

# Boltzmann's Entropy and Time's Arrow

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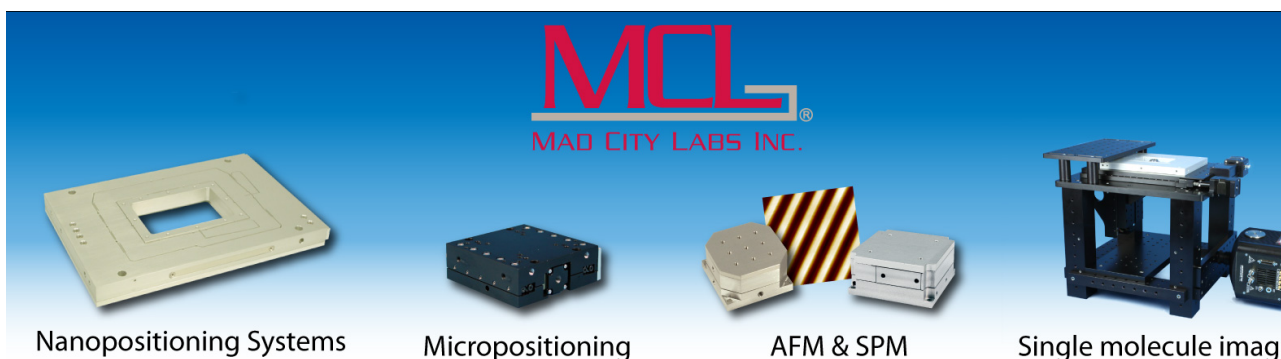
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# BOLTZMANN'S ENTROPY AND TIME'S ARROW

Given that microscopic physical laws are reversible, why do all macroscopic events have a preferred time direction? Boltzmann's thoughts on this question have withstood the test of time.

Joel L. Lebowitz

Given the success of Ludwig Boltzmann's statistical approach in explaining the observed irreversible behavior of macroscopic systems in a manner consistent with their reversible microscopic dynamics, it is quite surprising that there is still so much confusion about the problem of irreversibility. (See figure 1.) I attribute this confusion to the originality of Boltzmann's ideas: It made them difficult for some of his contemporaries to grasp. The controversies generated by the misunderstandings of Ernst Zermelo and others have been perpetuated by various authors. There is really no excuse for this, considering the clarity of Boltzmann's later writings.<sup>1</sup> Since next year, 1994, is the 150th anniversary of Boltzmann's birth, this is a fitting moment to review his ideas on the arrow of time. In Erwin Schrödinger's words, "Boltzmann's ideas really give an understanding" of the origin of macroscopic behavior. All claims of inconsistencies that I know of are, in my opinion, wrong; I see no need for alternate explanations. For further reading I highly recommend Boltzmann's works as well as references 2-7. (See also *PHYSICS TODAY*, January 1992, page 44, for a marvelous description by Boltzmann of his visit to California in 1906.)

Boltzmann's statistical theory of time-asymmetric, irreversible nonequilibrium behavior assigns to *each microscopic state* of a macroscopic system, be it solid, liquid, gas or otherwise, a number  $S_B$ , the "Boltzmann entropy" of that state. This entropy agrees (up to terms that are negligible for a large system) with the macroscopic thermodynamic entropy of Rudolf Clausius,  $S_{eq}$ , *when the system is in equilibrium*. It then also coincides with the Gibbs entropy  $S_G$ , which is defined not for an individual microstate but for a statistical ensemble (a collection of independent systems, all with the same Hamiltonian, distributed in different microscopic states consistent with some specified macroscopic constraints). However, unlike  $S_G$ , which does not change in time even for ensembles

describing (isolated) systems not in equilibrium,  $S_B$  typically increases in a way that explains the evolution toward equilibrium of such systems.

