1 Semi-group estimates

Pour $\sigma \in \mathbb{R}$ Considerons le système d'onde linéaire en dimension un

$$\partial_t p + \partial_x v = \sigma(v - p) \tag{1}$$
$$\partial_t v + \partial_x p = \sigma(p - v) \tag{2}$$

a) Determine $u = (p, v)$ en fonction de $u_0 = (p_0, v_0)$

(1)+(2)

$$\begin{aligned}
\partial_t(p+v) + \partial_x(p+v) &= \sigma(v-p) + \sigma(p-v) \\
&= \sigma(v - p + p - v) \\
&= \sigma(0) \\
\partial_t(p+v) + \partial_x(p+v) &= 0
\end{aligned}$$

(1)-(2)

$$\begin{aligned}
???\partial_t(p-v) + \partial_x(v-p) &= \sigma(v-p) - \sigma(p-v) \\
\partial_t(p-v) - \partial_x(p-v) &= \sigma(v - p - p + v) \\
&= \sigma(2v - 2p) \\
&= 2\sigma(v-p) \\
\partial_t(p-v) - \partial_x(p-v) &= -2\sigma(p-v)
\end{aligned}$$

$$\partial_t(p+v) + \partial_x(p+v) = 0 \tag{3}$$
$$\partial_t(p-v) - \partial_x(p-v) = -2\sigma(p-v) \tag{4}$$

On a deux équations de transport.

Resolution par la méthode des caractéristiques

(3)

$$\left\{\begin{aligned}
\tfrac{dx^*(t)}{dt} &= 1 \\
x^*(t^*) &= x_*
\end{aligned}\right. \Rightarrow$$

$$\begin{aligned}
x^*(t) &= t + c \\
x^*(t^*) &= x_* = t^* + c \\
c &= x_* - t^* \\
x^*(t) &= t + x_* - t^*
\end{aligned}$$

$$\begin{aligned}
(p+v)(x_*, t^*) &= (p_0 + v_0)(x_*, t^*) \\
(p+v)(x,t) &= p_0(x,t) + v_0(x,t)
\end{aligned}$$

(4)

$$\left\{\begin{aligned}
\tfrac{dx_1^*(t)}{dt} &= -1 \\
x_1^*(t^*) &= x_*
\end{aligned}\right. \Rightarrow$$

$$\begin{aligned}
x_1^*(t) &= -t + c_1 \\
x_1^*(t^*) &= x_* = -t^* + c_1 \\
c_1 &= x_* + t^* \\
x^*(t) &= -t + x_* + t^*
\end{aligned}$$

$$
\begin{array}{rcl}
\frac{d}{dt}(p-v)(x_1^*(t),t) & = & -2\sigma(p-v)(x_*(t),t) \\
(p-v)(x_1^*(t),t) & = & K\exp^{-2\sigma t} \\
K & = & K\exp^{-2\sigma t} \\
 & = & (p-v)(x_1^*(0),0) \\
 & = & (p_0-v_0)(x_1^*+t^*)
\end{array}
$$

$$
\begin{array}{rcl}
(p-v)(x_1^*(t),t) & = & (p_0-v_0)(x_*+t^*)\exp^{-2\sigma t^*} \\
(p-v)(x,t) & = & (p_0-v_0)(x,t)\exp^{-2\sigma t}
\end{array}
$$

$$
\left\{
\begin{array}{rcl}
(p+v) & = & (x-t)+u_0(x-t) \\
(p-v) & = & (x+t)\exp^{-2\sigma t} - u_0(x+t)\exp^{-2\sigma t}
\end{array}
\right.
$$

$$
\begin{array}{rcl}
u(x,t) & = & (p(x,t),v(x,t)) \qquad\qquad \forall t \geq 0 \\
avec & & \\
p(x,t) & = & 1/2\left[p_0(x-t)+u_0(x-t)+(p_0(x+t)-u_0(x+t))\exp^{-2\sigma t}\right] \\
v(x,t) & = & 1/2\left[p_0(x-t)+u_0(x-t)-(p_0(x+t)-u_0(x+t))\exp^{-2\sigma t}\right]
\end{array}
$$

