Première partie

Semi-group estimates

Soit $\sigma \in \mathbb{R}$ un coefficient donné. Consider the linear wave system in dimension one Consideron le système d'onde linéaire en

$$\partial_t p + \partial_x v = \sigma(v - p) \tag{A}$$
$$\partial_t v + \partial_x p = \sigma(p - v) \tag{B}$$

a) Determine une répresentation explicite de $U(x,t)$ en fonction des données initiales.

(A)+(B)

$$\begin{aligned}
\partial_t(p+v) + \partial_x(p+v) &= \sigma(v-p) + \sigma(p-v) \\
&= \sigma(v-p+p-v) \\
&= \sigma(0) \\
\partial_t(p+v) + \partial_x(p+v) &= 0
\end{aligned}$$

(A)-(B)

$$\begin{aligned}
\partial_t(p-v) - \partial_x(p-v) &= \sigma(v-p) - \sigma(p-v) \\
&= \sigma(v-p-p+v) \\
&= \sigma(2v-2p) \\
&= 2\sigma(v-p) \\
\partial_t(p-v) - \partial_x(p-v) &= -2\sigma(p-v)
\end{aligned}$$

On a obtenu deux équations de transport :

$$\partial_t(p+v) + \partial_x(p+v) = 0 \tag{1}$$
$$\partial_t(p-v) - \partial_x(p-v) = -2\sigma(p-v) \tag{2}$$

Résolution de (1) par la méthode des caractéristiques.

$$\begin{cases} \frac{\mathrm{d}x^*(t)}{\mathrm{d}t} &= 1 \\ x^*(t_*) &= x_* \end{cases}$$

$$\Rightarrow \begin{aligned}
x^*(t) &= t + c \\
x^*(t_*) &= t_* + c = x_* \\
c &= x_* - t_* \\
x^*(t) &= t + x_* - t_*
\end{aligned}$$

$$\begin{aligned}
(p+v)(x_*, t_*) &= (p_0 + v_0)(x_*, t_*) \\
(p+v)(x,t) &= p_0(x,t) + v_0(x,t)
\end{aligned}$$

Résolution de (2) par la méthode des caractéristiques.

$$\left\{\begin{array}{rcl} \frac{\mathrm{d}x_1^*(t)}{\mathrm{d}t} & = & -1 \\ x_1^*(t_*) & = & x_* \end{array}\right. \Rightarrow$$

$$\begin{array}{rcl} x_1^*(t) & = & -t + c_1 \\ x_1^*(t_*) & = & x_* = -t_* + c_1 \\ c_1 & = & x_* + t_* \\ x^*(t) & = & -t + x_* + t_* \end{array}$$

$$\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}(p-v)(x_1^*(t),t) & = & -2\sigma(p-v)(x_*(t),t) \\ (p-v)(x_1^*(t),t) & = & K\exp^{-2\sigma t} \\ K & = & K\exp^{-2\sigma t} \\ & = & (p-v)(x_1^*(0),0) \\ & = & (p_0-v_0)(x_1^*+t_*) \end{array}$$

$$\begin{array}{rcl} (p-v)(x_1^*(t),t) & = & (p_0-v_0)(x_*+t_*)\exp^{-2\sigma t_*} \\ (p-v)(x,t) & = & (p_0-v_0)(x,t)\exp^{-2\sigma t} \end{array}$$

Solutions de 1 et 2

$$\left\{\begin{array}{rcl} (p+v) & = & (x-t) + u_0(x-t) \\ (p-v) & = & (x+t)\exp^{-2\sigma t} - u_0(x+t)\exp^{-2\sigma t} \end{array}\right.$$

$$\begin{array}{rcl} u(x,t) & = & (p(x,t), v(x,t)) \qquad \forall t \geq 0 \\ \textit{avec} & & \\ p(x,t) & = & 1/2\left[p_0(x-t) + u_0(x-t) + (p_0(x+t) - u_0(x+t))\exp^{-2\sigma t}\right] \\ v(x,t) & = & 1/2\left[p_0(x-t) + u_0(x-t) - (p_0(x+t) - u_0(x+t))\exp^{-2\sigma t}\right] \end{array} \tag{3}$$

