Waves & Instabilities in space plasmas

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Preamble

In astrophysical environments, most plasmas are maid of protons and electrons. With few alpha particles in cosmic rays, or few heavy ions in ionospheres or lower layers of the magnetosphere/exosphere, non-hydrogen plasmas are very rare. In this manuscript, I hence only consider hydrogen plasmas. Consequently, the mass ratio denoted by μ is equal to 1836, and the charge of a proton is $+e$.

I note ω_{Ps} the angular plasma frequency[1] of the s specie (depending on the mass of the s specie) and Ω_s the gyrofrequency of the particles of s specie. This angular frequency is algebraic ; it is hence negative for electrons.
To characterize the temperature effects, I also introduce the thermal speed of each s specie

$$v_{Ts} = \left(\frac{k_B T_s}{m_s} \right)^{1/2} \tag{1}$$

as well as the sound speed

$$c_s = \left(\frac{\gamma p}{\rho} \right)^{1/2} \tag{2}$$

where p is the total kinetic pressure (including the contribution of both protons and electrons), ρ the mass density and γ the adiabatic index. As plasmas are always quasi-neutral and considering the large μ value, then $\rho \simeq m_p n$. Morever, the total pressure value depends on the scaling between protons temperature and electrons temperature. Hence, for $T_e \gg T_p$, one has $c_s = v_s = (\gamma k_B T_e / m_p)^{1/2}$, that is the ion acoustic speed. It is also convenient to introduce the β parameter, the ratio between total kinetic pressure and magnetic pressure :

$$\beta = \frac{2}{\gamma} \frac{c_s^2}{v_A^2} \tag{3}$$

where v_A is the Alfvén speed :

$$v_A^2 = \frac{B^2}{\mu_0 n m} \tag{4}$$

To study the polarization of a mode, the static magnetic field (when it exists) is in the z direction. The wave number $\mathbf{k}$ is in the xz plane and makes an angle Θ with the z axis. $\widehat{\mathbf{k}}$ is the unit vector in the $\mathbf{k}$ direction and $\widehat{\mathbf{t}}$ is the unit vector normal to $\widehat{\mathbf{k}}$ also in the xz plane (see Fig. 1).

We recall in the appendix **??** the definition of the relative dielectric tensor $\boldsymbol{\varepsilon}$ which connects the displacement vector $\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon}.\mathbf{E}$ to the electric field vector $\mathbf{E}$. We also show that Maxwell's equations in Fourier space provide the eigen mode dispersion relation with $N = kc/\omega$:

[1] one also speak about plasma frequency ... but we must not forget the factor 2 π which exists between the two.

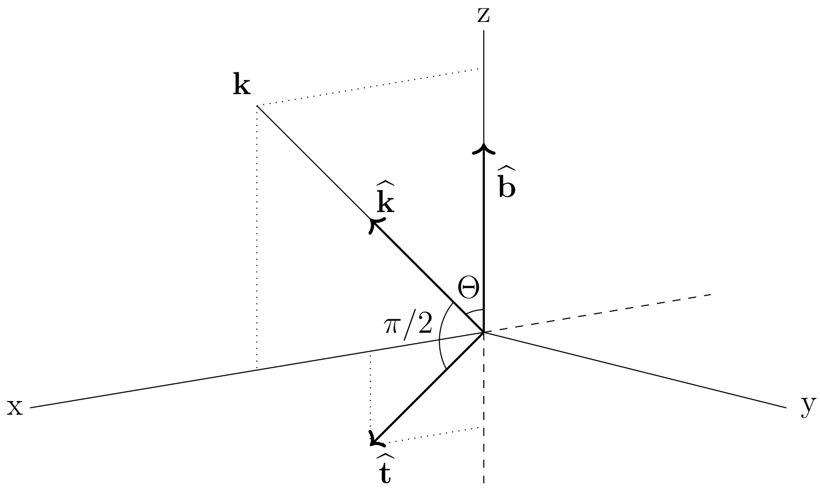

Figure 1: Definition of the vectors $\widehat{\mathbf{b}}$, $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{t}}$ defining the field aligned frame.

$$(\mathbf{NN} - N^2\mathbf{1} + \boldsymbol{\varepsilon})\ .\ \mathbf{E}(\mathbf{k},\omega) = 0 \tag{5}$$

The dispersion matrice is the term in front of $\mathbf{E}$. The couples (ω, k) for which the determinant is zero are the ones defining the eigen modes. This determinant is noted $D(\omega, k)$[2].

We introduce the decomposition of ω in its real (ω_r) and imaginary $(\imath\gamma)$ parts. The above determinant can also be split in its real and imaginary components

$$D(\omega_r, \gamma, \mathbf{k}) = D_r(\omega_r, \gamma, \mathbf{k}) + \imath D_i(\omega_r, \gamma, \mathbf{k}) \tag{6}$$

Using a Taylor expansion of $D(\omega_r, \gamma, \mathbf{k})$ around $\gamma = 0$ for the modes weakly damped or amplified, one obtains

$$D(\omega_r, \gamma, \mathbf{k}) = D_r(\omega_r, 0, \mathbf{k}) + \imath\gamma\ \frac{\partial D_r(\omega_r, \gamma, \mathbf{k})}{\partial \omega}\bigg|_{\gamma=0} + \imath D_i(\omega_r, 0, \mathbf{k}) = 0 \tag{7}$$

Then the dispersion relation and growth rate[3] of the mode are given by

$$D_r(\omega_r, 0, \mathbf{k}) = 0 \qquad\qquad \gamma(\omega, \mathbf{k}) = -\frac{D_i(\omega_r, 0, \mathbf{k})}{\partial D_r(\omega_r, \gamma, \mathbf{k})/\partial\omega|_{\gamma=0}} \tag{8}$$

[2]the determinant D should not be confused with the electric displacement vector $\mathbf{D}$

[3]it is clear that the determinant $D(\omega_r, \gamma, \mathbf{k})$ should have an imaginary part in order for the associated mode to be amplified or damped (see the associated discussion in appendix A).

— 1 —

Alfvén modes

In 1942, Hannes Alfvén wrote a seminal paper on plasma physics (*cf.* [Alfvén, 1942]). He showed that in a magnetized plasma, waves can propagate along the magnetic field. These wave have both acoustic and magnetic properties. One of these waves is often called the torsional Alfvén mode. By making an analogy with vibrating strings —for which the phase velocity of transverse waves is the square root of the ratio of the tension to the linear density— the expression of the magnetic tension and the linear density of a flux tube gives a phase velocity equal to the Alfvén velocity (see [Hasegawa and Uberoi, 1982]).

The Alfvén mode comes in many variants in astrophysical context, depending on the way it propagates and the level of approximation. These modes play an important role in heating and transporting energy. One of this mode could be part of the puzzle explaining the transport of magnetic energy in stellar winds, the transfer of angular momentum in molecular disks during star formation, the magnetic pulsations of planetary magnetospheres, or the scattering of cosmic rays during star formation, or their acceleration by diffusive shocks in supernovae remnants.

The four sections of this chapter expose some of these variations ; they are all made within the framework of a fluid approach, but could also be done with a kinetic approach, which would even be a necessity if we wanted to know their damping rate by Landau effect.

1.1 Alfvén wave in ideal MHD

We consider an adiabatic closure for a total iscalar pressure as well as an ideal Ohm's law. The system of fluid equations linearized at first order is then

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla}.\,(\rho_0 \mathbf{V}_1) = 0 \tag{1.1}$$

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} = -\boldsymbol{\nabla} p_1 + \mathbf{J}_1 \times \mathbf{B}_0 \tag{1.2}$$

The adiabatic closure $\mathrm{d}_t(pn^{-\gamma}) = 0$ is often justified when the phase speed of fluctuations is large compared to the thermal speed. In such case, the wave crosses the plasma too quickly to be able to exchange heat in an efficient way. With a linearization at first order,

$$p_1 = \frac{\gamma p_0}{\rho_0} \rho_1 \tag{1.3}$$

We must also include the two Maxwell equations, in which we neglect the transverse component of the displacement current. In the current $\mathbf{J}_1$ we hence do not neglect its longitudinal component

(which is an important remark !)

$$\begin{aligned} \frac{\partial \mathbf{B}_1}{\partial t} &= -\boldsymbol{\nabla} \times \mathbf{E}_1 & (1.4)\\ \mu_0 \mathbf{J}_1 &= \boldsymbol{\nabla} \times \mathbf{B}_1 & (1.5) \end{aligned}$$

Having thus lost the equation for the evolution of the electric field, one needs to write an Ohm's law. In ideal MHD,

$$\mathbf{E}_1 = -\mathbf{V}_1 \times \mathbf{B}_0 \tag{1.6}$$

Exercize 1. *What is the origin of this equation?*

We could take the Fourier transform of this linear system, then by substitution, reduce the number of equations. Then, the 3×3 matrix appears in a scalar product with a vector quantity : $\mathbf{V}_1$, $\mathbf{B}_1$ (if not electrostatic) or $\mathbf{E}_1$. Non-trivial solutions appear for the (ω, k) values for which the derminant is null, hence defining the dispersion relation of its eigen modes.

We remind that to simplify the notations, and without loss of generality, we consider that the magnetic field is along the z axis, and the wave number $\mathbf{k}$ is in the xz plane.

In the book by [Cramer, 2001], the approach is a bit different, and quite elegant : he only keeps the quantities $\kappa_1 = \boldsymbol{\nabla}.\mathbf{V}_1$, V_{1z}, B_{1z}, J_{1z}, ρ_1 and $\zeta_{1z} = (\boldsymbol{\nabla} \times \mathbf{V}_1)_z$. The full linearized system is then

$$\begin{aligned} \rho_0 \frac{\partial \zeta_{1z}}{\partial t} - B_0 \frac{\partial J_{1z}}{\partial z} &= 0 & (1.7)\\ \mu_0 \frac{\partial J_{1z}}{\partial t} - B_0 \frac{\partial \zeta_{1z}}{\partial z} &= 0 & (1.8)\\ \rho_0 \frac{\partial}{\partial t}\kappa_1 + \frac{B_0}{\mu_0}\nabla^2 B_{1z} + c_s^2 \nabla^2 \rho_1 &= 0 & (1.9)\\ \frac{\partial B_{1z}}{\partial t} + B_0 \left(\kappa_1 - \frac{\partial V_{1z}}{\partial z}\right) &= 0 & (1.10)\\ \rho_0 \frac{\partial V_{1z}}{\partial t} + c_s^2 \frac{\partial \rho_1}{\partial z} &= 0 & (1.11)\\ \frac{\partial \rho_1}{\partial t} + \rho_0 \kappa_1 &= 0 & (1.12) \end{aligned}$$

By taking the Fourier transform of this linear system, we obtain the relation between the angular frequency ω and the wave number $\mathbf{k}$ of the eigen modes which may exist. The dispersion relation writes

$$(\omega^2 - v_A^2 k_\parallel^2)[\omega^4 - \omega^2(c_s^2 + v_A^2)k^2 + v_A^2 c_s^2 k^2 k_\parallel^2] = 0 \tag{1.13}$$

where we introduced the speed of sound c_s and the Alfvén speed v_A, defined as

$$c_s^2 = \frac{\gamma p_0}{\rho_0}\ , \quad v_A^2 = \frac{B_0^2}{\mu_0 \rho_0} \tag{1.14}$$

Exercize 2. *Show that the solution of the system (1.7) - (1.12) is given by Eq. (1.13)*

It is clear in Eq. (1.13) that there are 2 uncoupled modes. We recall that Θ is the angle between the direction of the magnetic field (*i.e.* z) and the direction of the number $\mathbf{k}$.

The Alfvén mode. By canceling the first member of Eq. (1.13), one gets

$$\omega_A = k\, v_A |\cos\Theta| = |k_{\|}| v_A \tag{1.15}$$

This is the dispersion relation of the Alfvén mode. This mode only involves J_{1z} and ζ_{1z} which appear in Eq. (1.7) and (1.8). This underlines the transverse and incompressible character of this mode.

Exercize 3. *Show that this mode is polarized along* $\widehat{\mathbf{y}}$ *for its components* $\mathbf{B}_1$ *and* $\mathbf{V}_1$*, and that they are in phase.*

Moreover, its phase velocity $\mathbf{V}_\phi = \omega_A/\mathbf{k}$ is not depending on k, which means that this mode is non-dispersive. It is anisotropic in the sense that its phase speed depends on the angle Θ. Its group speed is always in the direction of the DC component of the magnetic field and is equal to $v_A \cos\Theta$. For these two reasons, the Alfvén wave makes it possible to efficiently transport magnetic energy along the field lines.

From its polarization, we also deduce $\boldsymbol{\nabla}.\mathbf{V}_1 = 0$, which means that this mode is not compressional ; this is the reason why it is called torsional. Eq. (1.12) shows that this mode is then not associated with a density modulation, *i.e.* $n_1 = 0$. This is an important remark when analyzing data (from probes or numerical simulations). Likewise, we have $E_1/B_1 = \omega/k = v_A$, which can also be observed in the data.

Remark 1. *In MHD, the time evolution of the electric field is missing (Darwin approximation). Then, an Ohm's Law is used to eliminate the electrical term from the Maxwell-Faraday equation. Therefore, the MHD system does not contain the electric field ; when necessary, it follows from Ohm's law.*

The component of $\mathbf{E}_1$ is polarized in the x direction. Except when $\Theta = 0$, it admits, in addition to its longitudinal component, a transverse component. There is naturally no component parallel to the magnetic field. The high mobility of the particles along the field lines would allow them to quickly smooth the associated potential gradient.

At first order, we have $B^2 = B_0^2 + 2\mathbf{B}_0.\mathbf{B}_1$. However the polarization in $\mathbf{B}$ is such that this scalar product is zero. For the Alfvén wave, there is neither density fluctuations, nor magnetic pressure fluctuations. Alfvén's mode is indeed strictly torsional.

Slow & Fast magnetosonic modes. Their dispersion relation is obtained by canceling the second term of Eq. (1.13). Among the two possible solutions, the fast mode is associated with the larger value of ω/k, the other one being the slow mode.

$$\omega_F^2 = \frac{k^2}{2}\left[v_A^2 + c_s^2 + \sqrt{(v_A^2+c_s^2)^2 - 4v_A^2c_s^2\cos^2\Theta}\right] \tag{1.16}$$
$$\omega_S^2 = \frac{k^2}{2}\left[v_A^2 + c_s^2 - \sqrt{(v_A^2+c_s^2)^2 - 4v_A^2c_s^2\cos^2\Theta}\right] \tag{1.17}$$

The $\mathbf{V}_1$ fluctuations is polarized in the $(\widehat{\mathbf{k}}, \widehat{\mathbf{t}})$ plane, $\mathbf{B}_1$ is along $\widehat{\mathbf{t}}$ and $\mathbf{E}_1$ is along $\widehat{\mathbf{y}}$. There is also a density fluctuation n_1 (since $\boldsymbol{\nabla}.\mathbf{V}_1 \neq 0$) and a magnetic field fluctuation with a z component of $\mathbf{B}_1$. These two modes are thus compressionals, even in the limit $c_s \to 0$. Within this limit (*i.e.* with $\beta \to 0$), the slow mode becomes evanescent, so the remaining mode is the fast mode whose dispersion relation reduces to $\omega = kv_A$.

For the fast mode, the magnetic and kinetic pressure fluctuations are in phase, while they are in phase opposition for the slow mode.

At $\Theta = 0$, the dispersion relation of the fast mode reduces to $\omega = |k|v_A$. It then looks like the Alfvén's mode, and can eventually be called the compressional Alfvén mode. In this case, $\mathbf{B}_1$ is also polarized according to $\widehat{\mathbf{y}}$. The slow mode becomes a pure sound wave.

1.2 Anisotropic instabilities

We here investigate how the anisotropy of the plasma distribution function modifies the properties of the MHD modes. We keep a fluid formalism and consider a gyrotropic distribution, *i.e.* that the distribution function does not depend on the gyrophase. To take into account the anisotropy, we introduce two closure equations for the parallel and perpendicular pressures. We introduce the notations

$$\lambda = \frac{\omega}{kv_A} \tag{1.18}$$
$$\Delta\beta = \beta_\parallel - \beta_\perp \tag{1.19}$$
$$\beta_\parallel^\star = \gamma_\parallel\beta_\parallel \tag{1.20}$$
$$\beta_\perp^\star = \gamma_\perp\beta_\perp \tag{1.21}$$

Then the dispersion relation of the MHD modes turns to be

$$\begin{vmatrix} \lambda^2 - 1 - {}^1\!/_2\beta_\perp^\star \sin^2\Theta + {}^1\!/_2\Delta\beta\cos^2\Theta & 0 & -{}^1\!/_2\beta_\perp^\star \sin\Theta\cos\Theta \\ 0 & \lambda^2 - (1 - {}^1\!/_2\Delta\beta)\cos^2\Theta & 0 \\ -{}^1\!/_2(\beta_\parallel^\star - \Delta\beta)\sin\Theta\cos\Theta & 0 & \lambda^2 - {}^1\!/_2\beta_\parallel^\star\cos^2\Theta \end{vmatrix} = 0 \tag{1.22}$$

This form results from the velocity dispersion relation. We will therefore have the polarization of the velocity fluctuations (resulting from the Maxwell-Faraday equation). We note that this matrix is sparse. In addition, one mode is decorelated with the two others. This is, as in the isotropic case, the Alfvén mode. The two magnetosonic modes remain coupled.

The Alfvén mode. It is given by the central term of Eq. (1.22). We recognize the torsional Alfvén mode, which dispersion equation is altered by the anisotropy

$$\omega = k_{\parallel} v_A \sqrt{1 - {}^1\!/_2 \Delta\beta} \tag{1.23}$$

One can note that for $\beta_{\parallel} > 2 + \beta_{\perp}$, the Alfvén's mode is no longer propagative. Otherwise, the fluctuations in velocity and magnetic field remain polarized in the y direction.

The mirror mode. In the quasi-perpendicular limit, $\cos^2\Theta \to 0$ and $\sin^2\Theta \to 1$. At order 0, Eq. (1.22) becomes

$$\begin{vmatrix} \lambda^2 - 1 - {}^1\!/_2\beta_{\perp}^{\star} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{vmatrix} = 0 \tag{1.24}$$

- The dispersion relation of the fast mode is $\lambda^2 = 1 + {}^1\!/_2\beta_{\perp}^{\star}$. The velocity fluctuations are polarized in the x direction.

- Slow mode requires a little more work ; the Taylor expansion at zeroth order was a little too rough. As the term at the bottom right contains a $\cos^2\Theta$ term, we suspect that first order will not be enough.

Without approximations, the 2×2 determinant gives

$$(\lambda^2 - 1 - {}^1\!/_2\beta_{\perp}^{\star}\sin^2\Theta + {}^1\!/_2\Delta\beta\cos^2\Theta)(\lambda^2 - {}^1\!/_2\beta_{\parallel}^{\star}\cos^2\Theta) - {}^1\!/_4(\beta_{\parallel}^{\star} - \Delta\beta)\beta_{\perp}^{\star}\sin^2\Theta\cos^2\Theta \tag{1.25}$$

At low frequency and in the quasi-perpendicular limit, one has $\lambda \to 0$ and $\cos\Theta \to 0$. We must therefore keep the lowest order terms in λ and $\cos\Theta$. One then obtains

$$\lambda^2 = \frac{\cos^2\Theta}{2 + \beta_{\perp}^{\star}}[\beta_{\parallel}^{\star} + {}^1\!/_2\Delta\beta\beta_{\perp}^{\star}] \tag{1.26}$$

Exercize 4. *Do the simplification to get the above dispersion relation.*

When $\Delta\beta < -2\beta_{\parallel}^{\star}/\beta_{\perp}^{\star}$, this mode becomes unstable ($\lambda^2 < 0$), *i.e.* when $\beta_{\perp}^{\star} \gg \beta_{\parallel}^{\star}$. This is the mirror mode. It is no longer propagative, its angular frequency being purely imaginary. The origin of the name of this mode does not seem totally justified, because in general, when drawing the magnetic field lines, they are not consistent with the fact that the wave number of this mode is perpendicular.

The fire-hose instability. We can also look at what happens in the quasi-parallel limit. Eq. (1.22) then becomes

$$\begin{vmatrix} \lambda^2 - 1 + {}^1\!/_2\Delta\beta & 0 & 0 \\ 0 & \lambda^2 - (1 - {}^1\!/_2\Delta\beta) & 0 \\ 0 & 0 & \lambda^2 - {}^1\!/_2\beta_{\parallel}^{\star} \end{vmatrix} = 0 \tag{1.27}$$

The top left term is the fast mode, the middle term is the Alfvén mode, and the bottom one is the slow mode. Clearly, the fast and the alfvén modes are degenerated. Moreover, If $\Delta\beta > 2$, these modes are unstable. The mode in which the velocity fluctuations are polarized in the y direction is then called the fire-hose mode, or garden-hose instability. This is indeed the same mechanism as when a garden-hose goes crazy and squirms after its reckless user drops it down while the tapper is open. The perpendicular pressure is then no longer sufficient to control the parallel pressure.

The physics of this instability is quite simple : when one disturbs a flux tube (which therefore admits a small curvature), this tube is subjected to 3 forces :

- The centrifugal force due to the parallel pressure of the plasma in the flux tube. If R is the curvature radius of the tube, this force is equal to $Mnv_\parallel^2/R$
- The thermal pressure force of the plasma outside the tube, equal to $p_\perp/R$
- The magnetic tension force in the flux tube, which writes $B_0^2/\mu_0 R$

The instability develops when the first term is greater than the sum of the two last ones, *i.e.* $p_\parallel > p_\perp + B_0^2/\mu_0$. We can then rewrite the growth rate of the fire-hose mode

$$\gamma = k_\parallel \frac{v_A}{\sqrt{2}} (\beta_\parallel - \beta_\perp - 2)^{1/2} \tag{1.28}$$

This growth rate increases with $k_\parallel$. In fact, at large $k_\parallel$, it is necessary to include the Hall term(which gives a correction in ω/Ω_p) and the electron pressure term (which gives a correction in $k_\perp \rho_e$). These corrective terms limit the growth of γ with $k_\parallel$. We then obtain a bell shaped curve $\gamma(k_\parallel)$.

The paper of [Bale et al., 2009] outlines the values of the instability threshold treated above. The approach is quite simple, while clever : they collected around 10^6 measurement points in the solar wind using the instruments on board the WIND probe. They measured the relative magnetic fluctuations B_1/B_0, as well as the value of the anisotropy ratio of the protons $T_\perp/T_\parallel$, and that of the $\beta_\parallel$ parameter. When fluctuations are measured, it means that there is an activity of associated waves. The idea is therefore to see in which domain the magnetic fluctuations can exist.

In dotted lines, the thresholds of mirror instabilities, firehose (oblique) and AIC (Alfvén Ion Cyclotron) are indicated. It is clear from this that mirror and firehose modes can only exist below their level of instability. Beyond that, the mode being unstable, it transfers its energy to the particles which in fact limits the development of magnetic fluctuations.

These results question the reason why AIC and parallel firehose modes do not limit the level of magnetic fluctuations that are observed. One answer is that mirror and firehose (oblique) modes are non-propagating, unlike AIC and firehose (parallel) modes. The question remains open...

1.3 Alfvén Ion Cyclotron mode

When the protons of a plasma are demagnetized (at least partially), *i.e.* when the perturbations have a frequency of the order of Ω_p or a wavelength of the order of the proton Larmor radius ρ_p, we

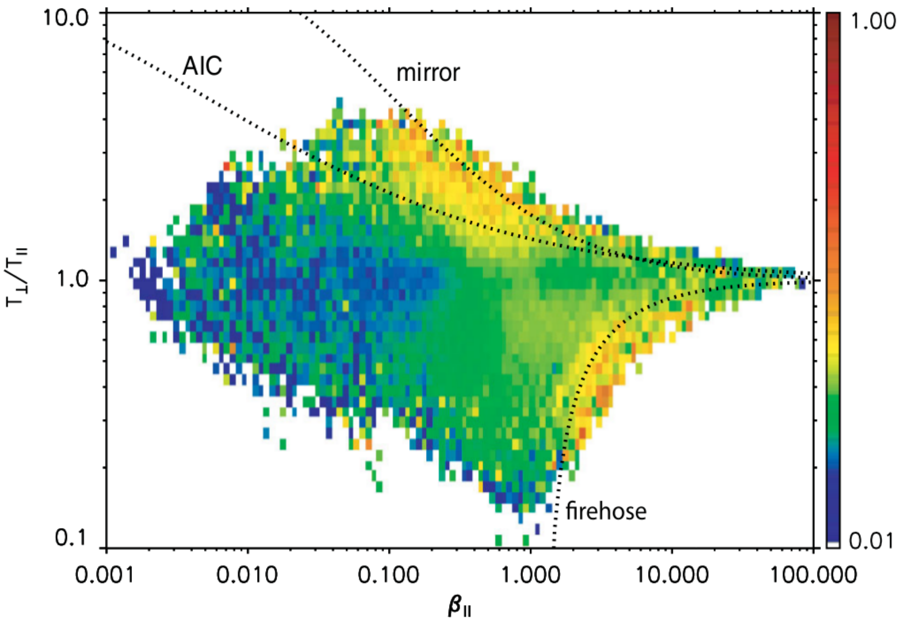

Figure 1.1: Magnitude of the relative magnetic fluctuations depending on $T_\perp/T_\parallel$ and $\beta_\parallel$ (see [Bale et al., 2009]).

must keep the second term of Eq. (A.44), called the Hall effect. The linearized form of the equations are then different in the sense that it contains additional terms. Eq. (1.7) to (1.12) thus become

$$\rho_0 \frac{\partial \zeta_{1z}}{\partial t} - B_0 \frac{\partial J_{1z}}{\partial z} = 0 \tag{1.29}$$

$$\mu_0 \frac{\partial J_{1z}}{\partial t} - B_0 \frac{\partial \zeta_{1z}}{\partial z} + \frac{v_A^2}{\Omega_p} \frac{\partial}{\partial z} \nabla^2 B_{1z} = 0 \tag{1.30}$$

$$\rho_0 \frac{\partial}{\partial t} \kappa_1 + \frac{B_0}{\mu_0} \nabla^2 B_{1z} + c_s^2 \nabla^2 \rho_1 = 0 \tag{1.31}$$

$$\frac{\partial B_{1z}}{\partial t} + B_0 \left(\kappa_1 - \frac{\partial V_{1z}}{\partial z} \right) + \frac{v_A^2}{\Omega_p} \frac{\partial J_{1z}}{\partial z} = 0 \tag{1.32}$$

$$\rho_0 \frac{\partial V_{1z}}{\partial t} + c_s^2 \frac{\partial \rho_1}{\partial z} = 0 \tag{1.33}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \kappa_1 = 0 \tag{1.34}$$

Exercize 5. *Write an Ohm's law for the electrons and deduce the linear form of Eq. (1.29) to (1.34) by including Hall term in Ohm's law.*

A cyclotronic term then appears in Eq. (1.30) which no longer makes it simply coupled to Eq. (1.29). These two equations leading the Alfvén mode, the first remark is that this mode is no longer decoupled from the two magnetosonic modes. Of course, this coupling disappears in the ideal MHD limit, when $\omega \ll \Omega_p$. Getting in the Fourier space, we obtain the dispersion relation

$$(\omega^2 - v_A k_z^2)[\omega^2(\omega^2 - c_s^2 k^2) - v_A^2 k^2(\omega^2 - c_s^2 k_z^2)] = \left(\frac{\omega}{\Omega_p}\right)^2 v_A^4 k_z^2 k^2 (\omega^2 - c_s^2 k^2) \tag{1.35}$$

It is a bi-squared equation of order six which therefore admits three solutions, hence bearing the same name as in ideal MHD. The left side of Eq. (1.30) is the same as in ideal MHD, but the right hand side is at the origin of the coupling between the Alfvén mode and the two magnetosonic modes. This nomenclature is still relevant insofar as the Hall term reduces the phase speed of the slow mode and increases that of the fast mode. Noting

$$\beta' = \frac{c_s^2}{v_A^2} \ , \quad \alpha = \frac{k v_A}{\Omega_p} \text{ et } \quad f = \frac{\omega}{\Omega_p} \tag{1.36}$$

Eq. (1.30) can be written

$$(f^2 - \alpha^2\cos^2\Theta)[f^2(f^2 - \alpha^2\beta') - \alpha^2(f^2 - \alpha^2\beta'\cos^2\Theta)] = f^2\alpha^4\cos^2\Theta(f^2 - \alpha^2\beta') \tag{1.37}$$

we can separate the cold case ($\beta' = 0$) from the hot case to discuss these modes.

