1 Semi-group estimates

Pour $\sigma \in \mathbb{R}$ Considerons le système d'onde linéaire en dimension un

$$\partial_t p + \partial_x v = \sigma(v - p) \tag{1}$$
$$\partial_t v + \partial_x p = \sigma(p - v) \tag{2}$$

a) Determine $u = (p, v)$ en fonction de $u_0 = (p_0, v_0)$

(1)+(2)

$$\begin{aligned}
\partial_t(p+v) + \partial_x(p+v) &= \sigma(v-p) + \sigma(p-v) \\
&= \sigma(v - p + p - v) \\
&= \sigma(0) \\
\partial_t(p+v) + \partial_x(p+v) &= 0
\end{aligned}$$

(1)-(2)

$$\begin{aligned}
\partial_t(p-v) + \partial_x(v-p) &= \sigma(v-p) - \sigma(p-v) \\
\partial_t(p-v) - \partial_x(p-v) &= \sigma(v - p - p + v) \\
&= \sigma(2v - 2p) \\
&= 2\sigma(v-p) \\
\partial_t(p-v) - \partial_x(p-v) &= -2\sigma(p-v)
\end{aligned}$$

$$\partial_t(p+v) + \partial_x(p+v) = 0 \tag{3}$$
$$\partial_t(p-v) - \partial_x(p-v) = -2\sigma(p-v) \tag{4}$$

On a deux équations de transport.

Resolution par la méthode des caractéristiques

Par l'équation (3)

$$\partial_t(p+v) + \partial_x(p+v) = 0$$

$$\left\{\begin{array}{rcl} \frac{dx^*(t)}{dt} &=& 1 \\ x^*(t^*) &=& x_* \end{array}\right. \Rightarrow$$

$$\begin{aligned}
x^*(t) &= t + c \\
x^*(t^*) &= x_* = t^* + c \\
c &= x_* - t^* \\
x^*(t) &= t + x_* - t^*
\end{aligned}$$

$$\begin{aligned}
(p+v)(x_*, t^*) &= (p_0 + v_0)(x_* - t^*) \\
(p+v)(x, t) &= p_0(x-t) + v_0(x-t) \\
p(x,t) + v(x,t) &= p_0(x-t) + v_0(x-t)
\end{aligned}$$

Par l'équation (4)

$$\partial_t(p-v) - \partial_x(p-v) = -2\sigma(p-v)$$

$$\left\{\begin{array}{rcl} \frac{dx_1^*(t)}{dt} &=& -1 \\ x_1^*(t^*) &=& x_* \end{array}\right. \Rightarrow$$

$$\begin{array}{rcl} x_1^*(t) & = & -t + c_1 \\ x_1^*(t^*) & = & x_* = -t^* + c_1 \\ c_1 & = & x_* + t^* \\ x^*(t) & = & -t + x_* + t^* \end{array}$$

$$\begin{array}{rcl} \frac{d}{dt}(p-v)(x_1^*(t),t) & = & -2\sigma(p-v)(x_*(t),t) \\ (p-v)(x_1^*(t),t) & = & K \exp^{-2\sigma t} \\ K & = & K \exp^{-2\sigma t} \\ & = & (p-v)(x_1^*(0),0) \\ & = & (p_0 - v_0)(x_1^* + t^*) \end{array}$$

$$\begin{array}{rcl} (p-v)(x_1^*(t),t) & = & (p_0 - v_0)(x_* + t^*) \exp^{-2\sigma t^*} \\ (p-v)(x,t) & = & (p_0 - v_0)(x+t) \exp^{-2\sigma t} \end{array}$$

$$\left\{ \begin{array}{rcl} p(x,t) + v(x,t) & = & p_0(x-t) + v_0(x-t) \\ p(x,t) - v(x,t) & = & p_0(x+t) \exp^{-2\sigma t} - v_0(x+t) \exp^{-2\sigma t} \end{array} \right.$$

On va résoudre le système.

