— A —

Recalls on waves and instabilities in plasmas

A.1 General

Phase velocity & group velocity

A wave can be characterized by its phase velocity V_ϕ. This vector is defined as

$$\mathbf{V}_\phi = \frac{\omega \mathbf{k}}{k^2} = \frac{\omega}{k}\widehat{\mathbf{k}} \tag{A.1}$$

This is the speed at which the front of a wave is moving. Depending on the ω and $\mathbf{k}$ values, this speed is hence defined for monochromatic waves. This velocity ca be larger than the speed of light in a vacuum. But as a consequence of the Heisenberg principle, we always have to deal with wave packets defined on a finite spectral band. A wave packet is the superposition of a set of monochromatic waves whose frequency is between ω and $\omega + d\omega$ while the associated wave number is between $\mathbf{k}$ and $\mathbf{k} + d\mathbf{k}$. The amplitude of each of these modes is of course a continuous function of ω and $\mathbf{k}$.

The group speed of a wave packet is the speed at which the energy in that packet travels. Most of this energy is where the superposition of these waves is constructive. The group speed will therefore be the speed at which the locus of these points moves. Considering the wave packet

$$\psi(\mathbf{r}, t) = \int \psi(\mathbf{k}, \omega) e^{\imath(\mathbf{k}.\mathbf{r} - \omega t)} d\omega d\mathbf{k} \tag{A.2}$$

we can compare the shape of this wave packet in $\mathbf{r}$ at time t with the shape of this same packet at a later time t' at $\mathbf{r}'$. At $t' = t + \delta t$ and $\mathbf{r}' = \mathbf{r} + \delta \mathbf{r}$.

$$\psi(\mathbf{r}, t) - \psi(\mathbf{r}', t') = \int \psi(\mathbf{k}, \omega) e^{\imath(\mathbf{k}.\mathbf{r} - \omega t)} [1 - e^{\imath(\mathbf{k}.\delta\mathbf{r} - \omega \delta t)}] d\omega d\mathbf{k} \tag{A.3}$$

For the interaction to be constructive, the relative phase between all the Fourier components of ψ must remain constant, *i.e.* its derivative has to be zero

$$d[\mathbf{k}.\delta\mathbf{r} - \omega \delta t] = 0 \tag{A.4}$$

Being in Fourier space, the unknowns are ω and $\mathbf{k}$. This derivative then writes

$$d\mathbf{k}\delta\mathbf{r} - d\omega \delta t = 0 \tag{A.5}$$

The energy of the disturbance will therefore move at the group speed

$$\mathbf{V}_g = \frac{\delta \mathbf{r}}{\delta t} = \frac{\partial \omega}{\partial \mathbf{k}} \tag{A.6}$$

The group speed is a vector which is not necessarily in the same direction as $\mathbf{k}$.

Parallel and perpendicular propagation

A magnetic field is a source of anisotropy. The magnetic field drives the gyromotion of the particles. It also often plays a role in the propagation of waves. Parallel and perpendicular propagation refer to the direction of the wave vector $\mathbf{k}$ relative to the DC magnetic field. Oblique propagation means $0 < \Theta < \pi/2$. As well, quasi-parallel propagation is when Θ is close to (but different) from 0, and quasi-perpendicular propagation is when Θ is close (but different) to $\pi/2$.

Electrostatic and electromagnetic modes

An electrostatic wave has only one electric component $\mathbf{E}$ and no magnetic component $\mathbf{B}$ [1]. For electrostatic modes, instead of treating the set of Maxwell's equations, only the Maxwell-Gauss (or Maxwell-Poisson) equation is sufficient. Conversely, when $\mathbf{k}$ and $\mathbf{E}$ are orthogonal, we speak of electromagnetic waves.

Longitudinal and transverse modes

This distinction refers to the relative direction of the electric field of the wave $\mathbf{E}$ with respect to the wave vector $\mathbf{k}$. When $\mathbf{k} \parallel \mathbf{E}$ we speak of longitudinal mode. When $\mathbf{k} \perp \mathbf{E}$ we speak of transverse mode. In Fourier space, the Maxwell-Faraday equation gives $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$. A longitudinal mode is therefore always electrostatic. Likewise,

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} = -k^2 \mathbf{E}_T \tag{A.7}$$

We will see that the dispersion equation is then written

$$\left(\frac{k^2 c^2}{\omega^2} \mathbf{1} - \boldsymbol{\varepsilon} \right) . \mathbf{E}_T = -\boldsymbol{\varepsilon} . \mathbf{E}_L \tag{A.8}$$

So, when $k \to \infty$, we have $\dfrac{k^2 c^2}{\omega^2} \gg \boldsymbol{\varepsilon}$, and E_T then goes to 0, that is the longitudinal term E_L therefore dominates. Close to the resonance, the waves are almost electrostatic. Generally in plasma, waves can have a longitudinal component and a transverse component at the same time. They are then neither purely electrostatic nor purely electromagnetic.

