# — 1 —

# Alfvén modes

In 1942, Hannes Alfvén wrote a seminal paper on plasma physics (*cf.* [Alfvén, 1942]). He showed that in a magnetized plasma, waves can propagate along the magnetic field. These wave have both acoustic and magnetic properties. One of these waves is often called the torsional Alfvén mode. By making an analogy with vibrating strings —for which the phase velocity of transverse waves is the square root of the ratio of the tension to the linear density— the expression of the magnetic tension and the linear density of a flux tube gives a phase velocity equal to the Alfvén velocity (see [Hasegawa and Uberoi, 1982]).

The Alfvén mode comes in many variants in astrophysical context, depending on the way it propagates and the level of approximation. These modes play an important role in heating and transporting energy. One of this mode could be part of the puzzle explaining the transport of magnetic energy in stellar winds, the transfer of angular momentum in molecular disks during star formation, the magnetic pulsations of planetary magnetospheres, or the scattering of cosmic rays during star formation, or their acceleration by diffusive shocks in supernovae remnants.

The four sections of this chapter expose some of these variations ; they are all made within the framework of a fluid approach, but could also be done with a kinetic approach, which would even be a necessity if we wanted to know their damping rate by Landau effect.

## 1.1 Alfvén wave in ideal MHD

We consider an adiabatic closure for a total iscalar pressure as well as an ideal Ohm's law. The system of fluid equations linearized at first order is then

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla}.\,(\rho_0 \mathbf{V}_1) = 0 \tag{1.1}$$

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} = -\boldsymbol{\nabla} p_1 + \mathbf{J}_1 \times \mathbf{B}_0 \tag{1.2}$$

The adiabatic closure $\mathrm{d}_t(pn^{-\gamma}) = 0$ is often justified when the phase speed of fluctuations is large compared to the thermal speed. In such case, the wave crosses the plasma too quickly to be able to exchange heat in an efficient way. With a linearization at first order,

$$p_1 = \frac{\gamma p_0}{\rho_0} \rho_1 \tag{1.3}$$

We must also include the two Maxwell equations, in which we neglect the transverse component of the displacement current. In the current $\mathbf{J}_1$ we hence do not neglect its longitudinal component

(which is an important remark !)

$$\begin{aligned}\frac{\partial \mathbf{B}_1}{\partial t} &= -\boldsymbol{\nabla} \times \mathbf{E}_1 &(1.4)\\ \mu_0 \mathbf{J}_1 &= \boldsymbol{\nabla} \times \mathbf{B}_1 &(1.5)\end{aligned}$$

Having thus lost the equation for the evolution of the electric field, one needs to write an Ohm's law. In ideal MHD,

$$\mathbf{E}_1 = -\mathbf{V}_1 \times \mathbf{B}_0 \tag{1.6}$$

**Exercize 1.** *What is the origin of this equation?*

This is the linearized form of the ideal Ohm's law, that is the momentum equation for the electrons. It is obtained in the limit $m_e \to 0$, and for the MHD hypothesis $k \to 0$ and $\omega \to 0$.

We could take the Fourier transform of this linear system, then by substitution, reduce the number of equations. Then, the $3 \times 3$ matrix appears in a scalar product with a vector quantity : $\mathbf{V}_1$, $\mathbf{B}_1$ (if not electrostatic) or $\mathbf{E}_1$. Non-trivial solutions appear for the $(\omega, k)$ values for which the derminant is null, hence defining the dispersion relation of its eigen modes.

We remind that to simplify the notations, and without loss of generality, we consider that the magnetic field is along the $z$ axis, and the wave number $\mathbf{k}$ is in the $xz$ plane.

In the book by [Cramer, 2001], the approach is a bit different, and quite elegant : he only keeps the quantities $\kappa_1 = \boldsymbol{\nabla}.\mathbf{V}_1$, $V_{1z}$, $B_{1z}$, $J_{1z}$, $\rho_1$ and $\zeta_{1z} = (\boldsymbol{\nabla} \times \mathbf{V}_1)_z$. The full linearized system is then

$$\begin{aligned}\rho_0 \frac{\partial \zeta_{1z}}{\partial t} - B_0 \frac{\partial J_{1z}}{\partial z} &= 0 &(1.7)\\ \mu_0 \frac{\partial J_{1z}}{\partial t} - B_0 \frac{\partial \zeta_{1z}}{\partial z} &= 0 &(1.8)\\ \rho_0 \frac{\partial}{\partial t}\kappa_1 + \frac{B_0}{\mu_0}\nabla^2 B_{1z} + c_s^2 \nabla^2 \rho_1 &= 0 &(1.9)\\ \frac{\partial B_{1z}}{\partial t} + B_0 \left(\kappa_1 - \frac{\partial V_{1z}}{\partial z}\right) &= 0 &(1.10)\\ \rho_0 \frac{\partial V_{1z}}{\partial t} + c_s^2 \frac{\partial \rho_1}{\partial z} &= 0 &(1.11)\\ \frac{\partial \rho_1}{\partial t} + \rho_0 \kappa_1 &= 0 &(1.12)\end{aligned}$$