Par somme on a :

$$p(x,t) \quad = \quad 1/2 \left[p_0(x-t) + v_0(x-t) + (p_0(x+t) - v_0(x+t)) \exp^{-2\sigma t} \right]$$

Par différence on a :

$$v(x,t) \quad = \quad 1/2 \left[p_0(x-t) + v_0(x-t) - (p_0(x+t) - v_0(x+t)) \exp^{-2\sigma t} \right]$$

Donc $\forall t > 0$

$$u(x,t) \quad = \quad (p(x,t), v(x,t))$$

Écrire explicitement l'operator A, tel que $u = \exp^{tA} u_0$

$$\begin{array}{rcl} u & = & (p,v) \\ \partial_t u & = & (\partial_t p, \partial_t v) \\ \partial_x u & = & (\partial_x p, \partial_x v) \end{array}$$

$$\begin{array}{rcl} \partial_x \begin{pmatrix} p \\ v \end{pmatrix} & = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x p \\ \partial_x v \end{pmatrix} \quad = \quad A_1 \partial_x u, \quad avec \\ A_1 & = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{rcl} \begin{pmatrix} \sigma(v-p) \\ \sigma(p-v) \end{pmatrix} & = & \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix} \quad = \quad Bu, \quad avec \\ B & = & \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix} \end{array}$$

$$\partial_t U = AU \quad \text{si on pose} \quad A = -A_1 \partial_x + B$$

$$\left\{ \begin{array}{rl} \partial_t U & = AU \\ U_0 & = (p_0, v_0). \end{array} \right. \Rightarrow U(t) = e^{At} u_0$$

b) $Y_1 = L^1(\mathbb{R})^2$

$$||(a,b)||_1 \quad = \quad ||a||_{L^1(\mathbb{R})} + ||b||_{L^1(\mathbb{R})}$$

Montrons que : $||e^{At}||_{\mathcal{L}(Y_p)} \leq (1 + e^{-2\sigma t})$ pour tout $t \geq 0$.

On a $U(x,t) = (p(x,t), v(x,t))$

$$\begin{aligned}
||U(t)||_1 &= ||p(t)||_{L^1(\mathbb{R})} + ||v(t)||_{L^1(\mathbb{R})} \\
&= \int_{\mathbb{R}} |p(x,t)|\, dx + \int_{\mathbb{R}} |v(x,t)|\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) + [p_0(x+t) - v_0(x+t)]e^{-2\sigma t}|\, dx \\
&+ \frac{1}{2}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) - [p_0(x+t) - v_0(x+t)]e^{-2\sigma t}|\, dx
\end{aligned}$$

$$\begin{aligned}
||U(t)||_1 \leq \frac{1}{2}\Bigg[&\int_{\mathbb{R}} |p_0(x-t)|\, dx + \int_{\mathbb{R}} |v_0(x-t)|\, dx + \\
&+ e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x+t)|\, dx + \int_{\mathbb{R}} |v_0(x+t)|\, dx\right)\Bigg] + \\
+ \frac{1}{2}\Bigg[&\int_{\mathbb{R}} |p_0(x-t)|\, dx + \int_{\mathbb{R}} |v_0(x-t)|\, dx + \\
&+ e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x+t)|\, dx + \int_{\mathbb{R}} |v_0(x+t)|\, dx\right)\Bigg]
\end{aligned}$$

On fait un changement de variable à chaque terme d'integrale :

$$\begin{aligned}
||U(t)||_1 \leq \frac{1}{2}\Bigg[&\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx + \\
&+ e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx\right)\Bigg] + \\
+ \frac{1}{2}\Bigg[&\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx + \\
&+ e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
||U(t)||_1 \leq &\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx + \\
&+ e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx\right)
\end{aligned}$$

$$||U(t)||_1 \leq ||p_0||_{L^1} + ||v_0||_{L^1} + e^{-2\sigma t}\left(||p_0||_{L^1} + ||v_0||_{L^1}\right)$$

$$||U(t)||_1 \leq ||U_0||_1 + e^{-2\sigma t}||U_0||_1$$

$$||U(t)||_1 \leq \left(1 + e^{-2\sigma t}\right)||U_0||_1 \tag{5}$$

$$||e^{At}||_{\mathcal{L}(Y_p)} = \sup_{||U_0||_1=1} \frac{||U(t)||_1}{||U_0||_1}$$

$$||e^{At}||_{\mathcal{L}(Y_p)} \leq 1 + e^{-2\sigma t}$$

c) $Y_2 = L^2(\mathbb{R})^2$

$$||(a,b)||_2 \quad = \quad \sqrt{||a||^2_{L^2(\mathbb{R})} + ||b||^2_{L^2(\mathbb{R})}}$$

Montrons que : $||e^{At}||_{\mathcal{L}(Y_2)} \leq max(1, e^{-2\sigma t})$ pour tout $t \geq 0$.