Polarization

The polarization of a wave is the direction of its electric field and its magnetic field. These 2 directions are very often different from each other. If the directions of the vectors $\mathbf{E}$ and $\mathbf{B}$ of the wave are fixed, then we speak of linear (or rectilinear) polarization. Otherwise, they are elliptical or circular (which is a specific case of the elliptical polarization). The extreme cases of parallel or perpendicular modes call for a new classification:

[1] be careful to distinguish the DC magnetic field with the fluctuation of the magnetic field associated with the wave. A wave can be electrostatic in magnetized plasma, or can be electromagnetic in a unmagnetized plasma.

- In parallel propagation, we speak of right and left mode. These are the 2 circularly polarized modes, rotating in the 2 possible directions (the right mode is the one which rotates with the electrons). Using the Stix notations (which will be detailed in the problem on the CMA diagram), we have $N = R$ for the right mode and $N = L$ for the left mode, where $N = kc/\omega$ is the optical index.

- In perpendicular propagation, we speak of ordinary and extraordinary mode. Using the Stix notations, we have $N = P$ for the ordinary mode and $N = RL/S$ for the extraordinary mode. Plasma oscillation is an example of ordinary mode.

More generally, the term polarization can be used to refer to any one of the disturbed vector quantities, which therefore includes the speed of order 1. In order to identify the direction of the vectors ($\mathbf{E}_1$, $\mathbf{B}_1$ or $\mathbf{V}_1$), we recall the unit vectors (in the Cartesian system) introduced in the preamble. By convention (in magnetized plasma), the DC component of the magnetic field is along the unit vector $+\widehat{\mathbf{z}}$ and the wave vector $\mathbf{k}$ is along a $\widehat{\mathbf{k}}$. Then, $\widehat{\mathbf{t}}$ the unit vector normal to $\widehat{\mathbf{k}}$ in the xz plane. These vectors are depicted in Fig. (1).

Helicity

Magnetic helicity is a quantity which makes it possible to quantify the degree of twisting and entwining of the magnetic field lines. Mathematically, it is defined by

$$K = \int \mathbf{A}.\mathbf{B}\,\mathrm{d}\mathbf{r} \tag{A.9}$$

where $\mathbf{A}$ is the vector potential associated with the magnetic field $\mathbf{B}$ by the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. For this, we always choose the Coulomb gauge $\boldsymbol{\nabla}.\mathbf{A} = 0$. For $\mathbf{A}$ and $\mathbf{B}$ getting null at infinity, the value of K is unique and characterizes the plasma.

It is important to note that this is a "spatial" concept, unlike polarization which is a "temporal" concept. Helicity will characterize the way a field line twists (at a given time) in space, while polarization characterize the way the magnetic field vector rotates (at a given location) with time.

A right-hand polarized mode is a mode for which the fluctuations of the magnetic field rotate in the same direction as the electrons around the DC magnetic field, whether this mode propagates parallel or anti-parallel to it. If considering two screws with opposite pitches, if one gives a right-hand polarized mode with a given direction of the wave vector, the other is also a right-hand polarized mode for a wave vector of opposite sign. The polarization therefore characterizes the direction of rotation with time, independently of the way in which the field lines are entangled. It is therefore the direction of $\mathbf{k}$ and not that of $\mathbf{B}$ that matters.

Helicity characterize the way a field line rolls up on itself. the notion of wave vector therefore has nothing to do with this definition, and it is here the vector $\mathbf{B}$ that matters. To illustrate this concept, a screw has negative helicity. And we screw by turning clockwise, or even in the dextrorotatory direction (if on the other hand this screw has a reverse pitch, it is the levogyre direction). By observing this screw, it appears that the winding is a characteristic of the screw, and is therefore independent of the direction in which you look at it.

If the structure of the field lines is the superposition of linear modes with the form $\exp \imath(\omega t - \mathbf{k}.\mathbf{r})$, then a right-polarized mode has a positive helicity (for $\mathbf{k}.\mathbf{B} > 0$) and vice versa.

The helicity being defined at a given time, it only quantifies the direction of winding of the field lines : looking at a spring from one side or the other, does not matter on how it is coiled. On the other hand, if you move along this spring, the direction in which you turn with time depends on your direction of propagation, *i.e.* on the sign of the wave vector $\mathbf{k}$.

In conclusion, for linear modes, polarization and magnetic helicity are related. But a plasmas in which there would be half left mode and half right mode, the helicity would be close to zero. In numerical simulation, the helicity can be normalized so as to vary only between -1 and +1.