By taking the Fourier transform of this linear system, we obtain the relation between the angular frequency $\omega$ and the wave number $\mathbf{k}$ of the eigen modes which may exist. The dispersion relation writes

$$(\omega^2 - v_A^2 k_\parallel^2)[\omega^4 - \omega^2 (c_s^2 + v_A^2) k^2 + v_A^2 c_s^2 k^2 k_\parallel^2] = 0 \tag{1.13}$$

where we introduced the speed of sound $c_s$ and the Alfvén speed $v_A$, defined as

$$c_s^2 = \frac{\gamma p_0}{\rho_0} \, , \quad v_A^2 = \frac{B_0^2}{\mu_0 \rho_0} \tag{1.14}$$

**Exercize 2.** *Show that the solution of the system (1.7) - (1.12) is given by Eq. (1.13)*

The first two equations with the unknown $J_{1z}$ and $\zeta_{1z}$ gives the first parenthesis associated to the Alfvén mode. The 4 other equations with the unknown $\rho_1$, $\kappa_1$, $B_{1z}$ and $V_{1z}$ gives the second term of order 4 in $\omega$. It is quite straightforward in the Fourier space.

It is clear in Eq. (1.13) that there are 2 uncoupled modes. We recall that $\Theta$ is the angle between the direction of the magnetic field (*i.e.* $z$) and the direction of the number $\mathbf{k}$.

**The Alfvén mode**. By canceling the first member of Eq. (1.13), one gets

$$\omega_A = k \, v_A |\cos\Theta| = |k_\parallel| v_A \tag{1.15}$$

This is the dispersion relation of the Alfvén mode. This mode only involves $J_{1z}$ and $\zeta_{1z}$ which appear in Eq. (1.7) and (1.8). This underlines the transverse and incompressible character of this mode.

**Exercize 3.** *Show that this mode is polarized along* $\widehat{\mathbf{y}}$ *for its components* $\mathbf{B}_1$ *and* $\mathbf{V}_1$*, and that they are in phase.*

$\mathbf{k}$ is in the $xz$, plane, so the $z$ component of a curl only involve the $y$ component of the associate vector. Of course, $\mathbf{J}$ and $\boldsymbol{\zeta}$ are the curl of $\mathbf{B}$ and $\mathbf{V}$, respectively. With $\partial_t \rightarrow -\imath\omega$ and $-\partial_z - \imath k_\perp$, the signs clearly show that $\mathbf{B}_1$ and $\mathbf{V}_1$ are in phase.

Moreover, its phase velocity $\mathbf{V}_\phi = \omega_A / \mathbf{k}$ is not depending on $k$, which means that this mode is non-dispersive. It is anisotropic in the sense that its phase speed depends on the angle $\Theta$. Its group speed is always in the direction of the DC component of the magnetic field and is equal to $v_A \cos\Theta$. For these two reasons, the Alfvén wave makes it possible to efficiently transport magnetic energy along the field lines.

From its polarization, we also deduce $\boldsymbol{\nabla}.\mathbf{V}_1 = 0$, which means that this mode is not compressional ; this is the reason why it is called torsional. Eq. (1.12) shows that this mode is then not associated with a density modulation, *i.e.* $n_1 = 0$. This is an important remark when analyzing data (from probes or numerical simulations). Likewise, we have $E_1/B_1 = \omega/k = v_A$, which can also be observed in the data.

**Remark 1.** *In MHD, the time evolution of the electric field is missing (Darwin approximation). Then, an Ohm's Law is used to eliminate the electrical term from the Maxwell-Faraday equation. Therefore, the MHD system does not contain the electric field ; when necessary, it follows from Ohm's law.*

The component of $\mathbf{E}_1$ is polarized in the $x$ direction. Except when $\Theta = 0$, it admits, in addition to its longitudinal component, a transverse component. There is naturally no component parallel to the magnetic field. The high mobility of the particles along the field lines would allow them to quickly smooth the associated potential gradient.

At first order, we have $B^2 = B_0^2 + 2\mathbf{B}_0.\mathbf{B}_1$. However the polarization in $\mathbf{B}$ is such that this scalar product is zero. For the Alfvén wave, there is neither density fluctuations, nor magnetic pressure fluctuations. Alfvén's mode is indeed strictly torsional.

**Slow & Fast magnetosonic modes**. Their dispersion relation is obtained by canceling the second term of Eq. (1.13). Among the two possible solutions, the fast mode is associated with the larger value of $\omega/k$, the other one being the slow mode.

$$\omega_F^2 = \frac{k^2}{2}\left[v_A^2 + c_s^2 + \sqrt{(v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 \cos^2\Theta}\right] \tag{1.16}$$

$$\omega_S^2 = \frac{k^2}{2}\left[v_A^2 + c_s^2 - \sqrt{(v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 \cos^2\Theta}\right] \tag{1.17}$$

The $\mathbf{V}_1$ fluctuations is polarized in the $(\widehat{\mathbf{k}}, \widehat{\mathbf{t}})$ plane, $\mathbf{B}_1$ is along $\widehat{\mathbf{t}}$ and $\mathbf{E}_1$ is along $\widehat{\mathbf{y}}$. There is also a density fluctuation $n_1$ (since $\boldsymbol{\nabla}.\mathbf{V}_1 \neq 0$) and a magnetic field fluctuation with a $z$ component of $\mathbf{B}_1$. These two modes are thus compressionals, even in the limit $c_s \to 0$. Within this limit (*i.e.* with $\beta \to 0$), the slow mode becomes evanescent, so the remaining mode is the fast mode whose dispersion relation reduces to $\omega = kv_A$.

