1 Semi-group estimates

Pour $\sigma \in \mathbb{R}$ Considerons le système d'onde linéaire en dimension un

$$\partial_t p + \partial_x v = \sigma(v - p) \tag{1}$$
$$\partial_t v + \partial_x p = \sigma(p - v) \tag{2}$$

a) Determine $u = (p, v)$ en fonction de $u_0 = (p_0, v_0)$

(1)+(2)

$$\begin{aligned}
\partial_t(p+v) + \partial_x(p+v) &= \sigma(v-p) + \sigma(p-v) \\
&= \sigma(v-p+p-v) \\
&= \sigma(0) \\
\partial_t(p+v) + \partial_x(p+v) &= 0
\end{aligned}$$

(1)-(2)

$$\begin{aligned}
???\partial_t(p-v) + \partial_x(v-p) &= \sigma(v-p) - \sigma(p-v) \\
\partial_t(p-v) - \partial_x(p-v) &= \sigma(v-p-p+v) \\
&= \sigma(2v-2p) \\
&= 2\sigma(v-p) \\
\partial_t(p-v) - \partial_x(p-v) &= -2\sigma(p-v)
\end{aligned}$$

$$\partial_t(p+v) + \partial_x(p+v) = 0 \tag{3}$$
$$\partial_t(p-v) - \partial_x(p-v) = -2\sigma(p-v) \tag{4}$$

On a deux équations de transport.

Resolution par la méthode des caractéristiques

(3)

$$\begin{cases} \frac{dx^*(t)}{dt} &= 1 \\ x^*(t^*) &= x_* \end{cases} \Rightarrow$$

$$\begin{aligned}
x^*(t) &= t + c \\
x^*(t^*) &= x_* = t^* + c \\
c &= x_* - t^* \\
x^*(t) &= t + x_* - t^*
\end{aligned}$$

$$\begin{aligned}
(p+v)(x_*, t^*) &= (p_0+v_0)(x_*, t^*) \\
(p+v)(x, t) &= p_0(x,t) + v_0(x,t)
\end{aligned}$$

(4)

$$\begin{cases} \frac{dx_1^*(t)}{dt} &= -1 \\ x_1^*(t^*) &= x_* \end{cases} \Rightarrow$$

$$\begin{aligned}
x_1^*(t) &= -t + c_1 \\
x_1^*(t^*) &= x_* = -t^* + c_1 \\
c_1 &= x_* + t^* \\
x^*(t) &= -t + x_* + t^*
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}(p-v)(x_1^*(t),t) &= -2\sigma(p-v)(x_*(t),t)\\
(p-v)(x_1^*(t),t) &= K\exp^{-2\sigma t}\\
K &= K\exp^{-2\sigma t}\\
&= (p-v)(x_1^*(0),0)\\
&= (p_0-v_0)(x_1^*+t^*)
\end{aligned}
$$

$$
\begin{aligned}
(p-v)(x_1^*(t),t) &= (p_0-v_0)(x_*+t^*)\exp^{-2\sigma t^*}\\
(p-v)(x,t) &= (p_0-v_0)(x,t)\exp^{-2\sigma t}
\end{aligned}
$$

$$
\begin{cases}
(p+v) = (x-t)+u_0(x-t)\\
(p-v) = (x+t)\exp^{-2\sigma t} - u_0(x+t)\exp^{-2\sigma t}
\end{cases}
$$

$$
\begin{aligned}
u(x,t) &= (p(x,t),v(x,t)) && \forall t\geq 0\\
&avec\\
p(x,t) &= 1/2\left[p_0(x-t)+u_0(x-t)+(p_0(x+t)-u_0(x+t))\exp^{-2\sigma t}\right]\\
v(x,t) &= 1/2\left[p_0(x-t)+u_0(x-t)-(p_0(x+t)-u_0(x+t))\exp^{-2\sigma t}\right]
\end{aligned}
$$

Écrire explicitement l'operator A, tel que $u=\exp^{tA}u_0$

$$
\begin{aligned}
u &= (p,v)\\
\partial_t u &= (\partial_t p,\partial_t v)\\
\partial_x u &= (\partial_x p,\partial_x v)
\end{aligned}
$$

$$
\begin{aligned}
\partial_x\begin{pmatrix} v\\ p\end{pmatrix} &= \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\partial_x p\\ \partial_x v\end{pmatrix} = A_1\partial_x u, \quad avec\\
A_1 &= \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\\
\begin{pmatrix}\sigma(v-p)\\ \sigma(p-v)\end{pmatrix} &= \begin{pmatrix}-\sigma & \sigma\\ \sigma & -\sigma\end{pmatrix}\begin{pmatrix} p\\ v\end{pmatrix} = Bu, \quad avec\\
B &= \begin{pmatrix}-\sigma & \sigma\\ \sigma & -\sigma\end{pmatrix}
\end{aligned}
$$

2 Numerical methods

a)