Cold plasmas. The Alfvén mode and the fast magnetosonic modes can exist. Their angular frequency can be written

$$\omega_\pm^2 = \frac{v_A^2 k^2}{2}\left[1 + (1+\alpha^2)\cos^2\Theta \pm \sqrt{1 - 2(1-\alpha^2)\cos^2\Theta + (1+\alpha^2)^2\cos^4\Theta}\right] \tag{1.38}$$

where we identify the minus sign in Eq. (1.38) with the Alfvén mode. In the parallel limit ($\Theta = 0$), this mode resonates with Ω_p because then, $k \to \infty$. This mode is often called the Alfvén Ion Cyclotron, or AIC. We can verify that the group velocity goes to zero, which means that the energy of the waves accumulates, until the dissipative or non-linear effects limit it. If we calculate the dielectric tensor (see appendix), we can study the polarization of the electric field, and show that this mode is no longer linear, but that it becomes elliptic then left circular at the resonance.

The plus sign in Eq. (1.38) gives the other mode which becomes the ideal fast mode in MHD. It therefore does not undergo cyclotronic resonance, but becomes circular for $\omega \gtrsim \Omega_p$. This is the whistler mode. Moreover this mode becomes dispersive insofar as ω is proportional to k^2. Finally, it is an electronic mode which does not depend on the mass of the electrons (which would tend to make it a strictly electromagnetic mode).

Hot plasmas. When c_s is no longer zero, the two magnetosonic modes are no more degenerated. The limit $\Theta = 0$ is the only one for which the Alfvén mode is separated from the two magnetosonic modes. As in cold plasmas, Alfvén mode resonates at Ω_p. At different Θ values, this mode is called intermediate mode rather than Alfvén mode. At low frequency, it is quite close to the slow mode. Also, it no longer resonates at Ω_p. Furthermore, the slow mode resonates with $\omega = k_\parallel v_A$

1.4 The quasi-perpendicular limit

There is another very important limit for space plasmas : quasi-perpendicular modes. Whether in the solar wind or in the magnetosheath of the Earth's magnetosphere, there is a high level of magnetic fluctuations ($B_1/B_0 \sim 0.2$) for which the wave numbers are essentially perpendicular to the DC component of the magnetic field. Besides the mirror mode, it is legitimate to wonder how the Alfvén wave is modified in this limit.

The form of Maxwell's equations in Fourier space is discussed in the appendix A. By introducing the dielectric tensor $\boldsymbol{\varepsilon}$,

$$\left(\frac{k^2c^2}{\omega^2}\mathbf{1} - \boldsymbol{\varepsilon}\right).\mathbf{E}_T = -\boldsymbol{\varepsilon}.\mathbf{E}_L \tag{1.39}$$

At large k values, the magnetic component becomes negligible ; at the vicinity of the resonance ($k \to \infty$), the waves turns to be essentially electrostatic ($\mathbf{E}_T \sim 0$). This is the case for the Alfvén mode at large values of $k_\perp$.

When the wave number becomes very large, the spatial gradients becomes small. It is then necessary to re-evaluate the form of the Ohm's law to keep the terms which may no longer become negligible within this limit. In Eq. (A.44), we must therefore keep, in addition to terms 1 and 2, terms 4 and 5. Term 3 is negligible because, as we will see, the frequency of this mode remains below the electron gyrofrequency. The term 6 is still negligible for a collisionless plasma.

The electric field appears in the Maxwell-Ampère and Maxwell-Faraday equations. The first of these equations gives Eq. (1.8) with an ideal Ohm's law. By keeping the terms 1, 2, 4 and 5, we get

$$\frac{\partial B_{1z}}{\partial t} - d_e^2\nabla^2\frac{\partial B_{1z}}{\partial t} + B_0\left(\boldsymbol{\nabla}.\mathbf{V}_1 - \frac{\partial V_{1z}}{\partial z}\right) + \frac{v_A^2}{\Omega_p}\frac{\partial J_{1z}}{\partial z} = 0 \tag{1.40}$$

where we have introduced the electron inertial length $d_e = c/\omega_{Pe}$. Likewise, Eq. (1.10) can be written

$$\mu_0\frac{\partial J_{1z}}{\partial t} - \mu_0 d_e^2\nabla^2\frac{\partial J_{1z}}{\partial t} - B_0\frac{\partial \zeta_{1z}}{\partial z} + \frac{v_A^2}{\Omega_p}\frac{\partial}{\partial z}\nabla^2 B_{1z} = 0 \tag{1.41}$$

As in the Hall MHD case, it appears that these two equations are no longer decoupled from the remaining equations of the system. The compressional nature of the plasma will therefore modify the Alfvén mode. Moreover, as discussed in the appendix (A), the importance of the term associated with electron compressibility depends on the electron temperature, *i.e.* on the value of β.

With Eq. (1.7), (1.9), (1.11) and (1.12), we can solve the system to find the dispersion relation of the eigen modes. After a few lines of calculations, we obtain

$$[\omega^2(1+d_e^2k^2) - v_A^2k_\parallel^2][\omega^2(1+d_e^2k^2)(\omega^2 - c_s^2k^2) - v_A^2k^2(\omega^2 - c_s^2k_\parallel^2)] = \left(\frac{\omega}{\Omega_p}\right)^2 v_A^2k^2k_\parallel^2(\omega^2 - c_s^2k^2) \tag{1.42}$$

Remember that we are in the quasi-perpendicular limit, $k_\perp \gg k_\parallel$, *i.e.* $k \simeq k_\perp$. Moreover, we can verify *a posteriori* that $\omega \lesssim v_A k_\parallel$. We introduce the thermal Larmor radius $\rho_s = c_s/\Omega_p$. Eq. (1.42) then simplifies

$$\left[1 + d_e^2 k_\perp^2 - \frac{v_A^2 k_\parallel^2}{\omega^2}\right]\left[1 + \frac{c_s^2}{v_A^2}\left(1 + d_e^2 k_\perp^2 - \frac{v_A^2 k_\parallel^2}{\omega^2}\right)\right] = \frac{v_A^2 k_\parallel^2}{\omega^2}\left(k_\perp^2 \rho_s^2 - \frac{\omega^2}{\Omega_p^2}\right) \tag{1.43}$$

Eq. (1.43) calls for some thoughts. The first term does not involve any compressibility terms ; it is the Alfvénic term. But the existence of a right hand side means that it is coupled to the second magnetosonic term, as in Hall MHD. A correction also appears due to the electron inertial length. Insofar, as we are interested in frequencies below the proton gyrofrequency, and large perpendicular wave number, we have $\omega/\Omega_p \ll k_\perp^2 \rho_s^2$. One can then neglect the cyclotron correction in the right-hand side of Eq. (1.43), which means that the coupling term becomes proportional to $k_\perp^2 \rho_s^2$.

In the low-β case (*i.e.* $c_s \ll v_A$), Eq. (1.42) simply reduces to

$$\omega^2 = v_A^2 k_\parallel^2 \frac{1 + k_\perp^2 \rho_s^2}{1 + k_\perp^2 d_e^2} \tag{1.44}$$

Before solving this equation, we identify two different regimes depending on the relative importance of $k_\perp \rho_s$ in front of $k_\perp d_e$. With the electron inertial length written as

$$d_e^2 = \frac{1}{\mu}\frac{v_A^2}{\Omega_p^2} = \frac{1}{\mu} d_p^2 \tag{1.45}$$

which gives

$$\frac{\rho_s^2}{d_e^2} = \frac{\gamma}{2}\beta\mu \tag{1.46}$$

we can discuss two limit cases, depending on how (weak) β compares to μ^{-1}. For these two cases, we always consider the limit $T_e \gg T_p$. Thus, the speed of sound c_s is equal to the ion acoustic speed v_s.

The Kinetic Alfvén Wave. In the case $\beta \gtrsim \mu^{-1}$, *i.e.* $v_{Te} \gtrsim v_A$, the dispersion relation writes

$$\omega^2 = v_A^2 k_\parallel^2 (1 + k_\perp^2 \rho_s^2) \tag{1.47}$$

The Alfvén mode is modified by the fact that $k_\perp$ is large enough to consider the contribution of the Larmor radius of thermal protons. This mode is called the kinetic Alfvén wave, hence the acronym KAW.

From the KAW mode dispersion equation, we can calculate its phase velocity. The dependence on k makes it a dispersive mode. To try to find this phase speed in the data measured by satellite, one technique is to reconstruct the E/B ratio. This work was done by [Sahraoui et al., 2009] with Cluster measurements in the solar wind.

In the MHD limit, the magnetic field is frozen in the plasma ; we then have $\mathbf{E} \sim -\mathbf{V} \times \mathbf{B}$. On the other hand, E^2 and B^2 both have spectra at $k^{-1.62}$. Their ratio is therefore constant, and independent of k. On the other hand, at scales above the Larmor radius of the protons, the dispersive effects give a E/B ratio which linearly depends on $k_\perp$. This can be seen in the Figure below, in which at $k\rho_p \geq 1$, the slope of E/B is in $k^{1.08}$. Electric and magnetic fluctuations therefore suggest that the turbulence in the solar wind is mainly due to the KAW modes.

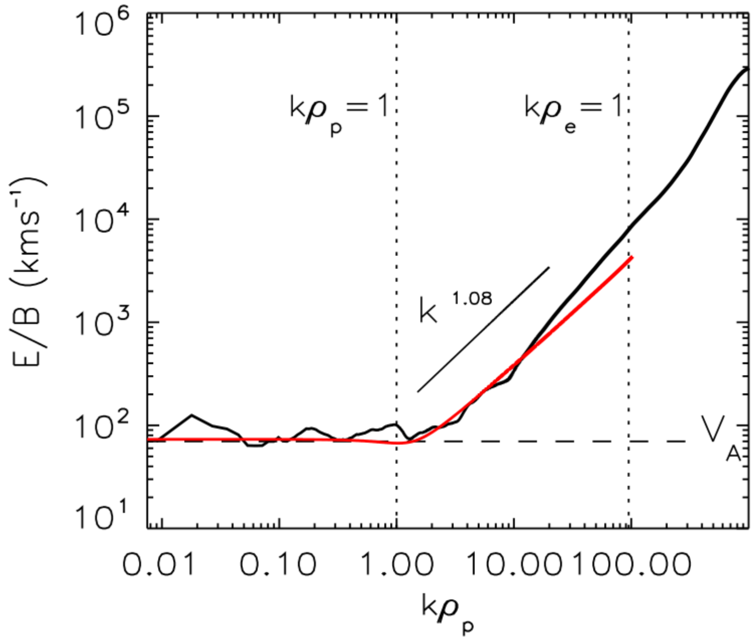

Figure 1.2: The solid line gives the E/B ratio measured by CLUSTER, depending on k. The red curve is the analytical expectation for the KAW mode (from [Sahraoui et al., 2009]).

The Inertial Alfvén Wave. In the other limit, $\beta \lesssim \mu^{-1}$ which is also equivalent to $v_{Te} \lesssim v_A$. A magnetized and sparingly dense plasma of this type is found, for example, in an ionosphere. The dispersion relation becomes

$$\omega^2 = \frac{v_A^2 k_\parallel^2}{1 + d_e^2 k_\perp^2} \tag{1.48}$$

In this case, the Alfvén mode is modified by the fact that $k_\perp$ is large enough so that $k_\perp d_e > 1$. This mode is called the Inertial Alfvén Wave, hence the acronym IAW [1].

The auroral zone, due to its very strong magnetization and the low temperature of the charged particles, is a region where the parameter β is much lower. IAW modes can therefore play an important role. In this case, the electric field (essentially electrostatic) of this wave can heat the ions, and thus ensure their escape. A study by [Stasiewicz et al., 2000] uses FREJA data at 1700 km altitude. Electric and magnetic fluctuations are at very low frequency ($f \sim 1$ Hz). The frequency of the mode is then linked to the associated wave number via the speed of the satellite, 6.8 km.s^{-1} in this case. The Figure below is the value of the ratio E_1/B_1 as a function of $V_{SC}/\omega \sim k_\perp^{-1}$ (Taylor hypothesis). The points are the theoretical values, including for the protons the correction associated to the Larmor radius effects. In this case, the dispersion equation of the IAW mode becomes

$$\omega = k_\parallel v_A \left(\frac{1 + k_\perp^2 \rho_p^2}{1 + k_\perp^2 d_e^2} \right)^{1/2} \tag{1.49}$$

These experimental results suggest that IAWs exist in auroral regions and may partly explain ionospheric heating and exhaust.

[1] be careful as this acronym also holds for Ion Acoustic Waves.

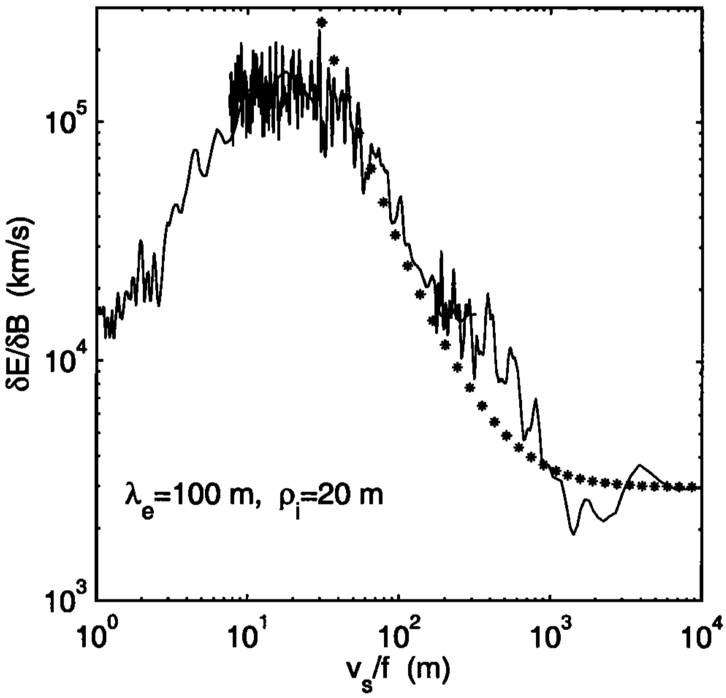

Figure 1.3: Ratio of the fluctuations E/B depending on V_{SC}/ω (see [Stasiewicz et al., 2000]).

For the two KAW & IAW modes, being in the electrostatic limit with $k_\perp \gg k_\parallel$, there is a component of the electric field along the magnetic field. As a consequence of this parallel electric field, these waves can interact efficiently with particles, and is therefore a good candidate for heating them. These two modes have been extensively studied in the formation of particle beams in the auroral ionosphere, as well as in energy transport in tokamaks.

Consequently, it is also possible to study their dispersion relation by introducing the two components (parallel and perpendicular) of the electric field in the fluid equations and the Maxwell equations (*cf.* [Hasegawa and Uberoi, 1982]).

— 2 —

Schocks and discontinuities

Shocks and discontinuities can exist in a plasma. These two physical structures are associated to the existence of a discontinuous spatial structure of a set of physical unknowns of the plasma. We will focus on the asymptotic values of these quantities on each sides of the discontinuity, while the detail of the kinetic structure is quite hard to access from an analytical point of view.

2.1 Impossibility of a stationary gradient

To illustrate this concept, we start by considering a neutral fluid behaving adiabatically. We then have $p\rho^{-\gamma} = s$ (s being a constant and $\gamma = \frac{5}{3}$). This fluid is flowing from the left and to the right, that is in the $+\hat{\mathbf{x}}$ direction. Considering an obstacle located at $x = 0$, we then have the boundary conditions $v_x = v_0$ for $x \to -\infty$ and $v_x = 0$ for $x > 0$. We then want to solve the fluid equations and obtain a stationary solution (if it exists) satisfying the boundary conditions.

In the simplified 1-dimensional case, the 3 unknowns are ρ, v (along x) and p. Continuity, momentum and closure equations are then enough to solve the problem. We introduce the sound speed

$$c_s = \sqrt{\frac{\gamma p}{\rho}} \tag{2.1}$$

which is not a constant value as both p and ρ depend on x. But these 2 quantities do not evolve independently regarding the closure equation. Hence, both p and ρ can be expressed as a function of c_s.

The set of the 3 differential equations on ρ, v and p can then reduce to the set of 2 differential equations on v and c_s :

$$\frac{2}{\gamma - 1}[\partial_t + v\partial_x]c_s + c_s\partial_x v = 0 \tag{2.2}$$

$$[\partial_t + v\partial_x]v + \frac{2}{\gamma - 1}c_s\partial_x c_s = 0 \tag{2.3}$$

We then impose 2 (non-independent) conditions for the solution :

- it is obtained in the frame where it is non-moving and quote this solution with a star index
- it is stationary, that is it satisfies $[\partial_t] \equiv 0$ (whatever the operand, $v_\star$ or c_s)

It should then satisfy the system

$$\frac{2}{\gamma-1}v_\star\partial_x c_s + c_s\partial_x v_\star = 0 \tag{2.4}$$

$$v_\star\partial_x v_\star + \frac{2}{\gamma-1}c_s\partial_x c_s = 0 \tag{2.5}$$

Substituting one equation in the other to get rid of the $\partial_x c_s$ term, we obtain

$$[v_\star^2 - c_s^2]\partial_x v^\star = 0 \tag{2.6}$$

The solution to this equation are $v_\star = \text{const}$ or $v_\star = \pm c_s$:

- the first one is incompatible with the boundary conditions of a flow with a velocity continuously decreasing from v_0 to 0
- the second one is no good either as to get $v_\star$ to zero would also imply $c_s = 0$, which is never the case in a plasma, whatever its density.

The conclusion of this example is then that there is no way to continuously decrease the velocity of a neutral fluid flow from a non-zero value to zero. The same goes for a plasma.

The linear solution of this set of equation is well known ; it is the propagation of an acoustic which satisfies

$$v(x,t) = v_0(x \pm c_s t, 0) \tag{2.7}$$

meaning that the "initial" profile (that is the one at the time $t = 0$) is advected at the sound speed in whatever the $+\hat{\mathbf{x}}$ or $-\hat{\mathbf{x}}$ direction. But then, $c_s(x,t)$ depends on both $p(x,t)$ and $\rho(x,t)$. As these two quantities are not in phase, it means that $c_s(x,t)$ can not be uniform and depends on x (and t).

Remark 2. *From the set of fluid equation for a neutral fluid, the relation $v + \frac{2}{\gamma-1}c_s = const$ along a line of flow (defined by $x(t)$ satisfying $\mathrm{d}_t x = v$) is straightforward.*

The profile of $v_0(x,t)$ is then going to be modified as each value of $v(x,t)$ will be advected at the local value of $c_s(x,t)$, hence perturbating the initial profile. Hence, an initially sinusoidal profile will get steeper in half of its fronts, and smoother in the other half. This point out the fact that a shock creation should be the fatal evolution of whatever acoustic wave. Why isn't it what we observe ? And could we expect a surging profile which would then not be a function anymore ?

2.2 Rankin-Hugoniot jump conditions

A discontinuity is a stationary 1-dimensional structure, that is with a gradient along a normal $\hat{\mathbf{n}}$. This doesn't suggest any constraint on the thickness of the discontinuity. Furthermore, each quantitie of the problem reach an asymptotic value on both side of the discontinuity.

Notation 1. *We note u_0 and u_1 the upstream and downstream (of the discontinuity) asymptotic value of u (for any quantity u), respectively.*

Notation 2. *We define $\Delta u = u_1 - u_0$ the jump of the quantity u across the discontinuity.*

Definition 1. *The jump equations are the equations on Δu satisfied across a discontinuity. They are also called* **Rankin-Hugoniot jump equations***.*

Remark 3. *Rankin-Hugoniot (RH) jump equations concern the neutral fluid, and are extended to the* **generalized Rankin-Hugoniot jump equations** *in the MHD case.*

The jump conditions are obtained from the Maxwell and plasma equations. Focusing on the ideal MHD case of a single magnetized fluid, we also make the hypothesis of an isotropic plasma, meaning that the pressure tensor reduces to a scalar p, and we will also consider the case of a polytropic closure $p = s\rho^\gamma$ which includes the isotherm and adiabatic cases.

Remark 4. *Far from the discontinuity, the magnetic diffusivity of a plasma $\eta = 0$. The role of turbulence in the η value could be questioned inside the discontinuity, but remind that the RH jump conditions are written asymptotically far enough from the discontinuity.*

With few algebra, many of the transport equations can be written in a conservative form, that is

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}.f = 0 \tag{2.8}$$

for any quantity u and its associated flux f.

Remark 5. *It is clear that if u is a tensor of order $n \in \{0, 1, 2 \dots\}$, f is a tensor of order $n+1$.*

For a stationary solution, Eq. 2.8 reduces to $\boldsymbol{\nabla}.f = 0$, that is $\hat{\mathbf{n}}\partial_x.f = \partial_x \hat{\mathbf{n}}.f = \partial_x f_n = 0$ where f_n is the normal component of the flux f[1]. This equation can be integrated across the discontinuity, that is on a domain wide enough to include each of the asymptotic values of the flux f on both sides of the discontinuity. We then get the simple relation

$$\Delta f_n = 0 \tag{2.9}$$

which can writes in the other simple form

$$f_{n0} = f_{n1} \tag{2.10}$$

The game is now to write the MHD equations (that is both fluid and Maxwell equations) in conservative form (if possible). The continuity of the normal component of the flux will then be straightforward. We can then remind the form of these equations that we obtained in the chapter dedicated to the Alfvén mode.

■ The first one is the **continuity equation** :

[1] f_n is hence a tensor of same order n as u

$$\partial_t \rho + \boldsymbol{\nabla}.\left(\rho \mathbf{V}\right) = 0 \tag{2.11}$$

which is already in the conservative form.

■ The second one is the **momentum equation** :

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}.\mathbf{V}\right] = -\boldsymbol{\nabla} p + \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} \tag{2.12}$$

which needs a little of work. Using the continuity equation and developping the double cross product, it writes

$$\partial_t(\rho \mathbf{V}) + \boldsymbol{\nabla}.\left(\rho \mathbf{V}\mathbf{V} + p\mathbf{1} - \frac{\mathbf{B}\mathbf{B}}{\mu_0} + \frac{B^2}{2\mu_0}\mathbf{1}\right) = 0 \tag{2.13}$$

■ The third equation is the **energy equation**. This equation is more complicated ; it includes the bulk flow energy, the thermal energy and the magnetic energy. Generally, each of these equations are written in a separate way and then added. The resulting equation in conservative form is

$$\partial_t\left(\frac{1}{2}\rho V^2 + \frac{3}{2}p + \frac{B^2}{2\mu_0}\right) + \boldsymbol{\nabla}.\left[\left(\frac{1}{2}\rho V^2 + \frac{5}{2}p\right)\mathbf{V} - \frac{(\mathbf{V} \times \mathbf{B}) \times \mathbf{B}}{\mu_0} + \mathbf{q}\right) = 0 \tag{2.14}$$

where $\mathbf{q}$ is the reduced heat flux which is a tensor of first order (that is a vector).

■ The fourth equation is the **Maxwell-Faraday** equation which includes the ideal Ohm's law as the electric field is not a "natural" unknown of the MHD system, but can rather be seen as resulting from the non-relativistic Lorentz transform. This equation then writes

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{B}) \tag{2.15}$$

With the relation of vectorial analysis $\boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{B}) = \mathbf{V}\boldsymbol{\nabla}.\mathbf{B} - \mathbf{V}.\boldsymbol{\nabla}\mathbf{B} + \mathbf{B}\boldsymbol{\nabla}.\mathbf{V} + \mathbf{B}.\boldsymbol{\nabla}\mathbf{V}$, this equation doesn't write in a simple conservative form, but the stationary condition simply means that the component of the electric field perpendicular to the normal $\hat{\mathbf{n}}$ (that is the tangential component labeled with index T) has to be continuous.

■ A fifth equation, the simplest one, is the **Maxwell-Thomson equation**

$$\boldsymbol{\nabla}.\mathbf{B} = 0 \tag{2.16}$$

which is already in the appropriate form.