For the fast mode, the magnetic and kinetic pressure fluctuations are in phase, while they are in phase opposition for the slow mode.

At $\Theta = 0$, the dispersion relation of the fast mode reduces to $\omega = |k|v_A$. It then looks like the Alfvén's mode, and can eventually be called the compressional Alfvén mode. In this case, $\mathbf{B}_1$ is also polarized according to $\widehat{\mathbf{y}}$. The slow mode becomes a pure sound wave.

## 1.2 Anisotropic instabilities

We here investigate how the anisotropy of the plasma distribution function modifies the properties of the MHD modes. We keep a fluid formalism and consider a gyrotropic distribution, *i.e.* that the distribution function does not depend on the gyrophase. To take into account the anisotropy, we introduce two closure equations for the parallel and perpendicular pressures. We introduce the notations

$$\lambda = \frac{\omega}{kv_A} \tag{1.18}$$

$$\Delta\beta = \beta_\parallel - \beta_\perp \tag{1.19}$$

$$\beta_\parallel^\star = \gamma_\parallel \beta_\parallel \tag{1.20}$$

$$\beta_\perp^\star = \gamma_\perp \beta_\perp \tag{1.21}$$

Then the dispersion relation of the MHD modes turns to be

$$\begin{vmatrix} \lambda^2 - 1 - {}^1\!/_2\beta_\perp^\star \sin^2\Theta + {}^1\!/_2\Delta\beta\cos^2\Theta & 0 & -{}^1\!/_2\beta_\perp^\star \sin\Theta\cos\Theta \\ 0 & \lambda^2 - (1 - {}^1\!/_2\Delta\beta)\cos^2\Theta & 0 \\ -{}^1\!/_2(\beta_\parallel^\star - \Delta\beta)\sin\Theta\cos\Theta & 0 & \lambda^2 - {}^1\!/_2\beta_\parallel^\star\cos^2\Theta \end{vmatrix} = 0 \tag{1.22}$$

This form results from the velocity dispersion relation. We will therefore have the polarization of the velocity fluctuations (resulting from the Maxwell-Faraday equation). We note that this matrix is sparse. In addition, one mode is decorelated with the two others. This is, as in the isotropic case, the Alfvén mode. The two magnetosonic modes remain coupled.

**The Alfvén mode.** It is given by the central term of Eq. (1.22). We recognize the torsional Alfvén mode, which dispersion equation is altered by the anisotropy

$$\omega = k_\parallel v_A \sqrt{1 - {}^1\!/_2\Delta\beta} \tag{1.23}$$

One can note that for $\beta_\parallel > 2 + \beta_\perp$, the Alfvén's mode is no longer propagative. Otherwise, the fluctuations in velocity and magnetic field remain polarized in the $y$ direction.

**The mirror mode.** In the quasi-perpendicular limit, $\cos^2\Theta \to 0$ and $\sin^2\Theta \to 1$. At order 0, Eq. (1.22) becomes

$$\begin{vmatrix} \lambda^2 - 1 - {}^1\!/_2\beta_\perp^\star & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{vmatrix} = 0 \tag{1.24}$$

- The dispersion relation of the fast mode is $\lambda^2 = 1 + {}^1\!/_2\beta_\perp^\star$. The velocity fluctuations are polarized in the $x$ direction.

- Slow mode requires a little more work ; the Taylor expansion at zeroth order was a little too rough. As the term at the bottom right contains a $\cos^2\Theta$ term, we suspect that first order will not be enough.

Without approximations, the $2 \times 2$ determinant gives

$$(\lambda^2 - 1 - {}^1\!/_2\beta_\perp^\star \sin^2\Theta + {}^1\!/_2\Delta\beta\cos^2\Theta)(\lambda^2 - {}^1\!/_2\beta_\parallel^\star\cos^2\Theta) - {}^1\!/_4(\beta_\parallel^\star - \Delta\beta)\beta_\perp^\star \sin^2\Theta\cos^2\Theta \tag{1.25}$$

At low frequency and in the quasi-perpendicular limit, one has $\lambda \to 0$ and $\cos\Theta \to 0$. We must therefore keep the lowest order terms in $\lambda$ and $\cos\Theta$. One then obtains

$$\lambda^2 = \frac{\cos^2\Theta}{2 + \beta_\perp^\star}[\beta_\parallel^\star + {}^1\!/_2\Delta\beta\beta_\perp^\star] \tag{1.26}$$

**Exercize 4.** *Do the simplification to get the above dispersion relation.*

We need to keep the second order term, that is in $\lambda^2$ and $\cos^2\Theta$. In the first parenthesis, they have to be neglected compared to $1 + {}^1\!/\!_2\beta^\star_\perp$ (with $\sin^2\Theta \to 1$). The result is straightforward.