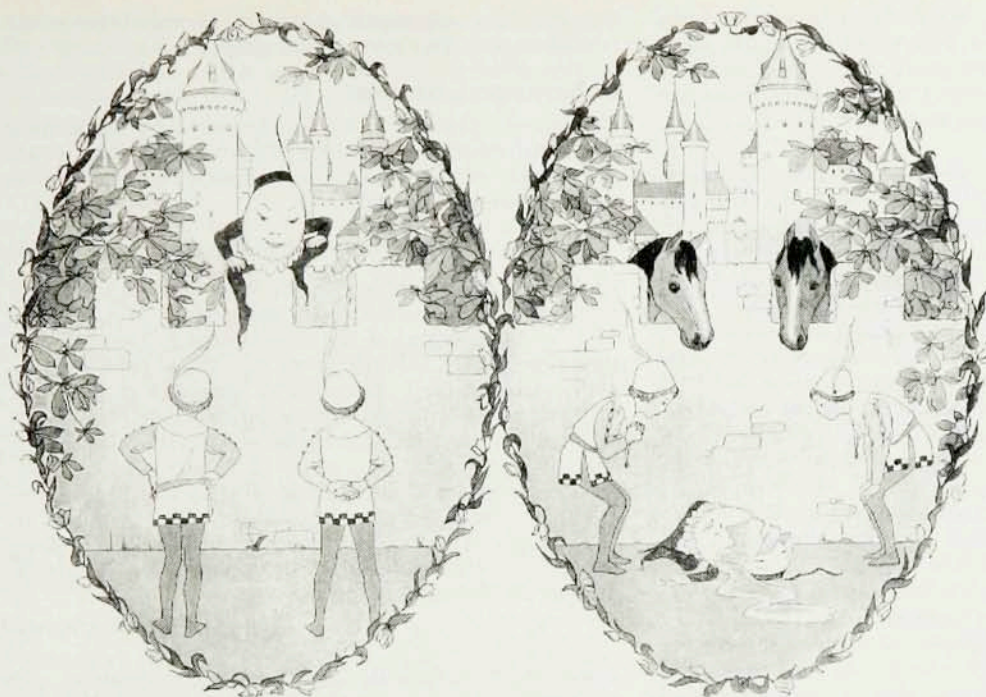
The superiority of  $S_B$  over  $S_G$  in this regard comes from the fact that unlike  $S_G$ ,  $S_B$  captures the separation between microscopic and macroscopic scales. This separation of scales, inherent in the very large number of degrees of freedom within any macroscopic system, is exactly what enables us to predict the evolution "typical" of a particular macroscopic system—where, after all, we actually observe irreversible behavior.<sup>6-8</sup> The "typicality" is very robust: We can expect to see unusual events, such as gases unmixing themselves, only if we wait for times inconceivably long compared with the age of the universe. In addition, the *essential* features of the evolution do not depend on specific dynamical properties such as the positivity of Lyapunov exponents, ergodicity or mixing (see my article with Oliver Penrose in *PHYSICS TODAY*, February 1973, page 23), nor on assumptions about the distribution of microstates, such as "equal *a priori* probabilities," being strictly satisfied. Particular ensembles with these properties, commonly used in statistical mechanics, are no more than mathematical tools for describing the macroscopic behavior of typical individual systems.

Macroscopic evolution is quantitatively described in many cases by time-asymmetric equations of a hydrodynamic variety, such as the diffusion equation. These equations are derived (rigorously, in some cases) from the (reversible) microscopic dynamics of a system by utilization of the large ratio of macroscopic to microscopic scales.<sup>7,8</sup> They describe the time-asymmetric behavior of individual macroscopic systems, not just that of ensemble averages. This behavior should be distinguished from the chaotic but *time-symmetric* behavior of systems with a few degrees of freedom, such as two hard spheres in a box. Given a sequence of photographs of a system of the latter sort, there is no way to order them in time. To call such behavior "irreversible" is, to say the least, confusing.

On the other hand, instabilities induced by chaotic microscopic behavior are responsible for the detailed

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**Irreversibility of macroscopic events:** "All the king's horses and all the king's men / Couldn't put Humpty together again." But why not? (Illustration by H. Willebeck Le Mair, from A. Moffat, *Our Old Nursery Rhymes*, Augener, London, 1911.) **Figure 1**

features of the macroscopic evolution. An example is the Lorentz gas, a macroscopically large number of noninteracting particles moving in a plane among a fixed, periodic array of convex scatterers, arranged so that a particle can travel no more than a specified distance between collisions. The chaotic nature of the (reversible) microscopic dynamics allows a simple deterministic description, via a diffusion equation, of the macroscopic density profile of this system to exist. This description is achieved in the hydrodynamic scaling limit, in which the ratio of macroscopic to microscopic scales goes to infinity.<sup>6,8</sup> Such a deterministic description also holds to a high accuracy when this ratio is finite but very large; it is, however, clearly impossible when the system contains only one or a few particles. I use this example, despite its highly idealized nature, because here all the mathematical  $i$ 's have been dotted. The Lorentz gas shows *ipso facto*, in a way that should convince even (as Mark Kac put it) an "unreasonable" person, not only that there is no conflict between reversible microscopic and irreversible macroscopic behavior but also that the latter follows from the former—in complete accord with Boltzmann's ideas.

Boltzmann's analysis was of course done in terms of classical, Newtonian mechanics, and I shall also use that framework. I believe, however, that his basic ideas carry over to quantum mechanics without essential modification. I do not agree with the viewpoint that in the quantum universe, measurement is a new source of irreversibility; rather, I believe that real measurements on quantum systems are time asymmetric because they involve, of necessity, systems, such as measuring apparatus, with a very large number of degrees of freedom. Quantum irreversibility should and can be understood using Boltzmann's ideas. I will discuss this briefly later.