Remembering the stationary condition for a discontinuity, the intergation of this equation across the discontinuity, that is replacing $\boldsymbol{\nabla}.\mathbf{f} \equiv \hat{\mathbf{n}}.\partial_x \mathbf{f}$ by $f_{n0} = f_{n1}$, we obtain the **Rankin-Hugoniot jump equations**

$$\rho_0 v_{n0} = \rho_1 v_{n1} \tag{2.17}$$

$$\rho_0 V_{n0}\mathbf{V}_0 + \left(p_0 + \frac{B_0^2}{2\mu_0}\right)\hat{\mathbf{n}} - \frac{B_{n0}\mathbf{B}_0}{\mu_0} = \rho_1 V_{n1}\mathbf{V}_1 + \left(p_1 + \frac{B_1^2}{2\mu_0}\right)\hat{\mathbf{n}} - \frac{B_{n1}\mathbf{B}_1}{\mu_0} \tag{2.18}$$

$$\frac{1}{2}\rho_0 V_0^2 V_{n0} + \frac{5}{2}p_0 V_{n0} - \frac{1}{\mu_0}\left[B_{n0}(\mathbf{B}_0.\mathbf{V}_0) - B_0^2 V_{n0}\right] = \frac{1}{2}\rho_1 V_1^2 V_{n1} + \frac{5}{2}p_1 V_{n1} - \frac{1}{\mu_0}\left[B_{n1}(\mathbf{B}_1.\mathbf{V}_1) - B_1^2 V_{n1}\right] \tag{2.19}$$

$$V_{n0}\mathbf{B}_{T0} - B_{n0}\mathbf{V}_{T0} = V_{n1}\mathbf{B}_{T1} - B_{n1}\mathbf{V}_{T1} \tag{2.20}$$

$$B_{n0} = B_{n1} \tag{2.21}$$

We already stated that these relations where obtained in the frame where the discontinuity is stationary. This condition unambiguously sets the normal component of the velocity of this frame. We then have the liberty to determine the value for its tangential component. For that issue we chose the "de Hoffmann-Teller frame".

Definition 2. *The* **de Hoffmann-Teller frame** *(HT) is the frame in which the fluid velocity is parallel to the magnetic field.*

Property 1. *In the HT frame, the electric field* $\mathbf{E}$ *are null.*

As a consequence, $\mathbf{E} = -\mathbf{V}\times\mathbf{B} = 0$ gives

$$B_n \mathbf{V}_T = V_n \mathbf{B}_T \tag{2.22}$$

Remark 6. *Using the non-relativistic Lorentz transform, there is always such a frame where the electric field is null, provided that this electric field does not have a parallel component. When it exists, such an electric field exists at spatial and temporal kinetic scales, and are then out of the scope of the MHD shocks and discontinuities.*

It is now time to solve these equations. But a focus on the RH jump equation associated to the momentum conservation put forward a difference in the 2 kinds of solution of this system, namely the **discontinuities** and the **shocks**.

The second RH equations can be projected along the normal and tangential directions to give the system

$$\rho_0 V_{n0}^2 + p_0 + \frac{B_{T0}^2}{2\mu_0} = \rho_1 V_{n1}^2 + p_1 + \frac{B_{T1}^2}{2\mu_0} \tag{2.23}$$

$$\rho_0 V_{n0}\mathbf{V}_{T0} - \frac{B_{n0}\mathbf{B}_{T0}}{\mu_0} = \rho_0 V_{n1}\mathbf{V}_{T1} - \frac{B_{n1}\mathbf{B}_{T1}}{\mu_0} \tag{2.24}$$

The second relation can be simplified as the tangential component of the fluid velocity $\mathbf{V}_T$ can be reformulated using the HT definition so

$$\mathbf{V}_T = \frac{V_n}{B_n}\mathbf{B}_T \tag{2.25}$$

and Eq. 2.24 also writes

$$\left(\rho_0 V_{n0}^2 - \frac{B_{n0}^2}{\mu_0}\right)\mathbf{B}_{T0} = \left(\rho_1 V_{n1}^2 - \frac{B_{n1}^2}{\mu_0}\right)\mathbf{B}_{T1} \tag{2.26}$$

2.3 Discontinuities

If the parenthesis in Eq. 2.26 is null, there is no more constraints on how the direction of tangential component of the magnetic field should evolve across the discontinuity. More precisely, under the condition

$$\rho_0 V_{n0}^2 - \frac{B_{n0}^2}{\mu_0} = \rho_0 V_{n1}^2 - \frac{B_{n1}^2}{\mu_0} \tag{2.27}$$

the tangential component of the magnetic field can rotate. Such a plasma topology is called a **rotational doscontinuity**. This name is self-explaining as this is the situation where the magnetic field can rotate. Considering Eq. 2.17 and Eq. 2.21, it is clear that ρ, V_n and B_n are conserved.

Using Eq. 2.23, we then also have

$$p_0 + \frac{B_{T0}^2}{2\mu_0} = p_1 + \frac{B_{T1}^2}{2\mu_0} \tag{2.28}$$

Then, using Eq. 2.19 as well as the definition of $\mathbf{V}_T$ in the HT frame given by Eq. 2.25, we finally also obtain the conservation of p and B_T (and then V_T) across the discontinuity. .

Property 2. *In a rotational discontinuity, ρ, V_n, V_T, p, B_n and B_T are conserved across the discontinuity. In other words every scalar quantity is conserved, as well as the modulus of any vectorial quantity.*

Property 3. *In a rotational discontinuity, both the fluid velocity and the magnetic field are rotating (in the way)*

From Eq. 2.27, it is also clear that in the HT frame,

$$V_n^2 = \frac{B_n^2}{\mu_0 \rho} \tag{2.29}$$

which means that the plasma flow is Alfvenic in the frame of the discontinuity, or equivalently that the discontinuity is propagating at the Alfvén speed relative to the plasma. This is thus the non-linear limit of the Alfvén wave.

Remark 7. *It is very important to point out that the Alfven speed of a rotational discontinuity is calculated only with the normal component of the magnetic field.*

Special case 1 : There is a very special case of rotational discontinuity that deserves a different name ; the one for which $B_n = 0$. For a strictly tangential magnetic field, Eq. 2.20 gives

$$V_{n0}\mathbf{B}_{T0} = V_{n1}\mathbf{B}_{T1} \tag{2.30}$$

But the ideal Ohm's law $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$ gives

$$\mathbf{E}_T = -V_n \hat{\mathbf{n}} \times \mathbf{B}_T \tag{2.31}$$

- If $V_n \neq 0$, there is no way to cancel the $\mathbf{E}_T$ component. Such a discontinuity can then not exist
- If $V_n = 0$, the HT frame can be defined. This special case of rotational discontinuity is called a **tangential discontinuity**

In such a discontinuity, both V_n and B_n are null. Hence, the scalar ρ and p are not conserved. From Eq. 2.18, it is then clear that the only constraint on this type of discontinuity is to conserve the total pressure across the discontinuity, that is

$$p_0 + \frac{B_0^2}{2\mu_0} = p_1 + \frac{B_1^2}{2\mu_0} \tag{2.32}$$

The tangential electric field $\mathbf{E}_T = 0$, but E_n is neither null, nor conserved across the discontinuity.
Special case 2 : Contrary to the tangential case, consider the case $B_n \neq 0$ but $V_n = 0$. Then, all the quantities should be conserved except the pressure p. As $p = nk_BT$, it means that both the density n and the temperature T should vary across the discontinuity, but keeping their product constant. This is a **contact discontinuity**.

2.4 Schocks

This other class of solution is the one for which the direction of $\mathbf{B}_T$ is conserved across the discontinuity. As a consequence, the magnetic field on both sides of the discontinuity is in the same plane, defined by the $\hat{\mathbf{n}}$ and $\hat{\mathbf{T}}$ directions. The plasma velocity $\mathbf{V}$ in the HT frame is also coplanar.

The RH jump equations given by Eq. 2.17 to 2.21 should then be solved in the general case. The way to solve this system is to calculate V_{n1} as a function of V_{n0}. Of course, the trivial solution $V_{n0} = V_{n1}$ is of no interest. In the same way, any solution $V_{n1} \geq V_{n0}$ is not physically interesting either ; what would then be the source (in the shock) to increase the flow energy ?

The solution has then to be solved numerically, and is depicted in Fig. ???.

The intersection of the solution with the first bisector $V_{n1} = V_{n0}$ gives 3 peculiar values of the velocity that we call V_s, V_i and V_f, these index standing for "slow, "intermediate" and "fast", respectively. V_s and V_f are the **upstream** values of the phase velocity of the slow magnetosonic mode and the fast magnetosonic mode, respectively.

Remark 8. *Very close to the bisector, V_{n0} and V_{n1} are very close, so the solution of this probleme should be very close to the solution of the associated linear problem. It is then obvious to recover the linear MHD modes.*

We then have

- A **slow shock** if $V_{s0} < V_{n0} < V_{i0}$
- An **intermediate shock** if $V_{i0} < V_{n0} < V_{\dagger 0}$
- A **fast shock** if $V_{f0} < V_{n0}$

where $V_\dagger$ is defined as where the tangent to the curve is vertical (that is the derivative of this curve diverge).

Definition 3. *The "intermediate" velocity V_i is given by*

$$V_i^2 = \frac{B_n^2}{\rho\mu_0} \tag{2.33}$$

meaning that it only depends on the normal component of the magnetic field (as for the speed of a tangential discontinuity)

The names of these regimes are important. The linear limit of the intermediate shock is not the Alfvén mode. As we already saw, the Alfvén mode is the linear limit of the rotational discontinuity. Furthermore, for the intermediate shock, $\mathbf{B}_T$ and $\mathbf{V}_T$ are reversed across the discontinuity while for a rotational discontinuity, any value of the rotation angle can be a solution (depending on the thickness of the discontinuity) as for a (torsional) Alfvén wave.

Property 4. *Out from the 3 ranges defined above, no shock-like solution can exist.*

From Eq. 2.26, we can write

$$\left(V_{n0} - \frac{V_{i0}^2}{V_{n0}}\right)\mathbf{B}_{T0} = \left(V_{n1} - \frac{V_{i1}^2}{V_{n1}}\right)\mathbf{B}_{T1} = \tag{2.34}$$

With the definition of the intermediate velocity

$$V_i = \frac{B_n^2}{\mu_0\rho} \tag{2.35}$$

the above relation can simply writes

$$\mathbf{B}_{T1} = F(V_{n1})\mathbf{B}_{T0} \tag{2.36}$$

with

$$F(V_{n1}) = \frac{V_{n0} - \frac{V_{i0}^2}{V_{n0}}}{V_{n1} - \frac{V_{i1}^2}{V_{n1}}} \tag{2.37}$$

Remark 9. *It important to note that the intermediate velocity does not depend on the total magnetic field, but only on its normal component.*

The factor F given by Eq. 2.37 is the one defining how the magnetic field is going to evolve. For an intermediate shock, it is negative. It can be shown that $|F(V_{n1})| < 1$ for slow shocks and $|F(V_{n1})| > 1$ for fasts shocks. We also have $|F(V_{n1})| < 1$ for intermediate shocks.
These results can be summerized :

Slow shock	Intermediate shock	Fast shock
ρ increases	ρ increases	ρ increases
V_n decreases	V_n decreases	V_n decreases
p increases	p increases	p increases
$\mathbf{B}_T$ decreases	$\mathbf{B}_T$ decreases	$\mathbf{B}_T$ increases
	$\mathbf{B}_T$ and $\mathbf{V}_T$ flip over	

— 3 —

Magnetic reconnection

Magnetic reconnection (MR) occurs in electrically conducting plasma where :

- magnetic topology of the magnetic field lines is rearranged
- magnetic ennergy is converted in particle energy, that is both thermal and flow energy

MR can disconnect formerly connected magnetic field lines like in the solar photosphere ([Giovanelli, 1947]) and MR can connect formerly disconnected magnetic field lines like in stellar wind-magnetosphere interaction ([Dungey, 1961]).

The first quantitative models of reconnection were based on the resistivity of the medium to break the frozen-in theorem of ideal MHD. The models of [Sweet, 1958] & [Parker, 1957] and [Petschek, 1964] were historically the most important. With the paper of [Harris, 1962], we had the first kinetic equilibrium of a current sheet associated to a magnetic field reversal, where magnetic reconnection can grow. Then, the paper of [Furth et al., 1963] laid the groundwork for the resistive tearing mode, and the letter of [Coppi et al., 1966] the one for the collisionless tearing mode.

3.1 The first reconnection models

A simple calculation can put forward what is needed (and mandatory) for magnetic reconnection to occur in a plasma.

A magnetic field line can be materialized by the center of gyration of a charged particle (when no E field is applied). The velocity of the B field is hence the one of the HT frame where $E_\perp$ is null. This velocity writes $\mathbf{V}_{\mathrm{HT}} = \mathbf{E} \times \mathbf{B}/B^2$ for a non-relativistic Lorentz transform. In ideal MHD,

$$\mathbf{E} = -\mathbf{U} \times \mathbf{B} \tag{3.1}$$

i.e. plasma and magnetic field are comoving : this is the Alfvén theorem of the frozen–in flux.

Notation 3. *Let* $\mathbf{E}^\star$ *be the non-ideal part of the electric field, ie* $\mathbf{E} = -\mathbf{U} \times \mathbf{B} + \mathbf{E}^\star$

What are the conditions on $\mathbf{E}^\star$ to allow reconnection ?

Let 2 (infinitely) close points A and C be on a same magnetic field line $\mathbf{B}$ and defining $\mathbf{L} = \overrightarrow{AC}$, that is $\mathbf{L} \times \mathbf{B} = 0$ (see Fig. 3.1). Under which condition do we have $\mathrm{d}_t(\mathbf{L} \times \mathbf{B}) = \mathrm{d}_t\mathbf{L} \times \mathbf{B} + \mathbf{L} \times \mathrm{d}_t\mathbf{B} = 0$?

The calculation should give you that in 2D reconnection can develop with a parallel irrotational electric field

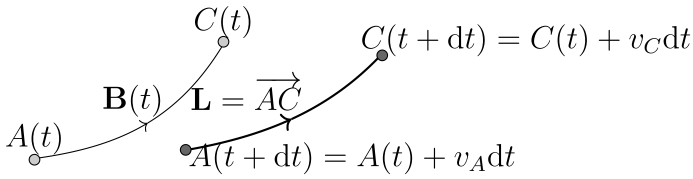

Figure 3.1: Advection of two close-by points on a magnetic field line.

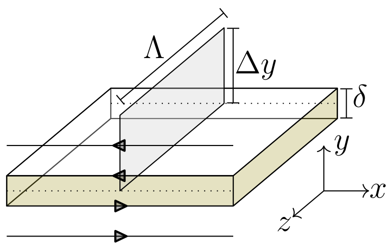

Figure 3.2: Geometrical limit of the magnetic flux above a current sheet.

$$\boldsymbol{\nabla} \times \mathbf{E}^{\star}_{\parallel} \neq 0 \tag{3.2}$$

This result is important because it puts forward that at the loci where magnetic field lines are de-connected/re-connected, such $E^{\star}$ should exist and its physical origin needs to be identified.

Before addressing these models, lets point-out how the efficiency of magnetic reconnection can be evaluated.

Reconnected flux and reconnection rate

Let $\mathrm{d}\Phi = \mathbf{B}.\mathrm{d}\mathbf{S}$ be the differential magnetic flux across a surface $\mathrm{d}\mathbf{S}$. Magnetic reconnection will only develop in current sheets where there is a large enough magnetic shear. The magnetic field reversal is given by

$$B_x(y) > 0 \quad \forall \quad y < 0 \tag{3.3}$$

$$B_x(y) < 0 \quad \forall \quad y > 0 \tag{3.4}$$

and is associated to a thin current sheet.

We focus on the magnetic flux through the surface $\Lambda\Delta y$ indicated in light gray in Fig. 3.2. The partial time derivative of the magnetic flux writes

$$\partial_t \Phi = \frac{B\Lambda\Delta y}{\Delta t} \tag{3.5}$$

The time-derivative of this flux is associated to the transport of B (along x) advected at the plasma velocity U (along y) by the E field (along z). The advection of B at the velocity U results

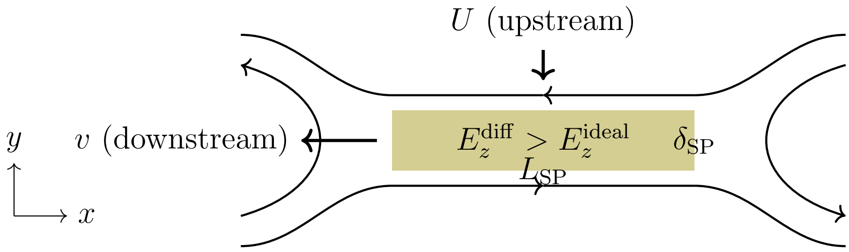

Figure 3.3: 2D geometry of the Sweet-Parker model.

from the frozen-in condition in the plasma, far above the CS where MHD is ideal. The Maxwell-Faraday equation gives

$$\partial_t \iint \mathbf{B}.\mathrm{d}\mathbf{S} = \iint (\boldsymbol{\nabla} \times \mathbf{E}).\mathrm{d}\mathbf{S} = \int \mathbf{E}.\mathrm{d}\mathbf{l} \tag{3.6}$$

i.e. $\partial_t \Phi = E\Lambda$. With the same upward flow below the $y = 0$ plane, we have the condition $E = 0$ at $y = 0$. Then, $E = B\Delta y/\Delta t = BU$ where U is also the fluid velocity in the frozen-in condition. This electric field E can be normalized using B_0 and A_0, *i.e.*

$$E' = \frac{E}{B_0 A_0} = \frac{U}{A_0} \tag{3.7}$$

Definition 4. *A_0 is the Alfvén velocity with the upstream asymptotic B_0 and ρ_0 values*

It clearly appears that this is a creation rate of flux upstream from the CS. For stationary reconnection, this flux is expelled by reconnection, so that creation rate $\equiv$ reconnection rate.

Definition 5. *E' is the reconnection rate and is not limited to the 2D case.*

Magnetic reconnection has to do with the decoupling between the magnetic field and the plasma. While such decoupling can be achieved by magnetic diffusivity, this phenomena is oftenly a very slow and ineffective process, because of the small magnetic diffusivity (the plasma resistivity, is very small in most space and astrophysical plasmas).

As a consequence, many efforts have been dedicated to reconnection model. The first resistive model, the one of Sweet & Parker involves the resistivity of the plasma, but is more than pure diffusion of the magnetic field.

The Sweet-Parker model

This is the first theoretical model of magnetic reconnection : Sweet suggested in a 1956 conference [Sweet, 1958] the role of resistive diffusion in a resistive current sheet and Parker wrote the associated equations [Parker, 1957] using conservation laws. The geometry of this model is depicted in Fig. 3.3.

As many theoretical models, the solution of this problem consist in finding the forme of the fluid flow in the upstream region where the ideal term in the Ohm's law is leading, the form of this same flow in the diffusion region where diffusive effects (that is collisions) are leading, and then find the way to continuously connect these flows together. Hence :

- In the upstream region, the frozen-in theorem is valid so the ideal Ohm's law gives $E_z = UB_0$
- In the diffusion region, plasma effects are dominated by collisions so $E_z = \sigma^{-1} J_z$

At MHD scales, $\omega/k \ll c$ meaning that the displacement current can be neglected. Hence,

$$(\nabla \times \mathbf{B}).\hat{\mathbf{z}} = \mu_0 J_z \simeq \frac{B_0}{\delta_{\mathrm{SP}}} \tag{3.8}$$

meaning that in the diffusion region,

$$E_z = \frac{B_0}{\mu_0 \sigma \delta_{\mathrm{SP}}} \tag{3.9}$$

For a stationnary 2D process, $\partial_t B_x = \partial_t B_y = 0$, so that E_z is constant across the current sheet : the electric field in the diffusion region $E_z^{\mathrm{diffusion}} = J_z/\sigma$ has then to be equal to the electric field far upstream from the current sheet, that is in the ideal region where $E_z^{\mathrm{ideal}} = UB_0$. As a consequence,

$$U = \frac{1}{\mu_0 \sigma \delta_{\mathrm{SP}}} = \frac{\eta}{\delta_{\mathrm{SP}}} \tag{3.10}$$

Remark 10. *This form of U is equal to the diffusion velocity, as $\eta = l^2/\tau$, then*

$$v_{\mathrm{diff}} = \frac{l}{\tau} = \frac{\eta}{l} \tag{3.11}$$

with $l = \delta_{\mathrm{SP}}$

The upstream velocity U is very important as it is quantifying the reconnection efficiency (see Eq. 3.7). It is related to the geometry of the flow in the continuity equation

$$UL_{\mathrm{SP}} = v\delta_{\mathrm{SP}} \tag{3.12}$$

We then need an extra condition to get the value of the downstream flow v. It is going this way because both L_{SP} and δ_{SP} are geometric parameters of the flow and the SP model don't expect to provide their quantitative value. The downstream flow v can then be obtained from the pressure balance. In the upstream region, the inflow velocity U is quite small so that the pressure is essentially magnetic

$$\frac{B_0^2}{2\mu_0} \tag{3.13}$$

and in the downstream region where the magnetic field is vanishing, the pressure is essentially kinetic

$$\frac{1}{2}\rho v^2 \tag{3.14}$$

The pressure balance then writes

$$\rho v^2 = \frac{B_0^2}{\mu_0} \tag{3.15}$$

that is $v = A_0$: the outflow is Alfvénic (assuming a uniform density across the CS)

Remark 11. *In magnetic reconnection, any reference to an Alfvén velocity is always done using the upstream conditions.*

The electrical conductivity σ is measured in S.m^{-1}. But as a purely personnal choice, the magnetic diffusivity η related to σ with the relation

$$\eta = \frac{1}{\mu_0 \sigma} \tag{3.16}$$

is more pleasans to manipulate : because its unity is in m^2.s^{-1}, and because it quantifies the diffusion of the magnetic field across the plasma. Then,

$$\delta_{\mathrm{SP}} = \frac{\eta}{U} \tag{3.17}$$

so that one obtains

$$\frac{U^2}{A_0^2} = \frac{\eta}{L_{\mathrm{SP}} A_0} = \frac{1}{S} \tag{3.18}$$

Definition 6. *S is the Lundquist number : it is the ratio between the alfvén time and the diffusive time. Ii is also the magnetic Reynolds number if one considers the Alfvén velocity as the caracteristic velocity of the flow.*

The reconnection rate is then

$$E' = \frac{U}{A_0} = \frac{1}{\sqrt{S}} \tag{3.19}$$

so the simple geometric relation $\delta_{\mathrm{SP}} = L_{\mathrm{SP}} E'$. To set orders of magnitude, $S \sim 10^8$ in solar flares and $S \sim 10^{11}$ in the solar wind/Earth magnetosphere, which are quite large values, then giving a very small reconneciton rate. For these reasons, such reconnecting mechanicm is called "slow reconnection".

Remark 12. *The SP regime is called collisional as the Alfvén time associated to the length L_{SP} is larger than $\tau_{\mathrm{collision}}$*

As a result, any length, hence the CS thickness, is larger than d_i : $\delta_{\mathrm{SP}} > d_i$. Furthermore, when the plasma is expelled, it is frozen in the plasma, so it drives the B field out of the reconnection region. As a consequence, L_{SP} is growing meaning that the CS is getting elongated. This is not part of the SP model, but it has been verified experimentally and numerically.

During the increase of L_{SP} (decrease of S), the CS elongates until being unstable to secondary islands at $S_{\mathrm{crit}} \sim 10^4$. This regime will be discussed later in this chapter. For a uniform resistivity, simulation always produce a SP reconnection topology with a highly elongated diffusion region. In large collisional full-PIC [Daughton et al., 2009], for $\delta > d_i$, ρ_s, the SP-like CS grows hence getting elongated.

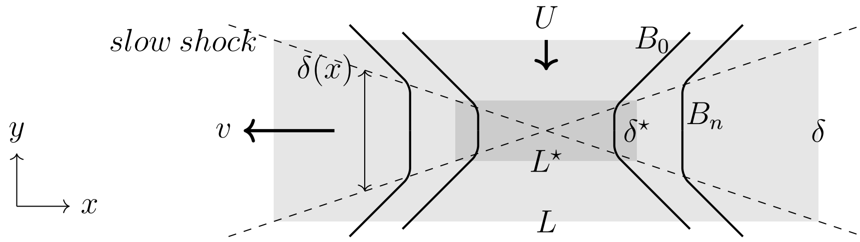

Figure 3.4: Schematic of the reconnection zone in the Petschek model

Remark 13. *The plasma inflow in SP is supposed laminar... if not, it could be the source of plasmoids [Lazarian and Vishniac, 1999].*

The Petscheck model

The Petscheck model starts with the topology of the Sweet-Parker model, but with an additional ingredient, which emphasizes the clear-sightedness of its author : in this model, there is still a dissipative zone in which the resistivity allows the topological reconfiguration of the magnetic field lines, but the plasma is not forced to pass through it! This eliminates the bottleneck of the Sweet-Parker model which is at the origin of its low reconnection rate. However, it will be necessary to find the process that will allow the acceleration of the plasma that no longer passes through the diffusion zone : it will be a slow shock.

Notation 4. *We introduce two characteristic scales in the plasma ejection direction : L is the characteristic dimension of the reconnection area, and $L^\star$ is the diffusion scale.*

In the Petschek model, the outflow region is much broader (between shock fronts of a slow shock) so the plasma is accelerated by the slow shock.

Notation 5. *In this model, the thickness $\delta(x)$ is an x-function, with a limit $\delta^\star$ in the diffusion region of length $L^\star \ll L$.*

Notation 6. *We define the dimensionless quantity $b_n = B_n/B_0 \ll 1$. We also remember that the reconnection rate is*

$$E' = \frac{U}{A_0} \tag{3.20}$$

Notation 7. *We also define two Alfvén velocity,*

$$A_n = \frac{B_n}{\sqrt{\mu_0 \rho}} \text{ and } A_T = \frac{B_T}{\sqrt{\mu_0 \rho}} \tag{3.21}$$

on each sides of the discontinuity (with respect to the normal)

Hypothesis 1. *We make the hypothesis of uniform ($\rho_0 \equiv \rho_1$) incompressible ($[V_n]_{\text{shock}} = 0$) plasma.*

The continuity equations writes

$$Ux = v(x)\delta(x) \tag{3.22}$$

for a uniform inflow velocity U (laminar hypothesis).

• Far from the diffusion region (upstream and downstream) but in the neighborhood of the shock :
The jump equations for the tangential momentum in the HT frame writes

$$[(V_n^2 - \frac{B_n^2}{\mu_0 \rho})\mathbf{B}_T]_{\text{shock}} = 0 \tag{3.23}$$

In the upstream region, $V_{n0} = U$, B_n is conserved and $B_{T0} \sim B_0$. In the downstream region, $\mathbf{B}_{T1} = 0$. As a result, the parenthesis in the upstream is null, that is

$$U^2 = \frac{B_n^2}{\mu_0 \rho_0} \tag{3.24}$$

A first equation is then $E' = |b_n|$ which depends (as B_n) on x and y.