Finally, note that we can also define the current helicity by

$$\int \mathbf{B}.(\boldsymbol{\nabla} \times \mathbf{B})\,\mathrm{d}\mathbf{r} \tag{A.10}$$

and the kinetic helicity by

$$\int \mathbf{V}.(\boldsymbol{\nabla} \times \mathbf{V})\,\mathrm{d}\mathbf{r} \tag{A.11}$$

The current helicity characterizes the winding of the current while the kinetic helicity characterizes the winding of the flow lines. These three helicities can be defined locally (at a point) by their respective integrand ; we then speak of helicity density.

A.2 Linear and non-linear modes

The system of equations (Maxwell + plasmas) which makes it possible to describe the response of the plasma to external fields is intrinsically non-linear, which constitutes a complication. In theory, one cannot therefore study the response to an excitation without taking into account the amplitude of this one.

In linear theory, we assume that the wave amplitudes are very low. The response to a superposition of excitations is the superposition of the responses to individual excitations. This way of proceeding is only acceptable if the electromagnetic energy density associated with the wave is low compared to the density of thermal energy of the undisturbed plasma :

$$n\frac{mv_T^2}{2} \gg \varepsilon_0\frac{E^2}{2} + \frac{B^2}{2\mu_0} \tag{A.12}$$

The equations at zero$^{\mathrm{th}}$ order are satisfied as they describe the equilibrium state. The first order equations describe the waves. Due to the temporally and spatially disspersive nature of the plasma, these calculations are done in the Fourier space $(\mathbf{k}, \omega)$. For causality reason, the transformation to be used in time should be a Laplace transform. But when interested in a mode and not in the process of its creation and/or destruction (because of its damping or instable nature), we nonetheless generally use a Fourier transform.

Otherwise, if the characteristics of the mode depends on its amplitude, it is hence necessary to do the calculations in a non-linear way while explicitly keeping the amplitude of the wave. There is another possible source of non-linearity : the inhomogeneity of the medium. In the presence of a gradient (of density, of temperature ...) the Fourier transform gets complicated and it is generally better to keep the explicit form of the gradient ; this leads to solving a partial differential equation.

A.3 Plasma dispersion relation

We here focus on the Maxwell's equations and not in the specific form of the conductivity tensor (which will be done later). We also only consider the 2 rotational equations which provides the time evolution of $\mathbf{E}$ and $\mathbf{B}$.

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{A.13}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{A.14}$$

Then,

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) &= \boldsymbol{\nabla} \times -\frac{\partial}{\partial t}\mathbf{B} && \text{(A.15)}\\
&= -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) && \text{(A.16)}\\
&= -\frac{\partial}{\partial t}\left(\mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}\right) && \text{(A.17)}\\
&= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} - \mu_0 \frac{\partial}{\partial t}\mathbf{J} && \text{(A.18)}
\end{aligned}$$

Developing the double vectorial product, one gets

$$\nabla^2 \mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{E}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \mu_0 \frac{\partial}{\partial t}\mathbf{J}, \tag{A.19}$$

which writes in Fourier space

$$\left[\left(\frac{\omega^2}{c^2} - k^2\right)\mathbf{1} + \mathbf{k}\mathbf{k}\right].\mathbf{E}(\mathbf{k},\omega) = +\imath\omega\mu_0 \mathbf{J}(\mathbf{k},\omega) \tag{A.20}$$

The current density $\mathbf{J}$ involved in this equation is total, meaning that it is the sum of the external current density $\mathbf{J}_f$ and of the induced (by the plasma) current density $\mathbf{J}_i$. The induced current density is related to the electric field by the conductivity tensor. In real space, we recall that the relation is of the form

$$\mathbf{J}_i(\mathbf{r},t) = \int_0^{+\infty} \iiint_{\mathbb{R}^3} \boldsymbol{\sigma}(\boldsymbol{\rho},t).\mathbf{E}(\mathbf{r}-\boldsymbol{\rho},t-\tau)d\boldsymbol{\rho}d\tau \tag{A.21}$$

Removing the induced part of the current density (by introducing the conductivity tensor),

$$\left[\left(\frac{\omega^2}{c^2} - k^2\right)\mathbf{1} + \mathbf{kk} - \imath\omega\mu_0\boldsymbol{\sigma}\right].\mathbf{E}(\mathbf{k},\omega) = +\imath\omega\mu_0\mathbf{J}_f(\mathbf{k},\omega) \tag{A.22}$$

We often consider the case where $\mathbf{J}_f = 0$ (which means that the wave dispersion equation that we obtain is only valid in a medium without current). Multiplying this equations by c^2/ω^2, we can introduce the vector form of the optical index defined by

$$\mathbf{N} = \frac{\mathbf{k}c}{\omega} \tag{A.23}$$

With the relation $\boldsymbol{\sigma} = -\imath\omega\varepsilon_0(\boldsymbol{\varepsilon} - \mathbf{1})$, the above equation then writes

$$(\mathbf{NN} - N^2\mathbf{1} + \boldsymbol{\varepsilon}).\mathbf{E}(\mathbf{k},\omega) = 0 \tag{A.24}$$

The eigen modes are those satisfying the above equation. A non-trivial solution ($\mathbf{E} \neq 0$) exists if the determinant of the left hand side is zero.