When $\Delta\beta < -2\beta^\star_\parallel/\beta^\star_\perp$, this mode becomes unstable ($\lambda^2 < 0$), *i.e.* when $\beta^\star_\perp \gg \beta^\star_\parallel$. This is the mirror mode. It is no longer propagative, its angular frequency being purely imaginary. The origin of the name of this mode does not seem totally justified, because in general, when drawing the magnetic field lines, they are not consistent with the fact that the wave number of this mode is perpendicular.

**The fire-hose instability.** We can also look at what happens in the quasi-parallel limit. Eq. (1.22) then becomes

$$\begin{vmatrix} \lambda^2 - 1 + {}^1\!/\!_2\Delta\beta & 0 & 0 \\ 0 & \lambda^2 - (1 - {}^1\!/\!_2\Delta\beta) & 0 \\ 0 & 0 & \lambda^2 - {}^1\!/\!_2\beta^\star_\parallel \end{vmatrix} = 0 \tag{1.27}$$

The top left term is the fast mode, the middle term is the Alfvén mode, and the bottom one is the slow mode. Clearly, the fast and the alfvén modes are degenerated. Moreover, If $\Delta\beta > 2$, these modes are unstable. The mode in which the velocity fluctuations are polarized in the $y$ direction is then called the fire-hose mode, or garden-hose instability. This is indeed the same mechanism as when a garden-hose goes crazy and squirms after its reckless user drops it down while the tapper is open. The perpendicular pressure is then no longer sufficient to control the parallel pressure.

The physics of this instability is quite simple : when one disturbs a flux tube (which therefore admits a small curvature), this tube is subjected to 3 forces :

- The centrifugal force due to the parallel pressure of the plasma in the flux tube. If $R$ is the curvature radius of the tube, this force is equal to $Mnv_\parallel^2/R$
- The thermal pressure force of the plasma outside the tube, equal to $p_\perp/R$
- The magnetic tension force in the flux tube, which writes $B_0^2/\mu_0 R$

The instability develops when the first term is greater than the sum of the two last ones, *i.e.* $p_\parallel > p_\perp + B_0^2/\mu_0$. We can then rewrite the growth rate of the fire-hose mode

$$\gamma = k_\parallel \frac{v_A}{\sqrt{2}}(\beta_\parallel - \beta_\perp - 2)^{1/2} \tag{1.28}$$

This growth rate increases with $k_\parallel$. In fact, at large $k_\parallel$, it is necessary to include the Hall term(which gives a correction in $\omega/\Omega_p$) and the electron pressure term (which gives a correction in $k_\perp\rho_e$). These corrective terms limit the growth of $\gamma$ with $k_\parallel$. We then obtain a bell shaped curve $\gamma(k_\parallel)$.

The paper of [Bale et al., 2009] outlines the values of the instability threshold treated above. The approach is quite simple, while clever : they collected around $10^6$ measurement points in the solar wind using the instruments on board the Wind probe. They measured the relative magnetic fluctuations $B_1/B_0$, as well as the value of the anisotropy ratio of the protons $T_\perp/T_\parallel$, and that of the $\beta_\parallel$ parameter. When fluctuations are measured, it means that there is an activity of associated waves. The idea is therefore to see in which domain the magnetic fluctuations can exist.

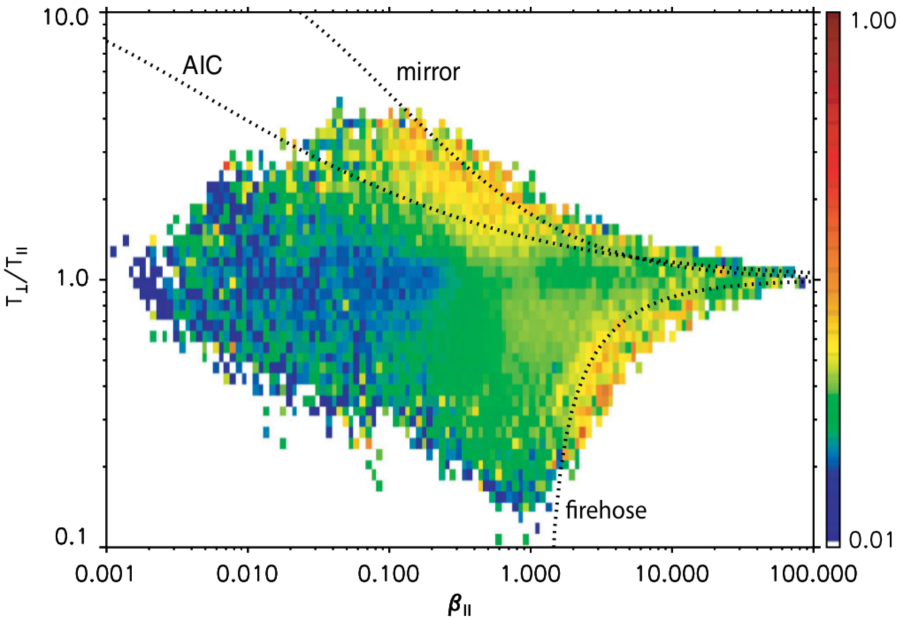


Figure 1.1: Magnitude of the relative magnetic fluctuations depending on $T_\perp/T_\parallel$ and $\beta_\parallel$ (see [Bale et al., 2009]).