L'operator A, tel que $u = \exp^{tA} u_0$

$$\begin{array}{rcl} u(x,t) & = & (p(x,t), v(x,t)) \\ \partial_t u(x,t) & = & (\partial_t p(x,t), \partial_t v(x,t)) \\ \partial_x u(x,t) & = & (\partial_x p(x,t), \partial_x v(x,t)) \end{array}$$

$$\partial_x \begin{pmatrix} p \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x p \\ \partial_x v \end{pmatrix}$$

$$\partial_x \begin{pmatrix} p \\ v \end{pmatrix} = A_0 \partial_x u, \quad \textit{avec}$$

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \sigma(v-p) \\ \sigma(p-v) \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}$$

$$\begin{pmatrix}\sigma(v-p)\\ \sigma(p-v)\end{pmatrix} = Bu, \quad avec$$

$$B = \begin{pmatrix}-\sigma & \sigma\\ \sigma & -\sigma\end{pmatrix}$$

Par $\partial_t U = AU$ et $A = -A_1\partial_x + B$ on a :

$$\begin{cases} \partial_t U &= AU\\ U_0 &= (p_0, v_0).\end{cases}$$

$$\Rightarrow U(t) = e^{At}U_0$$

b) $Y_1 = L^1(\mathbb{R})^2$

$$||(a,b)||_1 = ||a||_{L^1(\mathbb{R})} + ||b||_{L^1(\mathbb{R})}$$

Montrons que : $||e^{At}||_{\mathcal{L}(Y_p)} \leq (1 + e^{-2\sigma t})$ pour tout $t \geq 0$.

On a $U(x,t) = (p(x,t), v(x,t))$

$$\begin{aligned}||U(t)||_1 &= ||p(t)||_{L^1(\mathbb{R})} + ||v(t)||_{L^1(\mathbb{R})}\\ &= \int_{\mathbb{R}} |p(x,t)|\,dx + \int_{\mathbb{R}} |v(x,t)|\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) + [p_0(x+t) - v_0(x+t)]e^{-2\sigma t}|\,dx\\ &+ \frac{1}{2}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) - [p_0(x+t) - v_0(x+t)]e^{-2\sigma t}|\,dx\end{aligned}$$

$$\begin{aligned}||U(t)||_1 \leq \frac{1}{2}\Big[\int_{\mathbb{R}} |p_0(x-t)|\,dx + \int_{\mathbb{R}} |v_0(x-t)|\,dx +\\ + e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x+t)|\,dx + \int_{\mathbb{R}} |v_0(x+t)|\,dx\right)\Big] +\\ + \frac{1}{2}\Big[\int_{\mathbb{R}} |p_0(x-t)|\,dx + \int_{\mathbb{R}} |v_0(x-t)|\,dx +\\ + e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x+t)|\,dx + \int_{\mathbb{R}} |v_0(x+t)|\,dx\right)\Big]\end{aligned}$$

On fait un changement de variable à chaque terme d'integrale :

$$\begin{aligned}||U(t)||_1 \leq \frac{1}{2}\Big[\int_{\mathbb{R}} |p_0(x)|\,dx + \int_{\mathbb{R}} |v_0(x)|\,dx +\\ + e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\,dx + \int_{\mathbb{R}} |v_0(x)|\,dx\right)\Big] +\\ + \frac{1}{2}\Big[\int_{\mathbb{R}} |p_0(x)|\,dx + \int_{\mathbb{R}} |v_0(x)|\,dx +\\ + e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\,dx + \int_{\mathbb{R}} |v_0(x)|\,dx\right)\Big]\end{aligned}$$

$$
\begin{aligned}
||U(t)||_1 \leq \int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx + \\
+ e^{-2\sigma t}\left(\int_{\mathbb{R}} |p_0(x)|\, dx + \int_{\mathbb{R}} |v_0(x)|\, dx\right)
\end{aligned}
$$

$$
||U(t)||_1 \leq ||p_0||_{L^1} + ||v_0||_{L^1} + e^{-2\sigma t}\left(||p_0||_{L^1} + ||v_0||_{L^1}\right)
$$