I will, however, in this article completely ignore relativity, special or general. The phenomenon we wish to explain, namely the time-asymmetric behavior of spatially localized macroscopic objects, is for all practical purposes the same in the real, relativistic universe as in a model, nonrelativistic one. For similar reasons I will

also ignore the violation of time invariance in the weak interactions and focus on idealized versions of the problem of irreversibility, in the simplest contexts. The reasoning and conclusions will be the same whenever we deal with systems containing a great many atoms or molecules.

### Microscopic reversibility

Consider an isolated macroscopic system evolving in time, as in the (macroscopic) snapshots in figure 2. The blue color represents, for example, ink; the white, a colorless fluid; and the four frames are taken at different times during the undisturbed evolution of the system. The problem is to identify the time order in which the sequence of snapshots was taken.

The obvious answer, based on experience, is that time increases from figure 2a to 2b to 2c to 2d—any other order is clearly absurd. Now it would be very simple and nice if this answer followed directly from the microscopic laws of nature. But that is not the case: The microscopic laws, as we know them, tell a different story. If the sequence going from 2a to 2d is permissible, so is the one going from 2d to 2a.

This is easiest seen in classical mechanics, where the complete microscopic state of a system of  $N$  particles is represented by a point  $X = (\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, \dots, \mathbf{r}_N, \mathbf{v}_N)$  in its phase space  $\Gamma$ ,  $\mathbf{r}_i$  and  $\mathbf{v}_i$  being the position and velocity of the  $i$ th particle. The snapshots in figure 2 do not, however, specify the microstate  $X$  of the system; rather, each snapshot describes a macrostate, which we denote by  $M \equiv M(X)$ . To each macrostate  $M$  there corresponds a very large set of microstates making up a region  $\Gamma_M$  in the total phase space  $\Gamma$ . To specify the macrostate  $M$  we could, say, divide the 1-liter box in figure 2 into a billion little cubes and state, within some tolerance, the number of particles of each type in each cube. To completely specify  $M$  so that its further evolution can be predicted, we would also need to state the total energy of the system and any other macroscopically relevant constants of the motion, also within some tolerance. While this specifi-



cation of the macroscopic state clearly contains some arbitrariness, this need not concern us too much, since all the statements we are going to make about the evolution of  $M$  are independent of its precise definition as long as there is a large separation between the macro- and microscales.

Let us consider now the time evolution of microstates underlying that of the macrostate  $M$ . The evolution is governed by Hamiltonian dynamics, which connects a microstate  $X(t_0)$  at some time  $t_0$  to the microstate  $X(t)$  at any other time  $t$ . Let  $X(t_0)$  and  $X(t_0 + \tau)$ , with  $\tau$  positive, be two such microstates. Reversing (physically or mathematically) all velocities at time  $t_0 + \tau$ , we obtain a new microstate. If we now follow the evolution for another interval  $\tau$  we find that the new microstate at time  $t_0 + 2\tau$  is just the state  $X(t_0)$  with all velocities reversed. We shall call the microstate obtained from  $X$  by velocity reversal  $RX$ ; that is,  $RX = (\mathbf{r}_1, -\mathbf{v}_1, \mathbf{r}_2, -\mathbf{v}_2, \dots, \mathbf{r}_N, -\mathbf{v}_N)$ .

Returning now to the snapshots in figure 2, it is clear that they would not change if we reversed the velocities of all the particles; hence if  $X$  belongs to  $\Gamma_M$ , then so does  $RX$ . Now we see the problem with our definite assignment of a time order to the snapshots in the figure: Going from a macrostate  $M_a$  at time  $t_a$  to another macrostate  $M_b$  at time  $t_b$  ( $t_b = t_a + \tau, \tau > 0$ ) means that there is a microstate  $X$  in  $\Gamma_{M_a}$  that gives rise to a microstate  $Y$  in  $\Gamma_{M_b}$  (at time  $t_b$ ); but then  $RY$  is also in  $\Gamma_{M_b}$ , and following the evolution of  $RY$  for a time  $\tau$  would produce the state  $RX$ —which would then be in  $\Gamma_{M_a}$ . Hence the snapshots depicting  $M_a, M_b, M_c$  and  $M_d$  in figure 2 could, as far as the laws of nature go, correspond to a sequence of times going in either direction.