On a $U(x,t) = (p(x,t), v(x,t))$

$$\begin{aligned}
||U(t)||_2^2 &= ||p(t)||^2_{L^2(\mathbb{R})} + ||v(t)||^2_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} |p(x,t)|^2\,dx + \int_{\mathbb{R}} |v(x,t)|^2\,dx \\
&= \frac{1}{4}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) + (p_0(x+t) - v_0(x+t))e^{-2\sigma t}|^2\,dx \\
&\quad + \frac{1}{4}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) - (p_0(x+t) - v_0(x+t))e^{-2\sigma t}|^2\,dx
\end{aligned}$$

$$\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{4}\Big[&\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2\,dx \\
&+ \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 e^{-4\sigma t}\,dx \\
&+2\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)||p_0(x+t) - v_0(x+t)|e^{-2\sigma t}\,dx\Big] \\
&+ \frac{1}{4}\Big[\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2\,dx \\
&+ \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 e^{-4\sigma t}\,dx \\
&-2\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)||p_0(x+t) - v_0(x+t)|e^{-2\sigma t}\,dx\Big]
\end{aligned}$$

$$\begin{aligned}
||U(t)||_2^2 &\leq \frac{1}{2}\Big[\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2\,dx \\
&\quad + \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 e^{-4\sigma t}\,dx\Big] \\
&\leq \frac{1}{2}\Big[max(1, e^{-4\sigma t})\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2\,dx \\
&\quad + max(1, e^{-4\sigma t})\int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2\,dx\Big] \\
&\leq \frac{1}{2}max(1, e^{-4\sigma t})\left[\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2\,dx + \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2\,dx\right]
\end{aligned}$$

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{2} max(1, e^{-4\sigma t}) \Big[\int_{\mathbb{R}} |p_0(x-t)|^2 \, dx + \int_{\mathbb{R}} |v_0(x-t)|^2 \, dx \\
+2 \int_{\mathbb{R}} |p_0(x-t)||v_0(x-t)| \, dx \\
+ \int_{\mathbb{R}} |p_0(x+t)|^2 \, dx + \int_{\mathbb{R}} |v_0(x+t)|^2 \, dx \\
-2 \int_{\mathbb{R}} |p_0(x+t)||v_0(x+t)| \, dx \Big]
\end{aligned}
$$

On fait un changement de variable à chaque terme d'integrale :

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{2} max(1, e^{-4\sigma t}) \Big[\int_{\mathbb{R}} |p_0(x)|^2 \, dx + \int_{\mathbb{R}} |v_0(x)|^2 \, dx \\
+2 \int_{\mathbb{R}} |p_0(x)||v_0(x)| \, dx \\
+ \int_{\mathbb{R}} |p_0(x)|^2 \, dx + \int_{\mathbb{R}} |v_0(x)|^2 \, dx \\
-2 \int_{\mathbb{R}} |p_0(x)||v_0(x)| \, dx \Big]
\end{aligned}
$$

$$
||U(t)||_2^2 \leq max(1, e^{-4\sigma t}) \left[\int_{\mathbb{R}} |p_0(x)|^2 \, dx + \int_{\mathbb{R}} |v_0(x)|^2 \, dx \right]
$$

$$
||U(t)||_2^2 \leq max(1, e^{-4\sigma t}) ||U_0||_2^2
$$

$$
||U(t)||_2 \leq max(1, e^{-2\sigma t}) ||U_0||_2
$$

$$
||e^{At}||_{\mathcal{L}(Y_2)} = \sup_{||U_0||_2 = 1} \frac{||U(t)||_2}{||U_0||_2}
$$

$$
||e^{At}||_{\mathcal{L}(Y_2)} \leq max(1, e^{-2\sigma t})
$$

2 Numerical methods

$$
\left\{
\begin{array}{rcll}
\partial_t u - \partial_{xx} u & = & 0, & x \in \mathbb{R}, \quad t > 0, \\
u(0,x) & = & u_0(x), & x \in \mathbb{R}
\end{array}
\right.
\tag{6}
$$