With the incompressible hypothesis, $[V_n]_{\text{shock}} = 0$ and $[B_n]_{\text{shock}} = 0$, so we have $[A_n]_{\text{shock}} = 0$. We then have the condition $V_n[\mathbf{V}_T]_{\text{shock}} = A_n[\mathbf{A}_T]_{\text{shock}}$ from the transverse momentum equation. V_n and A_n being conserved, we get $\mathbf{V}_{T0} - \mathbf{A}_{T0} = \mathbf{V}_{T1} - \mathbf{A}_{T1}$. With $V_{T0} \sim 0$, $A_{T0} \sim A_0$, $V_{T1} \sim v$ and $A_{T1} \sim 0$, we obtain

$$v = A_0 \tag{3.25}$$

as in the SP model. Note that v is then constant, *i.e.* does not depend on x. With the help of the continuity equation, $\delta(x) = E'|x|$ is linear.

• In the diffusion region :
The diffusion velocity writes

$$v_{\text{diff}} = \frac{\eta}{\delta^\star} = U \tag{3.26}$$

like in the SP model, to ensure the stationarity condition. A first condition is then

$$E' = \frac{\eta}{\delta^\star A_0} \tag{3.27}$$

so that

$$\delta^\star = \frac{\eta}{E' A_0} \tag{3.28}$$

which then defines the value of $\delta^\star$. A second condition is the relation between $L^\star$ and $\delta^\star$: for geometrical reasons,

$$\delta^\star = |b_n| L^\star = E' L^\star \tag{3.29}$$

In the diffusion region, b_n has to be an odd function of x, so we assume a linear form with

$$b_n(L^\star) = E' = \frac{\delta^\star}{L^\star} \tag{3.30}$$

hence

$$b_n(x) = \frac{(E')^3 A_0}{\eta} x \tag{3.31}$$

which is a first order approximation at the corner of the diffusion region, as well as

$$L^\star = \frac{\eta}{(E')^2 A_0} \tag{3.32}$$

The general solution has to continuously connect these 2 solutions in and out of the diffusion region. The form of the magnetic field $\mathbf{B}$ is solution of a Laplace equation with boundary conditions given by Eq. 3.31 at the shock.

The solution of the Laplace equation $\Delta\mathbf{B} = 0$ on D with $\mathbf{B}^\star(\mathbf{r}')$ given on $\mathbf{r}' \in \partial D$ is

$$\mathbf{B}(\mathbf{r}) = -\oint \mathrm{d}S' \mathbf{B}^\star(\mathbf{r}').\boldsymbol{\nabla} G_{\mathrm{HD}}(\mathbf{r}, \mathbf{r}') \tag{3.33}$$

where

$$G_{\mathrm{HD}}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{3.34}$$

is the Green function for homogeneous Dirichlet BC. The shock equation (in the first quadrant) being $y = E'x$, the integration gives

$$B_x(x, y) = -\frac{4E'B_0}{\pi} \ln \frac{L}{\sqrt{x^2 + y^2}} \text{ and } B_y(x, y) = \frac{4E'B_0}{\pi} \arctan \frac{y}{x} \tag{3.35}$$

We then have

$$B_x(L^\star, \delta^\star) = -\frac{4B_0 E'}{\pi} \ln \frac{L}{L^\star} \tag{3.36}$$

Eq. 3.37 relates the reconnection rate E', the geometry of the system (L and $L^\star$) and the magnitude of B_x at $(L^\star, \delta^\star)$ which is unknown.
The reconnection rate is then

$$E' = \frac{\pi B_x(L^\star, \delta^\star)}{4B_0} \frac{1}{\ln L/L^\star} \tag{3.37}$$

With $L^\star$ given by Eq. 3.32, and the arbitrary value $B_x(0, \delta^\star) = -\frac{1}{2}B_0$, one obtains

$$E' = \frac{\pi}{16} \frac{1}{\ln(E'S)} \tag{3.38}$$

Being in the Log, the E' in the rhs weakly contributes, so we have

$$E' \sim \frac{1}{\ln S} \tag{3.39}$$

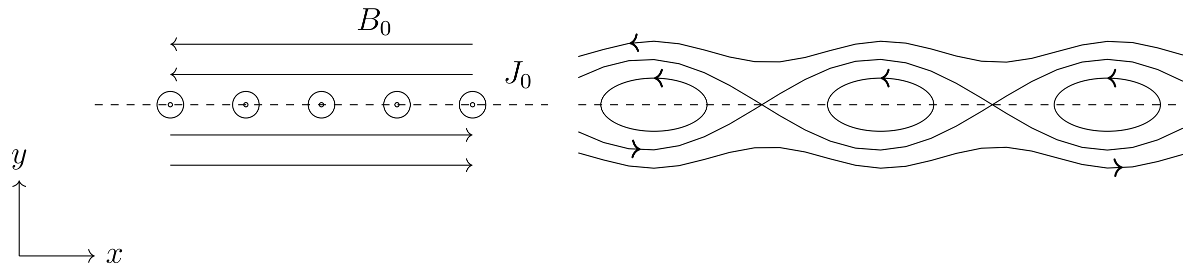

Figure 3.5: Schematic of a Harris sheet (left) and its evolution with the tearing mode (right)

It is the magnetic tension, important at the angular point, which ensures the acceleration of the plasma at the Alfvén speed. Through a slow shock, the normal velocity and tangential components of the magnetic field decrease, and the pressure and density increase. In the downstream flow zone, the magnetic field decreases by conservation of the flux. The kinetic pressure must therefore increase to satisfy the pressure balance, thus the heating of the plasma.

To increase the reconnection rate, it is necessary to increase the inflow speed U. However, this speed is not defined in a self-consistent way, and its relation with the magnetic field does not necessarily make it possible to reach a fast reconnection regime.

Petschek reconnection has been observed in-situ in the far geomagnetic tail ([Machida et al., 1994]) and is observed in simulation only for a highly localized resistivity (see eg. [Krauss-Varban and Omidi, 1995], [Ugai, 1999]).

3.2 The Harris kinetic equilibrium

In order to study magnetic reconnection, one must first describe the current sheet associated with the magnetic field reversal. The paper of [Harris, 1962], describes such a layer in the one-dimensional case. The left panel of Fig. 3.5 displays the shape of the magnetic field and the associated current.

A kinetic equilibrium is determined by the distribution function of each specie s as well as that of the associated electric and magnetic fields. This set must obviously satisfy the Vlasov equation (for each species s), as well as the stationary Maxwell equations. The resulting system is thus of the form

$$\mathbf{v}.\frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\frac{\partial f_s}{\partial \mathbf{v}} = 0 \tag{3.40}$$

$$\boldsymbol{\nabla}.\,\mathbf{E} = \frac{1}{\varepsilon_0}\sum_s \int_{\mathbb{R}^3} q_s f_s \, d\mathbf{v} \tag{3.41}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \sum_s \int_{\mathbb{R}^3} q_s f_s \mathbf{v} \, d\mathbf{v} \tag{3.42}$$

For a 1-dimensional equilibrium, the operators ∂_x and ∂_z are identically null. So the generalized canonical momentum P_x and P_z are constants of the motion. Moreover, the system is isolated (constant energy) and so the Hamiltonian H is also a constant of motion. The magnetic field is of the form $\mathbf{B} = B(y)\widehat{\mathbf{x}}$ and thus derives from a vector potential of the form

$$\mathbf{A} = A(y)\widehat{\mathbf{z}} \tag{3.43}$$

For each s particle specie, the invariants can be written (omitting the s index)

$$\begin{aligned} H &= {}^1\!/\!_2 m(v_x^2 + v_y^2 + v_z^2) + q\phi && (3.44)\\ P_x &= mv_x && (3.45)\\ P_z &= mv_z + qA(y) && (3.46) \end{aligned}$$

so we can deduce three new constants of motion for each s specie

$$\begin{aligned} \alpha_3 &= \frac{P_x}{m} = v_x && (3.47)\\ \alpha_2 &= \frac{P_z}{m} = v_z + q\frac{A_z}{m} && (3.48)\\ \alpha_1^2 &= 2\frac{H}{m} - \alpha_3^2 - \alpha_2^2 = v_y^2 - 2qv_y\frac{A_z}{m} - \frac{q^2A_z^2}{m^2} + 2q\frac{\phi}{m} && (3.49) \end{aligned}$$

For a system with three degrees of freedom (the three speed coordinates) a distribution function which only depends on the three constants α_1, α_2 and α_3 is by construction solution of the Vlasov equation. For a Maxwellian distribution with a density n_s, a bulk speed V_s (in the z direction) and a temperature T_s,

$$f_s(\alpha_1, \alpha_2, \alpha_3) = n_s\left(\frac{m_s}{2\pi k_B T_s}\right)^{3/2} \exp - \left[\frac{m_s}{2k_B T_s}(\alpha_1^2 + (\alpha_2 - V_s)^2 + \alpha_3^2)\right] \tag{3.50}$$

with

$$\alpha_1^2 + (\alpha_2 - V_s)^2 + \alpha_3^2 = v_x^2 + v_y^2 + (v_z - V_s)^2 + \frac{2q_s\phi}{m_s} - \frac{2q_sAV_s}{m_s} \tag{3.51}$$

The form of these distribution functions for each specie s is then integrated on $\mathbb{R}^3$ in the two Maxwell equations. These equations depend on the scalar and vector potentials. For the scalar potential,

$$\boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \frac{d^2A}{dy^2} \tag{3.52}$$

and for the scalar potential,

$$\boldsymbol{\nabla}.\,\mathbf{E} = \boldsymbol{\nabla}.\,(-\boldsymbol{\nabla}\phi) = -\frac{d^2\phi}{dy^2} \tag{3.53}$$

For a hydrogen plasma consisting of protons p and electrons e, of same density n and whose modulus of charge is e, after the integration of Eq. (3.41), the equation on ϕ is rewritten

$$\frac{d^2\phi}{dy^2} = \frac{en}{\varepsilon_0}\left\{\exp\left[-\frac{e\phi}{k_BT_p} + \frac{eAV_p}{k_BT_p}\right] - \exp\left[\frac{e\phi}{k_BT_e} - \frac{eAV_e}{k_BT_e}\right]\right\} \tag{3.54}$$

and Eq. (3.42) on A is rewritten

$$\frac{d^2A}{dy^2} = \mu_0 en\left\{V_p\exp\left[-\frac{e\phi}{k_BT_p} + \frac{eAV_p}{k_BT_p}\right] - V_e\exp\left[\frac{e\phi}{k_BT_e} - \frac{eAV_e}{k_BT_e}\right]\right\} \tag{3.55}$$

In the de Hoffman-Teller frame, the electric field is null : the temperatures and drift velocities of each specie s then satisfy

$$-\frac{V_e}{T_e} = \frac{V_p}{T_p} \tag{3.56}$$

Exercize 6. *Using a first order linearization of the momentum equation of specie s, show Eq. (3.56)*

With the new hypothesis $T_e = T_p$, the system writes for ϕ

$$\frac{d^2\phi}{dy^2} = \frac{en}{\varepsilon_0}\exp\left[\frac{eAV}{k_BT}\right]\left\{\exp\left[-\frac{e\phi}{k_BT}\right] - \exp\left[\frac{e\phi}{k_BT}\right]\right\} \tag{3.57}$$

and for A

$$\frac{d^2A}{dy^2} = \mu_0 enV\exp\left[\frac{eAV}{k_BT}\right]\left\{\exp\left[-\frac{e\phi}{k_BT}\right] + \exp\left[\frac{e\phi}{k_BT}\right]\right\} \tag{3.58}$$

$\phi = 0$ is solution for the first equation, so the second equation can be written

$$\frac{d^2A}{dy^2} = 2\mu_0 enV\exp\left[\frac{eAV}{k_BT}\right] \tag{3.59}$$

A solution for this equation satisfying $A = 0$ and $B = 0$ at $y = 0$ is of the form

$$A(y) = \Gamma\ln[\cosh(\eta y)] \tag{3.60}$$

By deriving twice and identifying the terms, we get

$$\Gamma = -\frac{2k_BT}{eV}\ , \quad \eta = \left(\frac{\mu_0 ne^2V^2}{4k_BT}\right)^{1/2} \tag{3.61}$$

that is, introducing the Debye length λ_D,

$$\eta = \frac{V}{2c\lambda_D} \tag{3.62}$$

The magnetic field is of the form

$$\mathbf{B} = (\mu_0 nk_BT)^{1/2}\tanh\left(\frac{Vy}{2c\lambda_D}\right)\widehat{\mathbf{x}} \tag{3.63}$$

By isolating the density term n in Eq. (3.59), one obtains

$$n(y) = n\cosh^{-2}\left(\frac{Vy}{2c\lambda_D}\right) \tag{3.64}$$

For the forthcoming calculations, we write L the half-width of the Harris sheet

$$L = \frac{2c\lambda_D}{V} \tag{3.65}$$

In this result, we can discuss several hypotheses, including choosing $\phi = 0$. This implies that in the current layer, the equilibrium is associated with no electric field. This point can be discussed, and the satellite measurements do not show it clearly. But it is a mathematically sympathetic hypothesis, often made by the few models of kinetic equilibrium of the current layer.

3.3 The collisional tearing mode

The collisional tearing mode is an example of resistive MHD instabilities. The idea is to write the minimum set of equations in order to describe the plasma configuration, and to find a solution as purely growig mode (in time) for a given wavenumber. The system of coordinate is the same as for the Harris equilibrium ; X is the main direction of the anti-parallel (unperturbed) magnetic field, Y is the normal direction where the gradient are explicit, and Z is the direction of the current.

We start withthe linear phase of the collisional tearing mode firstly discussed by [Furth et al., 1963]. In order to decrease the number of unknowns, and thus the number of equations, we describe the magnetic field using the Euler potential. The magnetic field is thus written using the two scalar potential $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta \tag{3.66}$$

Exercize 7. *Can you justify this choice ? Can we describe whatever magnetic field with such potential description ? Can you say why the magnetic field lines are then given by the intersection of the two surfaces $\alpha = Cst.$ and $\beta = Cst.$?*

The unperturbed magnetic field has a component in the X direction and the first order perturbations also have a component in the Y direction. Hence, the magnetic field can be written

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\psi \times \hat{\mathbf{z}} + B_G\hat{\mathbf{z}} \tag{3.67}$$

In the above equation, B_G is a guide field, which plays no role in this instability. Noting B_0 the assymptotic magnetic field and $f(y)$ the analytical profile of the unberturbed magnetic field, we have $B_0 f(y) = \partial_y\psi_0$, and a first order ψ_1 quantity will also develop. The total current is then given by

$$\mathbf{J} = \frac{1}{\mu_0}\nabla^2\psi\hat{\mathbf{z}} \tag{3.68}$$

As we are looking for a purely growing mode, ψ_0 depends on y, and ψ_1 evolves as $e^{\imath kx}$. The laplacian operator then writes $\nabla^2 = \partial^2_{y^2} - k^2$.

In the same spirit, we chose to describe the velocity using a potential. As we search for a velocity in the XY plan (i.e. without a Z component), then

$$\mathbf{v} = \boldsymbol{\nabla}\phi \times \hat{\mathbf{z}} \tag{3.69}$$

Exercize 8. *Can you explain this choice ? Do we have enough degrees of freedom with this form ? Is there any associated constraint ?*

We then have a parametric form of the magnetic field, while the electric field is given by the resistive Ohm's law. We do not need any continuity equation for an incompressible plasma, and we will see later on that we do not need any closing equation for the pressure. We then have all the needed parameters. The next step is to write the Maxwell-Ampère equation :

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \left[-\mathbf{v} \times \mathbf{B} + \frac{\mathbf{J}}{\sigma} \right] \tag{3.70}$$

Using the Euler potentials, this equation can be written

$$\frac{\partial \psi}{\partial t} - \mathbf{v}.\boldsymbol{\nabla}\psi = \frac{1}{\mu_0 \sigma} \nabla^2 \psi \tag{3.71}$$

Exercize 9. *Demonstrate the form of Eq. (3.71).*

As the plasma velocity $\mathbf{v}$ appears in this equation, we need to write an equation giving its time evolution ; it is of course the momentum equation. Its curl gives

$$nm \left(\frac{\partial \nabla^2 \phi}{\partial t} + \mathbf{v}.\boldsymbol{\nabla}\nabla^2\phi \right) = \frac{\hat{\mathbf{z}}}{\mu_0} \left[\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}(\nabla^2 \psi) \right] \tag{3.72}$$

Exercize 10. *Is it now clear why we do not need any closure equation for the pressure ? is there any underlying assumption(s) ?*

The system of Eq. (3.71) and Eq. (3.72) drives the time evolution of the two scalars ϕ and ψ. As is, this system is not that simple to solve as it is differential in both space and time. But as already mentioned, we look at a solution with a wavenumber k in the X direction, we keep the explicit derivative in the Y direction (as we need to consider the form of $f(y)$), and we solve this system for a purely growing mode in $e^{\gamma t}$. After linearizing this sytem at first order, one obtains

$$\gamma \psi_1 + \imath k B_0 \phi_1 f(y) = \frac{1}{\mu_0 \sigma} \left(\frac{\partial^2}{\partial y^2} - k^2 \right) \psi_1 \tag{3.73}$$

$$\gamma nm \left(\frac{\partial^2}{\partial y^2} - k^2 \right) \phi_1 = \frac{\imath k B_0}{\mu_0} \left[\psi_1 \frac{\partial^2 f(y)}{\partial y^2} - f(y) \left(\frac{\partial^2}{\partial y^2} - k^2 \right) \psi_1 \right] \tag{3.74}$$

We then need to distinguish two regions : far from the current sheet where the system is somewhat in ideal MHD conditions (i.e. $\sigma \to \infty$) and inside the current sheet, where the plasma is dominated by the diffusive processes.

In the MHD region : From Eq. (3.73), we have

$$\phi_1 = \imath \frac{\gamma}{kB_0} \frac{\psi_1}{f(y)} \tag{3.75}$$

combining with Eq. (3.74) we get

$$\left[\frac{d^2}{dy^2} - k^2 - \frac{1}{f(y)} \frac{d^2 f(y)}{dy^2} \right] \psi_1 = -\frac{(\gamma \tau_A)^2}{f(y)} \psi_1 \sim 0 \tag{3.76}$$

where we have introduced $\tau_A = (kv_A)^{-1}$, the characteristic Alfven time. As $f(y)$, ψ_1 and ϕ_1 are only depending on the Y coordinate, we change the partial derivative with respect to y in total derivative. Eq. (3.76) is close to zero, as we make the hypothesis that $\gamma \tau_A \ll 1$ (which will be verified at the end of the calculation).

It then appears that we have a singular point at $y = 0$ because $f(0) = 0$. As a consequence, ψ_1 is discontinuous at the origin $y = 0$. We then introduce a parameter to characterize this discontinuity, defined by

$$\Delta' = \frac{d_y \psi_1(0^+) - d_y \psi_1(0^-)}{\psi_1(0)} \tag{3.77}$$

Exercize 11. *With this definition, one can verify that $d_y B_y = \Delta' B_y$ at $y = 0$.*

In the diffusive region : ψ_1 and ϕ_1 are such that their second derivative with respect to y is very large compared to their respective product with k^2 (because of the discontinuities of their derivatives). Hence, the system of Eq. (3.73) and (3.74) can be rewritten

$$\gamma \psi_1 + \imath k B_0 \phi_1 \frac{y}{L} \sim \frac{1}{\mu_0 \sigma} \frac{d^2 \psi_1}{dy^2} \tag{3.78}$$

$$\gamma n m \frac{d^2 \phi_1}{dy^2} \sim -\imath \frac{k B_0}{\mu_0} \frac{y}{L} \frac{d^2 \psi_1}{dy^2} \tag{3.79}$$

because we approximate $f(y) \sim y/L$ (valid in the very middle of the current sheet). We then need to make the connection between the two solutions, inside the diffusive region and in the MHD region.

At the edge of the layer : we note $y = W$ the edge of the diffusive layer. The second derivative of both ψ_1 and ϕ_1 can be written

$$\frac{d^2 \psi_1}{dy^2} = \frac{\Delta'}{W} \psi_1 \tag{3.80}$$

$$\frac{d^2 \phi_1}{dy^2} = \frac{1}{W^2} \psi_1 \tag{3.81}$$

At the edge $y = W$ of the layer, the three terms in the Maxwell-Ampère equation are then comparable, so

$$\gamma\psi_1 \sim \imath k B_0 \phi_1 \frac{W}{L} \sim \frac{1}{\mu_0 \sigma}\frac{\Delta'}{W}\psi_1 \tag{3.82}$$

The momentum equation given by Eq. (3.79) can also be approximated by

$$nm\frac{\gamma}{W^2}\phi_1 \sim -\imath\frac{kB_0}{\mu_0}\frac{W}{L}\frac{\Delta'}{W}\psi_1 \tag{3.83}$$

With Eq. (3.82) and (3.83), we have $\left(\frac{W}{L}\right)^5 = \frac{\Delta' L}{S^2}$ where

$$S = \frac{\tau_R}{\tau_A} \tag{3.84}$$

is the Lundquist number, $\tau_A = L/v_A$ is the Alfven time and $\tau_R = L^2/\eta$ the resistive time. Then, the growth rate of the resistive tearing mode is

$$\gamma = \frac{(\Delta' L)^{4/5}}{\tau_R} S^{2/5} \tag{3.85}$$

It is important to note that in most of the space and astrophysical plasmas, S is very large, meaning that $\gamma^{-1} \ll \tau_R$. Even if the origin of the tearing mode is resistivity, it means that the associated reconnection process is much faster than the simple magnetic diffusivity. It is also important to note that the growth rate of the collisional tearing mode increase with L, i.e. the length of the current sheet.

In his seminal paper, [Loureiro et al., 2005] described in a quantitative way how this linear phase evolves with time. He studied the case of a large Δ'/low η current sheet, destabilized for $\Delta' > 0$. In such a strongly driven regime, the linear tearing mode develops as described by [Furth et al., 1963]. Then, its following non-linear phase is well described by [Rutherford, 1973] as a square law with time. During this phase, the two islands bordering the X-line have a width W that is growing until a critical value W_{critical}. At this stage, the X-line collapses, meaning that a current sheet is developing and elongating between the two islands. The fastest growth of W is observed during this last phase. The length L of this current sheet then increases with time, and during this phase, the reconnection process is the one described by [Sweet, 1958] and [Parker, 1957] with a growth rate $\gamma_{\text{SP}} \sim \eta^{\frac{1}{2}}$. The thickness of the current sheet $\delta \sim \eta^{\frac{1}{2}}$ leads to an aspcet ratio L/δ that grows with time, until a critical value around 50, observed numerically. In term of the Lundquist number, elongated SP layers are unstable to secondary magnetic islands (MATTHAEUS1985) for $S > S_{\text{critical}} \sim 10^4$,

Then, for such an elongated current sheet, the collisional tearing turns to be unstable, so that a new island (surrounded by two X-lines) is created in the elongated Sweet-Parker current sheet. This secondary island is attracted by the primary one, so will merge with the closest/larger of their neighbour. The process continues until no more magnetic flux is available, or when "monster plasmoids" will get this region too far from being a current sheet ready for reconnection. This happen for a critical value of W, at which the instability saturates, ending this chained process.

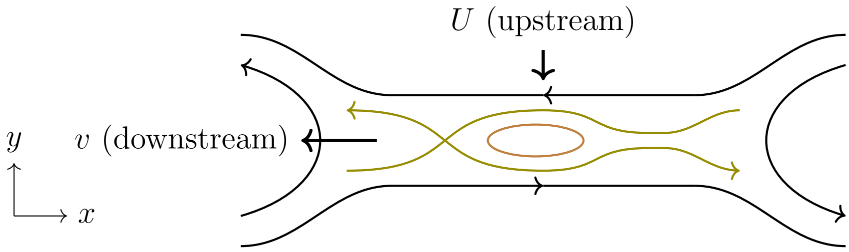

Figure 3.6: Time evolution of the topology of a SP current-sheet unstable to plasmoids.

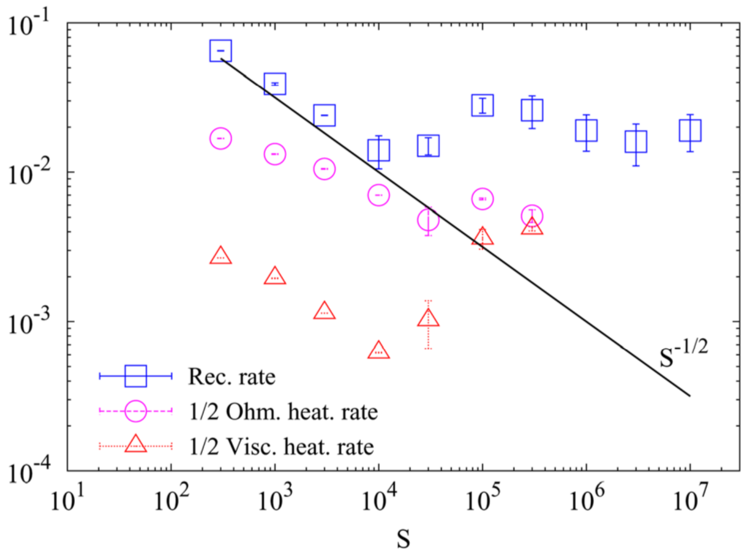

Figure 3.7: Time evolution of the reconnection rate (blue line) depending on the Lundqvits number S (see [Loureiro et al., 2012]).

[Loureiro et al., 2012] showed that for $S > 10^4$, the reconnection rate no longer decreases with a law in $S^{-1/2}$ as in the model of Sweet- Parker, but saturates at a value of about 10^{-2}, associated to $S_{\text{critical}} \sim 10^4$. This is illustrated in Fig. 3.7. Note that the definition of the reconnection rate must be reviewed in a system for which there is more than one reconnection site. Although not discussed in this document, the resistive tearing mode can therefore play a role in the formation of “ plasma chain ” in a thin and long current layer, even for a medium for which $S \to \infty$. The numerical results of [Loureiro et al., 2012] also show the possibility during the coalescence of these structures to create “ monster plasmoids ”.