So our judgment of the time order in figure 2 is not based on the dynamical laws of evolution alone; these permit either order. Rather, it is based on experience:

One direction is common and easily arranged; the other is never seen. *But why should this be so?*

## Boltzmann's answer

Boltzmann (depicted in figure 3) starts by associating with each macroscopic state  $M$ —and thus with every microscopic state  $X$  in  $\Gamma_M$ —an entropy, known now as the Boltzmann entropy,

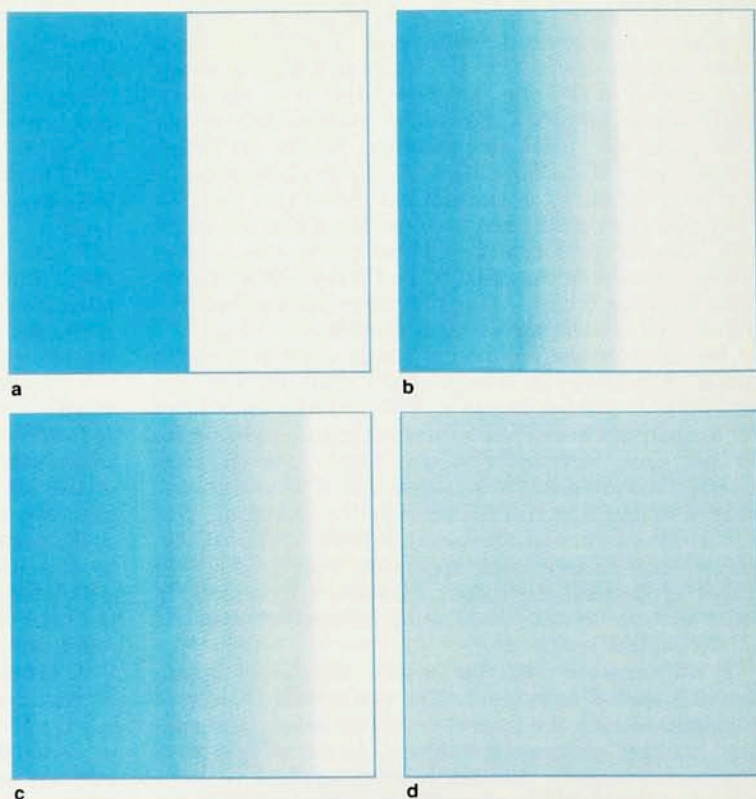
$$S_B(M(X)) = k \log |\Gamma_{M(X)}| \quad (1)$$

where  $k$  is Boltzmann's constant and  $|\Gamma_M|$  is the phase space volume associated with macrostate  $M$ . That is,  $|\Gamma_M|$  is the integral of the time-invariant Liouville volume element  $\prod_{i=1}^N d^3r_i d^3v_i$  over  $\Gamma_M$ . ( $S_B$  is defined up to additive constants.) Boltzmann's stroke of genius was, first, to make a direct connection between this microscopically defined function  $S_B(M)$  and the thermodynamic entropy of Clausius,  $S_{eq}$ , which is a macroscopically defined, operationally measurable (also up to additive constants), extensive property of macroscopic systems in equilibrium. For a system in equilibrium having a given energy  $E$  (within some tolerance), volume  $V$  and particle number  $N$ , Boltzmann showed that

$$S_{eq}(E, V, N) = S_B(M_{eq}) \quad (2)$$

where  $M_{eq}(E, V, N)$  is the corresponding macrostate. By the symbol " $\approx$ " we mean that for large  $N$ , such that the system is really macroscopic, the equality holds up to negligible terms when both sides of equation 2 are divided by  $N$  and the additive constant is suitably fixed. We require here that the size of the cells used to define  $M_{eq}$ , that is, the macroscale, be very large compared with the microscale.

Boltzmann's second contribution was to use equation 1 to define entropy for nonequilibrium systems and thus identify increases in Clausius (macroscopic) entropy with



How would you order this sequence of "snapshots" in time? Each represents a macroscopic state of a system containing, for example, two fluids. **Figure 2**



increases in the volume of the phase space region  $\Gamma_{M(X)}$ . This identification explains in a natural way the observation, embodied in the second law of thermodynamics, that when a constraint is lifted from an isolated macroscopic system, it evolves toward a state with greater entropy. To see how the explanation works, consider what happens when a wall dividing the box in figure 2 is removed at time  $t_a$ . The phase space volume available to the system now becomes fantastically enlarged; for 1 mole of fluid in a 1-liter container the volume ratio of the unconstrained region to the constrained one is of order  $2^N$  or  $10^{10^{23}}$ . For the system in figure 2 this corresponds roughly to the ratio  $|\Gamma_{M_d}|/|\Gamma_{M_a}|$ . We can then expect that when the constraint is removed the dynamical motion of the phase point  $X$  will with very high "probability" move into the newly available regions of phase space, for which  $|\Gamma_M|$  is large. This will continue until  $X(t)$  reaches  $\Gamma_{M_{eq}}$ . After that time we can expect to see only small fluctuations from equilibrium unless we wait for times that are much larger than the age of our universe. (Such times correspond roughly to the Poincaré recurrence time of an isolated macroscopic system.)

