DEVOIR MAISON - MÉTHODES NUMÉRIQUES POUR LES EDP INSTATIONNAIRES: DIFFÉRENCES FINIES ET VOLUMES FINIS

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1 Semi-group estimates

Pour $\sigma \in \mathbb{R}$ Considerons le système d'onde linéaire en dimension un

$$\partial_t p + \partial_x v = \sigma(v - p) \tag{1}$$
$$\partial_t v + \partial_x p = \sigma(p - v) \tag{2}$$

a) Determine $u = (p, v)$ en fonction de $u_0 = (p_0, v_0)$

(1)+(2)

$$\begin{array}{rcl} \partial_t(p+v) + \partial_x(p+v) & = & \sigma(v-p) + \sigma(p-v) \\ & = & \sigma(v-p+p-v) \\ & = & \sigma(0) \\ \partial_t(p+v) + \partial_x(p+v) & = & 0 \end{array}$$

(1)-(2)

$$\begin{array}{rcl} \partial_t(p-v) + \partial_x(v-p) & = & \sigma(v-p) - \sigma(p-v) \\ \partial_t(p-v) - \partial_x(p-v) & = & \sigma(v-p-p+v) \\ & = & \sigma(2v-2p) \\ & = & 2\sigma(v-p) \\ \partial_t(p-v) - \partial_x(p-v) & = & -2\sigma(p-v) \end{array}$$

$$\partial_t(p+v) + \partial_x(p+v) = 0 \tag{3}$$
$$\partial_t(p-v) - \partial_x(p-v) = -2\sigma(p-v) \tag{4}$$

On a deux équations de transport.

Resolution par la méthode des caractéristiques

Par l'équation (3)

$$\partial_t(p+v) + \partial_x(p+v) = 0$$

$$\left\{ \begin{array}{rcl} \frac{dx^*(t)}{dt} & = & 1 \\ x^*(t^*) & = & x_* \end{array} \right. \Rightarrow$$

$$
\begin{array}{rcl}
x^*(t) &=& t+c \\
x^*(t^*) &=& x_* = t^* + c \\
c &=& x_* - t^* \\
x^*(t) &=& t + x_* - t^*
\end{array}
$$

$$
\begin{array}{rcl}
(p+v)(x_*,t^*) &=& (p_0+v_0)(x_* - t^*) \\
(p+v)(x,t) &=& p_0(x-t) + v_0(x-t) \\
p(x,t)+v(x,t) &=& p_0(x-t) + v_0(x-t)
\end{array}
$$

Par l'équation (4)

$$
\partial_t(p-v) - \partial_x(p-v) = -2\sigma(p-v)
$$

$$
\left\{
\begin{array}{rcl}
\frac{dx_1^*(t)}{dt} &=& -1 \\
x_1^*(t^*) &=& x_*
\end{array}
\right. \Rightarrow
$$

$$
\begin{array}{rcl}
x_1^*(t) &=& -t + c_1 \\
x_1^*(t^*) &=& x_* = -t^* + c_1 \\
c_1 &=& x_* + t^* \\
x^*(t) &=& -t + x_* + t^*
\end{array}
$$

$$
\begin{array}{rcl}
\frac{d}{dt}(p-v)(x_1^*(t),t) &=& -2\sigma(p-v)(x_*(t),t) \\
(p-v)(x_1^*(t),t) &=& K\exp^{-2\sigma t} \\
K &=& K\exp^{-2\sigma t} \\
&=& (p-v)(x_1^*(0),0) \\
&=& (p_0-v_0)(x_1^* + t^*)
\end{array}
$$

$$
\begin{array}{rcl}
(p-v)(x_1^*(t),t) &=& (p_0-v_0)(x_* + t^*)\exp^{-2\sigma t^*} \\
(p-v)(x,t) &=& (p_0-v_0)(x+t)\exp^{-2\sigma t}
\end{array}
$$

$$
\left\{
\begin{array}{rcl}
p(x,t)+v(x,t) &=& p_0(x-t) + v_0(x-t) \\
p(x,t)-v(x,t) &=& p_0(x+t)\exp^{-2\sigma t} - v_0(x+t)\exp^{-2\sigma t}
\end{array}
\right.
$$

On va résoudre le système.