3.4 The collisionless tearing mode

In a Harris sheet, the magnetic field depends only on the y coordinate. The stationary Maxwell-Ampère equation provides the current density required to maintain this magnetic topology,

$$\mu_0 \mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B} \tag{3.86}$$

This magnetic field is associated with a current sheet in the direction $+\widehat{\mathbf{z}}$ located at $y = 0$ and of infinite extension in x and z directions as shown in the left panel of Fig. 3.5. This layer can be seen as an assembly of wires of parallel currents. There is hence an attractive force between each of these wires, whose modulus is inversely proportional to the square of their distance. An initial disturbance consisting in bringing together two close-by wires is therefore amplified, which reflects an unstable situation. However, it must be ensured that the quantity of energy available is sufficient. In addition, the current density accumulation involves a change in the topology of the field lines as shown in the right panel of Fig. 3.5, which is a priori not allowed by the frozen-in theorem.

We can give a qualitative answer to these two problems before demonstrating it analytically. To analyze the stability of the phenomenon, it is necessary to know if the energy released is higher than the one necessary to topologicaly modify the magnetic field lines. The energy released by the coalescence of current filaments depends on the length of the current sheet (in the X direction), whereas the modification of the shape of the magnetic field occurs on the thickness of the layer (in the Y direction). Since the length of the sheet is always greater than its thickness, there will always be an instability threshold for a sufficiently thin and long layer.

One then has to identify the physical argument(s) that no longer freeze the magnetic field in the plasma. Since the work of Landau, the effect that bears his name (and by which the particles whose distribution function f is such that $\partial_\mathbf{v} f > 0$ excite a wave with a positive growth rate) is a good candidate to produce diffusion. The situation in which a wave excites particles whose velocity is equal to or greater than the phase velocity of the wave also exists, and is often called the Čerenkov effect. It is by this effect that the waves produced by the tearing mode will accelerate the particles, and produce the diffusion sufficient to modify the magnetic field line connections.

For the development of the Čerenkov effect, the particles must be unmagnetized. In the opposite case, the cyclotronic motion associated with a magnetic field does not allow the resonance of the particles (or more exactly not the one associated with the Čerenkov effect). This is so because at y (the reference $y = 0$ is taken at the center of the layer) the local Larmor radius must be larger than y. By approximating the shape of the magnetic field at the center of the layer by a linear law, this inequality is written

$$|y| < l_s = \sqrt{\rho_s L} \tag{3.87}$$

where $\rho_s = v_{Ts}/\Omega_s$ is the Larmor radius of the thermal particles of species s, $v_{Ts} = \sqrt{k_B T_s / m_s}$ is the thermal velocity of the particles of species s and $\Omega_s = q_s B_0 / m_s$ is the cyclotron frequency of the species s in the asymptotic field B_0.

The problem is therefore simple : considering a perturbation of the magnetic field whose vector potential is of the form [1]

[1] This choice results from the shape of the magnetic field of order 0 and 1 that we are looking for

$$\mathbf{A}_1(x, y, t) = A_1(y) e^{-\imath(\omega t - kx)} \widehat{\mathbf{z}} \tag{3.88}$$

it has to satisfy the Maxwell-Faraday equation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}_1) = \mu_0 \mathbf{J}_1 \tag{3.89}$$

in which the right-hand side must include all the currents of order 1 : the one associated with the current density perturbation $\mathbf{J}_1^\star$, and the one associated to the resonant particles $\mathbf{J}_1^\dagger$ (since the plasma will respond to the perturbation).

The shape of the distribution function in a Harris layer is given by the Eq. (3.50). It is still valid for the case of tearing mode because the canonical momentum P_z is always a Hamiltonian invariant. It is then necessary to take into account the perturbed part of the vector potential. This one being small, one can make a Taylor expansion at first order of the distribution given by Eq. (3.50) to get the distribution function of the specie s at first order $f_{1s}^\star$

$$f_{1s}^\star = \frac{q_s}{k_B T_s} \mathbf{V}_s. \, \mathbf{A}_1(x, y, t) f_{0s}(y, \mathbf{v}) \tag{3.90}$$

Exercize 12. *Using the form of $A_0(y)$ given by Eq. (3.60), show that the sum on the species s of the moments of order 1 of $f_{1s}^\star$ gives*

$$\mathbf{J}_1^\star = \frac{2}{\mu_0 L^2 \cosh^2(y/L)} A_1 \widehat{\mathbf{z}} \tag{3.91}$$

To get $\mathbf{J}_1^\dagger$ associated with the Čerenkov effect, you have to solve the Vlasov equation to get $f_{1s}^\dagger$. This calculation can only be done for the species s in the region $|y| < l_s$: the magnetic field of order 0 is very small (the particles are demagnetized). The Vlasov equation of order 1 can be solved with the method of the characteristics ; it is written [2]

$$\frac{\partial f_{1s}}{\partial t} + \mathbf{v}.\frac{\partial f_{1s}}{\partial \mathbf{r}} = -\frac{q_s}{m_s}\left[-\frac{\partial \mathbf{A}_1}{\partial t} + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}_1)\right].\frac{\partial f_{0s}}{\partial \mathbf{v}} \tag{3.92}$$

The last term of the right-hand side is associated with the evolution of $f_{1s}^\star$ (the magnetic force does not work). The first is therefore associated with the electric force, and thus with the evolution of $f_{1s}^\dagger$. This equation is therefore written

$$\frac{\partial f_{1s}^\dagger}{\partial t} + \mathbf{v}.\frac{\partial f_{1s}^\dagger}{\partial \mathbf{r}} = \frac{q_s}{m_s}\frac{\partial \mathbf{A}_1}{\partial t}.\frac{\partial f_{0s}}{\partial \mathbf{v}} \tag{3.93}$$

The right-hand side is integrated using the characteristics method along an undisturbed orbit :

$$f_{1s}^\dagger = \int_{-\infty}^{t} \frac{\imath\omega q_s}{k_B T_s} A_1(y) v_z f_{0s}(y, \mathbf{v}) e^{-\imath[\omega\tau - kx(\tau)]} d\tau \tag{3.94}$$

[2] As for the Harris sheet, an electrostatic potential that is identically zero in the whole domain is a solution ... and this is the one that we choose therefore

The orbit $x(\tau)$ of a particle in the undisturbed field is simple : being demagnetized, the equation of its motion is $x(\tau) = v_x \tau$. It is further assumed that for each specie s, the resonant particles are in the $|y| < l_s$ layer, which implies that $A_1(y)$ can be considered as constant. We can therefore take this term out of the integral, as well as the order 0 of the distribution function (for the same reason). The integral therefore relates only to the exponential term,

$$f_{1s}^{\dagger} = \frac{-\omega q_s}{k_B T_s} A_1(y) v_z f_{0s}(y, \mathbf{v}) \frac{1}{\omega - k v_x} e^{-\imath(\omega - k v_x)t} \tag{3.95}$$

The function $f_{1s}^{\dagger}$ has a single pole at the Čerenkov resonnance ($v_x = \omega/k$). To calculate its first order moment, one has to define this function on all $\mathbb{C}$ by analytic continuation, and use the residue theorem to calculate the integral of $f_{1s}^{\dagger}$ to get the current. By summing on the species s and using Plemelj's formula,

$$J_{1z}^{\dagger} = \sum_s \frac{\omega q_s^2}{k_B T_s} A_1(y) \iiint_{\mathbb{R}^3} v_z^2 f_{0s}(y, \mathbf{v}) \imath \pi \delta(\omega - k v_x) d\mathbf{v} \tag{3.96}$$

Exercize 13. *Show that this integral can be written*

$$\mathbf{J}_1^{\dagger} = \sum_s -\imath \pi^{1/2} \frac{\omega_{Ps}^2}{\mu_0 c^2} \frac{\omega}{|k| v_{Ts}} A_1(y) \widehat{\mathbf{z}} \tag{3.97}$$

The form of $J_{1z}^{\dagger}$ is only valid for $|y| < l_s$. Outside this layer, the particles are magnetized and can not satisfy the Čerenkov effect, ie $\mathbf{J}_{1s}^{\dagger} = 0$. Returning to the Maxwell-Faraday equation,

$$\frac{d^2 A_1(y)}{dy^2} - \left[k^2 + \Psi_0(y) + \sum_s \Psi_s(y, \mathbf{k}, \omega) \right] A_1(y) = 0 \tag{3.98}$$

with

$$\Psi_0 = -\frac{2}{L^2 \cosh^2(y/L)} \ , \ \Psi_s = \begin{cases} -\imath \pi^{1/2} \dfrac{\omega}{|k| v_{Ts}} \dfrac{\omega_{Ps}^2}{c^2} & |y| < l_s \\ 0 & |y| > l_s \end{cases} \tag{3.99}$$

In Eq. (3.98), the first term in the brackets is the Laplacian in the x direction, the second one is equal to $\mu_0 \mathbf{J}_1^{\star}$ and the third one is equal to $\mu_0 \mathbf{J}_1^{\dagger}$. The problem is thus to solve a Schrödinger equation in a potential Ψ_0 supperposed to a thin potential well $\sum_s \Psi_s$ in its center. We then have to find the form of $A_1(y)$ in and out of Ψ_s, and then connect these 2 solutions. In the region where Ψ_s is null (out of the current layer), the solution must satisfy $A_1(y) \to 0$ for $y \to \pm\infty$, and is (see [White et al., 1977])

$$A_1(y) = A_1(0) \left[1 + \frac{\tanh(|y|/L)}{kL} \right] e^{-k|y|} \tag{3.100}$$

The solution of $A_1(y)$ in the region where at least one of the Ψ_s is not negligible is more hard, and can only be done in simple cases. Considering that the potential Ψ_e is constant in $|y| < l_e$, the solution of the Maxwell-Faraday equation is (by obviously omitting adiabatic contributions Ψ_0 as well as those of protons Ψ_p, which is justified if Ψ_e is large enough)

$$A_1(y) = A_1(0)\cosh(\Psi_e^{1/2} y) \tag{3.101}$$

which is a solution of $d^2_{y^2} A_1 - \Psi_e A_1 = 0$.

It remains to ensure the continuity of the two solutions (3.100) and (3.101) at the border $y = \pm l_e$, which can be done by equalizing the logarithmic derivatives

$$\frac{1 - k^2 L^2}{kL^2 + l_e} = \Psi_e^{1/2} \tanh(\Psi_e^{1/2} l_e) \tag{3.102}$$

We can do a Taylor's expansion, l_e being small, to get the value of Ψ_e

$$\Psi_e = \frac{1 - k^2 L^2}{kL^2} \frac{1}{l_e} \tag{3.103}$$

By matching this expression to that obtained in the Eq. (3.99) for $\Psi_s (s = e)$

$$\Psi_e = -\imath \pi^{1/2} \frac{\omega}{|k| v_{Te}} \frac{\omega_{Pe}^2}{c^2} \tag{3.104}$$

This expression is real only if $\omega = \imath\gamma$ is purely imaginary. The growth rate γ of the tearing mode is thus given by

$$\pi^{1/2} \frac{\gamma}{v_{Te}} \frac{\omega_{Pe}^2}{c^2} = \frac{1 - k^2 L^2}{L^2} \frac{1}{l_e} \tag{3.105}$$

Exercize 14. *Show that the pressure balance across the layer can be rewritten as a function of c/ω_{Pe} (the electron inertial length)*

$$\left[\frac{c}{\omega_{Pe}}\right]^2 = 2\left[1 + \frac{T_p}{T_e}\right] \rho_e^2 \tag{3.106}$$

In developing, we obtain the expression of the growth rate given by [Coppi et al., 1966]

$$\gamma = \left[1 + \frac{T_p}{T_e}\right]^2 \left[\frac{c}{\omega_{Pe}}\right]^{5/2} \frac{\Omega_e}{\pi^{1/2}} (1 - k^2 L^2) \tag{3.107}$$

Many scientific communities paid attention to this result. In particular, it was a scenario to explain the triggering of the magnetospheric substorms associated with the injection of particles in auroral zones responsible for the northern lights. But this enthusiasm has been restrained.

The growth rate of the tearing mode is large at small k, but decreases to saturate at $kL = 1$. During the growth of this mode, the magnetic islands of Fig. 3.5 will therefore reach a quasi-circular form. The current layer being thin, this mode will saturate very quickly, not allowing to reconnect a lot of magnetic flux. Moreover, [Lembege and Pellat, 1982] have shown that a normal component of the magnetic field (in the Y direction), even weak, allows to magnetize the electrons and to thus stabilize the configuration.

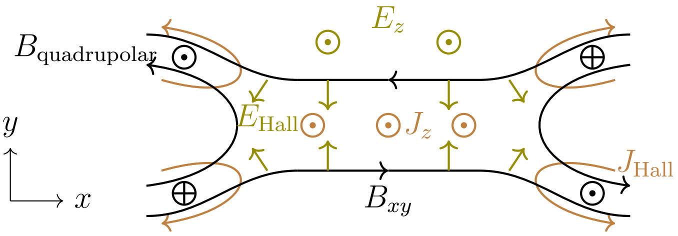

Figure 3.8: Structure of the magnetic field, electric field and current during fast reconnection.

The limit between collisional and collisionless tearing (also called two-fluid transition criteria, see *e.g.* [Cassak et al., 2005]) is $\delta \lesssim d_i$ for a weak guide-field or $\delta \lesssim \rho_s$ for a strong guide-field. This limit has also been predicted numerically by [Simakov and Chacón, 2008].

As a consequence, the thining of the CS can mean that reconnection will fall in the kinetic regime.

The Hall term in the Ohm's law is then fundamental in controlling the (fast) reconnection rate (BIRN2001). For $\delta \lesssim d_i$, Hall term will lead, so the reconnection will be whistler mediated.

Inside the CS, E field is mainly associated to the field reversal, ie $E_y = J_0 B_0$ (directed toward the mid-plane). At the tips of the separatrices (outer from the CS), this E-field is no more curl-free. In polar coordinates, $\partial_t B_z = +\partial_\theta E_r (\equiv E_y)$ so a quadrupolar B_z pattern develops. There is an in-plane current J_{xy} associated to the Hall quadrupolar B-field, carried by the e$^-$ along the separatrices. Then e$^-$ are flowing toward the X-point along the upstream side of the separatrices. They then get out from the X-point flowing along the downstream side of the separatrices. Close enough to the mid-plane, U vanishes so E_z is mainly resulting from the Hall $J_{xy} \times B_{xy}$ term.

Large-scale full-PIC simulations ([Shay et al., 2007]) showed that the asymptotic reconnection rate is independent of m_e and of the system size. Such Hall reconnection observed both in-situ ([Nagai et al., 2001]) and in laboratory ([Yamada et al., 2006]).

Fast reconnection is also at play in electron-positron plasma where Hall effect is not operative ([Bessho and Bhattacharjee, 2005]), which keeps open the question of the importance of the Hall effect in fast reconnection.

Aside from the Hall quadrupolar magnetic field, there is an other important observational feature associated to fast reconnection : an "electron depletion layer" develops along the separatrices where the e$^-$ and p$^+$ density strongly decrease ([Shay, 1998]).

— 4 —

Beam-plasma instabilities

Beam-plasma instability is one of the best prototype of microscopic (or kinetic) instability, *i.e.* which origin depends on the distribution function. In order to feed the electric and possibly magnetic fluctuations for the modes that may exist, a source of free energy is needed. For this type of instability, it is the bulk flow energy of the beam.

4.1 Electrostatic modes

The electrostatic modes are the easiest to derive from an analytical point of view, because in addition to the Vlasov equation, the Maxwell-Gauss equation is the only equation needed because of the electrostatic nature of the instability. Thus, for a magnetized plasma and for the s specie,

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\frac{\partial f_s}{\partial \mathbf{v}} = 0 \tag{4.1}$$

The Maxwell-Gauss equation is written

$$\boldsymbol{\nabla}.\mathbf{E} = \frac{1}{\varepsilon_0}\sum_s n_s q_s \tag{4.2}$$

where, moreover, the electrostatic field derives from a scalar potential

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \tag{4.3}$$

By omitting the s index, Eq. (4.1) linearized at first order gives

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\frac{\partial f_1}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0).\frac{\partial f_1}{\partial \mathbf{v}} + \frac{q}{m}\mathbf{E}_1.\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{4.4}$$

The sum of the 3 first terms is equal to $d_t f_1$, provided that we are along the trajectory

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \tag{4.5}$$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0) \tag{4.6}$$

i.e. along the unperturbed orbit of a particle in the $\mathbf{B}_0$ field. Along this orbit,

$$\frac{df_1}{dt} = -\frac{q}{m}\mathbf{E}_1.\frac{\partial f_0}{\partial \mathbf{v}} \tag{4.7}$$

We are looking for an electric field of the form $\mathbf{E}_1(\mathbf{r},t) = \mathtt{E}_1 e^{\imath(\mathbf{k}.\mathbf{r}-\omega t)}$. By integrating Eq. (4.7), one gets f_1 :

$$f_1 = -\frac{q}{m}\int_{-\infty}^{t} e^{\imath(\mathbf{k}.\mathbf{r}'-\omega t')}\mathtt{E}_1 \, . \, \frac{\partial f_0}{\partial \mathbf{v}}\, dt' \tag{4.8}$$

where $\mathbf{r}'(t')$ and $\mathbf{v}'(t')$ are the position and velocity of the particle arriving at $\mathbf{r}$ at the velocity $\mathbf{v}$ at time t, along the unperturbed orbit.
By noting φ the phase at the origin of the particle at time t and $\tau = t' - t$, $\mathbf{v}'(\tau)$ is written

$$v'_x = -v_\perp \sin(\Omega_c \tau + \varphi) \tag{4.9}$$
$$v'_y = +v_\perp \cos(\Omega_c \tau + \varphi) \tag{4.10}$$
$$v'_z = +v_\parallel \tag{4.11}$$

for electrons[1]. One gets $\mathbf{r}'(t')$ by integration

$$x' = x + v_\perp/\Omega_c[\cos(\Omega_c\tau + \varphi) - \cos\varphi] \tag{4.12}$$
$$y' = y + v_\perp/\Omega_c[\sin(\Omega_c\tau + \varphi) - \sin\varphi] \tag{4.13}$$
$$z' = z + v_\parallel \tau \tag{4.14}$$

We are looking for a solution of the form

$$f_1(\mathbf{r},\mathbf{v},t) = f_1(\mathbf{k},\mathbf{v},\omega)e^{\imath(\mathbf{k}.\mathbf{r}-\omega t)} \tag{4.15}$$

We only focus on parallel modes for which $\mathbf{k} = k_\parallel \hat{\mathbf{z}}$ (we will write k instead of $k_\parallel$ by simplification). So, $\mathbf{k}.\mathbf{r} = kv_\parallel t$ and, for the unperturbed orbit, the cyclotron motion has no effect.

Remark 14. *This work is necessary. For non-electrostatic modes, the wave number $\mathbf{k}$ has a perpendicular component. Its scalar product with $\mathbf{r}'(t')$ then involves the cyclotron motion of the particle. The integral over τ which results from it is then a little more difficult because it involves Bessel functions. The most classical example is surely that of Bernstein's modes, whether associated to electrons or protons.*

Eq. (4.8) can then be written

$$f_1(k,\mathbf{v},\omega) = -\frac{q}{m}\int_{-\infty}^{0} e^{\imath(kv_\parallel - \omega)\tau}\, d\tau \mathtt{E}_1 \, . \, \frac{\partial f_0}{\partial \mathbf{v}} \tag{4.16}$$

Moreover, in Fourier space, we have for an electrostatic mode $\mathtt{E}_1 = \imath \mathbf{k}\phi_1$ polarized in the direction of $\mathbf{B}_0$ *i.e.* along z. Then the integral can be written

[1] The calculation indeed shows that the modes we are interested in are electronic. For a mode associated with protons, it would be necessary to change the direction of rotation around the field $\mathbf{B}_0$, that is considering a positive Ω_c.

$$f_1(k,\mathbf{v},\omega) = -\frac{q}{m}\frac{1}{kv_\parallel - \omega}k\phi_1\frac{\partial f_0}{\partial v_\parallel} \tag{4.17}$$

This form of $f_1(k,\mathbf{v},\omega)$ does not involve the static magnetic field $\mathbf{B}_0$. We therefore suspect that the form of this instability is the same as that which can exist in unmagnetized plasma. It can be checked ; in “ field-free ”, the differential equation which governs f_1 no longer involves a derivative with respect to the velocity. We can then obtain the form of f_1 by simply using a Fourier transform, without any need of the characteristic method. The form of the Vlasov equation is simplified by losing the magnetic term

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\frac{\partial f_1}{\partial \mathbf{r}} + \frac{q}{m}\mathbf{E}_1.\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{4.18}$$

One can get in the Fourier space to obtain f_1

$$-\imath(\omega - \mathbf{k}.\mathbf{v})f_1(\mathbf{k},\mathbf{v},\omega) = \frac{q}{m}\phi_1(\mathbf{k},\omega)\imath\mathbf{k}.\frac{\partial f_0}{\partial \mathbf{v}} \tag{4.19}$$

Eq. (4.17) and (4.19) are clearly the same. This is a fairly general result ; the dynamics of a plasma in the parallel direction is quite often the same as it would be in the same unmagnetized system.

By integration over velocity space, we get the first order density fluctuation

$$n_1(k,\omega) = \frac{q}{m}\phi_1(k,\omega)\int_{\mathbb{R}^3}\frac{d\mathbf{v}}{kv_\parallel - \omega}k\frac{\partial f_0}{\partial v_\parallel} \tag{4.20}$$

Writing Eq. (4.2) in the Fourier space, one gets

$$k^2\phi_1 = \frac{1}{\varepsilon_0}\sum_s q_s n_{1s} \tag{4.21}$$

With these two equations, we can write the dispersion relation of the electrostatic modes :

$$1 - \sum_s \frac{q_s^2}{m_s\varepsilon_0 k^2}\int_{\mathbb{R}^3}\frac{d\mathbf{v}}{kv_\parallel - \omega}k\frac{\partial f_0}{\partial v_\parallel} = 0 \tag{4.22}$$

As often, we assume a Maxwell-Boltzmann distribution, with a bulk flow directed along the magnetic field, *i.e.* in the same direction as the wave number $\mathbf{k}$. Being only interested in the electrostatic modes, everything happens in the z direction. As this distribution function is separable, we can write

$$f_0(v_\parallel,\mathbf{v}_\perp) = n_0 F_{0\parallel}(v_\parallel)F_{0\perp}(\mathbf{v}_\perp) \tag{4.23}$$

The normalization conditions is

$$\int_{\mathbb{R}} F_{0\parallel}(v_\parallel)\,dv_\parallel = \int_{\mathbb{R}^2} F_{0\perp}(\mathbf{v}_\perp)\,d\mathbf{v}_\perp = 1 \tag{4.24}$$

In Eq. (4.22), we simplify the integral by calculating it on the two directions of $\mathbf{v}_\perp$. Thus,

$$1-\sum_s \frac{n_{0s}q_s^2}{m_s\varepsilon_0 k}\int_{\mathbb{R}} \frac{dv_\parallel}{kv_\parallel-\omega}\frac{\partial F_{0\parallel}}{\partial v_\parallel}=0 \tag{4.25}$$

To continue this computation, one needs to give the form of the zeroth order distribution function in the parallel direction in order to be able to clarify the integral. For a Maxwell-Boltzmann distribution with a bulk-flow velocity v_0 in the parallel direction,

$$F_{0\parallel}(v_\parallel)F_{0\perp}(v_\perp)=\left[\frac{\alpha_\parallel}{\pi}\right]^{1/2} e^{-\alpha_\parallel(v_\parallel-v_0)^2}\left[\frac{\alpha_\perp}{\pi}\right] e^{-\alpha_\perp v_\perp^2} \tag{4.26}$$

with

$$\alpha_\parallel=\frac{m}{2k_BT_\parallel}\ ,\ \alpha_\perp=\frac{m}{2k_BT_\perp} \tag{4.27}$$

One then gets

$$\frac{\partial F_{0\parallel}}{\partial v_\parallel}=-2\alpha_\parallel(v_\parallel-v_0)F_{0\parallel} \tag{4.28}$$

so the dispersion relation writes

$$1-\sum_s \frac{\omega_{Ps}^2}{k}\int_{\mathbb{R}} \frac{dv_\parallel}{kv_\parallel-\omega}[-2\alpha_\parallel(v_\parallel-v_0)F_{0\parallel}]=0 \tag{4.29}$$

We note $\mathcal{I}$ the integral of Eq. (4.29). By setting $u^2=\alpha_\parallel(v_\parallel-v_0)^2$, $\mathcal{I}$ can be rewritten

$$\mathcal{I}=-2\alpha_\parallel\left[\frac{1}{\pi}\right]^{1/2}\int_{\mathbb{R}}\frac{du}{ku+\alpha_\parallel^{1/2}(kv_0-\omega)}ue^{-u^2} \tag{4.30}$$

Using the notation

$$\zeta=\left[\frac{m}{2k_BT_\parallel}\right]^{1/2}\left(\frac{\omega}{k}-v_0\right) \tag{4.31}$$

then

$$\mathcal{I}=-\frac{\alpha_\parallel}{k}\left[\frac{1}{\pi}\right]^{1/2}\int_{\mathbb{R}}\frac{2ue^{-u^2}\,du}{u-\zeta} \tag{4.32}$$

This form makes it possible to naturally introduce the Fried & Conte function [2]. This function appears naturally in the dispersion relation of the eigenmodes in a hot plasmas with a Maxwell-Boltzmann distribution. The Fried & Conte function is defined by

$$Z(\zeta)=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\frac{e^{-u^2}}{u-\zeta}\,du \tag{4.33}$$

One recognize the Hilbert transform of the Gauss function. Deriving under the sum sign, one gets

[2] That one is also called the plasma dispersion function

$$Z'(\zeta) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{e^{-u^2}}{(u-\zeta)^2}\, du \tag{4.34}$$

so that an integration by part gives

$$Z'(\zeta) = -\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{2ue^{-u^2}}{u-\zeta}\, du \tag{4.35}$$

hence giving the relation

$$Z'(\zeta) = -2[1 + \zeta Z(\zeta)] \tag{4.36}$$

Then,

$$\mathcal{I} = \frac{\alpha_\parallel}{k} Z'(\zeta) \tag{4.37}$$

so that the disperison relation of this mode is

$$1 - \sum_s \frac{\omega_{Ps}^2}{k^2} \alpha_\parallel Z'(\zeta_s) = 0 \tag{4.38}$$

which can be simply written

$$1 + \sum_s \frac{2}{k^2 \lambda_{Ds\parallel}^2} [1 + \zeta_s Z(\zeta_s)] = 0 \tag{4.39}$$

where $\lambda_{Ds\parallel}$ is the Debye length for the s specie along the DC magnetic field, and

$$\zeta_s = \left[\frac{m}{2k_B T_{s\parallel}}\right]^{1/2} \left(\frac{\omega}{k} - v_0\right) \tag{4.40}$$

Introducing the thermel speed v_{Ts} of the s specie, one gets

$$\zeta_s = \frac{1}{\sqrt{2} v_{Ts\parallel}} \left(\frac{\omega}{k} - v_0\right) \tag{4.41}$$

In the next section, we will discuss the consequences of the growth of the Langmuir modes in type-3 solar radio bursts.