## Notions of probability

Boltzmann's analysis implies the existence of a relation between phase space volume and probability. In particular, his explanation of the second law depends upon identifying a small fraction of the phase space volume with small probability. This is in the spirit of (but does not require) the assumption that an isolated "aged" macroscopic system should be found in different macroscopic states  $M$  for fractions of time that equal the ratio of  $|\Gamma_M|$  to the total phase space volume  $|\Gamma|$  having the same energy. Unless there are reasons to the contrary (such as extra additive constants of the motion), the latter statement, a mild form of Boltzmann's ergodic hypothesis, seems very plausible for all macroscopic systems. Its application to "small fluctuations" from equilibrium is consistent with observations.

Thus not only did Boltzmann's great insights give a microscopic interpretation of the mysterious thermodynamic entropy of Clausius; they also gave a natural generalization of entropy to nonequilibrium macrostates  $M$ , and with it an explanation of the second law of thermodynamics—the formal expression of the time-asymmetric evolution of macroscopic states occurring in nature. In particular, Boltzmann realized that for a macroscopic system the fraction of microstates for which the evolution leads to macrostates with larger  $S_B$  is so close to 1, in terms of their phase space volume, that such behavior is what should "always" happen. In mathematical language, we say this behavior is "typical."

Boltzmann's ideas are, as David Ruelle<sup>5</sup> says, at the same time simple and rather subtle. They introduce into the "laws of nature" notions of probability, which, certainly in Boltzmann's time, were quite alien to the scientific outlook. Physical laws were supposed to hold without any exceptions, not just "almost" always, and indeed no exceptions to the second law were (or are) known; nor should we expect any, as Richard Feynman<sup>2</sup> rather conservatively says, "in a million years."

## Initial conditions

Once we accept Boltzmann's explanation of why macroscopic systems evolve in a manner that makes  $S_B$  increase with time, there remains the nagging problem (of which Boltzmann was well aware) of what we mean by "with time." Since the microscopic dynamical laws are symmetric, the two directions of the time variable are *a priori*



Boltzmann lecturing, by a contemporary Viennese cartoonist. (Courtesy of the University of Vienna.) **Figure 3**

equivalent and thus must remain so *a posteriori*.<sup>9</sup> Consider a system with a nonuniform macroscopic density profile, such as  $M_b$ , shown in figure 2b. If its microstate at time  $t_b$  is typical of  $\Gamma_{M_b}$ , then almost surely its macrostate at both times  $t_b + \tau$  and  $t_b - \tau$  would be like  $M_c$ . This is inevitable: Since the phase space volume corresponding to  $M_b$  at some time  $t_b$  gives equal weight to microstates  $X$  and  $RX$ , it must make the same prediction for  $t_b - \tau$  as for  $t_b + \tau$ . Yet experience shows that the assumption of typicality at time  $t_b$  will give the correct behavior only for times  $t > t_b$  and not for times  $t < t_b$ . In particular, given just  $M_b$  and  $M_c$ , we have no hesitation in ordering  $M_b$  before  $M_c$ .

If we think further about our ordering of  $M_b$  and  $M_c$ , it seems to derive from our assumption that  $M_b$  is itself so unlikely that it must have evolved from an initial state of even lower entropy like  $M_a$ . From such an initial state, which can be readily created by an experimentalist having a microstate typical of  $M_a$ , we get a monotonic be-