Par somme on a :

$$
p(x,t) = 1/2\left[p_0(x-t) + v_0(x-t) + (p_0(x+t) - v_0(x+t))\exp^{-2\sigma t}\right]
$$

Par différence on a :

$$
v(x,t) = 1/2\left[p_0(x-t) + v_0(x-t) - (p_0(x+t) - v_0(x+t))\exp^{-2\sigma t}\right]
$$

Donc $\forall t > 0$

$$
u(x,t) = (p(x,t), v(x,t))
$$

Écrire explicitement l'operator A, tel que $u = \exp^{tA} u_0$

$$\begin{aligned} u &= (p, v) \\ \partial_t u &= (\partial_t p, \partial_t v) \\ \partial_x u &= (\partial_x p, \partial_x v) \end{aligned}$$

$$\begin{aligned} \partial_x \begin{pmatrix} p \\ v \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x p \\ \partial_x v \end{pmatrix} = A_1 \partial_x u, \quad avec \\ A_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} \sigma(v-p) \\ \sigma(p-v) \end{pmatrix} &= \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix} = Bu, \quad avec \\ B &= \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix} \end{aligned}$$

$$\partial_t U = AU \quad \text{si on pose} \quad A = -A_1 \partial_x + B$$

$$\left\{ \begin{array}{ll} \partial_t U & = AU \\ U_0 & = (p_0, v_0). \end{array} \right. \Rightarrow U(t) = e^{At} u_0$$

b) $\quad Y_1 = L^1(\mathbb{R})^2$

$$||(a,b)||_1 = ||a||_{L^1(\mathbb{R})} + ||b||_{L^1(\mathbb{R})}$$

Montrons que : $||e^{At}||_{\mathcal{L}(Y_p)} \leq (1 + e^{-2\sigma t}) \qquad$ pour tout $\qquad t \geq 0.$

On a $U(x,t) = (p(x,t), v(x,t))$

$$\begin{aligned} ||U(t)||_1 &= ||p(t)||_{L^1(\mathbb{R})} + ||v(t)||_{L^1(\mathbb{R})} \\ &= \int_{\mathbb{R}} |p(x,t)|\, dx + \int_{\mathbb{R}} |v(x,t)|\, dx \\ &= \frac{1}{2} \int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) + [p_0(x+t) - v_0(x+t)] e^{-2\sigma t}|\, dx \\ &+ \frac{1}{2} \int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) - [p_0(x+t) - v_0(x+t)] e^{-2\sigma t}|\, dx \end{aligned}$$

$$\begin{aligned} ||U(t)||_1 \leq \frac{1}{2} \Bigg[& \int_{\mathbb{R}} |p_0(x-t)|\, dx + \int_{\mathbb{R}} |v_0(x-t)|\, dx + \\ & + e^{-2\sigma t} \left(\int_{\mathbb{R}} |p_0(x+t)|\, dx + \int_{\mathbb{R}} |v_0(x+t)|\, dx \right) \Bigg] + \\ + \frac{1}{2} \Bigg[& \int_{\mathbb{R}} |p_0(x-t)|\, dx + \int_{\mathbb{R}} |v_0(x-t)|\, dx + \\ & + e^{-2\sigma t} \left(\int_{\mathbb{R}} |p_0(x+t)|\, dx + \int_{\mathbb{R}} |v_0(x+t)|\, dx \right) \Bigg] \end{aligned}$$

On fait un changement de variable à chaque terme d'integrale :

$$\begin{aligned}||U(t)||_1 \leq \frac{1}{2}\Bigg[\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx + \\ +e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx\right)\Bigg] + \\ + \frac{1}{2}\Bigg[\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx + \\ +e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx\right)\Bigg]\end{aligned}$$

$$\begin{aligned}||U(t)||_1 \leq \int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx + \\ + e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx\right)\end{aligned}$$

$$||U(t)||_1 \leq ||p_0||_{L^1} + ||v_0||_{L^1} + e^{-2\sigma t}\left(||p_0||_{L^1} + ||v_0||_{L^1}\right)$$