Using the wave vector components, $\mathbf{k} = k\sin\Theta\widehat{x} + k\cos\Theta\widehat{z}$, the above determinant writes

$$\begin{vmatrix} \varepsilon_{xx} - N^2\cos^2\Theta & \varepsilon_{xy} & \varepsilon_{xz} + N^2\sin\Theta\cos\Theta \\ \varepsilon_{yx} & \varepsilon_{yy} - N^2 & \varepsilon_{yz} \\ \varepsilon_{zx} + N^2\sin\Theta\cos\Theta & \varepsilon_{zy} & \varepsilon_{zz} - N^2\sin^2\Theta \end{vmatrix} = 0 \tag{A.25}$$

To finish this developments, it is necessary to explain the shape of the dielectric tensor. This can be done using kinetic formalism or fluid formalism. In kinetics, we write and linearize a kinetic equation (genrally Vlasov). In fluid, we write the fluid equations (for each fluid). In both cases, the plasma equations then need to be linearized in the same way as the Maxwell equations.

A.4 Magnetic permeability of a plasma

In plasma physics, we generally only talk about magnetic field that we note $\mathbf{B}$. But when studying electromagnetism in continuous medium, we name $\mathbf{B}$ the "magnetic induction field" (in Teslas), and $\mathbf{H}$ the "magnetization field" (in Amperes per meter). For a media in linear regime, these two fields are linked by the magnetic permeability of the medium μ

$$\mathbf{B} = \mu\mathbf{H} \tag{A.26}$$

Premeability then characterizes for a material its ability to modify a magnetization field $\mathbf{H}$. Recall that $B < \mu_0 H$ for a diamagnetic material, $B > \mu_0 H$ for a paramagnetic material, and $B \gg \mu_0 H$ for a ferromagnetic material, $\mu_0 = 4\pi 10^{-7}$ TmA $^{-1}$ being the permeability of the vacuum.

In magnetostatic (to simplify), the current has a component $\mathbf{J}_c$ due to conduction and a $\mathbf{J}_m$ component due to magnetization, ie to the response of the medium to the magnetization field $\mathbf{H}$. This magnetization current is linked to the magnetization of the medium $\mathbf{M}$ by $\mathbf{J}_m = \boldsymbol{\nabla} \times \mathbf{M}$. This classical statement actually contains a quantum behavior of matter, by which a current is associated with the orbital angular momentum of the atoms constituting the medium. Likewise, a moment due to the spin of the electrons influences this current density, even if this image is then fragile in a non-quantum context.

A question therefore arises; Why in plasma physics, the magnetic permeability is always worth μ_0? The answer is quite simple : while the orbital or spin angular moments have a privileged direction, it is necessary that the atoms which carry them be aligned. This is the case in a cristal as the atoms have a fixed position on the cristal. This could also be the case in a plasma where atoms are freely moving, but statistical physics clearly show that the temperature associated to nuclear spin is very small (at meast for astrophysicl plasmas) meaning that the spin contribution to the hamiltonian of the atom is very small, compared to the translation component. The resulting magnetization current can then be neglected.

In conclusion, in a plasma, we always have $\mu = \mu_0$ and therefore $\mathbf{B} = \mu_0\mathbf{H}$, so that the vector $\mathbf{H}$ is of no interest (and therefore never mentioned).

A.5 Conductivity tensor and dielectric tensor

Because of an external electric field, a charged particle moves. This can be seen in 2 different ways :

- a current $\mathbf{J}$ is associated with the movement of the particle. The medium is then a conductor, characterized by its conductivity tensor $\boldsymbol{\sigma}$. The current density $\mathbf{J}$ is given by the general relation $\mathbf{J} = \boldsymbol{\sigma}.\mathbf{E}$. In the simplest cases, $\boldsymbol{\sigma}$ is a diagonal tensor whose elements (of the diagonal) are all equal to each other (and therefore equal to one third of the trace).