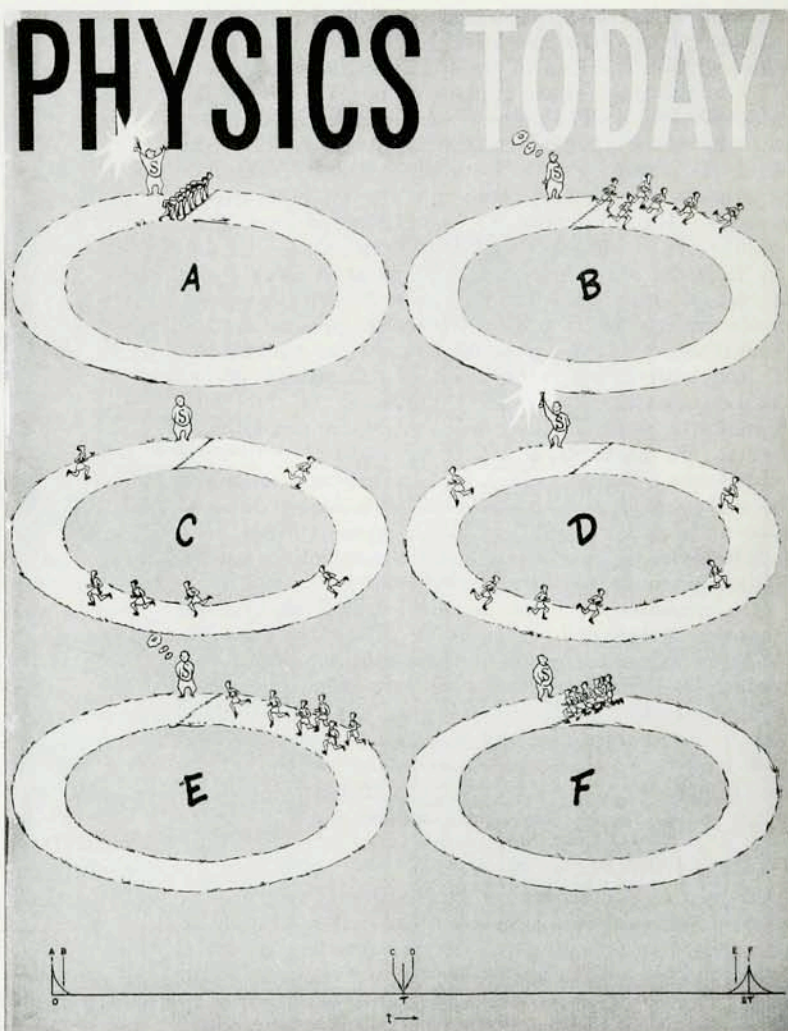
havior of  $S_B$  with the time ordering  $M_a, M_b, M_c$  and  $M_d$ . If, by contrast, the system in figure 2 had been completely isolated for a very long time compared with its hydrodynamic relaxation time, then we would expect to see it in its equilibrium state  $M_d$ . Presented with the four pictures, we would (in this very, very unlikely case) have no basis for assigning an order to them; microscopic reversibility assures that fluctuations from equilibrium are typically symmetric about times at which there is a local minimum of  $S_B$ . In the absence of any knowledge about the history of the system before and after the sequence, we use our experience to conclude that the low-entropy state  $M_a$  must have been an initial prepared state. In the words of Roger Penrose,<sup>4</sup> "The time-asymmetry comes merely from the fact that the system has been *started off* in a very special (i.e., low-entropy) state, and having so started the system, we have watched it evolve in the *future* direction." The point is that the microstate corresponding to  $M_b$  coming from  $M_a$  must be *atypical* in some respects of points in  $\Gamma_{M_b}$ . This is because, if we consider all microstates in  $M_b$  at time  $t_b$  that were in  $M_a$  at time  $t_a$ , then by Liouville's theorem, the set of all such phase points has a volume  $\Gamma_{M_a}$  that is much smaller than  $\Gamma_{M_b}$ .

### Origin of low-entropy states

The creation of low-entropy initial states poses no problem in laboratory situations such as the one depicted in figure 2. Laboratory systems are prepared in states of

low Boltzmann entropy by experimentalists who are themselves in low-entropy states. Like other living beings, they are born in such states and maintained there by eating low-entropy foods, which in turn are produced by plants using low-entropy radiation coming from the Sun, and so on.<sup>4</sup> In addition these experimentalists are able to prepare systems in particular macrostates with low values of  $S_B(M)$ , like our state  $M_a$ . Of course the total entropy  $S_B$ , including the entropy of the experimentalists and that of their instruments, must always increase: There are no Maxwell demons. But what are the origins of these complex creatures the experimentalists? Or what about events in which there is no human participation—for example, if instead of figure 2 we are given snapshots of a meteor and the Moon before and after their collision? Surely the time direction is just as obvious.

Trying to answer this question along the Boltzmann chain of reasoning, we are led more or less inevitably to cosmological considerations of an initial "state of the universe" having a very small Boltzmann entropy. That is, the universe is pictured to be born in an initial macrostate  $M_0$  for which  $|\Gamma_{M_0}|$  is a very small fraction of the "total available" phase space volume. Roger Penrose takes that initial state, the macrostate of the universe just after the Big Bang, to be one in which the energy density is approximately spatially uniform. He then estimates that if  $M_f$  is the macrostate of the final "Big Crunch," having a phase space volume of  $|\Gamma_{M_f}|$ ,



**Reversing time.** PHYSICS TODAY cover from November 1953 shows athletes on a racetrack. At the first gunshot, they start running; at the second, they reverse and run back, ending up again in a line. The drawing, by Kay Kaszas, refers to an article by Erwin L. Hahn on the spin echo effect on page 4 of that issue. **Figure 4**



$S = k \log W$  is Boltzmann's epitaph.  $W$  is what we call  $|\Gamma_M|$  (see equation 1). Boltzmann killed himself on 5 September 1906. Some speculate (but I rather doubt) that it was because of the hostility his ideas on statistical mechanics met with from positivists like Ernst Mach and formalists like Ernst Zermelo. Boltzmann is buried in Vienna, the city of his birth in 1844. (Courtesy of the University of Vienna.) **Figure 5**

then  $|\Gamma_{M_0}| / |\Gamma_{M_f}| \approx 10^{-10^{23}}$ .