In dotted lines, the thresholds of mirror instabilities, firehose (oblique) and AIC (Alfvén Ion Cyclotron) are indicated. It is clear from this that mirror and firehose modes can only exist below their level of instability. Beyond that, the mode being unstable, it transfers its energy to the particles which in fact limits the development of magnetic fluctuations.

These results question the reason why AIC and parallel firehose modes do not limit the level of magnetic fluctuations that are observed. One answer is that mirror and firehose (oblique) modes are non-propagating, unlike AIC and firehose (parallel) modes. The question remains open...

## 1.3 Alfvén Ion Cyclotron mode

When the protons of a plasma are demagnetized (at least partially), *i.e.* when the perturbations have a frequency of the order of $\Omega_p$ or a wavelength of the order of the proton Larmor radius $\rho_p$, we must keep the second term of Eq. (A.44), called the Hall effect. The linearized form of the equations are then different in the sense that it contains additional terms. Eq. (1.7) to (1.12) thus become

$$
\begin{aligned}
\rho_0 \frac{\partial \zeta_{1z}}{\partial t} - B_0 \frac{\partial J_{1z}}{\partial z} &= 0 && (1.29)\\
\mu_0 \frac{\partial J_{1z}}{\partial t} - B_0 \frac{\partial \zeta_{1z}}{\partial z} + \frac{v_A^2}{\Omega_p}\frac{\partial}{\partial z}\nabla^2 B_{1z} &= 0 && (1.30)\\
\rho_0 \frac{\partial}{\partial t}\kappa_1 + \frac{B_0}{\mu_0}\nabla^2 B_{1z} + c_s^2 \nabla^2 \rho_1 &= 0 && (1.31)\\
\frac{\partial B_{1z}}{\partial t} + B_0\left(\kappa_1 - \frac{\partial V_{1z}}{\partial z}\right) + \frac{v_A^2}{\Omega_p}\frac{\partial J_{1z}}{\partial z} &= 0 && (1.32)\\
\rho_0 \frac{\partial V_{1z}}{\partial t} + c_s^2 \frac{\partial \rho_1}{\partial z} &= 0 && (1.33)\\
\frac{\partial \rho_1}{\partial t} + \rho_0 \kappa_1 &= 0 && (1.34)
\end{aligned}
$$

**Exercize 5.** *Write an Ohm's law for the electrons and deduce the linear form of Eq. (1.29) to (1.34) by including Hall term in Ohm's law.*

The Ohm's law in Hall MHD writes $\mathbf{E} = -\mathbf{V}_e \times \mathbf{B}$ that is with $\mathbf{V}_e = \mathbf{V}_i - \frac{1}{ne}\mathbf{J}$ we obtain $\mathbf{E} = -\mathbf{V}_i \times \mathbf{B} + \frac{1}{en}\mathbf{J} \times \mathbf{B}$.

A cyclotronic term then appears in Eq. (1.30) which no longer makes it simply coupled to Eq. (1.29). These two equations leading the Alfvén mode, the first remark is that this mode is no longer decoupled from the two magnetosonic modes. Of course, this coupling disappears in the ideal MHD limit, when $\omega \ll \Omega_p$. Getting in the Fourier space, we obtain the dispersion relation

$$
(\omega^2 - v_A k_z^2)[\omega^2(\omega^2 - c_s^2 k^2) - v_A^2 k^2(\omega^2 - c_s^2 k_z^2)] = \left(\frac{\omega}{\Omega_p}\right)^2 v_A^4 k_z^2 k^2 (\omega^2 - c_s^2 k^2) \qquad (1.35)
$$

It is a bi-squared equation of order six which therefore admits three solutions, hence bearing the same name as in ideal MHD. The left side of Eq. (1.30) is the same as in ideal MHD, but the right hand side is at the origin of the coupling between the Alfvén mode and the two magnetosonic modes. This nomenclature is still relevant insofar as the Hall term reduces the phase speed of the slow mode and increases that of the fast mode. Noting

$$
\beta' = \frac{c_s^2}{v_A^2}\ , \quad \alpha = \frac{k v_A}{\Omega_p} \ \text{ et } \ f = \frac{\omega}{\Omega_p} \qquad (1.36)
$$

Eq. (1.30) can be written

$$
(f^2 - \alpha^2 \cos^2\Theta)[f^2(f^2 - \alpha^2\beta') - \alpha^2(f^2 - \alpha^2\beta'\cos^2\Theta)] = f^2\alpha^4\cos^2\Theta(f^2 - \alpha^2\beta') \qquad (1.37)
$$

we can separate the cold case ($\beta' = 0$) from the hot case to discuss these modes.