- the movement of the particle can be seen as the creation —in addition to the initial charge of the particle— of a dipole formed of a charge equal to the initial charge, placed in its new position, and a charge of opposite sign to the initial charge placed in its old position. The medium is then a dielectric characterized by its dielectric tensor $\boldsymbol{\varepsilon}$ or by its susceptibility tensor $\boldsymbol{\chi}$. The dielectric displacement vector[2] $\mathbf{D}$ is given by the relation $\mathbf{D} = \varepsilon_0\boldsymbol{\varepsilon}.\mathbf{E} = \varepsilon_0(\mathbf{1} + \boldsymbol{\chi}).\mathbf{E}$[3].

From this purely formal choice, we have two way to write the Maxwell-Ampere equation. This equation involves the current density $\mathbf{J}$. For a conductor, this density is associated with the current resulting from the displacement of free charges (these are electrons in a metal, but it can also be ions in a plasma). For any material, this current density $\mathbf{J}$ results from the sum of the current density associated with the free charges $\mathbf{J}_f$ (associated with the free charge carriers of the material) and the current density $\mathbf{J}_i$ induced by the polarization of the medium. So,

[2]the vector $\mathbf{D}$ is also called the ≪ electric induction ≫

[3]In these notes, we will use the *relative* dielectric tensor $\boldsymbol{\varepsilon}$.

- for a conductor,

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} = \mu_0 \boldsymbol{\sigma}.\mathbf{E} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{A.27}$$

- for a non-magnetic dielectric

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{\varepsilon}.\mathbf{E}) \tag{A.28}$$

For $\mathbf{E}$ and $\mathbf{B}$ fields depending on time as $e^{\imath\omega t}$, the relation between these two tensors is

$$\boldsymbol{\sigma} = -\imath\omega\varepsilon_0 \boldsymbol{\chi} \tag{A.29}$$

with $\boldsymbol{\varepsilon} = \mathbf{1} + \boldsymbol{\chi}$.

Among these 2 formalisms, we often use the dielectric description because all the physics is hence contained in a single tensor $\boldsymbol{\varepsilon}$.

In addition, the electric polarization vector $\mathbf{P}$ is the response of a dielectric medium to an external electric field,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{A.30}$$

We remember the electromagnetism results giving the induced charge density and the induced current density:

$$\frac{\partial \mathbf{P}}{\partial t} = \mathbf{J}_i \quad , \quad \boldsymbol{\nabla}.\mathbf{P} = -\rho_i \tag{A.31}$$

whose compatibility is ensured by the continuity equation. But the relation between $\mathbf{P}$ and $\mathbf{E}$ is not always simple, and wuite different from the ones encountered in dielectrics.

A.6 Spatial & temporal dispersion

In optics, the relation between $\mathbf{D}$ and $\mathbf{E}$ is often non-instantaneous. Thus, the value of $\mathbf{D}$ $(\mathbf{r}, t)$ depends on the value of $\mathbf{E}$ $(\mathbf{r}, t')$, for all $t' \leq t$. The same goes for plasmas. But the motion of a charged particle depends on the value of the field that it undergoes all along its trajectory. The relation between $\mathbf{D}$ and $\mathbf{E}$ is therefore non-local, and can be written (for causality reason) in the form

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \varepsilon_0 \int_{-\infty}^{t} \iiint_{\mathbb{R}^3} \boldsymbol{\chi}(\mathbf{r} - \mathbf{r}', t - t').\mathbf{E}(\mathbf{r}', t')\, \mathrm{d}\mathbf{r}'\, \mathrm{d}t' \tag{A.32}$$

One recognize the convolution product in the last member, which suggests that we should go in the Fourier space (in space and time). Noting $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}'$, $\tau = t - t'$, and[4]

$$\boldsymbol{\chi}(\mathbf{k}, \omega) = \int_{0}^{+\infty} \iiint_{\mathbb{R}^3} \boldsymbol{\chi}(\boldsymbol{\rho}, \tau) e^{\imath(\omega\tau - \mathbf{k}.\boldsymbol{\rho})}\, \mathrm{d}\boldsymbol{\rho}\, \mathrm{d}\tau \tag{A.33}$$

[4]We use the same notation for the susceptibility tensor in real space and in Fourier space. There is no possible confusion when the variables of the considered space are indicated

one gets the relation

$$\mathbf{D}(\mathbf{k},\omega) = \varepsilon_0 \boldsymbol{\varepsilon}(\mathbf{k},\omega).\mathbf{E}(\mathbf{k},\omega) \tag{A.34}$$

with $\boldsymbol{\varepsilon} = \mathbf{1} + \boldsymbol{\chi}$ in the $(\mathbf{k},\omega)$ space. In a plasma, the dielectric tensor depends on the wave number and on the frequency. This is called spatial dispersion and temporal dispersion.

