windows such as the 12-month version offer more stable and smoother signals by averaging over a longer period, but they may lag when markets reverse sharply. Regarding the rebalancing frequency, we observe that too frequent rebalancing introduces high turnover, which inevitably leads to the investor paying more transaction fees. On the other hand, too infrequent rebalancing may miss key inflection points in market trends, resulting in delayed adjustments.

The comparison with the equally weighted portfolio shows that we do not beat it. This is due the nature of the time frame we have selected. We can see that over the whole period, a lot of the assets increase their value tremendously. This of course drives the return of the equally weighted portfolio much higher than what we obtain by using our strategy, which will invest more heavily in the more stable assets since it is based on risk parity. The fact, however, that we do not beat the EW portfolio is not necessarily a bad thing. The EW portfolio holds equal positions in all assets, meaning it is exposed to the very volatile assets a lot. We clearly see that during total down trends (between PP and Tr), the EW portfolio takes a massive hit (-53% as discussed in question 2), while our HRP portfolios are able to navigate the crisis better, and approximately retain their value, or even increase it slightly in some cases. Since HRP gives us portfolios less exposed to the very volatile assets, they are much stabler. But we should still be careful, as with the wrong parameters we end up losing money over the time period. This is the main problem with our strategy: we cannot guarantee optimal performance, as a slight tweak in the parameters might lead to better results. This is extremely dependant on the specific time frame we use. Choose a high rebalancing frequency and the model might miss the optimal rebalancing moment by a few days. Choose a low rebalancing frequency and the model might persist in using the wrong allocations for too long, leading to performance loss. All in all, what we can see from our graphs is that a low lookback period is generally bad, and that extreme rebalancing frequencies generally perform worse than moderate values. But again, this highly depends on the data chosen. Below is an example where we started on 2018-01-08 instead of the first date available.

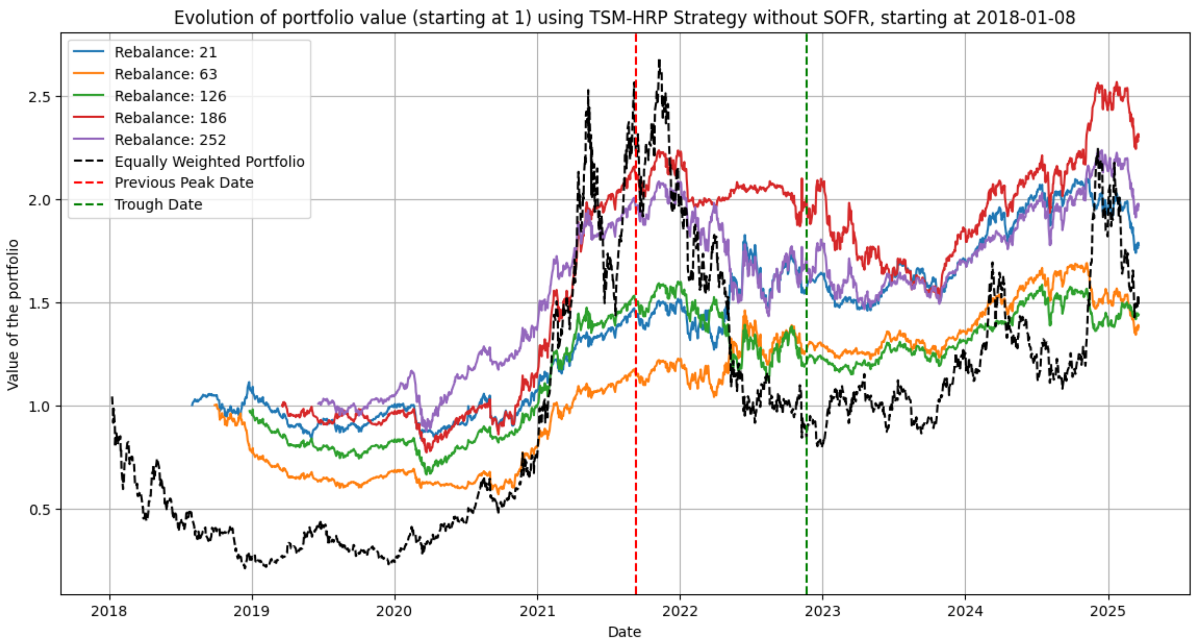


Figure 35: Portfolio value (starting at 1) calculated with time-series momentum and Copula-based Hierarchical Risk Parity diversification strategy when we change the starting date to 2018-01-08

We clearly see that our strategies now look better than the equally weighted portfolio returns, which reinforces the point we made about the importance of the time frame used.

In conclusion, we obtain a strategy that seems to be reliable. It manages to navigate crises well, while also taking advantage of price increases. Since we are managing risk and investing according to risk parity, our portfolio returns are relatively stable but we still manage to capture good returns due to cryptocurrencies (which are notoriously volatile). All in all, this portfolio construction seems adequate for moderately risk averse investors, who seek exposure to the crypto market.

References

[1] Joël Bun, Jean-Philippe Bouchaud, and Marc Potters. "Cleaning correlation matrices". In: *Risk.net* (Apr. 2016). Reprinted from Risk. URL: `https://www.risk.net`.

[2] Antonello Cirulli, Michal Kobak, and Urban Ulrych. *Portfolio Construction with Hierarchical Momentum.* 2024.

[3] Paul Embrechts, Alexander McNeil, and Daniel Straumann. "Correlation and Dependence in Risk Management: Properties and Pitfalls". In: *Risk Management: Value at Risk and Beyond* (1999).

[4] Paul Embrechts, Alexander McNeil, and Daniel Straumann. "Correlation: Pitfalls and Alternatives". In: *Department of Mathematics, ETH Zurich* (1999). Unpublished manuscript. URL: `https://www.math.ethz.ch/`.

[5] G. Giecold and L. Ouaknin. *pyRMT.* `https://github.com/GGiecold/pyRMT`. 2017.

[6] Marcos López de Prado. "Building Diversified Portfolios that Outperform Out-of-Sample". In: *The Journal of Portfolio Management* 42.4 (2016), pp. 59–69.

[7] Attilio Meucci. *Linear vs. Compounded Returns: Common Pitfalls in Portfolio Management.* 2009.

[8] Attilio Meucci. *Managing Diversification.* Available at SSRN: `https://ssrn.com/abstract=1358533`. 2009.

[9] Tobias J Moskowitz, Yao Hua Ooi, and Lasse Heje Pedersen. "Time Series Momentum". In: *Journal of Financial Economics* 104.2 (2010), pp. 228–250.