**Cold plasmas**. The Alfvén mode and the fast magnetosonic modes can exist. Their angular frequency can be written

$$\omega_\pm^2 = \frac{v_A^2 k^2}{2}\left[1 + (1+\alpha^2)\cos^2\Theta \pm \sqrt{1 - 2(1-\alpha^2)\cos^2\Theta + (1+\alpha^2)^2\cos^4\Theta}\right] \tag{1.38}$$

where we identify the minus sign in Eq. (1.38) with the Alfvén mode. In the parallel limit ($\Theta = 0$), this mode resonates with $\Omega_p$ because then, $k \to \infty$. This mode is often called the Alfvén Ion Cyclotron, or AIC. We can verify that the group velocity goes to zero, which means that the energy of the waves accumulates, until the dissipative or non-linear effects limit it. If we calculate the dielectric tensor (see appendix), we can study the polarization of the electric field, and show that this mode is no longer linear, but that it becomes elliptic then left circular at the resonance.

> Use the scaling law in the Ohm's law where the order of magnitude of $\mathbf{V}_p$ is given by the zeroth order of the Ohù's law, that is $E/B$. Then Using Maxwell-Faraday Eq., $E/B \sim \omega/k$.

The plus sign in Eq. (1.38) gives the other mode which becomes the ideal fast mode in MHD. It therefore does not undergo cyclotronic resonance, but becomes circular for $\omega \gtrsim \Omega_p$. This is the whistler mode. Moreover this mode becomes dispersive insofar as $\omega$ is proportional to $k^2$. Finally, it is an electronic mode which does not depend on the mass of the electrons (which would tend to make it a strictly electromagnetic mode).

**Hot plasmas**. When $c_s$ is no longer zero, the two magnetosonic modes are no more degenerated. The limit $\Theta = 0$ is the only one for which the Alfvén mode is separated from the two magnetosonic modes. As in cold plasmas, Alfvén mode resonates at $\Omega_p$. At different $\Theta$ values, this mode is called intermediate mode rather than Alfvén mode. At low frequency, it is quite close to the slow mode. Also, it no longer resonates at $\Omega_p$. Furthermore, the slow mode resonates with $\omega = k_\parallel v_A$

## 1.4 The quasi-perpendicular limit

There is another very important limit for space plasmas : quasi-perpendicular modes. Whether in the solar wind or in the magnetosheath of the Earth's magnetosphere, there is a high level of magnetic fluctuations ($B_1/B_0 \sim 0.2$) for which the wave numbers are essentially perpendicular to the DC component of the magnetic field. Besides the mirror mode, it is legitimate to wonder how the Alfvén wave is modified in this limit.

The form of Maxwell's equations in Fourier space is discussed in the appendix A. By introducing the dielectric tensor $\boldsymbol{\varepsilon}$,

$$\left(\frac{k^2c^2}{\omega^2}\mathbf{1} - \boldsymbol{\varepsilon}\right).\mathbf{E}_T = -\boldsymbol{\varepsilon}.\mathbf{E}_L \tag{1.39}$$

At large $k$ values, the magnetic component becomes negligible ; at the vicinity of the resonance ($k \to \infty$), the waves turns to be essentially electrostatic ($\mathbf{E}_T \sim 0$). This is the case for the Alfvén mode at large values of $k_\perp$.

When the wave number becomes very large, the spatial gradients becomes small. It is then necessary to re-evaluate the form of the Ohm's law to keep the terms which may no longer become negligible within this limit. In Eq. (A.44), we must therefore keep, in addition to terms 1 and 2, terms 4 and 5. Term 3 is negligible because, as we will see, the frequency of this mode remains below the electron gyrofrequency. The term 6 is still negligible for a collisionless plasma.

The electric field appears in the Maxwell-Ampère and Maxwell-Faraday equations. The first of these equations gives Eq. (1.8) with an ideal Ohm's law. By keeping the terms 1, 2, 4 and 5, we get

$$\frac{\partial B_{1z}}{\partial t} - d_e^2\nabla^2\frac{\partial B_{1z}}{\partial t} + B_0\left(\boldsymbol{\nabla}.\mathbf{V}_1 - \frac{\partial V_{1z}}{\partial z}\right) + \frac{v_A^2}{\Omega_p}\frac{\partial J_{1z}}{\partial z} = 0 \tag{1.40}$$

where we have introduced the electron inertial length $d_e = c/\omega_{Pe}$. Likewise, Eq. (1.10) can be written

$$\mu_0\frac{\partial J_{1z}}{\partial t} - \mu_0 d_e^2\nabla^2\frac{\partial J_{1z}}{\partial t} - B_0\frac{\partial \zeta_{1z}}{\partial z} + \frac{v_A^2}{\Omega_p}\frac{\partial}{\partial z}\nabla^2 B_{1z} = 0 \tag{1.41}$$

As in the Hall MHD case, it appears that these two equations are no longer decoupled from the remaining equations of the system. The compressional nature of the plasma will therefore modify the Alfvén mode. Moreover, as discussed in the appendix (A), the importance of the term associated with electron compressibility depends on the electron temperature, *i.e.* on the value of $\beta$.