The motion of a charged particle depends on the value of the electric field met during its whole history, i.e. along its trajectory. However, there is a distance l_{cor} beyond which the shape of the electric field in r has little consequence on the electric displacement vector in $r + l_{cor}$. In other words, the state of the particle at $r + l_{cor}$ is poorly correlated with that which it had at r. This characteristic length can only depend on 2 processes: collisions and fluctuations in electric and magnetic fields due to collective effects.

In a solid dielectric, the locality of the relationship is clear: atoms have a location in the solid which does not change with time. If we apply a field $\mathbf{E}$ at $\mathbf{r}$ where an atom is located, the a dipole will develop at $\mathbf{r}$. This one is grounded to its atom, and therefore will not travel. The field $\mathbf{E}$ in $\mathbf{r}$ will induce a displacement vector $\mathbf{D}$ in $\mathbf{r}$ and not elsewhere. In this case, the relations are local and do not require a convolution product as in the equation (A.32).

In a plasma, the charges are free and therefore move. This induces non-locality, but in a limited way: in the equation (A.32), the integral over the position space can be limited to a volume centered on $\mathbf{r}$ because the function $\boldsymbol{\chi}$ is decreasing. We can evaluate the characteristic distance over which $\boldsymbol{\chi}$ will tend towards 0. For the case of collisions, we consider an electric field constant over time (i.e. for which $\omega \to 0$). At $(\mathbf{r}, t)$, a field $\mathbf{E}$ will induce a polarization as well as a displacement vector $\mathbf{D}$. Since the particle can move freely, it will be able to transport this dipole moment as long as it does not undergo a collision (the effect of this could be to modify the dipole). It can therefore move over a time of the order of ν^{-1} where ν is the collision frequency. The non-locality will therefore be important over a distance $l_c < v_T/\nu$ where v_T is the thermal speed.

In the case of collective effects, the electric field felt by the particle is no longer constant through time. It varies on a time scale of the order of ω^{-1} where ω is the wave angular frequency. The polarized particle will therefore be able to transport its dipole moment over a time at most of the order of ω^{-1}, ie over a distance $l_f = v_T/\omega$. Beyond this distance, the moving dipole will be modified by the shape of the electric field, which will have significantly changed with time. With these 2 arguments, we can define a correlation length $l_{cor} = \min(l_c, l_f)$ beyond which $\boldsymbol{\chi}$ becomes negligible.

Spatial dispersion is important when $kl_{cor} \geq 1$. Otherwise, it is negligible: the term in $e^{-\imath \mathbf{k}.\boldsymbol{\rho}}$ is very close to 1 in the relation (A.33) and the integral no longer depends on $\mathbf{k}$. A medium is therefore no longer spatially dispersive when $\omega, \nu \gg kv_T$. In the equation (A.33), $\boldsymbol{\chi}(\boldsymbol{\rho}, \tau)$ is a function which is maximum in $\tau = 0$ and $\boldsymbol{\rho} = 0$. This function will then decrease (in τ and in $\boldsymbol{\rho}$), and l_{cor} is the distance over which the value of $\boldsymbol{\chi}(\boldsymbol{\rho}, \tau)$ will be substantially decreased.

A.7 Dissipation

The dissipation for a given mode in a plasma can be calculated from the conservation equation of electromagnetic energy. To take into account the plasma, the current density which occurs must contain the current density associated with the free charges as well as the polarization current density. It is then very easy to write Maxwell's equations in a dielectric

$$\boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B} \tag{A.35}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{free}} - \partial_t \mathbf{D} \tag{A.36}$$

so the conservation equation of electromagnetic energy for a dielectric is

$$\left(\mathbf{E}.\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}.\frac{\partial}{\partial t}\mathbf{B}\right) + \boldsymbol{\nabla}.(\mathbf{E} \times \mathbf{H}) = \mathbf{J}_{\text{free}}.\mathbf{E} \tag{A.37}$$

which reveals the new form of the Poynting vector which we denote by $\mathbf{S}$. In the absence of free charge current density, the variation of the electromagnetic energy is given by $\boldsymbol{\nabla}.\mathbf{S}$. In addition, the dissipation of electromagnetic energy can only be done by exchanging energy with the plasma. However, $\mathbf{H} = \mathbf{B}/\mu_0$ does not involve the properties of the plasma. So only the first term $\mathbf{E}.\partial_t\mathbf{D}$ contains the dissipative term. Its calculation requires to retain only the real part of $\mathbf{E}$ and $\mathbf{D}$, and to calculate a time average. The dissipation that we quantify by the scalar Q therefore writes