$$
||U(t)||_1 \leq ||U_0||_1 + e^{-2\sigma t}||U_0||_1
$$

$$
||U(t)||_1 \leq \left(1 + e^{-2\sigma t}\right)||U_0||_1 \tag{4}
$$

$$
||e^{At}||_{\mathcal{L}(Y_p)} = \sup_{||U_0||_1=1} \frac{||U(t)||_1}{||U_0||_1}
$$

$$
||e^{At}||_{\mathcal{L}(Y_p)} \leq 1 + e^{-2\sigma t}
$$

c) $Y_2 = L^2(\mathbb{R})^2$

$$
||(a,b)||_2 \quad = \quad \sqrt{||a||^2_{L^2(\mathbb{R})} + ||b||^2_{L^2(\mathbb{R})}}
$$

Montrons que : $||e^{At}||_{\mathcal{L}(Y_2)} \leq max(1, e^{-2\sigma t})$ pour tout $t \geq 0$.

On a $U(x,t) = (p(x,t), v(x,t))$

$$
\begin{aligned}
||U(t)||_2^2 &= ||p(t)||^2_{L^2(\mathbb{R})} + ||v(t)||^2_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} |p(x,t)|^2\, dx + \int_{\mathbb{R}} |v(x,t)|^2\, dx \\
&= \frac{1}{4}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) + (p_0(x+t) - v_0(x+t))e^{-2\sigma t}|^2\, dx \\
&\quad + \frac{1}{4}\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t) - (p_0(x+t) - v_0(x+t))e^{-2\sigma t}|^2\, dx
\end{aligned}
$$

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{4}\Bigg[&\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2\, dx \\
&+ \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 e^{-4\sigma t}\, dx \\
&+2\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)||p_0(x+t) - v_0(x+t)|e^{-2\sigma t}\, dx\Bigg] \\
&+ \frac{1}{4}\Bigg[\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2\, dx \\
&+ \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 e^{-4\sigma t}\, dx \\
&-2\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)||p_0(x+t) - v_0(x+t)|e^{-2\sigma t}\, dx\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{2} \Bigg[\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2 \, dx \\
+ \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 e^{-4\sigma t} \, dx \Bigg] \\
\leq \frac{1}{2} \Bigg[max(1, e^{-4\sigma t}) \int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2 \, dx \\
+ max(1, e^{-4\sigma t}) \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 \, dx \Bigg] \\
\leq \frac{1}{2} max(1, e^{-4\sigma t}) \Bigg[\int_{\mathbb{R}} |p_0(x-t) + v_0(x-t)|^2 \, dx + \int_{\mathbb{R}} |p_0(x+t) - v_0(x+t)|^2 \, dx \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{2} max(1, e^{-4\sigma t}) \Bigg[\int_{\mathbb{R}} |p_0(x-t)|^2 \, dx + \int_{\mathbb{R}} |v_0(x-t)|^2 \, dx \\
+2 \int_{\mathbb{R}} |p_0(x-t)||v_0(x-t)| \, dx \\
+ \int_{\mathbb{R}} |p_0(x+t)|^2 \, dx + \int_{\mathbb{R}} |v_0(x+t)|^2 \, dx \\
-2 \int_{\mathbb{R}} |p_0(x+t)||v_0(x+t)| \, dx \Bigg]
\end{aligned}
$$

On fait un changement de variable à chaque terme d'integrale :

$$
\begin{aligned}
||U(t)||_2^2 \leq \frac{1}{2} max(1, e^{-4\sigma t}) \Bigg[\int_{\mathbb{R}} |p_0(x)|^2 \, dx + \int_{\mathbb{R}} |v_0(x)|^2 \, dx \\
+2 \int_{\mathbb{R}} |p_0(x)||v_0(x)| \, dx \\
+ \int_{\mathbb{R}} |p_0(x)|^2 \, dx + \int_{\mathbb{R}} |v_0(x)|^2 \, dx \\
-2 \int_{\mathbb{R}} |p_0(x)||v_0(x)| \, dx \Bigg]
\end{aligned}
$$

$$
||U(t)||_2^2 \leq max(1, e^{-4\sigma t}) \left[\int_{\mathbb{R}} |p_0(x)|^2 \, dx + \int_{\mathbb{R}} |v_0(x)|^2 \, dx \right]
$$