$$||U(t)||_1 \leq ||U_0||_1 + e^{-2\sigma t}||U_0||_1$$

$$||U(t)||_1 \leq \left(1 + e^{-2\sigma t}\right)||U_0||_1 \tag{5}$$

$$||e^{At}||_{\mathcal{L}(Y_p)} = \sup_{||U_0||_1=1} \frac{||U(t)||_1}{||U_0||_1}$$

$$||e^{At}||_{\mathcal{L}(Y_p)} \leq 1 + e^{-2\sigma t}$$

c) $Y_2 = L^2(\mathbb{R})^2$

$$||(a,b)||_2 \quad = \quad \sqrt{||a||^2_{L^2(\mathbb{R})} + ||b||^2_{L^2(\mathbb{R})}}$$

Montrons que : $||e^{At}||_{\mathcal{L}(Y_2)} \leq max(1, e^{-2\sigma t})$ pour tout $t \geq 0$.

On a $U(x,t) = (p(x,t), v(x,t))$

$$\begin{aligned}||U(t)||_2^2 &= ||p(t)||^2_{L^2(\mathbb{R})} + ||v(t)||^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} |p(x,t)|^2\, dx + \int_{\mathbb{R}} |v(x,t)|^2\, dx \\ &= \frac{1}{4}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) + (p_0(x+t) - v_0(x+t))e^{-2\sigma t}|^2\, dx \\ &+ \frac{1}{4}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) - (p_0(x+t) - v_0(x+t))e^{-2\sigma t}|^2\, dx\end{aligned}$$

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{4}\Big[&\int_{\mathbb{R}} |p_0(x-t)+v_0(x-t)|^2\,dx \\
&+\int_{\mathbb{R}} |p_0(x+t)-v_0(x+t)|^2 e^{-4\sigma t}\,dx \\
&+2\int_{\mathbb{R}} |p_0(x-t)+v_0(x-t)||p_0(x+t)-v_0(x+t)|e^{-2\sigma t}\,dx\Big] \\
&+\frac{1}{4}\Big[\int_{\mathbb{R}} |p_0(x-t)+v_0(x-t)|^2\,dx \\
&+\int_{\mathbb{R}} |p_0(x+t)-v_0(x+t)|^2 e^{-4\sigma t}\,dx \\
&-2\int_{\mathbb{R}} |p_0(x-t)+v_0(x-t)||p_0(x+t)-v_0(x+t)|e^{-2\sigma t}\,dx\Big]
\end{aligned}
$$

$$
\begin{aligned}
||U(t)||_2^2 &\leq \frac{1}{2}\Big[\int_{\mathbb{R}} |p_0(x-t)+v_0(x-t)|^2\,dx \\
&\quad+\int_{\mathbb{R}} |p_0(x+t)-v_0(x+t)|^2 e^{-4\sigma t}\,dx\Big] \\
&\leq \frac{1}{2}\Big[max(1,e^{-4\sigma t})\int_{\mathbb{R}} |p_0(x-t)+v_0(x-t)|^2\,dx \\
&\quad+max(1,e^{-4\sigma t})\int_{\mathbb{R}} |p_0(x+t)-v_0(x+t)|^2\,dx\Big] \\
&\leq \frac{1}{2}max(1,e^{-4\sigma t})\Big[\int_{\mathbb{R}} |p_0(x-t)+v_0(x-t)|^2\,dx+\int_{\mathbb{R}} |p_0(x+t)-v_0(x+t)|^2\,dx\Big]
\end{aligned}
$$

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{2}max(1,e^{-4\sigma t})\Big[&\int_{\mathbb{R}} |p_0(x-t)|^2\,dx+\int_{\mathbb{R}} |v_0(x-t)|^2\,dx \\
&+2\int_{\mathbb{R}} |p_0(x-t)||v_0(x-t)|\,dx \\
&+\int_{\mathbb{R}} |p_0(x+t)|^2\,dx+\int_{\mathbb{R}} |v_0(x+t)|^2\,dx \\
&-2\int_{\mathbb{R}} |p_0(x+t)||v_0(x+t)|\,dx\Big]
\end{aligned}
$$

On fait un changement de variable à chaque terme d'integrale :