$$Q = \langle \mathbf{E}_r.\dot{\mathbf{D}}_r \rangle \tag{A.38}$$

where the brackets indicate a time average calculated on $\mathbf{E}_r$ and $\mathbf{D}_r$ which are the real[5] expressions of electric field and electric displacement vectors. With an electric field proportional to $e^{\imath(\mathbf{k}.\mathbf{r}-\omega t)}$, one has $\mathbf{E}_r = \frac{1}{2}(\mathbf{E} + \mathbf{E}^*)$ and $\dot{\mathbf{D}}_r = \frac{1}{2}\imath\omega\varepsilon_0[-\boldsymbol{\varepsilon}(\mathbf{k},\omega).\mathbf{E} + \boldsymbol{\varepsilon}^\star(\mathbf{k},\omega).\mathbf{E}^*]$ [6], so

$$Q = -\frac{\imath\omega\varepsilon_0}{4}\mathbf{E}^T.(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{*T}).\mathbf{E}^* \tag{A.39}$$

Because the (complex) dielectric tensor $\boldsymbol{\varepsilon}$ can be spread in an hermitian $\frac{1}{2}(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{*T})$ and an anti-hermitian part $\frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{*T})$ it is noted that the dissipation in a plasma is due only to the anti-Hermitian part of the dielectric tensor. This is an important point which leads to several remarks:

- in the case of electrostatic (or longitudinal) modes, everything happens in one dimension. The dielectric tensor is then reduced to a scalar. In such a case, the dissipation can only occur if $\boldsymbol{\varepsilon}$ has an imaginary component. This is the case when one studies Landau damping.

[5] We cannot do the calculations in complex because the multiplication and real part operations do not commute between them. This remark becomes important for non-linear terms like this one.

[6] this being the consequence of the definition of $\boldsymbol{\varepsilon}$ by a Fourier transform, and that $\mathbf{D}_r = \frac{1}{2}\varepsilon_0(\boldsymbol{\varepsilon}\mathbf{E} + \boldsymbol{\varepsilon}^\star\mathbf{E}^\star)$. Morever, we have by construction (for the derivative) $\boldsymbol{\varepsilon}^\star(\mathbf{k},\omega) = \boldsymbol{\varepsilon}(-\mathbf{k},-\omega)$

- For cold plasmas (without temperature effect), we can establish the shape of the dielectric tensor (see Eq. **??**, **??** and **??**). There is only one complex term that appears, of Hermitian symmetry. So all the modes that we can study in the context of magnetized cold plasmas will be non-dissipative.

- For these same magnetized cold plasmas, if we take into account a collision term (for example a term in $-nm\nu\mathbf{V}$ where ν is a collision frequency), we easily show that we modified the terms of the dielectric tensor by revealing an anti-Hermitian component.

Thus, to make an anti-Hermitian component appear in the dielectric tensor, either collisions or temperature effects are needed:

- The case of collisions is the most intuitive. During collisions (without discussing their nature), there is a transfer of momentum and energy which therefore makes it possible to dissipate directed energy into thermal energy.

- In the case of temperature effects, consider a particle with a velocity v in the same direction as the wave number k of a wave with angular frequency ω. The angular frequency of this wave seen by the particle is, by Doppler effect, $\omega - kv$. If this angular frequency is zero, it means that the particle sees a constant phase, and can therefore work with a non-zero average value. The particle will then gain or lose energy, which means that the wave will lose or gain some. This is another form of possible dissipation, which is handled correctly in kinetic theory and is called the Landau effect.

For magnetized plasmas, the temperature effects have somewhat different consequences. The particles gyrate around the field lines at the ω_c angular frequency. The above condition can be rewritten in magnetized plasma $\omega - k_\parallel v_\parallel = \omega_c$. In this case, the Doppler effect to be considered is indeed that in the direction parallel to the magnetic field. This formula can even be extended to all the harmonics of the gyrofrequency, and we obtain for $n \in \mathbb{N}$, $\omega - k_\parallel v_\parallel = n\omega_c$. This is referred to as the Landau cyclotron effect. If one wants to consider the Landau effect, we can no longer treat the problem with a fluid formalism (the shape of the distribution function becomes important), and the kinetic formalism becomes necessary.

A.8 Recalls of MHD

In the MHD approach, only one fluid is considered: the MHD fluid. Its density is the one of protons (or electrons by quasi-neutrality), its fluid velocity is the barycentric mean of the fluid velocities of its constituents weighted by their respective masses[7], and its pressure (isotropic) is the sum of the partial kinetic pressures of all species. The framework of the MHD imposes a few assumptions that we recall.

[7]the mass ratio $\mu = 1836$ causes that the MHD velocity is roughly that of protons.