With Eq. (1.7), (1.9), (1.11) and (1.12), we can solve the system to find the dispersion relation of the eigen modes. After a few lines of calculations, we obtain

$$[\omega^2(1+d_e^2k^2) - v_A^2k_\parallel^2][\omega^2(1+d_e^2k^2)(\omega^2 - c_s^2k^2) - v_A^2k^2(\omega^2 - c_s^2k_\parallel^2)] = \left(\frac{\omega}{\Omega_p}\right)^2 v_A^2k^2k_\parallel^2(\omega^2 - c_s^2k^2) \tag{1.42}$$

Remember that we are in the quasi-perpendicular limit, $k_\perp \gg k_\parallel$, *i.e.* $k \simeq k_\perp$. Moreover, we can verify *a posteriori* that $\omega \lesssim v_Ak_\parallel$. We introduce the thermal Larmor radius $\rho_s = c_s/\Omega_p$. Eq. (1.42) then simplifies

$$\left[1 + d_e^2k_\perp^2 - \frac{v_A^2k_\parallel^2}{\omega^2}\right]\left[1 + \frac{c_s^2}{v_A^2}\left(1 + d_e^2k_\perp^2 - \frac{v_A^2k_\parallel^2}{\omega^2}\right)\right] = \frac{v_A^2k_\parallel^2}{\omega^2}\left(k_\perp^2\rho_s^2 - \frac{\omega^2}{\Omega_p^2}\right) \tag{1.43}$$

Eq. (1.43) calls for some thoughts. The first term does not involve any compressibility terms ; it is the Alfvénic term. But the existence of a right hand side means that it is coupled to the second magnetosonic term, as in Hall MHD. A correction also appears due to the electron inertial length. Insofar, as we are interested in frequencies below the proton gyrofrequency, and large perpendicular wave number, we have $\omega/\Omega_p \ll k_\perp^2\rho_s^2$. One can then neglect the cyclotron correction in the right-hand side of Eq. (1.43), which means that the coupling term becomes proportional to $k_\perp^2\rho_s^2$.

In the low-$\beta$ case (*i.e.* $c_s \ll v_A$), Eq. (1.42) simply reduces to

$$\omega^2 = v_A^2k_\parallel^2\frac{1 + k_\perp^2\rho_s^2}{1 + k_\perp^2d_e^2} \tag{1.44}$$

Before solving this equation, we identify two different regimes depending on the relative importance of $k_\perp\rho_s$ in front of $k_\perp d_e$. With the electron inertial length written as

$$d_e^2 = \frac{1}{\mu}\frac{v_A^2}{\Omega_p^2} = \frac{1}{\mu}d_p^2 \tag{1.45}$$

which gives

$$\frac{\rho_s^2}{d_e^2} = \frac{\gamma}{2}\beta\mu \tag{1.46}$$

we can discuss two limit cases, depending on how (weak) $\beta$ compares to $\mu^{-1}$. For these two cases, we always consider the limit $T_e \gg T_p$. Thus, the speed of sound $c_s$ is equal to the ion acoustic speed $v_s$.

**The Kinetic Alfvén Wave**. In the case $\beta \gtrsim \mu^{-1}$, *i.e.* $v_{Te} \gtrsim v_A$, the dispersion relation writes

$$\omega^2 = v_A^2 k_\parallel^2(1 + k_\perp^2\rho_s^2) \tag{1.47}$$

The Alfvén mode is modified by the fact that $k_\perp$ is large enough to consider the contribution of the Larmor radius of thermal protons. This mode is called the kinetic Alfvén wave, hence the acronym KAW.

From the KAW mode dispersion equation, we can calculate its phase velocity. The dependence on $k$ makes it a dispersive mode. To try to find this phase speed in the data measured by satellite, one technique is to reconstruct the $E/B$ ratio. This work was done by [Sahraoui et al., 2009] with Cluster measurements in the solar wind.

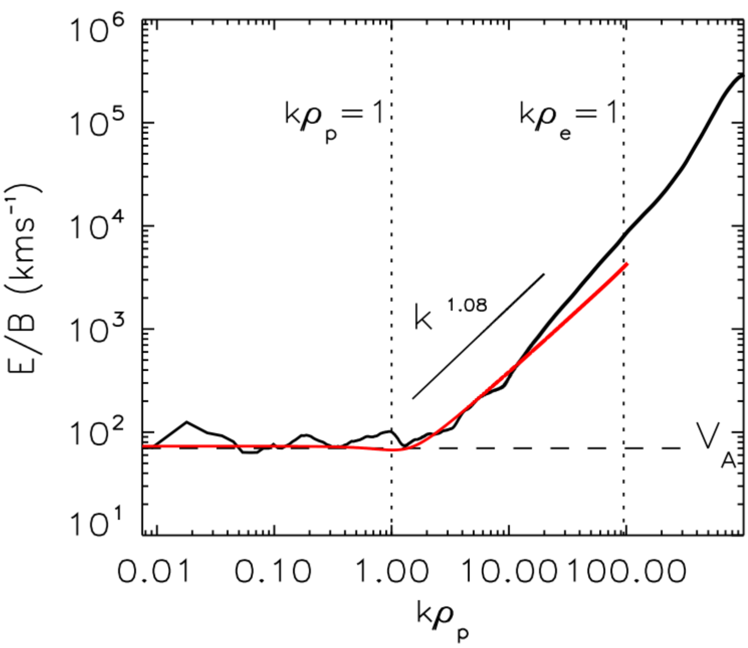


Figure 1.2: The solid line gives the $E/B$ ratio measured by CLUSTER, depending on $k$. The red curve is the analytical expectation for the KAW mode (from [Sahraoui et al., 2009]).