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{2}max(1,e^{-4\sigma t})\Big[&\int_{\mathbb{R}} |p_0(x)|^2\,dx+\int_{\mathbb{R}} |v_0(x)|^2\,dx \\
&+2\int_{\mathbb{R}} |p_0(x)||v_0(x)|\,dx \\
&+\int_{\mathbb{R}} |p_0(x)|^2\,dx+\int_{\mathbb{R}} |v_0(x)|^2\,dx \\
&-2\int_{\mathbb{R}} |p_0(x)||v_0(x)|\,dx\Big]
\end{aligned}
$$

$$
||U(t)||_2^2 \leq max(1,e^{-4\sigma t})\Big[\int_{\mathbb{R}} |p_0(x)|^2\,dx+\int_{\mathbb{R}} |v_0(x)|^2\,dx\Big]
$$

$$
||U(t)||_2^2 \leq max(1,e^{-4\sigma t})||U_0||_2^2
$$

$$||U(t)||_2 \leq max(1, e^{-2\sigma t})||U_0||_2$$

$$||e^{At}||_{\mathcal{L}(Y_2)} = \sup_{||U_0||_2=1} \frac{||U(t)||_2}{||U_0||_2}$$

$$||e^{At}||_{\mathcal{L}(Y_2)} \leq max(1, e^{-2\sigma t})$$

d) $Y_\infty = L^\infty(\mathbb{R})^2$

$$\begin{aligned} ||(a,b)||_\infty &= max(||a||_{L^\infty(\mathbb{R})}, ||b||_{L^\infty(\mathbb{R})}) \\ \sigma = 0, \quad et \quad t \geq 0 \end{aligned}$$

Montrons que $||e^{At}||_{\mathcal{L}(Y_\infty)} \leq 2$.

Par définition, nous avons :

$$||U(t)||_\infty = \max\Big(||p(t)||_{L^\infty(\mathbb{R})}, ||v(t)||_{L^\infty(\mathbb{R})}\Big)$$

avec

$$||p(t)||_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |p(x,t)| \quad \text{et} \quad ||v(t)||_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |v(x,t)|.$$

$$\begin{aligned} ||U(t)||_\infty = \max\Bigg(&\frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x+t)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x+t)|\ , \\ &\frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x+t)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x+t)|\Bigg) \end{aligned}$$

Par un changement de variable, on a :

$$\begin{aligned} ||U(t)||_\infty = \max\Bigg(&\frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x)|\ , \\ &\frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}}|v_0(x)|\Bigg) \end{aligned}$$

$$||U(t)||_\infty = \max\left(\sup_{x\in\mathbb{R}}|p_0(x)| + \sup_{x\in\mathbb{R}}|v_0(x)|\ , \sup_{x\in\mathbb{R}}|p_0(x)| + \sup_{x\in\mathbb{R}}|v_0(x)|\right)$$

$$||U(t)||_\infty = \max\Big(||p_0||_{L^\infty(\mathbb{R})} + ||v_0||_{L^\infty(\mathbb{R})}\ , ||p_0||_{L^\infty(\mathbb{R})} + ||v_0||_{L^\infty(\mathbb{R})}\Big)$$

$$||p_0||_{L^\infty(\mathbb{R})} + ||v_0||_{L^\infty(\mathbb{R})} \leq 2||U_0||_\infty$$

$$||U(t)||_\infty \leq \max\left(2||U_0||_\infty\ , 2||U_0||_\infty\right)$$

$$||U(t)||_\infty \leq 2||U_0||_\infty$$

$$||e^{At}||_{\mathcal{L}(Y_\infty)} = \sup_{||U_0||_\infty=1} \frac{||U(t)||_\infty}{||U_0||_\infty} \leq 2$$

$$||e^{At}||_{\mathcal{L}(Y_\infty)} \leq 2$$

Condition initial tel que $||e^{At}||_{\mathcal{L}(Y_\infty)} = 2$

Prenons $p_0 = 1$ et

$$\begin{cases} v_0 = -1 & si \quad x > 0, \\ v_0 = 1 & si \quad x < 0, \\ v_0 = 0 & si \quad x = 0. \end{cases}$$

$$||e^{At}||_{\mathcal{L}(Y_\infty)} = 2$$