$$
||U(t)||_2^2 \leq max(1, e^{-4\sigma t}) ||U_0||_2^2
$$

$$
||U(t)||_2 \leq max(1, e^{-2\sigma t}) ||U_0||_2
$$

$$
||e^{At}||_{\mathcal{L}(Y_2)} = \sup_{||U_0||_2 = 1} \frac{||U(t)||_2}{||U_0||_2}
$$

$$
||e^{At}||_{\mathcal{L}(Y_2)} \leq max(1, e^{-2\sigma t})
$$

d) $Y_\infty = L^\infty(\mathbb{R})^2$

$$\begin{aligned} ||(a,b)||_\infty &= max(||a||_{L^\infty(\mathbb{R})}, ||b||_{L^\infty(\mathbb{R})}) \\ \sigma = 0, & \quad et \quad t \geq 0 \end{aligned}$$

Montrons que $||e^{At}||_{\mathcal{L}(Y_\infty)} \leq 2$.

Par définition, nous avons :

$$||U(t)||_\infty = \max\Big(||p(t)||_{L^\infty(\mathbb{R})}, ||v(t)||_{L^\infty(\mathbb{R})}\Big)$$

avec

$$||p(t)||_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |p(x,t)| \quad \text{et} \quad ||v(t)||_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |v(x,t)|.$$

$$\begin{aligned} ||U(t)||_\infty = \max \Bigg(& \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x+t)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x+t)|\,, \\ & \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x-t)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x+t)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x+t)| \Bigg) \end{aligned}$$

Par un changement de variable, on a :

$$\begin{aligned} ||U(t)||_\infty = \max \Bigg(& \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x)|\,, \\ & \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |p_0(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |v_0(x)| \Bigg) \end{aligned}$$

$$||U(t)||_\infty = \max\left(\sup_{x\in\mathbb{R}} |p_0(x)| + \sup_{x\in\mathbb{R}} |v_0(x)|\,,\ \sup_{x\in\mathbb{R}} |p_0(x)| + \sup_{x\in\mathbb{R}} |v_0(x)|\right)$$

$$||U(t)||_\infty = \max\Big(||p_0||_{L^\infty(\mathbb{R})} + ||v_0||_{L^\infty(\mathbb{R})}\,,\ ||p_0||_{L^\infty(\mathbb{R})} + ||v_0||_{L^\infty(\mathbb{R})}\Big)$$

$$||p_0||_{L^\infty(\mathbb{R})} + ||v_0||_{L^\infty(\mathbb{R})} \leq 2||U_0||_\infty$$

$$||U(t)||_\infty \leq \max\left(2||U_0||_\infty\,,\ 2||U_0||_\infty\right)$$

$$||U(t)||_\infty \leq 2||U_0||_\infty$$

$$||e^{At}||_{\mathcal{L}(Y_\infty)} = \sup_{||U_0||_\infty = 1} \frac{||U(t)||_\infty}{||U_0||_\infty} \leq 2$$

$$||e^{At}||_{\mathcal{L}(Y_\infty)} \leq 2$$

Condition initial tel que $||e^{At}||_{\mathcal{L}(Y_\infty)} = 2$

Prenons :

$$\begin{cases} p_0 = 1 \\ v_0 = -1 \quad si \quad x > 0, \\ v_0 = 1 \quad\; si \quad x < 0, \\ v_0 = 0 \quad\; si \quad x = 0. \end{cases}$$

$$||e^{At}||_{\mathcal{L}(Y_\infty)} = 2$$

Deuxième partie

Numerical methods

$$\left\{ \begin{array}{rcll} \partial_t u - \partial_{xx} u & = & 0, & x \in \mathbb{R}, \quad t > 0, \\ u(0,x) & = & u_0(x), & x \in \mathbb{R} \end{array} \right. \tag{5}$$

La discrétisation de type Différences Finis explicite avec un schéma sur la forme :