4.2 Type-3 radio bursts

The sun is a tremendous particle accelerator. “ Solar flares ” are manifested by the appearance of intense flashes near the surface of the sun. These are events in which broad spectrum electromagnetic emissions occur, often accompanied by “ Coronal Mass Ejection ”, *i.e.* violent and massive ejections of energetic particles. The most consensual scenario is one in which magnetic arches having their 2 feet in the chromosphere swell and then open by magnetic reconnection. They thus release the stored magnetic energy in kinetic energy of the particles, which are accelerated and heated. The released energy can reach 10^{25} J in order to accelerate up to 10^{36} electrons. The interaction of these particles with the chromospheric plasma then produces intense electromagnetic radiation, from the radio domain to γ rays. However, they are classified by their emission maximum in the 100-800 pm band, *i.e.* in the X band.

During a solar flare, the energetic electrons that are produced are non-thermal, and have a power law distribution of their speed with a negative slope. By time-of-flight effect, the fastest electrons overtake the slower. Thus, at a finite distance, this population of fast electrons will produce a velocity distribution with a positive slope. Such a distribution is unstable and will produce Langmuir waves by Čerenkov effect. There is therefore a minimum distance necessary for this electron distribution to be unstable. The local electron density makes it possible to calculate the plasma frequency at which the first emissions occur. Then, during the transport of these structures, the regions visited being less and less dense, the plasma frequency at which the emissions occur is increasingly low. We thus observe a drift in the frequency of radio waves measured on the ground over time. An example is given in Fig. 4.1. The paper by [Ginzburg and Zheleznyakov, 1958] is the founding paper for the theory of type-3 radio bursts.

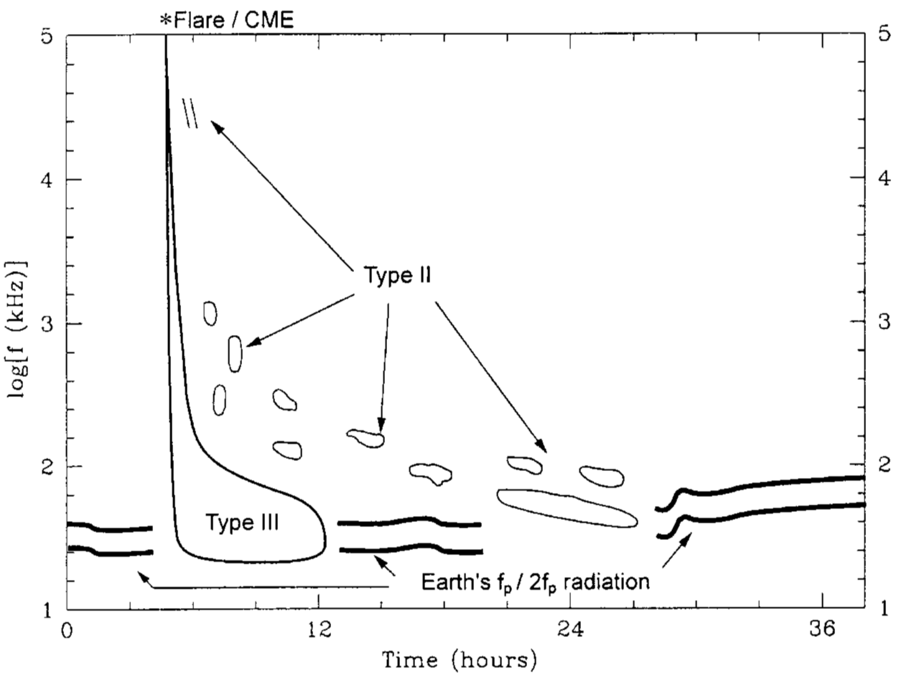

Figure 4.1: Schematic of a dynamical spectra for a type 3 radio burst (see [Cairns and Robinson, 1999])

The essential mechanism foreseen is still relevant today. The beam-plasma electrostatic instability produces Langmuir waves. A conversion mechanism, which remains debated, then ensures the conversion of these electrostatic waves into electromagnetic modes, at the same frequency (fundamental) or at a double frequency (second harmonic). There are different *scenari* for this :

— Langmuir waves scatter protons, thus modifying their distribution function. The mechanism described in the previous section is purely linear, but the amplitude of Langmuir waves will be limited within the framework of the quasi-linear theory, which jointly describes the evolution of particles and wave energy. These scattered protons will in turn be able to induce the emission of electromagnetic waves at the fundamental frequency.

— By a process similar to Brillouin scattering, the Langmuir wave can decrease producing a scattered Langmuir wave as well as an ion acoustic wave. By non-linear coupling, this ion acoustic wave can again couple with the mother Langmuir wave to produce a fundamental electromagnetic wave.

— By the same Brillouin scattering process, the scattered Langmuir wave can also couple with the mother wave to produce the 2^{nd} harmonic of an electromagnetic wave.

— In the previous mechanism, the scattered Langmuir wave can also result, not from Brillouin scattering, but from the scattering of ions by the mother Langmuir wave. The same coalescence mechanism as previously then also produces the 2^{nd} harmonic of an electromagnetic wave.

To close this brief presentation of type-3 radio bursts[3], we must also underline the remarkable persistence of these electron beams. Indeed, within the framework of the quasi-linear theory, the diffusion of electrons by Langmuir waves should lead to the disappearance of this beam after the electrons have traveled a few meters ! However, the observations show that these beams can exist on distances of the order of the astronomical unit. Let's quote in particular the numerical work of [Takakura & Shibahashi, 1976], showing that Langmuir waves can be continuously produced at the nose of the beam, and reabsorbed at its end, which has been proposed analytically by [Zhelenyakov & Zaitsev, 1970]. An alternative process would be the existence of local density fluctuations, by which the associated Langmuir frequency could no longer resonate with the electron beam (see [Smith and Sime, 1979]).

4.3 Electromagnetic modes

The calculations as presented below are very similar to those published by [Gary and Feldman, 1978]. The Vlasov equation for the specie s is,

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}[\mathbf{E} + \mathbf{v} \times \mathbf{B}].\frac{\partial f_s}{\partial \mathbf{v}} = 0 \tag{4.42}$$

Since these are electromagnetic modes, we also need the Maxwell's equations (the " curl one "). By forgetting Maxwell-Thomson and Maxwell-Gauss, which are implicit in the other 2 equations, the Maxwell-Ampère equation is

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{4.43}$$

and the Maxwell-Faraday equation is

[3]For a recent review on the physics of type-3 solar bursts, you can read the paper by [Reid and Ratcliffe, 2014].

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.44}$$

Exercize 15. *From the zeroth order Vlasov equation, show that a stationary and homogeneous distribution in a magnetic field is independent of the gyrophase φ.*

By omitting the s index, the first order linearization of Eq. (4.42) gives

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\frac{\partial f_1}{\partial \mathbf{r}} + \frac{q}{m}\mathbf{E}_1.\frac{\partial f_0}{\partial \mathbf{v}} + \frac{q}{m}(\mathbf{v}\times\mathbf{B}_1).\frac{\partial f_0}{\partial \mathbf{v}} + \frac{q}{m}(\mathbf{v}\times\mathbf{B}_0).\frac{\partial f_1}{\partial \mathbf{v}} = 0 \tag{4.45}$$

The sum of the first 2 terms and the last is equal to $d_t f_1$, provided that we are along the trajectory

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{4.46}$$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v}\times\mathbf{B}_0) \tag{4.47}$$

i.e. along the unperturbed orbit of a particle in the $\mathbf{B}_0$ field. Along this orbit, we therefore have

$$\frac{df_1}{dt} + \frac{q}{m}(\mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1).\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{4.48}$$

You will notice how elegant it is[4] : it allows to transform a partial differential equation into an ordinary differential equation ... the second class of equations being much simpler to solve than the first one.

In Eq. (4.48), we can make $\mathbf{B}_1$ disappear thanks to its relation to $\mathbf{E}_1$ via Eq. (4.44). Its linearization in Fourier space gives $\omega\mathbf{B}_1 = \mathbf{k}\times\mathbf{E}_1$. We can then rewrite the term $\mathbf{v}\times\mathbf{B}_1$ as a function of $\mathbf{E}_1$. By developing the double cross product,

$$\mathbf{v}\times\mathbf{B}_1 = \left(\frac{\mathbf{v}.\,\mathbf{E}_1}{\omega}\right)\mathbf{k} - \left(\frac{\mathbf{k}\,.\mathbf{v}}{\omega}\right)\mathbf{E}_1 \tag{4.49}$$

so Eq. (4.48) can be written

$$\frac{df_1}{dt} + \frac{q}{m}\mathbf{E}_1\,.\left[\mathbf{1} + \frac{\mathbf{vk}}{\omega} - \left(\frac{\mathbf{k}\,.\mathbf{v}}{\omega}\right)\mathbf{1}\right].\,\frac{\partial f_0}{\partial \mathbf{v}}f_0 = 0 \tag{4.50}$$

We are still looking for an electric field of the form $\mathbf{E}_1 = \mathrm{E}_1 e^{\imath(\mathbf{k}.\mathbf{r}-\omega t)}$. By integrating Eq. (4.50), we get f_1:

$$f_1 = -\frac{q}{m}\int_{-\infty}^{t} e^{\imath(\mathbf{k}.\mathbf{r}'-\omega t')}\mathrm{E}_1\,.\left[\mathbf{1}\left(1 - \frac{\mathbf{v}'.\,\mathbf{k}}{\omega}\right) + \frac{\mathbf{v}'\mathbf{k}}{\omega}\right].\,\frac{\partial f_0}{\partial \mathbf{v}}\,dt' \tag{4.51}$$

where $\mathbf{r}'(t')$ and $\mathbf{v}'(t')$ are the position and velocity of the particle arriving at $\mathbf{r}$ at the speed $\mathbf{v}$ at time t, along an unperturbed orbit.

[4]this method is called “ the method of characteristics ”.

We are only interested in parallel modes. We therefore have $\mathbf{k} = k_\parallel \hat{\mathbf{z}}$ which simplifies the calculations. But the essential source of simplification, compared to the classical calculation of the dielectric tensor in hot plasma, is the fact that the unperturbed orbit of the paticle is much simpler. We have $\mathbf{v}'(t')|_z = v_\parallel$ [5] and therefore $\mathbf{r}'(t')|_z = v_\parallel(t'-t)+z$. We can therefore rewrite the exponential argument in Eq. (4.51) by introducing $\tau = t' - t$

$$e^{\imath(\mathbf{k}.\mathbf{r}'-\omega t')} = e^{\imath(kv_\parallel-\omega)\tau} e^{\imath(kz-\omega t)} \tag{4.52}$$

Looking for a solution of the form

$$f_1(\mathbf{r},\mathbf{v},t) = f_1(k,\mathbf{v},\omega)e^{\imath(kz-\omega t)} \tag{4.53}$$

Eq. (4.51) can then be written

$$f_1(k,\mathbf{v},\omega) = -\frac{q}{m}\int_{-\infty}^{0} e^{\imath(kv_\parallel-\omega)\tau}\, d\tau \mathtt{E}_1 \cdot \left[\mathbf{1}\left(1-\frac{v_\parallel k}{\omega}\right)+\frac{\mathbf{v}'\mathbf{k}}{\omega}\right]\cdot\frac{\partial f_0}{\partial \mathbf{v}} \tag{4.54}$$

Calling $\mathtt{T}$ the double scalar product,

$$\mathtt{T} = \left(1-\frac{v_\parallel k}{\omega}\right)\left(\mathtt{E}_1 \cdot \frac{\partial f_0}{\partial \mathbf{v}}\right) + \mathtt{E}_1.\mathbf{v}'\frac{k}{\omega}\frac{\partial f_0}{\partial v_\parallel} \tag{4.55}$$

We then make a hypothesis, or rather an important choice : we are only interested in purely electromagnetic modes (*i.e.* no electrostatic contributions, $\mathbf{k}.\mathbf{E}_1 = 0$). The electric field is consequently polarized in the plane normal to the wave vector $\mathbf{k}$, *i.e.* in the (x,y) plan.

The second parenthesis of the first term of Eq. (4.55) can be rewritten, with a cartesian decomposition of the two vectors. For this, we use

$$v'_x = v_\perp \cos\varphi \tag{4.56}$$
$$v'_y = v_\perp \sin\varphi \tag{4.57}$$

where $\varphi(t')$ is a function of t'. One then gets the derivatives

$$dv_\perp = \cos\varphi\, dv'_x + \sin\varphi\, dv'_y \tag{4.58}$$
$$-v_\perp\, d\varphi = \sin\varphi\, dv'_x - \cos\varphi\, dv'_x \tag{4.59}$$

so we can write the derivatives of f_0 in cartesian coordinates

$$\frac{\partial f_0}{\partial v_x} = \cos\varphi\frac{\partial f_0}{\partial v_\perp} - \frac{1}{v_\perp}\sin\varphi\frac{\partial f_0}{\partial \varphi} \tag{4.60}$$
$$\frac{\partial f_0}{\partial v_y} = \sin\varphi\frac{\partial f_0}{\partial v_\perp} + \frac{1}{v_\perp}\cos\varphi\frac{\partial f_0}{\partial \varphi} \tag{4.61}$$

[5]The value of the parallel speed, in field-free, is constant.

from which we deduce (the last term of these two equations being null)

$$\mathtt{E}_1 \,.\frac{\partial f_0}{\partial \mathbf{v}} = \frac{\mathtt{E}_1 \,.\mathbf{v}'}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \tag{4.62}$$

We can then rewrite Eq. (4.54) in the form

$$f_1(k,\mathbf{v},\omega) = -\frac{q}{m\omega}\int_{-\infty}^{0} e^{\imath(kv_\parallel-\omega)\tau}\,d\tau(\mathtt{E}_1\,.\mathbf{v}')\left[k\frac{\partial f_0}{\partial v_\parallel} + \frac{\omega - kv_\parallel}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\right] \tag{4.63}$$

To carry the integral, it is necessary to clarify the shape of the orbit. By using the gyrofrequency, we can rewrite Eq. (4.46) and (4.47)

$$\frac{d\mathbf{x}'}{dt'} = \mathbf{v}' \tag{4.64}$$

$$\frac{d\mathbf{v}'}{dt'} = \Omega(\mathbf{v}'\times\widehat{\mathbf{z}}) \tag{4.65}$$

The resolution is direct in the complex plane by choosing the new unit vectors in the perpendicular plane

$$\widehat{\mathbf{e}}_\pm = \frac{1}{\sqrt{2}}(\widehat{\mathbf{x}} \pm \imath\widehat{\mathbf{y}}) \tag{4.66}$$

We then define the two velocity vectors

$$v'_\pm = \frac{1}{\sqrt{2}}(v'_x \pm \imath v'_y) \tag{4.67}$$

which allows to express $\mathbf{v}'_\perp(\tau) = v'_+\widehat{\mathbf{e}}_- + v'_-\widehat{\mathbf{e}}_+$. The solution of the unperturbed orbit is then

$$v'_\parallel(\tau) = v_\parallel \tag{4.68}$$

$$v'_+(\tau) = v_+e^{-\imath\Omega\tau} \tag{4.69}$$

$$v'_-(\tau) = v_-e^{+\imath\Omega\tau} \tag{4.70}$$

Remark 15. *Note that the vectors of the new basis being complex, they have an unusual scalar product, i.e.* $\hat{\mathbf{e}}_+.\,\hat{\mathbf{e}}_- = 1$ *and* $\hat{\mathbf{e}}_+.\,\hat{\mathbf{e}}_+ = \hat{\mathbf{e}}_-.\,\hat{\mathbf{e}}_- = 0$.

One then deduce

$$\mathtt{E}_1\,.\mathbf{v}' = E_-v_+e^{-\imath\Omega\tau} + E_+v_-e^{+\imath\Omega\tau} \tag{4.71}$$

We can then calculate the integral in Eq. (4.63)

$$\int_{-\infty}^{0} e^{\imath(kv_\parallel-\omega)\tau}[E_-v_+e^{-\imath\Omega\tau} + E_+v_-e^{+\imath\Omega\tau}]\,d\tau = \frac{E_-v_+}{\imath(kv_\parallel-\omega-\Omega)} + \frac{E_+v_-}{\imath(kv_\parallel-\omega+\Omega)} \tag{4.72}$$

so we can write

$$f_1(k, \mathbf{v}, \omega) = \frac{\imath q}{m\omega}\left[k\frac{\partial f_0}{\partial v_\parallel} + \frac{\omega - kv_\parallel}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\right]\left(\frac{E_- v_+}{kv_\parallel - \omega - \Omega} + \frac{E_+ v_-}{kv_\parallel - \omega + \Omega}\right) \tag{4.73}$$

The form of the distribution function linearized at first order makes it possible to deduce the form of the conduction current by integration. This is the source term that naturally appears in the Maxwell-Ampère equation. As a reminder,

$$\mathbf{J}_1 = q\int_{\mathbb{R}^3} f_1 \mathbf{v}\, d\mathbf{v} \tag{4.74}$$

For the sake of simplification, we introduce the new variables

$$\begin{aligned}
\mathtt{A} &= \frac{\imath q^2}{m\omega} && (4.75)\\
\mathtt{B} &= k\frac{\partial f_0}{\partial v_\parallel} + \frac{\omega - kv_\parallel}{v_\perp}\frac{\partial f_0}{\partial v_\perp} && (4.76)\\
\mathtt{C} &= kv_\parallel - \omega - \Omega && (4.77)\\
\mathtt{D} &= kv_\parallel - \omega + \Omega && (4.78)
\end{aligned}$$

so we can write

$$\mathbf{J}_1 = \mathtt{A}\int_{\mathbb{R}} \mathtt{B}\left[\frac{v_+^2 E_-}{\mathtt{C}}\hat{\mathbf{e}}_- + \frac{v_+ v_- E_+}{\mathtt{D}}\hat{\mathbf{e}}_- + \frac{v_+ v_- E_-}{\mathtt{C}}\hat{\mathbf{e}}_+ + \frac{v_-^2 E_+}{\mathtt{D}}\hat{\mathbf{e}}_+\right] d\mathbf{v} \tag{4.79}$$

In addition, the combination of the Maxwell-Ampère and Maxwell-Faraday equations, linearized in Fourier space gives

$$\left(1 - \frac{k^2c^2}{\omega^2}\right)\mathtt{E}_1 = -\frac{\imath}{\omega\varepsilon_0}\mathbf{J}_1 \tag{4.80}$$

Using the decomposition $\mathtt{E}_1 = E_+\hat{\mathbf{e}}_- + E_-\hat{\mathbf{e}}_+$ and then projecting on the two vectors $\hat{\mathbf{e}}_+$ and $\hat{\mathbf{e}}_-$, we get the system

$$\begin{aligned}
\left(1 - \frac{k^2c^2}{\omega^2}\right)E_+ &= \mathtt{A}\int_{\mathbb{R}} \mathtt{B}\left[\frac{v_+^2}{\mathtt{C}}E_- + \frac{v_+ v_-}{\mathtt{D}}E_+\right] d\mathbf{v} && (4.81)\\
\left(1 - \frac{k^2c^2}{\omega^2}\right)E_- &= \mathtt{A}\int_{\mathbb{R}} \mathtt{B}\left[\frac{v_+ v_-}{\mathtt{C}}E_- + \frac{v_-^2}{\mathtt{D}}E_+\right] d\mathbf{v} && (4.82)
\end{aligned}$$

At this stage, we will consider that the equilibrium distribution function is a Maxwellian. Without this assumption, there is no *a priori* reason to be able to decouple the left and right modes. We therefore set $f_0 = n_0 F_0$ with

$$F_0(v_\parallel, v_\perp) = \left[\frac{\alpha_\parallel}{\pi}\right]^{1/2}\left[\frac{\alpha_\perp}{\pi}\right] e^{-\alpha_\parallel (v_\parallel - v_0)^2} e^{-\alpha_\perp v_\perp^2} \tag{4.83}$$

and

$$\alpha_\parallel = \frac{m}{2k_BT_\parallel}\ ,\ \alpha_\perp = \frac{m}{2k_BT_\perp} \tag{4.84}$$

so we have

$$\frac{\partial F_0}{\partial v_\perp} = -2\alpha_\perp v_\perp F_0 \tag{4.85}$$

and the term B becomes independent of $v_\perp$. In Eq. (4.81), we will show that the integral of the first term of the right hand side is zero. To do this, we decompose $2v_+^2 = v_x^2 - v_y^2 + 2\imath v_x v_y$. As F_0 is gyrotropic in the perpendicular plane (v_x, v_y), we have

$$\int_{\mathbb{R}} F_0 v_x^2 dv_x = \int_{\mathbb{R}} F_0 v_y^2 dv_y \tag{4.86}$$

Moreover, the cross term $2\imath v_x v_y$ is an odd power of v_x and v_y. F_0 being even, its integral on a symmetric support is zero. Thus, all the first term of the right hand side of Eq. (4.81) has a vanishing integral. Symmetrically, it is the same in Eq. (4.82). The two polarizations in E_+ and E_- are therefore decoupled.

For the calculation of the remaining integral, with the equality $2v_+v_- = v_\perp^2$, we deduce the dispersion relation for the two modes

$$1 - \frac{k^2c^2}{\omega^2} - \sum_s \frac{\omega_{Ps}^2}{2\omega^2}\int_{\mathbb{R}} d\mathbf{v}\, v_\perp \frac{1}{\omega - kv_\parallel \pm \Omega_s}\left[kv_\perp \frac{\partial F_{s0}}{\partial v_\parallel} + (\omega - kv_\parallel)\frac{\partial F_{s0}}{\partial v_\perp}\right] = 0 \tag{4.87}$$

The + sign in the denominator is associated with the E_+ mode (right mode) and the - sign with the E_- mode (left mode). The difficulty now lies in calculating the integral $\mathcal{I}$ defined by

$$\mathcal{I} = \int_{\mathbb{R}} d\mathbf{v}\, v_\perp \frac{1}{\omega - kv_\parallel \pm \Omega}\left[kv_\perp \frac{\partial F_{s0}}{\partial v_\parallel} + (\omega - kv_\parallel)\frac{\partial F_{s0}}{\partial v_\perp}\right] \tag{4.88}$$

With a Maxwell distribution (Eq. 4.83), we have, as for Eq. (4.85),

$$\frac{\partial F_0}{\partial v_\parallel} = -2\alpha_\parallel (v_\parallel - v_0) F_0 \tag{4.89}$$

By rearranging the terms, we thus have

$$kv_\perp \frac{\partial F_0}{\partial v_\parallel} + (\omega - kv_\parallel)\frac{\partial F_0}{\partial v_\perp} = 2v_\perp F_0[kv_\parallel(\alpha_\perp - \alpha_\parallel) + k\alpha_\parallel v_0 - \omega\alpha_\perp] \tag{4.90}$$

Using this form in the expression of $\mathcal{I}$, it appears that this integral depends on $v_\perp$ only through F_0. By not omitting the term in $v_\perp^2$, we calculate the integral

$$\int_{\mathbb{R}} d\mathbf{v}_\perp v_\perp^2 F_0 = 2\pi \int_0^\infty v_\perp^3 F_0\, dv_\perp \tag{4.91}$$

This calculation then needs a classical result that we remind here:

$$I_0 = \int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad , \quad I_1 = \int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha} \quad , \quad I_n = \int_0^\infty x^n e^{-\alpha x^2} dx \tag{4.92}$$

as well as the recurrence relation

$$I_n = \frac{n-1}{2\alpha} I_{n-2} \tag{4.93}$$

These expressions make it possible in particular to calculate the shape of the moments of order n of a Maxwell-Boltzmann distribution. We can deduce

$$\int_{\mathbb{R}^2} d\mathbf{v}_\perp v_\perp^2 F_0 = \frac{1}{\alpha_\perp}\left[\frac{\alpha_\parallel}{\pi}\right]^{1/2} e^{-\alpha_\parallel (v_\parallel - v_0)^2} \tag{4.94}$$

We can then write the form of the integral $\mathcal{I}$,

$$\mathcal{I} = \frac{2}{\alpha_\perp}\left[\frac{\alpha_\parallel}{\pi}\right]^{1/2} \mathcal{J} \tag{4.95}$$

with

$$\mathcal{J} = \int_{\mathbb{R}} dv_\parallel \frac{1}{\omega - kv_\parallel \pm \Omega}[kv_\parallel(\alpha_\perp - \alpha_\parallel) + k\alpha_\parallel v_0 - \omega\alpha_\perp] e^{-\alpha_\parallel (v_\parallel - v_0)^2} \tag{4.96}$$

A change of variable $u^2 = \alpha_\parallel (v_\parallel - v_0)^2$ allows to center this integral and to have only the square of the integrant in the exponential. We introduce

$$\zeta = \frac{\alpha_\parallel^{1/2}}{k}(\omega - kv_0 \pm \Omega) \tag{4.97}$$

so the integral can be written

$$\mathcal{J} = \frac{\alpha_\parallel - \alpha_\perp}{\alpha_\parallel^{1/2}} \int_{\mathbb{R}} \frac{u}{u-\zeta} e^{-u^2}\, du + \frac{\alpha_\perp}{\alpha_\parallel^{1/2}} \int_{\mathbb{R}} \frac{1}{u-\zeta} e^{-u^2}\, du \tag{4.98}$$

As before, this form allows the Fried & Conte function to appear naturally. The integral $\mathcal{J}$ can then be rewritten

$$\mathcal{J} = \left[\frac{\pi}{\alpha_\parallel}\right]^{1/2} \frac{\alpha_\perp - \alpha_\parallel}{2} Z'(\zeta) - \left[\frac{\pi}{\alpha_\parallel}\right]^{1/2} \alpha_\perp \left(v_0 - \frac{\omega}{k}\right) Z(\zeta) \tag{4.99}$$

with the explicit form of $\zeta^\pm$

$$\zeta = \alpha_\parallel^{1/2}\left(\frac{\omega}{k} - v_0 \pm \frac{\Omega}{k}\right) \tag{4.100}$$

that can be written depending on the parallel thermal velocity

$$\zeta_s^\pm = \frac{1}{\sqrt{2}kv_{Ts\parallel}}(\omega - kv_{s0} \pm \Omega_s) \tag{4.101}$$

We can then write the dispersion relation of this mode gien by Eq. (4.87), with Eq. (4.88), (4.95) and (4.99)

$$1 - \frac{k^2c^2}{\omega^2} - \sum_s \frac{\omega_{Ps}^2}{\omega^2}\left[\frac{\alpha_\perp - \alpha_\parallel}{2\alpha_\perp} Z'(\zeta_s^\pm) + \alpha_\parallel^{1/2}\left(v_{s0} - \frac{\omega}{k}\right) Z(\zeta_s^\pm)\right] = 0 \tag{4.102}$$

so with the explicit iparallel and perpendicular α values

$$1 - \frac{k^2c^2}{\omega^2} - \sum_s \frac{\omega_{Ps}^2}{\omega^2}\left[\frac{T_\parallel - T_\perp}{2T_\parallel} Z'(\zeta_s^\pm) + \frac{1}{\sqrt{2}v_{Ts\parallel}}\left(v_{s0} - \frac{\omega}{k}\right) Z(\zeta_s^\pm)\right] = 0 \tag{4.103}$$

Eq. (4.103) is the parallel electromagnetic mode dispersion equation for a hot Maxwellian plasma. For an isotropic plasma ($T_\parallel = T_\perp$) it can be rewritten in the more classical form

$$\omega^2 - k^2c^2 + \sum_s \omega_{Ps}^2 \frac{\omega - kv_{s0}}{\sqrt{2}kv_{Ts\parallel}} Z(\zeta_s^\pm) = 0 \tag{4.104}$$

For a cold plasma, *i.e.* $v_T \to 0$, we then have $\zeta \to \infty$. We can then use the asymptotic expansion $Z(\zeta) = -1/\zeta$. The dispersion equation is then

$$\omega^2 - k^2c^2 - \sum_s \omega_{Ps}^2 \frac{\omega - kv_{s0}}{\omega - kv_{s0} \pm \Omega_s} = 0 \tag{4.105}$$

In space plasmas, the instabilities associated with Eq. (4.103) can be encountered in a 3 populations plasma : a dense population of protons with a quasi-zero drift speed called “ core ”, a more tenuous population of protons with a field-aligned drift speed, so-called “ beam ”, and a population of electrons which ensures neutrality. For the beam, we can commensurate the thermal speed v_{Tb} to that of drift v_{0b}. If $v_{Tb} \ll v_{0b}$, the beam is cold, if $v_{Tb} \sim v_{0b}$, it is warm, and if $v_{Tb} \gg v_{0b}$ it is hot.