Écrire explicitement l'operator A, tel que $u = \exp^{tA} u_0$

$$
\begin{array}{rcl}
u & = & (p,v) \\
\partial_t u & = & (\partial_t p, \partial_t v) \\
\partial_x u & = & (\partial_x p, \partial_x v)
\end{array}
$$

$$
\begin{array}{rcl}
\partial_x \begin{pmatrix} v \\ p \end{pmatrix} & = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x p \\ \partial_x v \end{pmatrix} \quad = \quad A_1 \partial_x u, \quad avec \\
A_1 & = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\begin{pmatrix} \sigma(v-p) \\ \sigma(p-v) \end{pmatrix} & = & \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix} \quad = \quad Bu, \quad avec \\
B & = & \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix}
\end{array}
$$

2 Numerical methods

$$
\left\{
\begin{array}{rcll}
\partial_t u - \partial_{xx} u & = & 0, & x \in \mathbb{R}, \quad t > 0, \\
u(0,x) & = & u_0(x), & x \in \mathbb{R}
\end{array}
\right. \tag{5}
$$

La discrétisation de type Différences Finis explicite avec un schéma sur la forme :

$$
\frac{u_j^{n+1}-u_j^n}{\Delta t} - \frac{4}{3}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{1}{12}\frac{u_{j+2}^n - 2u_j^n + u_{j-2}^n}{\Delta x^2} - \frac{\Delta t^2}{2}\frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n}{\Delta x^4} = 0 \tag{6}
$$

a) Determination du symbol du schéma

Le symbole du schéma (6) est donné par :

$$\lambda(\theta) = \sum_{r=-2}^{2} \alpha_r e^{\mathbf{i}\theta r}, \quad \theta \in \mathbb{R}$$

Par développement de ce schéma on a :

$$u_j^{n+1} \quad = \quad \alpha_0 u_j^n + \alpha_{-1} u_{j-1}^n + \alpha_{-2} u_{j-2}^n + \alpha_1 u_{j+1}^n + \alpha_2 u_{j+2}^n, \quad \textit{où}$$

$$\left\{ \begin{array}{rcl} \alpha_0 & = & 1 - \frac{15}{6}\nu + 3\nu^2 \\ \alpha_{-1} & = & \frac{4}{3}\nu - 2\nu^2 \\ \alpha_{-2} & = & \frac{-1}{12}\nu + \frac{1}{2}\nu^2 \\ \alpha_1 & = & \frac{4}{3}\nu - 2\nu^2 \\ \alpha_2 & = & \frac{-1}{12}\nu + \frac{1}{2}\nu^2 \end{array} \right. ,$$

$$\textit{avec} \quad \begin{array}{rcl} \nu & = & \Delta t / \Delta x^2 \\ \nu^2 & = & \Delta t^2 / \Delta x^4 \end{array}$$

3 Consistence du schéma

Par développement de Taylor du schéma (6) on a :

$$u(x_{j-1}, t^n) = u(x_j, t^n) - \Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(\Delta x)^3}{3!} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j+1}, t^n) = u(x_j, t^n) - \Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(\Delta x)^3}{3!} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j-2}, t^n) = u(x_j, t^n) - 2\Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(2\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(2\Delta x)^3}{6} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j+2}, t^n) = u(x_j, t^n) - 2\Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(2\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(2\Delta x)^3}{6} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_j, t^{n+1}) = u(x_j, t^n) + \Delta t \frac{\partial}{\partial t} u(x_j, t^n) + \frac{(\Delta t)^2}{2} \frac{\partial^2}{\partial t^2} u(x_j, t^n) + \mathcal{O}((\Delta t)^3)$$