The high value of  $|\Gamma_{M_f}|$  compared with  $|\Gamma_{M_0}|$  comes from the vast amount of phase space corresponding to a universe collapsed into a black hole in the Big Crunch. I do not know whether these initial and final states are reasonable, but in any case one has to agree with Feynman's statement<sup>2</sup> that "it is necessary to add to the physical laws the hypothesis that in the past the universe was more ordered, in the technical sense, than it is today . . . to make an understanding of the irreversibility." "Technical sense" clearly refers to the initial state  $M_0$  having a smaller  $S_B$  than  $M_f$ . Once we accept such an initial macrostate  $M_0$ , then the initial microstate can be assumed to be typical of  $\Gamma_{M_0}$ . We can then apply our statistical reasoning to the further evolution of this initial state, despite the fact that it was a very unlikely one, and use phase-space-volume arguments to predict the future behavior of macroscopic systems—but not to determine the past.

## Velocity reversal

The microscopic physical laws don't preclude having a microstate  $X$  for which  $S_B(M)$  would be decreasing as  $t$  increased. An experimentalist could, *in principle*, reverse all velocities of the system in figure 2b (see also figure 4) and then watch the system unmix itself. In practice, however, it seems impossible: Even if he or she managed to perfectly reverse the velocities, as was done (imperfectly) in spin echo experiments,<sup>10</sup> we would not expect to see the system in figure 2 go from  $M_b$  to  $M_a$ . This would require that *both* the velocity reversal *and* the system isolation be absolutely perfect. We would need such perfection now but did not need it before because while the macroscopic behavior of a system with microstate  $Y$  in the state  $M_b$ —that came from a microstate  $X$  typical of  $\Gamma_{M_a}$ —is stable against perturbations in its future evolution, it is very unstable as far as its past is concerned. That is the same as saying that the forward evolution of  $RY$  is unstable to perturbations. This difference in stability is a consequence of the fact that almost any perturbation of a microstate in  $\Gamma_{M_b}$  will tend to make it more typical of its macrostate  $M_b$ . Perturbation will thus not interfere with behavior typical of  $\Gamma_{M_b}$ . But the forward evolution of the unperturbed  $RY$  is by construction heading toward a smaller phase space volume and is thus untypical of  $\Gamma_{M_b}$ . Such an evolution therefore requires "perfect aiming" and will be derailed even by small imprecisions in the reversal or by tiny outside influences. This situation is analogous to pinball-machine-type puzzles where one is supposed to get a small metal ball into a particular small region. You have to do things just right to get it in, but almost anything you do gets it out into larger regions. For the macroscopic systems we are considering, the disparity between the sizes of the comparable regions of the phase space is unimaginably larger.

The behavior of all macroscopic systems, whether ideally isolated or realistically subject to unavoidable interactions with the outside world, can therefore be confidently predicted to be in accordance with the second law. Even if we deliberately create, say by velocity



reversal, a microstate that is very atypical of its macrostate, we will find that after a *very short* time in which  $S_B$  decreases, the imperfections in the reversal and outside perturbations, as from a solar flare, a starquake in a distant galaxy a long time ago or a butterfly beating its wings,<sup>5</sup> will make it increase again. The same thing happens also in the usual computer simulations, where velocity reversal is easy to accomplish but where roundoff errors play the role of perturbations. It is possible, however, to avoid this effect in simulations by the use of discrete-time integer arithmetic and clearly see  $RY$  go to  $RX$  (as in figure 4) following perfect velocity reversal in the now truly isolated system.<sup>11</sup>