La discrétisation de type Différences Finis explicite avec un schéma sur la forme :

$$
\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{4}{3} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{1}{12} \frac{u_{j+2}^n - 2u_j^n + u_{j-2}^n}{\Delta x^2} - \frac{\Delta t^2}{2} \frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n}{\Delta x^4} = 0
\tag{7}
$$

a) Determination du symbol du schéma

Le symbole du schéma (**??**) est donné par :

$$\lambda(\theta) = \sum_{r=-2}^{2} \alpha_r e^{\mathbf{i}\theta r}, \quad \theta \in \mathbb{R}$$

Par développement de ce schéma on a :

$$u_j^{n+1} \quad = \quad \alpha_0 u_j^n + \alpha_{-1} u_{j-1}^n + \alpha_{-2} u_{j-2}^n + \alpha_1 u_{j+1}^n + \alpha_2 u_{j+2}^n, \quad \textit{où}$$

$$\left\{ \begin{array}{rcl} \alpha_0 & = & 1 - \frac{15}{6}\nu + 3\nu^2 \\ \alpha_{-1} & = & \frac{4}{3}\nu - 2\nu^2 \\ \alpha_{-2} & = & \frac{-1}{12}\nu + \frac{1}{2}\nu^2 \\ \alpha_1 & = & \frac{4}{3}\nu - 2\nu^2 \\ \alpha_2 & = & \frac{-1}{12}\nu + \frac{1}{2}\nu^2 \end{array} \right. ,$$

$$avec \quad \begin{array}{rcl} \nu & = & \Delta t/\Delta x^2 \\ \nu^2 & = & \Delta t^2/\Delta x^4 \end{array}$$

3 Consistence du schéma

On définit l'erreur de troncature par :

$$r_j^t = u(x_j, t^{n+1}) - \alpha_0 u(x_j, t^n) - \alpha_{-1} u(x_{j-1}, t^n) - \alpha_{-2} u(x_{j-2}, t^n) - \alpha_{+1} u(x_{j+1}, t^n) - \alpha_{+2} u(x_{j+2}, t^n)$$

Par développement de Taylor du schéma (**??**) on a :

$$u(x_{j-1}, t^n) = u(x_j, t^n) - \Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(\Delta x)^3}{3!} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j+1}, t^n) = u(x_j, t^n) - \Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(\Delta x)^3}{3!} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j-2}, t^n) = u(x_j, t^n) - 2\Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(2\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(2\Delta x)^3}{6} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j+2}, t^n) = u(x_j, t^n) - 2\Delta x \frac{\partial}{\partial x} u(x_j, t^n) + \frac{(2\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j, t^n) - \frac{(2\Delta x)^3}{6} \frac{\partial^3}{\partial x^3} u(x_j, t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_j, t^{n+1}) = u(x_j, t^n) + \Delta t \frac{\partial}{\partial t} u(x_j, t^n) + \mathcal{O}((\Delta t)^2)$$

On obtient :

$$
\begin{aligned}
&\frac{u(x_j,t^{n+1})-u(x_j,t^n)}{\Delta t}-\\
&-\frac{4}{3}\frac{u(x_{j+1},t^n)-2u(x_j,t^n)+u(x_{j-1},t^n)}{\Delta x^2}+\\
&+\frac{1}{12}\frac{u(x_{j+2},t^n)-2u(x_j,t^n)+u(x_{j-2},t^n)}{\Delta x^2}-\\
&-\frac{\Delta t^2}{2}\frac{u(x_{j+2},t^n)-4u(x_{j+1},t^n)+6u(x_j,t^n)-4u(x_{j-1},t^n)+u(x_{j-2},t^n)}{\Delta x^4}=\\
&=\mathcal{O}((\Delta t)^2+(\Delta x)^4)
\end{aligned}
$$

Alors le schéma consistant et d'ordre 2 en temps et d'ordre 4 en espace.

a) Stabilité

La stabilité au sens de Von Neumann est une condition suffisante et nécessaire pour la stabilité uniforme en norme quadratique

On obtient le résultat de convergence ? ? ?

b)