The Figure below, is taken from [Gary, 1993], and shows the 3 types of instabilities that can occur. From top to bottom, these are Right Resonant Mode, Non-Resonant Mode, and Left Resonant Mode.

Right resonnant mode. This is the mode which has the lowest instability threshold. It appears for an isotropic or weakly anisotropic cold beam. For $v_{0b} = 0$, we find the whistler mode, *i.e.* the high frequency part of the compressional alfvén mode. It is therefore right polarized, hence the first part of its name. This instability develops for $|\zeta_b \lesssim 1|$, which can only occur for the beam. So we have $\omega_r \simeq k_\parallel v_{0b} - \Omega_c$, which justifies the second part of its name. We then have $\omega_r > 0$, $k_\parallel > 0$ and $v_{0b} > 0$, so the mode propagates in the same direction as the beam.

The numerical solutions of this mode show that for $10^{-2} < n_b/n_e < 10^{-1}$, one has $\omega_r \simeq \gamma$ (at the maximum of the growth rate, *i.e.* where γ is maximum).

Exercize 16. *From Eq. (4.103), show that when $\omega_r \simeq \gamma$, the growth rate of this mode is given by*

$$\gamma = \Omega_c \left(\frac{n_b}{2n_e}\right)^{1/3} \tag{4.106}$$

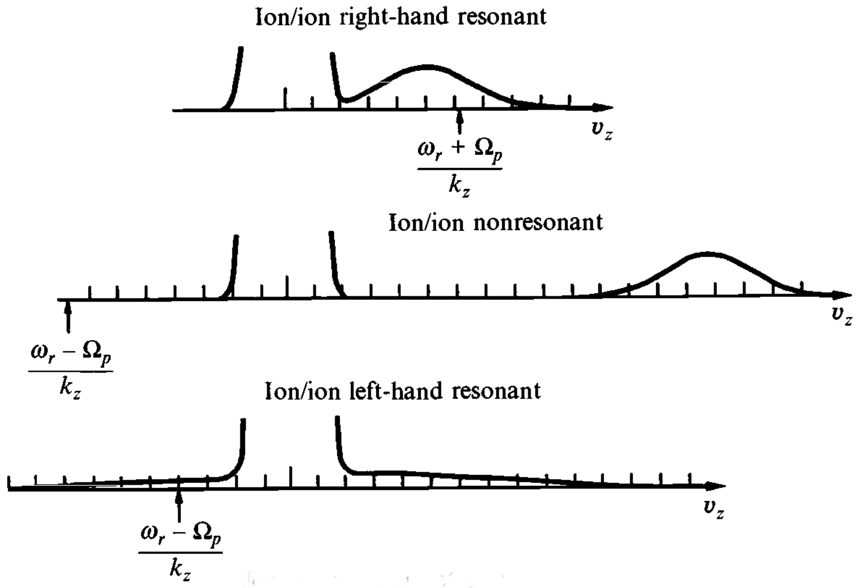

Figure 4.2: Schematic exhibiting the relation between ω_r, k_z, and the component of the particle velocity along the DC field (see [Gary, 1993]).

Left resonant mode. This mode appears for a sufficiently hot beam, typically $T_b/T_c \sim 10^2$. When v_{0b} approaches zero, this mode approaches AIC mode. It is therefore polarized to the left. As for the previous mode, $|\zeta_c \ll 1|$ and $|\zeta_e \ll 1|$. So only the protons of the beam can resonate ; A very hot beam is needed to have enough of them at $\omega_r \sim k_\parallel v_{0b} + \Omega_c$. So, its growth rate can be as large as the one of the right resonant mode. This mode also propagates in the direction of the beam.

Non-resonnant mode. This instability is similar to the firehose mode. For a fast and rather cold beam, we have $|\zeta_s^{-1}| \gg 1$ for the three s species of the plasma. This mode is then non-resonant. But the second order moment associated with the two proton distributions has the same consequences as a high parallel temperature, which favors the firehose mode. Unlike the other 2 modes, this mode propagates in the opposite direction to the beam. In order for its growth rate to become comparable to the right resonant mode, the ratios n_b/n_e and v_{0b}/v_A must be sufficiently large.

4.4 The ion fore-schock

Around a magnetized planet plunged into the supersonic and / or superAlfvénic flow of the solar wind, a bow shock is created. A shock is the non-linear limit of a (solitary) wave getting infinitely stiff. One never reaches a profile (of magnetic field for example) whose derivative is infinite, because the dissipative effects often limit this gradient (this is the difference with a soliton for which the non-linearities are counterbalanced by the dispersive effects).

In the case of the Earth, the Mach number (sonic or Alfvenic) is of the order of 8 and the shock is around 15 Earth radii. Downstream of this shock, the plasma still has the characteristics of the solar wind (for its temperature and its density) but a smaller bulk flow velocity. As the plasma downstream of the shock is hotter and denser (this compressive nature shows that it cannot be slow or intermediate shock, so it is a fast shock) and its magnetic field is increased. The physics of collisionless shocks is complex. We can retain that the structure of a shock depends a lot on the angle Θ_{Bn} between the direction of the magnetic field and that of the normal $\widehat{\mathbf{n}}$ to this shock. One speaks of quasi-parallel shocks for $0 < \Theta_{Bn} < \pi/4$ and of quasi-perpendicular shocks for $\pi/4 < \Theta_{Bn} < \pi/2$. We will only talk about quasi-perpendicular shocks to simplify things.

If we look at the profile of the magnetic field when crossing a quasi-perpendicular shock with $M \gtrsim 3$ (super-critical shocks), we identify 3 characteristic regions : the foot, the ramp and the overshoot. The ramp has a characteristic thickness of around 20 km. But during a gyroperiod, the protons flowing at the speed of the solar wind travel 5000 km while the electrons travel 3 km. Note that being at $\beta \sim 1$, $d_s \sim \rho_s$. The protons therefore behave, when passing through the shock, like demagnetized particles, while the electrons perform several cyclotronic turns. Consequently, the accumulation of protons downstream of the shock (and therefore the " missing ones " upstream) creates an electrostatic electric field from downstream to upstream. This field therefore has the consequence of slowing down the protons and accelerating the electrons. In addition, in the frame of reference drifting at the speed of the solar wind, the shock is not stationary. During their reflection, the particles undergo a Fermi acceleration, which increases their perpendicular energy, especially as the solar wind is rapid.

On the other hand, this electrostatic electric field is in the direction normal to the shock. So for $\Theta_{Bn} \neq 0$, this implies a component of the parallel electric field which will be able to efficiently accelerate the reflected protons. With the Fermi process, this is another mechanism to produce a beam. There is a third one ; depending on the value of their gyrophase, the protons which interact with the shock can be trapped. These protons will be efficiently heated. During this trapping, they therefore produce a current, which is on the origin of the deformation of the magnetic field profile. This is the reason why a foot is created in the structure of the shock. Numerous numerical simulations like the one of [Biskamp and Welter, 1972] show that this foot grows and can form a new ramp, while the old ramp becomes evanescent. This double magnetic structure is visible in Fig. 4.3 taken from [Biskamp and Welter, 1972] (1-dimensional PIC simulations). Protons can be trapped there, which is the third mechanism for heating them.

The protons whose gyrophase allows trapping and heating constitute a population that is often called " gyrating protons " ; it is a spot of protons which will rotate in the perpendicular plane. When we observe such populations in the foot or the overshoot, we speak of supercritical shock. But the presence of this type of population is also intrinsically linked to the Mach number. Below an Alfvenic Mach number of the order of 2 or 3, the profile of the magnetic field is rather laminar, beyond this it becomes turbulent. This distinction is only valid for quasi-perpendicular shocks. For quasi-parallel shocks, the level of magnetic fluctuations is very high ($\delta B/B \sim 1$). The region of the pre-shock is therefore very extensive.

The reflected protons will therefore create a reflected population upstream of the shock, having a distribution of their perpendicular velocity collected around zero. We can consider that 10 to 20%

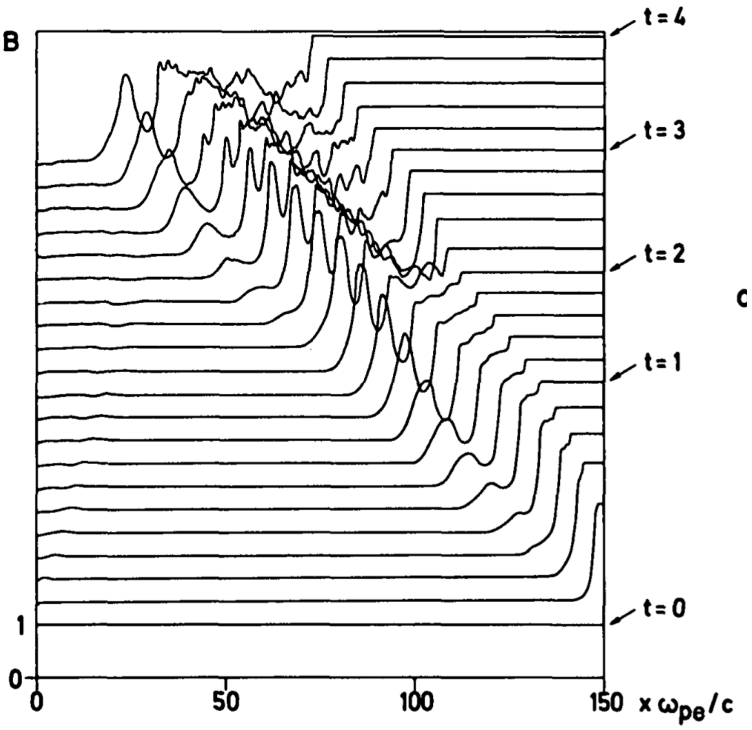

Figure 4.3: Time evolution of the magnetic field then exhibiting the shock reformation from the growth of the foot (see [Biskamp and Welter, 1972]).

of the protons of the solar wind are backscattered by the potential barrier of the shock. On the other hand, to see these protons at a certain distance from the shock, they must be able to get there because of their parallel speed. But the speed (in a perpendicular direction) of the solar wind competes to advect these protons as well as the field line on which they are frozen. This is called the " time of flight " effects. Thus, in Fig. **??** taken from [Cairns and Robinson, 1999], at the distance R from the intersection of the field line with the shock, we can only find protons reflected at the depth D_f if their parallel speed is such that

$$v_{\parallel} \geq \frac{R}{D_f} V_{SW} \tag{4.107}$$

where V_{SW} is the solar wind speed.

The observations in the pre-shock show that the parameter β_p (of the protons) is high, and that the density of the beam is low compared to that of the core, $n_b \sim 10^{-2} n_e$. As a result, the resonant upright mode is the most likely to grow.

Electromagnetic waves are of very low frequency, in the ULF range, with a period of 20s to 40s. They are often called the " 30s waves ". Sometimes observed in the form of " shocklets ", they are often almost monochromatic. In the satellite coordinate system, they are often observed as being left modes (see [Fairfield, 1969]). To see it, Fig. 4.5 represents a hodogram of the magnetic field in the plane xy, where the direction z is that of the magnetic field. The anti-clockwise direction of rotation is the signature of a right-hand polarized mode. In addition, 90% of observations of this

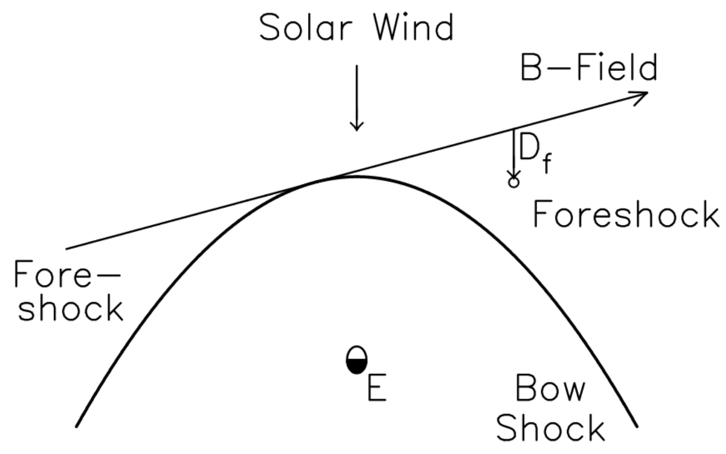

Figure 4.4: Représentation schématique de la position du pré-choc pour illustrer les effets de temps-de-vol. Voir [Cairns and Robinson, 1999].

type of mode are when the magnetic field is locally connected to the pre-shock, which reinforces the hypothesis of a non-local source of these modes.

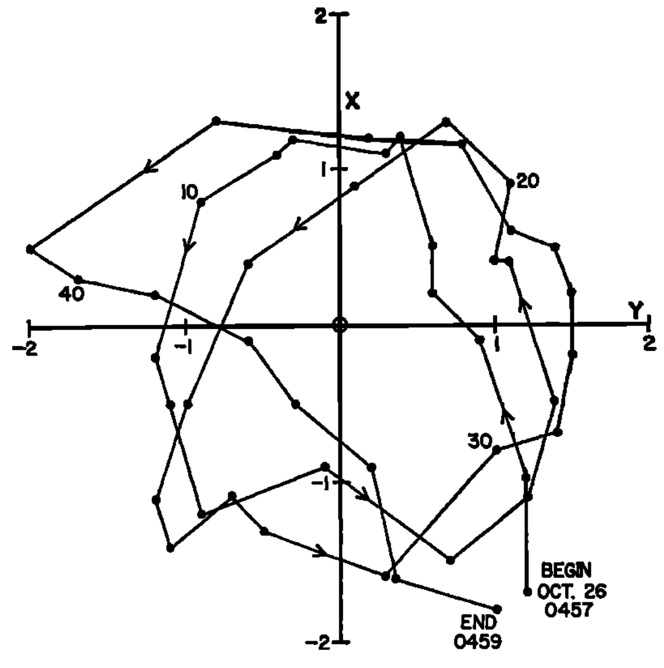

Figure 4.5: Hodograme du champ magnétique dans le plan xy. Voir [Fairfield, 1969].

In addition to the speed of the satellite, we must especially consider that of the solar wind. This speed is often significantly greater, as well as that of the satellite than that of the excited mode. As a result, the right mode moving " slowly " in the anti-solar direction, will be seen as a left mode moving in the direction of the solar wind, because the solar wind will knock it down quickly. This was shown in the study of [Hoppe et al., 1981]. Considering the Doppler effect, they show that these waves are right-hand polarized and propagate against the solar wind (see [Hoppe et al., 1981]).

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— A —

Recalls on waves and instabilities in plasmas

A.1 General

Phase velocity & group velocity

A wave can be characterized by its phase velocity V_ϕ. This vector is defined as

$$\mathbf{V}_\phi = \frac{\omega \mathbf{k}}{k^2} = \frac{\omega}{k}\widehat{\mathbf{k}} \tag{A.1}$$

This is the speed at which the front of a wave is moving. Depending on the ω and $\mathbf{k}$ values, this speed is hence defined for monochromatic waves. This velocity ca be larger than the speed of light in a vacuum. But as a consequence of the Heisenberg principle, we always have to deal with wave packets defined on a finite spectral band. A wave packet is the superposition of a set of monochromatic waves whose frequency is between ω and $\omega + d\omega$ while the associated wave number is between $\mathbf{k}$ and $\mathbf{k} + d\mathbf{k}$. The amplitude of each of these modes is of course a continuous function of ω and $\mathbf{k}$.

The group speed of a wave packet is the speed at which the energy in that packet travels. Most of this energy is where the superposition of these waves is constructive. The group speed will therefore be the speed at which the locus of these points moves. Considering the wave packet

$$\psi(\mathbf{r}, t) = \int \psi(\mathbf{k}, \omega) e^{\imath(\mathbf{k}.\mathbf{r} - \omega t)} d\omega d\mathbf{k} \tag{A.2}$$

we can compare the shape of this wave packet in $\mathbf{r}$ at time t with the shape of this same packet at a later time t' at $\mathbf{r}'$. At $t' = t + \delta t$ and $\mathbf{r}' = \mathbf{r} + \delta\mathbf{r}$.

$$\psi(\mathbf{r}, t) - \psi(\mathbf{r}', t') = \int \psi(\mathbf{k}, \omega) e^{\imath(\mathbf{k}.\mathbf{r} - \omega t)} [1 - e^{\imath(\mathbf{k}.\delta\mathbf{r} - \omega\delta t)}] d\omega d\mathbf{k} \tag{A.3}$$

For the interaction to be constructive, the relative phase between all the Fourier components of ψ must remain constant, *i.e.* its derivative has to be zero

$$d[\mathbf{k}.\delta\mathbf{r} - \omega\delta t] = 0 \tag{A.4}$$

Being in Fourier space, the unknowns are ω and $\mathbf{k}$. This derivative then writes

$$d\mathbf{k}\delta\mathbf{r} - d\omega\delta t = 0 \tag{A.5}$$

The energy of the disturbance will therefore move at the group speed

$$\mathbf{V}_g = \frac{\delta \mathbf{r}}{\delta t} = \frac{\partial \omega}{\partial \mathbf{k}} \tag{A.6}$$

The group speed is a vector which is not necessarily in the same direction as $\mathbf{k}$.

Parallel and perpendicular propagation

A magnetic field is a source of anisotropy. The magnetic field drives the gyromotion of the particles. It also often plays a role in the propagation of waves. Parallel and perpendicular propagation refer to the direction of the wave vector $\mathbf{k}$ relative to the DC magnetic field. Oblique propagation means $0 < \Theta < \pi/2$. As well, quasi-parallel propagation is when Θ is close to (but different) from 0, and quasi-perpendicular propagation is when Θ is close (but different) to $\pi/2$.

Electrostatic and electromagnetic modes

An electrostatic wave has only one electric component $\mathbf{E}$ and no magnetic component $\mathbf{B}$ [1]. For electrostatic modes, instead of treating the set of Maxwell's equations, only the Maxwell-Gauss (or Maxwell-Poisson) equation is sufficient. Conversely, when $\mathbf{k}$ and $\mathbf{E}$ are orthogonal, we speak of electromagnetic waves.

Longitudinal and transverse modes

This distinction refers to the relative direction of the electric field of the wave $\mathbf{E}$ with respect to the wave vector $\mathbf{k}$. When $\mathbf{k} \parallel \mathbf{E}$ we speak of longitudinal mode. When $\mathbf{k} \perp \mathbf{E}$ we speak of transverse mode. In Fourier space, the Maxwell-Faraday equation gives $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$. A longitudinal mode is therefore always electrostatic. Likewise,

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} = -k^2 \mathbf{E}_T \tag{A.7}$$

We will see that the dispersion equation is then written

$$\left(\frac{k^2 c^2}{\omega^2} \mathbf{1} - \boldsymbol{\varepsilon} \right) . \mathbf{E}_T = -\boldsymbol{\varepsilon} . \mathbf{E}_L \tag{A.8}$$

So, when $k \to \infty$, we have $\dfrac{k^2 c^2}{\omega^2} \gg \boldsymbol{\varepsilon}$, and E_T then goes to 0, that is the longitudinal term E_L therefore dominates. Close to the resonance, the waves are almost electrostatic. Generally in plasma, waves can have a longitudinal component and a transverse component at the same time. They are then neither purely electrostatic nor purely electromagnetic.

Polarization

The polarization of a wave is the direction of its electric field and its magnetic field. These 2 directions are very often different from each other. If the directions of the vectors $\mathbf{E}$ and $\mathbf{B}$ of the wave are fixed, then we speak of linear (or rectilinear) polarization. Otherwise, they are elliptical or circular (which is a specific case of the elliptical polarization). The extreme cases of parallel or perpendicular modes call for a new classification:

[1] be careful to distinguish the DC magnetic field with the fluctuation of the magnetic field associated with the wave. A wave can be electrostatic in magnetized plasma, or can be electromagnetic in a unmagnetized plasma.

- In parallel propagation, we speak of right and left mode. These are the 2 circularly polarized modes, rotating in the 2 possible directions (the right mode is the one which rotates with the electrons). Using the Stix notations (which will be detailed in the problem on the CMA diagram), we have $N = R$ for the right mode and $N = L$ for the left mode, where $N = kc/\omega$ is the optical index.

- In perpendicular propagation, we speak of ordinary and extraordinary mode. Using the Stix notations, we have $N = P$ for the ordinary mode and $N = RL/S$ for the extraordinary mode. Plasma oscillation is an example of ordinary mode.

More generally, the term polarization can be used to refer to any one of the disturbed vector quantities, which therefore includes the speed of order 1. In order to identify the direction of the vectors ($\mathbf{E}_1$, $\mathbf{B}_1$ or $\mathbf{V}_1$), we recall the unit vectors (in the Cartesian system) introduced in the preamble. By convention (in magnetized plasma), the DC component of the magnetic field is along the unit vector $+\widehat{\mathbf{z}}$ and the wave vector $\mathbf{k}$ is along a $\widehat{\mathbf{k}}$. Then, $\widehat{\mathbf{t}}$ the unit vector normal to $\widehat{\mathbf{k}}$ in the xz plane. These vectors are depicted in Fig. (1).

Helicity

Magnetic helicity is a quantity which makes it possible to quantify the degree of twisting and entwining of the magnetic field lines. Mathematically, it is defined by

$$K = \int \mathbf{A}.\mathbf{B}\,\mathrm{d}\mathbf{r} \tag{A.9}$$

where $\mathbf{A}$ is the vector potential associated with the magnetic field $\mathbf{B}$ by the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. For this, we always choose the Coulomb gauge $\boldsymbol{\nabla}.\mathbf{A} = 0$. For $\mathbf{A}$ and $\mathbf{B}$ getting null at infinity, the value of K is unique and characterizes the plasma.

It is important to note that this is a "spatial" concept, unlike polarization which is a "temporal" concept. Helicity will characterize the way a field line twists (at a given time) in space, while polarization characterize the way the magnetic field vector rotates (at a given location) with time.

A right-hand polarized mode is a mode for which the fluctuations of the magnetic field rotate in the same direction as the electrons around the DC magnetic field, whether this mode propagates parallel or anti-parallel to it. If considering two screws with opposite pitches, if one gives a right-hand polarized mode with a given direction of the wave vector, the other is also a right-hand polarized mode for a wave vector of opposite sign. The polarization therefore characterizes the direction of rotation with time, independently of the way in which the field lines are entangled. It is therefore the direction of $\mathbf{k}$ and not that of $\mathbf{B}$ that matters.

Helicity characterize the way a field line rolls up on itself. the notion of wave vector therefore has nothing to do with this definition, and it is here the vector $\mathbf{B}$ that matters. To illustrate this concept, a screw has negative helicity. And we screw by turning clockwise, or even in the dextrorotatory direction (if on the other hand this screw has a reverse pitch, it is the levogyre direction). By observing this screw, it appears that the winding is a characteristic of the screw, and is therefore independent of the direction in which you look at it.

If the structure of the field lines is the superposition of linear modes with the form $\exp \imath(\omega t - \mathbf{k}.\mathbf{r})$, then a right-polarized mode has a positive helicity (for $\mathbf{k}.\mathbf{B} > 0$) and vice versa.

The helicity being defined at a given time, it only quantifies the direction of winding of the field lines : looking at a spring from one side or the other, does not matter on how it is coiled. On the other hand, if you move along this spring, the direction in which you turn with time depends on your direction of propagation, *i.e.* on the sign of the wave vector $\mathbf{k}$.

In conclusion, for linear modes, polarization and magnetic helicity are related. But a plasmas in which there would be half left mode and half right mode, the helicity would be close to zero. In numerical simulation, the helicity can be normalized so as to vary only between -1 and +1.

Finally, note that we can also define the current helicity by

$$\int \mathbf{B}.(\boldsymbol{\nabla} \times \mathbf{B})\, \mathrm{d}\mathbf{r} \tag{A.10}$$

and the kinetic helicity by

$$\int \mathbf{V}.(\boldsymbol{\nabla} \times \mathbf{V})\, \mathrm{d}\mathbf{r} \tag{A.11}$$

The current helicity characterizes the winding of the current while the kinetic helicity characterizes the winding of the flow lines. These three helicities can be defined locally (at a point) by their respective integrand ; we then speak of helicity density.