## Boltzmann vs Gibbs entropies

The Boltzmannian focus on the evolution of a particular macroscopic system is conceptually different from Gibbs's approach, which focuses more on ensembles. This difference shows up strikingly when we compare Boltzmann's entropy—defined for a microstate  $X$  of a macroscopic system—with the more commonly used (and misused) entropy of Gibbs, defined by

$$S_G(\rho) = -k \int \rho(X) [\log \rho(X)] dX \quad (3)$$

Here  $\rho(X) dX$  is the ensemble density, or the probability



for the microscopic state of the system to be found in the phase space volume element  $dX$ , and the integral is over the total phase space  $\Gamma$ . Of course if we take  $\rho(X)$  to be the generalized microcanonical ensemble corresponding to a macrostate  $M$ —that is, if  $\rho_{M(X)} = 1/|\Gamma_M|$  if  $X$  belongs to  $\Gamma_M$ , and  $\rho_{M(X)} = 0$  otherwise—then clearly,

$$S_G(\rho_M) = k \log |\Gamma_M| = S_B(M) \quad (4)$$

More generally, the two entropies agree with each other, and with the generalized Clausius entropy, for systems in which the particle density, energy density and momentum density vary slowly on a microscopic scale and in which we have local equilibrium. Note, however, that unless the system is in complete equilibrium and there is no further systematic change in  $M$  or  $\rho$ , the time evolutions of  $S_B$  and  $S_G$  are very different. As is well known, as long as  $X$  evolves according to the Hamiltonian evolution (that is,  $\rho$  evolves according to the Liouville equation), the volume of any phase space region remains unchanged even though its shape changes, and  $S_G(\rho)$  never changes in time.  $S_B(M)$  certainly does change. Consider the situation in figure 2. At the initial time  $t_a$ ,  $S_G$  equals  $S_B$ , but subsequently  $S_B$  increases while  $S_G$  remains constant.  $S_G$  therefore does not give any indication that the system is evolving toward equilibrium. Thus the relevant entropy for understanding the time evolution of macroscopic systems is  $S_B$  and not  $S_G$ . (This great discovery of Boltzmann's is memorialized on his tombstone; see figure 5.)

## Quantum mechanics

The analysis given above in terms of classical mechanics applies also to quantum mechanics if we take the microstate  $X$  to correspond to the wavefunction  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_N)$ ; the time evolution  $X(t)$ , to the unitary Schrödinger evolution  $\psi(t)$ ; the velocity reversal  $RX$ , to the complex conjugation  $\bar{\psi}$ ; and the phase space volume of the macrostate  $|\Gamma_M|$ , to the dimension, that is, to the number of linearly independent quantum states in the subspace defined by  $M$ .

This correspondence preserves equation 2 as well as the time symmetry of classical mechanics. It does not, however, take into account the nonunitary, or "wavefunction collapse," part of quantum mechanics, which on the face of it looks time asymmetric. It seems to me that there is no necessity or room in quantum mechanics for such an extra measurement postulate.<sup>12</sup> Instead one should be able to deduce everything from some single time-symmetric theory describing the evolution of the state of the system and its environment; together these might encompass the whole universe.<sup>12,13</sup> The conventional wavepacket reduction formalism should then arise from the macroscopically large number of degrees of freedom affecting the time evolution of a nonisolated system. In fact, because of the intrinsically nonlocal character of quantum mechanics,<sup>12</sup> the whole notion of conceptually isolating a system, which serves us so well on the classical macroscopic level, requires much more care on the microscopic quantum level (see the Reference Frame column by David Mermin in PHYSICS TODAY, June 1990, page 9.)

Even aside from the standard quantum measurement formalism, it has been argued<sup>14</sup> that one should not introduce, via a measurement postulate, a time-symmetric element into the foundations of quantum mechanics. Rather one should and can conceptually and usefully separate the measurement formalism of conventional quantum theory into two parts, a time-symmetric part and a second-law-type asymmetric part. One can then, using Boltzmann-type reasoning, trace the latter back to

the initial low-entropy state of the universe.

I believe the same is true of other arrows of time such as the psychological one or that given by retarded versus advanced electromagnetic potentials. Ultimately all arrows of time and our being here to discuss them are manifestations of our (very large, macroscopic) universe's having started in a very low-entropy state.

## Typical vs averaged behavior

I conclude by emphasizing again that having results for typical microstates rather than averages is not just a mathematical nicety but is at the heart of understanding the microscopic origin of observed macroscopic behavior: *We neither have nor do we need ensembles when we carry out observations like those in figure 2.* What we do need and can expect to have is typical behavior. Ensembles are merely mathematical tools, useful for computing typical behavior as long as the dispersion in the quantities of interest is sufficiently small.

There is no such typicality with respect to ensembles describing the time evolution of a system with only a few degrees of freedom. This is an essential difference (unfortunately frequently overlooked or misunderstood) between the irreversible and the chaotic behavior of Hamiltonian systems. The latter, which can be observed already in systems consisting of only a few particles, will not have a unidirectional time behavior in any particular realization. Thus if we had only a few hard spheres in a box, we would get plenty of chaotic dynamics and very good ergodic behavior, but we could not tell the time order of any sequence of snapshots.

\* \* \*

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