In the MHD limit, the magnetic field is frozen in the plasma ; we then have $\mathbf{E} \sim -\mathbf{V} \times \mathbf{B}$. On the other hand, $E^2$ and $B^2$ both have spectra at $k^{-1.62}$. Their ratio is therefore constant, and independent of $k$. On the other hand, at scales above the Larmor radius of the protons, the dispersive effects give a $E/B$ ratio which linearly depends on $k_\perp$. This can be seen in the Figure below, in which at $k\rho_p \geq 1$, the slope of $E/B$ is in $k^{1.08}$. Electric and magnetic fluctuations therefore suggest that the turbulence in the solar wind is mainly due to the KAW modes.

**The Inertial Alfvén Wave**. In the other limit, $\beta \lesssim \mu^{-1}$ which is also equivalent to $v_{Te} \lesssim v_A$. A magnetized and sparingly dense plasma of this type is found, for example, in an ionosphere. The dispersion relation becomes

$$\omega^2 = \frac{v_A^2 k_\parallel^2}{1 + d_e^2 k_\perp^2} \tag{1.48}$$

In this case, the Alfvén mode is modified by the fact that $k_\perp$ is large enough so that $k_\perp d_e > 1$. This mode is called the Inertial Alfvén Wave, hence the acronym IAW [1].

The auroral zone, due to its very strong magnetization and the low temperature of the charged particles, is a region where the parameter $\beta$ is much lower. IAW modes can therefore play an important role. In this case, the electric field (essentially electrostatic) of this wave can heat the ions, and thus ensure their escape. A study by [Stasiewicz et al., 2000] uses FREJA data at 1700 km altitude. Electric and magnetic fluctuations are at very low frequency ($f \sim 1$ Hz). The frequency of the mode is then linked to the associated wave number via the speed of the satellite, 6.8 km.s$^{-1}$ in this case. The Figure below is the value of the ratio $E_1/B_1$ as a function of $V_{SC}/\omega \sim k_\perp^{-1}$ (Taylor hypothesis). The points are the theoretical values, including for the protons the correction associated to the Larmor radius effects. In this case, the dispersion equation of the IAW mode becomes

$$\omega = k_\parallel v_A \left( \frac{1 + k_\perp^2 \rho_p^2}{1 + k_\perp^2 d_e^2} \right)^{1/2} \tag{1.49}$$

These experimental results suggest that IAWs exist in auroral regions and may partly explain ionospheric heating and exhaust.

For the two KAW & IAW modes, being in the electrostatic limit with $k_\perp \gg k_\parallel$, there is a component of the electric field along the magnetic field. As a consequence of this parallel electric field, these waves can interact efficiently with particles, and is therefore a good candidate for heating them. These two modes have been extensively studied in the formation of particle beams in the auroral ionosphere, as well as in energy transport in tokamaks.

Consequently, it is also possible to study their dispersion relation by introducing the two components (parallel and perpendicular) of the electric field in the fluid equations and the Maxwell equations (*cf.* [Hasegawa and Uberoi, 1982]).

[1] be careful as this acronym also holds for Ion Acoustic Waves.

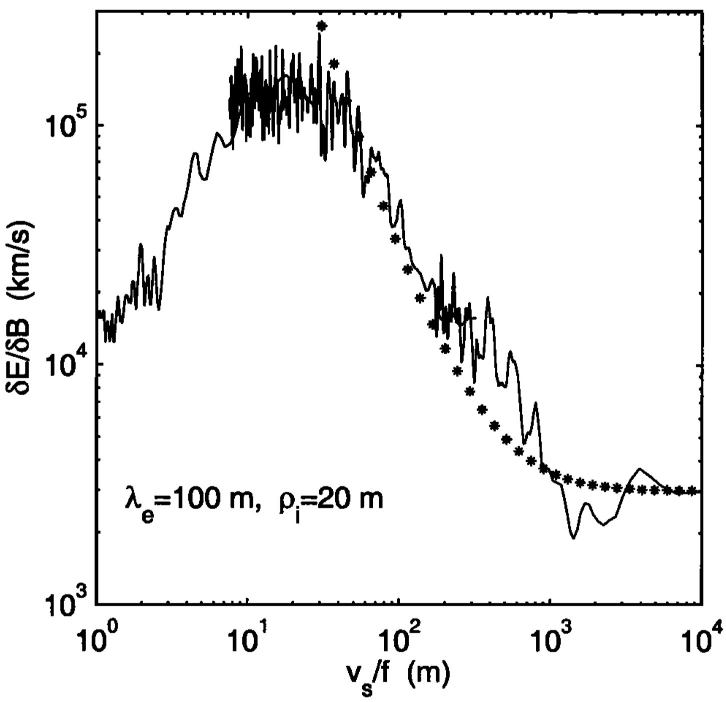


Figure 1.3: Ratio of the fluctuations $E/B$ depending on $V_{SC}/\omega$ (see [Stasiewicz et al., 2000]).