A.2 Linear and non-linear modes

The system of equations (Maxwell + plasmas) which makes it possible to describe the response of the plasma to external fields is intrinsically non-linear, which constitutes a complication. In theory, one cannot therefore study the response to an excitation without taking into account the amplitude of this one.

In linear theory, we assume that the wave amplitudes are very low. The response to a superposition of excitations is the superposition of the responses to individual excitations. This way of proceeding is only acceptable if the electromagnetic energy density associated with the wave is low compared to the density of thermal energy of the undisturbed plasma :

$$n\frac{mv_T^2}{2} \gg \varepsilon_0\frac{E^2}{2} + \frac{B^2}{2\mu_0} \tag{A.12}$$

The equations at zero$^{\text{th}}$ order are satisfied as they describe the equilibrium state. The first order equations describe the waves. Due to the temporally and spatially disspersive nature of the plasma, these calculations are done in the Fourier space $(\mathbf{k}, \omega)$. For causality reason, the transformation to be used in time should be a Laplace transform. But when interested in a mode and not in the process of its creation and/or destruction (because of its damping or instable nature), we nonetheless generally use a Fourier transform.

Otherwise, if the characteristics of the mode depends on its amplitude, it is hence necessary to do the calculations in a non-linear way while explicitly keeping the amplitude of the wave. There is another possible source of non-linearity : the inhomogeneity of the medium. In the presence of a gradient (of density, of temperature ...) the Fourier transform gets complicated and it is generally better to keep the explicit form of the gradient ; this leads to solving a partial differential equation.

A.3 Plasma dispersion relation

We here focus on the Maxwell's equations and not in the specific form of the conductivity tensor (which will be done later). We also only consider the 2 rotational equations which provides the time evolution of $\mathbf{E}$ and $\mathbf{B}$.

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{A.13}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{A.14}$$

Then,

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) &= \boldsymbol{\nabla} \times -\frac{\partial}{\partial t}\mathbf{B} && \text{(A.15)}\\
&= -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) && \text{(A.16)}\\
&= -\frac{\partial}{\partial t}\left(\mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}\right) && \text{(A.17)}\\
&= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} - \mu_0 \frac{\partial}{\partial t}\mathbf{J} && \text{(A.18)}
\end{aligned}$$

Developing the double vectorial product, one gets

$$\nabla^2 \mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{E}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \mu_0 \frac{\partial}{\partial t}\mathbf{J}, \tag{A.19}$$

which writes in Fourier space

$$\left[\left(\frac{\omega^2}{c^2} - k^2\right)\mathbf{1} + \mathbf{kk}\right].\mathbf{E}(\mathbf{k},\omega) = +\imath\omega\mu_0 \mathbf{J}(\mathbf{k},\omega) \tag{A.20}$$

The current density $\mathbf{J}$ involved in this equation is total, meaning that it is the sum of the external current density $\mathbf{J}_f$ and of the induced (by the plasma) current density $\mathbf{J}_i$. The induced current density is related to the electric field by the conductivity tensor. In real space, we recall that the relation is of the form

$$\mathbf{J}_i(\mathbf{r},t) = \int_0^{+\infty} \iiint_{\mathbb{R}^3} \boldsymbol{\sigma}(\boldsymbol{\rho},t).\mathbf{E}(\mathbf{r}-\boldsymbol{\rho},t-\tau)d\boldsymbol{\rho}d\tau \tag{A.21}$$

Removing the induced part of the current density (by introducing the conductivity tensor),

$$\left[\left(\frac{\omega^2}{c^2}-k^2\right)\mathbf{1}+\mathbf{kk}-\imath\omega\mu_0\boldsymbol{\sigma}\right].\mathbf{E}(\mathbf{k},\omega)=+\imath\omega\mu_0\mathbf{J}_f(\mathbf{k},\omega) \tag{A.22}$$

We often consider the case where $\mathbf{J}_f=0$ (which means that the wave dispersion equation that we obtain is only valid in a medium without current). Multiplying this equations by c^2/ω^2, we can introduce the vector form of the optical index defined by

$$\mathbf{N}=\frac{\mathbf{k}c}{\omega} \tag{A.23}$$

With the relation $\boldsymbol{\sigma}=-\imath\omega\varepsilon_0(\boldsymbol{\varepsilon}-\mathbf{1})$, the above equation then writes

$$(\mathbf{NN}-N^2\mathbf{1}+\boldsymbol{\varepsilon}).\mathbf{E}(\mathbf{k},\omega)=0 \tag{A.24}$$

The eigen modes are those satisfying the above equation. A non-trivial solution ($\mathbf{E}\neq 0$) exists if the determinant of the left hand side is zero.

Using the wave vector components, $\mathbf{k}=k\sin\Theta\widehat{x}+k\cos\Theta\widehat{z}$, the above determinant writes

$$\begin{vmatrix} \varepsilon_{xx}-N^2\cos^2\Theta & \varepsilon_{xy} & \varepsilon_{xz}+N^2\sin\Theta\cos\Theta \\ \varepsilon_{yx} & \varepsilon_{yy}-N^2 & \varepsilon_{yz} \\ \varepsilon_{zx}+N^2\sin\Theta\cos\Theta & \varepsilon_{zy} & \varepsilon_{zz}-N^2\sin^2\Theta \end{vmatrix}=0 \tag{A.25}$$

To finish this developments, it is necessary to explain the shape of the dielectric tensor. This can be done using kinetic formalism or fluid formalism. In kinetics, we write and linearize a kinetic equation (genrally Vlasov). In fluid, we write the fluid equations (for each fluid). In both cases, the plasma equations then need to be linearized in the same way as the Maxwell equations.

A.4 Magnetic permeability of a plasma

In plasma physics, we generally only talk about magnetic field that we note $\mathbf{B}$. But when studying electromagnetism in continuous medium, we name $\mathbf{B}$ the “magnetic induction field” (in Teslas), and $\mathbf{H}$ the “magnetization field” (in Amperes per meter). For a media in linear regime, these two fields are linked by the magnetic permeability of the medium μ

$$\mathbf{B}=\mu\mathbf{H} \tag{A.26}$$

Premeability then characterizes for a material its ability to modify a magnetization field $\mathbf{H}$. Recall that $B<\mu_0 H$ for a diamagnetic material, $B>\mu_0 H$ for a paramagnetic material, and $B\gg\mu_0 H$ for a ferromagnetic material, $\mu_0=4\pi 10^{-7}$ TmA $^{-1}$ being the permeability of the vacuum.

In magnetostatic (to simplify), the current has a component $\mathbf{J}_c$ due to conduction and a $\mathbf{J}_m$ component due to magnetization, ie to the response of the medium to the magnetization field $\mathbf{H}$. This magnetization current is linked to the magnetization of the medium $\mathbf{M}$ by $\mathbf{J}_m=\boldsymbol{\nabla}\times\mathbf{M}$. This classical statement actually contains a quantum behavior of matter, by which a current is associated with the orbital angular momentum of the atoms constituting the medium. Likewise, a moment due to the spin of the electrons influences this current density, even if this image is then fragile in a non-quantum context.

A question therefore arises; Why in plasma physics, the magnetic permeability is always worth μ_0? The answer is quite simple : while the orbital or spin angular moments have a privileged direction, it is necessary that the atoms which carry them be aligned. This is the case in a cristal as the atoms have a fixed position on the cristal. This could also be the case in a plasma where atoms are freely moving, but statistical physics clearly show that the temperature associated to nuclear spin is very small (at meast for astrophysicl plasmas) meaning that the spin contribution to the hamiltonian of the atom is very small, compared to the translation component. The resulting magnetization current can then be neglected.

In conclusion, in a plasma, we always have $\mu = \mu_0$ and therefore $\mathbf{B} = \mu_0\mathbf{H}$, so that the vector $\mathbf{H}$ is of no interest (and therefore never mentioned).

A.5 Conductivity tensor and dielectric tensor

Because of an external electric field, a charged particle moves. This can be seen in 2 different ways :

- a current $\mathbf{J}$ is associated with the movement of the particle. The medium is then a conductor, characterized by its conductivity tensor $\boldsymbol{\sigma}$. The current density $\mathbf{J}$ is given by the general relation $\mathbf{J} = \boldsymbol{\sigma}.\mathbf{E}$. In the simplest cases, $\boldsymbol{\sigma}$ is a diagonal tensor whose elements (of the diagonal) are all equal to each other (and therefore equal to one third of the trace).

- the movement of the particle can be seen as the creation —in addition to the initial charge of the particle— of a dipole formed of a charge equal to the initial charge, placed in its new position, and a charge of opposite sign to the initial charge placed in its old position. The medium is then a dielectric characterized by its dielectric tensor $\boldsymbol{\varepsilon}$ or by its susceptibility tensor $\boldsymbol{\chi}$. The dielectric displacement vector[2] $\mathbf{D}$ is given by the relation $\mathbf{D} = \varepsilon_0\boldsymbol{\varepsilon}.\mathbf{E} = \varepsilon_0(\mathbf{1} + \boldsymbol{\chi}).\mathbf{E}$[3].

From this purely formal choice, we have two way to write the Maxwell-Ampere equation. This equation involves the current density $\mathbf{J}$. For a conductor, this density is associated with the current resulting from the displacement of free charges (these are electrons in a metal, but it can also be ions in a plasma). For any material, this current density $\mathbf{J}$ results from the sum of the current density associated with the free charges $\mathbf{J}_f$ (associated with the free charge carriers of the material) and the current density $\mathbf{J}_i$ induced by the polarization of the medium. So,

[2]the vector $\mathbf{D}$ is also called the « electric induction »

[3]In these notes, we will use the *relative* dielectric tensor $\boldsymbol{\varepsilon}$.

- for a conductor,

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} = \mu_0 \boldsymbol{\sigma}.\mathbf{E} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{A.27}$$

- for a non-magnetic dielectric

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{\varepsilon}.\mathbf{E}) \tag{A.28}$$

For $\mathbf{E}$ and $\mathbf{B}$ fields depending on time as $e^{\imath\omega t}$, the relation between these two tensors is

$$\boldsymbol{\sigma} = -\imath\omega\varepsilon_0 \boldsymbol{\chi} \tag{A.29}$$

with $\boldsymbol{\varepsilon} = \mathbf{1} + \boldsymbol{\chi}$.

Among these 2 formalisms, we often use the dielectric description because all the physics is hence contained in a single tensor $\boldsymbol{\varepsilon}$.

In addition, the electric polarization vector $\mathbf{P}$ is the response of a dielectric medium to an external electric field,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{A.30}$$

We remember the electromagnetism results giving the induced charge density and the induced current density:

$$\frac{\partial \mathbf{P}}{\partial t} = \mathbf{J}_i \quad , \quad \boldsymbol{\nabla}.\mathbf{P} = -\rho_i \tag{A.31}$$

whose compatibility is ensured by the continuity equation. But the relation between $\mathbf{P}$ and $\mathbf{E}$ is not always simple, and wuite different from the ones encountered in dielectrics.

A.6 Spatial & temporal dispersion

In optics, the relation between $\mathbf{D}$ and $\mathbf{E}$ is often non-instantaneous. Thus, the value of $\mathbf{D}\ (\mathbf{r}, t)$ depends on the value of $\mathbf{E}\ (\mathbf{r}, t')$, for all $t' \leq t$. The same goes for plasmas. But the motion of a charged particle depends on the value of the field that it undergoes all along its trajectory. The relation between $\mathbf{D}$ and $\mathbf{E}$ is therefore non-local, and can be written (for causality reason) in the form

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \varepsilon_0 \int_{-\infty}^{t} \iiint_{\mathbb{R}^3} \boldsymbol{\chi}(\mathbf{r} - \mathbf{r}', t - t').\mathbf{E}(\mathbf{r}', t')\, \mathrm{d}\mathbf{r}'\, \mathrm{d}t' \tag{A.32}$$

One recognize the convolution product in the last member, which suggests that we should go in the Fourier space (in space and time). Noting $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}'$, $\tau = t - t'$, and[4]

$$\boldsymbol{\chi}(\mathbf{k}, \omega) = \int_{0}^{+\infty} \iiint_{\mathbb{R}^3} \boldsymbol{\chi}(\boldsymbol{\rho}, \tau) e^{\imath(\omega\tau - \mathbf{k}.\boldsymbol{\rho})}\, \mathrm{d}\boldsymbol{\rho}\, \mathrm{d}\tau \tag{A.33}$$

[4]We use the same notation for the susceptibility tensor in real space and in Fourier space. There is no possible confusion when the variables of the considered space are indicated

one gets the relation

$$\mathbf{D}(\mathbf{k},\omega) = \varepsilon_0 \boldsymbol{\varepsilon}(\mathbf{k},\omega).\mathbf{E}(\mathbf{k},\omega) \tag{A.34}$$

with $\boldsymbol{\varepsilon} = \mathbf{1} + \boldsymbol{\chi}$ in the $(\mathbf{k},\omega)$ space. In a plasma, the dielectric tensor depends on the wave number and on the frequency. This is called spatial dispersion and temporal dispersion.

The motion of a charged particle depends on the value of the electric field met during its whole history, i.e. along its trajectory. However, there is a distance l_{cor} beyond which the shape of the electric field in r has little consequence on the electric displacement vector in $r + l_{cor}$. In other words, the state of the particle at $r + l_{cor}$ is poorly correlated with that which it had at r. This characteristic length can only depend on 2 processes: collisions and fluctuations in electric and magnetic fields due to collective effects.

In a solid dielectric, the locality of the relationship is clear: atoms have a location in the solid which does not change with time. If we apply a field $\mathbf{E}$ at $\mathbf{r}$ where an atom is located, the a dipole will develop at $\mathbf{r}$. This one is grounded to its atom, and therefore will not travel. The field $\mathbf{E}$ in $\mathbf{r}$ will induce a displacement vector $\mathbf{D}$ in $\mathbf{r}$ and not elsewhere. In this case, the relations are local and do not require a convolution product as in the equation (A.32).

In a plasma, the charges are free and therefore move. This induces non-locality, but in a limited way: in the equation (A.32), the integral over the position space can be limited to a volume centered on $\mathbf{r}$ because the function $\boldsymbol{\chi}$ is decreasing. We can evaluate the characteristic distance over which $\boldsymbol{\chi}$ will tend towards 0. For the case of collisions, we consider an electric field constant over time (i.e. for which $\omega \to 0$). At $(\mathbf{r}, t)$, a field $\mathbf{E}$ will induce a polarization as well as a displacement vector $\mathbf{D}$. Since the particle can move freely, it will be able to transport this dipole moment as long as it does not undergo a collision (the effect of this could be to modify the dipole). It can therefore move over a time of the order of ν^{-1} where ν is the collision frequency. The non-locality will therefore be important over a distance $l_c < v_T/\nu$ where v_T is the thermal speed.

In the case of collective effects, the electric field felt by the particle is no longer constant through time. It varies on a time scale of the order of ω^{-1} where ω is the wave angular frequency. The polarized particle will therefore be able to transport its dipole moment over a time at most of the order of ω^{-1}, ie over a distance $l_f = v_T/\omega$. Beyond this distance, the moving dipole will be modified by the shape of the electric field, which will have significantly changed with time. With these 2 arguments, we can define a correlation length $l_{cor} = \min(l_c, l_f)$ beyond which $\boldsymbol{\chi}$ becomes negligible.

Spatial dispersion is important when $kl_{cor} \geq 1$. Otherwise, it is negligible: the term in $e^{-\imath \mathbf{k}.\boldsymbol{\rho}}$ is very close to 1 in the relation (A.33) and the integral no longer depends on $\mathbf{k}$. A medium is therefore no longer spatially dispersive when $\omega, \nu \gg kv_T$. In the equation (A.33), $\boldsymbol{\chi}(\boldsymbol{\rho}, \tau)$ is a function which is maximum in $\tau = 0$ and $\boldsymbol{\rho} = 0$. This function will then decrease (in τ and in $\boldsymbol{\rho}$), and l_{cor} is the distance over which the value of $\boldsymbol{\chi}(\boldsymbol{\rho}, \tau)$ will be substantially decreased.

A.7 Dissipation

The dissipation for a given mode in a plasma can be calculated from the conservation equation of electromagnetic energy. To take into account the plasma, the current density which occurs must contain the current density associated with the free charges as well as the polarization current density. It is then very easy to write Maxwell's equations in a dielectric

$$\boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B} \tag{A.35}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{free}} - \partial_t \mathbf{D} \tag{A.36}$$

so the conservation equation of electromagnetic energy for a dielectric is

$$\left(\mathbf{E}.\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}.\frac{\partial}{\partial t}\mathbf{B}\right) + \boldsymbol{\nabla}.(\mathbf{E} \times \mathbf{H}) = \mathbf{J}_{\text{free}}.\mathbf{E} \tag{A.37}$$

which reveals the new form of the Poynting vector which we denote by $\mathbf{S}$. In the absence of free charge current density, the variation of the electromagnetic energy is given by $\boldsymbol{\nabla}.\mathbf{S}$. In addition, the dissipation of electromagnetic energy can only be done by exchanging energy with the plasma. However, $\mathbf{H} = \mathbf{B}/\mu_0$ does not involve the properties of the plasma. So only the first term $\mathbf{E}.\partial_t\mathbf{D}$ contains the dissipative term. Its calculation requires to retain only the real part of $\mathbf{E}$ and $\mathbf{D}$, and to calculate a time average. The dissipation that we quantify by the scalar Q therefore writes

$$Q = \langle \mathbf{E}_r.\dot{\mathbf{D}}_r \rangle \tag{A.38}$$

where the brackets indicate a time average calculated on $\mathbf{E}_r$ and $\mathbf{D}_r$ which are the real[5] expressions of electric field and electric displacement vectors. With an electric field proportional to $e^{\imath(\mathbf{k}.\mathbf{r}-\omega t)}$, one has $\mathbf{E}_r = \frac{1}{2}(\mathbf{E} + \mathbf{E}^*)$ and $\dot{\mathbf{D}}_r = \frac{1}{2}\imath\omega\varepsilon_0[-\boldsymbol{\varepsilon}(\mathbf{k},\omega).\mathbf{E} + \boldsymbol{\varepsilon}^\star(\mathbf{k},\omega).\mathbf{E}^*]$ [6], so

$$Q = -\frac{\imath\omega\varepsilon_0}{4}\mathbf{E}^T.(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{*T}).\mathbf{E}^* \tag{A.39}$$

Because the (complex) dielectric tensor $\boldsymbol{\varepsilon}$ can be spread in an hermitian $\frac{1}{2}(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{*T})$ and an anti-hermitian part $\frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{*T})$ it is noted that the dissipation in a plasma is due only to the anti-Hermitian part of the dielectric tensor. This is an important point which leads to several remarks:

- in the case of electrostatic (or longitudinal) modes, everything happens in one dimension. The dielectric tensor is then reduced to a scalar. In such a case, the dissipation can only occur if $\boldsymbol{\varepsilon}$ has an imaginary component. This is the case when one studies Landau damping.

[5] We cannot do the calculations in complex because the multiplication and real part operations do not commute between them. This remark becomes important for non-linear terms like this one.

[6] this being the consequence of the definition of ε by a Fourier transform, and that $\mathbf{D}_r = \frac{1}{2}\varepsilon_0(\boldsymbol{\varepsilon}\mathbf{E} + \boldsymbol{\varepsilon}^\star\mathbf{E}^\star)$. Morever, we have by construction (for the derivative) $\boldsymbol{\varepsilon}^\star(\mathbf{k},\omega) = \boldsymbol{\varepsilon}(-\mathbf{k},-\omega)$

- For cold plasmas (without temperature effect), we can establish the shape of the dielectric tensor (see Eq. **??**, **??** and **??**). There is only one complex term that appears, of Hermitian symmetry. So all the modes that we can study in the context of magnetized cold plasmas will be non-dissipative.
- For these same magnetized cold plasmas, if we take into account a collision term (for example a term in $-nm\nu\mathbf{V}$ where ν is a collision frequency), we easily show that we modified the terms of the dielectric tensor by revealing an anti-Hermitian component.

Thus, to make an anti-Hermitian component appear in the dielectric tensor, either collisions or temperature effects are needed:

- The case of collisions is the most intuitive. During collisions (without discussing their nature), there is a transfer of momentum and energy which therefore makes it possible to dissipate directed energy into thermal energy.
- In the case of temperature effects, consider a particle with a velocity v in the same direction as the wave number k of a wave with angular frequency ω. The angular frequency of this wave seen by the particle is, by Doppler effect, $\omega - kv$. If this angular frequency is zero, it means that the particle sees a constant phase, and can therefore work with a non-zero average value. The particle will then gain or lose energy, which means that the wave will lose or gain some. This is another form of possible dissipation, which is handled correctly in kinetic theory and is called the Landau effect.

For magnetized plasmas, the temperature effects have somewhat different consequences. The particles gyrate around the field lines at the ω_c angular frequency. The above condition can be rewritten in magnetized plasma $\omega - k_\parallel v_\parallel = \omega_c$. In this case, the Doppler effect to be considered is indeed that in the direction parallel to the magnetic field. This formula can even be extended to all the harmonics of the gyrofrequency, and we obtain for $n \in \mathbb{N}$, $\omega - k_\parallel v_\parallel = n\omega_c$. This is referred to as the Landau cyclotron effect. If one wants to consider the Landau effect, we can no longer treat the problem with a fluid formalism (the shape of the distribution function becomes important), and the kinetic formalism becomes necessary.

A.8 Recalls of MHD

In the MHD approach, only one fluid is considered: the MHD fluid. Its density is the one of protons (or electrons by quasi-neutrality), its fluid velocity is the barycentric mean of the fluid velocities of its constituents weighted by their respective masses[7], and its pressure (isotropic) is the sum of the partial kinetic pressures of all species. The framework of the MHD imposes a few assumptions that we recall.

[7] the mass ratio $\mu = 1836$ causes that the MHD velocity is roughly that of protons.

Weak variations hypothesis. The MHD equations are only valid for "small fluctuations". The spatial and temporal gradients must be at "large scale" $\partial_t \sim 1/\tau \ll \omega_c, \omega_p$ and $\nabla \sim 1/L \ll r_L^{-1}, \lambda_D^{-1}$. By getting all the gradients null in the fluid and Maxwell equations,

$$\mathbf{E} + \mathbf{V}_s \times \mathbf{B} \rightarrow 0 \tag{A.40}$$

$$\rho \rightarrow 0 \tag{A.41}$$

$$\mathbf{J} \rightarrow 0 \tag{A.42}$$

— The first equation means that the species all have the same perpendicular fluid velocity. This fluid average speed is the MHD speed. It is equal to $\mathbf{E}\times\mathbf{B}/B^2$ which is also the speed of the magnetic field, *i.e.* the speed of the frame in which the electric field is zero[8]. A first consequence is that the parallel electric field is zero. For an electrostatic field, the field lines are then equipotential. A second is that for a non-relativistic plasma ($E_\perp/B \ll c$), the electrical energy is negligible compared to the magnetic energy: $\varepsilon_0 E^2/2 \ll B^2/2\mu_0$.

— The second relation is the quasi-neutrality approximation. It means that the charge density gradients of each species α are small compared to the charge densities of these species: $\delta n_\alpha / n_\alpha \ll 1$. As a consequence, in the Maxwell-Gauss equation, the divergence of the electric field can be used to estimate the total charge density, but not the opposite.

— The third relation means that the total current is weak compared to the currents carried by each species (There is no "slip" between the particles [9]). These last two relationships are the causes of the collective behavior of the plasma for movements parallel to the magnetic field.

The displacement current. Maxwell-Ampère equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{A.43}$$

contains two terms in the right hand side. The last term of this member is the displacement current. We can see simply see (as an exercise) that when the phase speed of the modes is small compared to the speed of light ($\omega/k < c$) then the displacement current is negligible.

As the two diverging equations do not give additional information on electric and magnetic fields, the electric field only appears in the Maxwell-Faraday equation. This is the reason why we must introduce another equation involving the electric field.

The Ohm's law. The electric field appears in the momentum conservation equation of each fluid. We can therefore use one of these equations to determine the form of $\mathbf{E}$. Writing it for electrons, with some substitutions and isolating the term $\mathbf{E}$, we get

$$\mathbf{E} = -\mathbf{V}_p \times \mathbf{B} + \frac{1}{ne}\mathbf{J}\times\mathbf{B} - \frac{m_e}{e}\mathrm{d}_t \mathbf{V}_p + \frac{m_e}{e^2}\mathrm{d}_t\left(\frac{\mathbf{J}}{n}\right) - \frac{1}{ne}\boldsymbol{\nabla}.\mathbf{P}_e + \eta\mathbf{J} \tag{A.44}$$

[8] This is a known result of the Lorentz transform (relativistic or not) of the fields **E** and **B**

[9] When this must be the case, we must then do at least Hall MHD.

The first term of the right hand side is of order 0; all the others are of order 1. To know in which spatial or temporal domain it is necessary to keep them, it is necessary to study their "scaling":

- The first term on the right hand side of Eq. (A.44) is of order 0, hence never neglected. When the electric field is given by this term alone, Ohm's law is said to be ideal. In MHD, $\mathbf{V}_p = \mathbf{V}$ at order m_e/m_p.

- The second term (Hall effect) has a scaling in $k^2 v_A^2/\omega\Omega_p$. For an Alfvenic perturbation, the time scale is in Ω_p^{-1}, and the spatial scale is in v_A/Ω_p (which is also equal to the inertial length of the protons). It is a term that is associated with the slip between electrons and protons, when the latter are no longer magnetized. We keep this term in MHD Hall.

- The third term (electron inertia - 1) has a scaling in $\omega/\mu\Omega_p$, that is to say a time scale in $(\mu\Omega_p)^{-1}$. It could become important in the vicinity of the gyrofrequency of the electrons. This term is very generally neglected.

- The fourth term (inertia of electrons - 2) has a scaling in d_e (length of inertia of electrons) and only becomes important at electronic spatial scales.

- The fifth term (compressibility of electrons) has a scaling in $k^2\rho_e^2\Omega_p/\omega$ ie a spatial scale in ρ_e (Larmor radius of electrons) and a time scale in Ω_p. This effect can therefore be important for hot electrons.

- The sixth term (resistive effects) has a scaling in $nek^2/\omega\mu_0\sigma$. To compare the importance of the resistive terms with the convective term of ideal Ohm's law, we often use the Lundquist number S, the ratio between the convection time and the diffusion time. Except in very dense plasmas, L is often infinite for astrophysical plasmas.

As an illustration, the nature of Alfvén waves therefore depends on Ohm's law that we keep. It also depends on the closure of the hierarchy (often at the level of pressure, scalar or tensorial).