Weak variations hypothesis. The MHD equations are only valid for "small fluctuations". The spatial and temporal gradients must be at "large scale" $\partial_t \sim 1/\tau \ll \omega_c, \omega_p$ and $\nabla \sim 1/L \ll r_L^{-1}, \lambda_D^{-1}$. By getting all the gradients null in the fluid and Maxwell equations,

$$\mathbf{E} + \mathbf{V}_s \times \mathbf{B} \rightarrow 0 \tag{A.40}$$

$$\rho \rightarrow 0 \tag{A.41}$$

$$\mathbf{J} \rightarrow 0 \tag{A.42}$$

— The first equation means that the species all have the same perpendicular fluid velocity. This fluid average speed is the MHD speed. It is equal to $\mathbf{E} \times \mathbf{B}/B^2$ which is also the speed of the magnetic field, *i.e.* the speed of the frame in which the electric field is zero[8]. A first consequence is that the parallel electric field is zero. For an electrostatic field, the field lines are then equipotential. A second is that for a non-relativistic plasma ($E_\perp/B \ll c$), the electrical energy is negligible compared to the magnetic energy: $\varepsilon_0 E^2/2 \ll B^2/2\mu_0$.

— The second relation is the quasi-neutrality approximation. It means that the charge density gradients of each species α are small compared to the charge densities of these species: $\delta n_\alpha/n_\alpha \ll 1$. As a consequence, in the Maxwell-Gauss equation, the divergence of the electric field can be used to estimate the total charge density, but not the opposite.

— The third relation means that the total current is weak compared to the currents carried by each species (There is no "slip" between the particles [9]). These last two relationships are the causes of the collective behavior of the plasma for movements parallel to the magnetic field.

The displacement current. Maxwell-Ampère equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{A.43}$$

contains two terms in the right hand side. The last term of this member is the displacement current. We can see simply see (as an exercise) that when the phase speed of the modes is small compared to the speed of light ($\omega/k < c$) then the displacement current is negligible.

As the two diverging equations do not give additional information on electric and magnetic fields, the electric field only appears in the Maxwell-Faraday equation. This is the reason why we must introduce another equation involving the electric field.

The Ohm's law. The electric field appears in the momentum conservation equation of each fluid. We can therefore use one of these equations to determine the form of $\mathbf{E}$. Writing it for electrons, with some substitutions and isolating the term $\mathbf{E}$, we get

$$\mathbf{E} = -\mathbf{V}_p \times \mathbf{B} + \frac{1}{ne}\mathbf{J} \times \mathbf{B} - \frac{m_e}{e}\mathrm{d}_t \mathbf{V}_p + \frac{m_e}{e^2}\mathrm{d}_t\left(\frac{\mathbf{J}}{n}\right) - \frac{1}{ne}\boldsymbol{\nabla}.\mathbf{P}_e + \eta\mathbf{J} \tag{A.44}$$

[8] This is a known result of the Lorentz transform (relativistic or not) of the fields $\mathbf{E}$ and $\mathbf{B}$

[9] When this must be the case, we must then do at least Hall MHD.

The first term of the right hand side is of order 0; all the others are of order 1. To know in which spatial or temporal domain it is necessary to keep them, it is necessary to study their "scaling":

- The first term on the right hand side of Eq. (A.44) is of order 0, hence never neglected. When the electric field is given by this term alone, Ohm's law is said to be ideal. In MHD, $\mathbf{V}_p = \mathbf{V}$ at order m_e/m_p.
- The second term (Hall effect) has a scaling in $k^2 v_A^2/\omega\Omega_p$. For an Alfvenic perturbation, the time scale is in Ω_p^{-1}, and the spatial scale is in v_A/Ω_p (which is also equal to the inertial length of the protons). It is a term that is associated with the slip between electrons and protons, when the latter are no longer magnetized. We keep this term in MHD Hall.
- The third term (electron inertia - 1) has a scaling in $\omega/\mu\Omega_p$, that is to say a time scale in $(\mu\Omega_p)^{-1}$. It could become important in the vicinity of the gyrofrequency of the electrons. This term is very generally neglected.
- The fourth term (inertia of electrons - 2) has a scaling in d_e (length of inertia of electrons) and only becomes important at electronic spatial scales.
- The fifth term (compressibility of electrons) has a scaling in $k^2\rho_e^2\Omega_p/\omega$ ie a spatial scale in ρ_e (Larmor radius of electrons) and a time scale in Ω_p. This effect can therefore be important for hot electrons.
- The sixth term (resistive effects) has a scaling in $nek^2/\omega\mu_0\sigma$. To compare the importance of the resistive terms with the convective term of ideal Ohm's law, we often use the Lundquist number S, the ratio between the convection time and the diffusion time. Except in very dense plasmas, L is often infinite for astrophysical plasmas.

As an illustration, the nature of Alfvén waves therefore depends on Ohm's law that we keep. It also depends on the closure of the hierarchy (often at the level of pressure, scalar or tensorial).