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} - \frac{4}{3}\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2} + \frac{1}{12}\frac{u_{j+2}^n-2u_j^n+u_{j-2}^n}{\Delta x^2} - \frac{\Delta t^2}{2}\frac{u_{j+2}^n-4u_{j+1}^n+6u_j^n-4u_{j-1}^n+u_{j-2}^n}{\Delta x^4} = 0 \tag{6}$$

a) Determination du symbol du schéma

Le symbole du schéma (6) est donné par :

$$\lambda(\theta) = \sum_{r=-2}^{2} \alpha_r e^{\mathbf{i}\theta r}, \quad \theta \in \mathbb{R}$$

Par développement de ce schéma on a :

$$u_j^{n+1} = \alpha_0 u_j^n + \alpha_{-1} u_{j-1}^n + \alpha_{-2} u_{j-2}^n + \alpha_1 u_{j+1}^n + \alpha_2 u_{j+2}^n, \quad où$$

$$\left\{ \begin{array}{rcl} \alpha_0 & = & 1 - \frac{15}{6}\nu + 3\nu^2 \\ \alpha_{-1} & = & \frac{4}{3}\nu - 2\nu^2 \\ \alpha_{-2} & = & \frac{-1}{12}\nu + \frac{1}{2}\nu^2 \\ \alpha_1 & = & \frac{4}{3}\nu - 2\nu^2 \\ \alpha_2 & = & \frac{-1}{12}\nu + \frac{1}{2}\nu^2 \end{array} \right. ,$$

$$avec \quad \begin{array}{rcl} \nu & = & \Delta t/\Delta x^2 \\ \nu^2 & = & \Delta t^2/\Delta x^4 \end{array}$$

b) Consistence du schéma

On définit l'erreur de troncature par :

$$r_j^t = u(x_j, t^{n+1}) - \alpha_0 u(x_j, t^n) - \alpha_{-1} u(x_{j-1}, t^n) - \alpha_{-2} u(x_{j-2}, t^n) - \alpha_{+1} u(x_{j+1}, t^n) - \alpha_{+2} u(x_{j+2}, t^n)$$

Par développement de Taylor du schéma (6) on a :

$$u(x_{j-1},t^n) = u(x_j,t^n) - \Delta x \frac{\partial}{\partial x} u(x_j,t^n) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j,t^n) - \frac{(\Delta x)^3}{3!} \frac{\partial^3}{\partial x^3} u(x_j,t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j+1},t^n) = u(x_j,t^n) - \Delta x \frac{\partial}{\partial x} u(x_j,t^n) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j,t^n) - \frac{(\Delta x)^3}{3!} \frac{\partial^3}{\partial x^3} u(x_j,t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j-2},t^n) = u(x_j,t^n) - 2\Delta x \frac{\partial}{\partial x} u(x_j,t^n) + \frac{(2\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j,t^n) - \frac{(2\Delta x)^3}{6} \frac{\partial^3}{\partial x^3} u(x_j,t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_{j+2},t^n) = u(x_j,t^n) - 2\Delta x \frac{\partial}{\partial x} u(x_j,t^n) + \frac{(2\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} u(x_j,t^n) - \frac{(2\Delta x)^3}{6} \frac{\partial^3}{\partial x^3} u(x_j,t^n) + \mathcal{O}((\Delta x)^4)$$

$$u(x_j,t^{n+1}) = u(x_j,t^n) + \Delta t \frac{\partial}{\partial t} u(x_j,t^n) + \mathcal{O}((\Delta t)^2)$$

On obtient :

$$\begin{aligned}
&\frac{u(x_j,t^{n+1}) - u(x_j,t^n)}{\Delta t} - \\
&- \frac{4}{3} \frac{u(x_{j+1},t^n) - 2u(x_j,t^n) + u(x_{j-1},t^n)}{\Delta x^2} + \\
&+ \frac{1}{12} \frac{u(x_{j+2},t^n) - 2u(x_j,t^n) + u(x_{j-2},t^n)}{\Delta x^2} - \\
&- \frac{\Delta t^2}{2} \frac{u(x_{j+2},t^n) - 4u(x_{j+1},t^n) + 6u(x_j,t^n) - 4u(x_{j-1},t^n) + u(x_{j-2},t^n)}{\Delta x^4} = \\
&= \mathcal{O}((\Delta t)^2 + (\Delta x)^4)
\end{aligned}$$

Alors le schéma consistant et d'ordre 2 en temps et d'ordre 4 en espace.