

categorical_picture

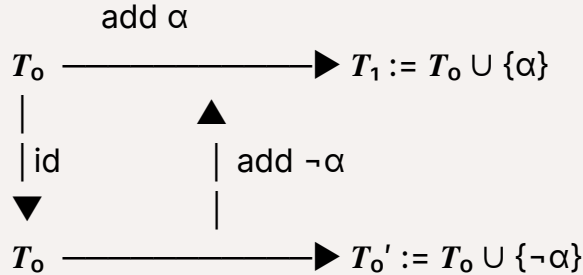
(1) Injection \rightarrow (2) Push-out/Synthesis \rightarrow (3) Semantic realisation

I first show the abstract categorical picture, then walk a concrete mini-example ("Birds fly / Penguins don't") all the way through.

1 Injection: turning $\alpha \wedge \neg\alpha$ into a morphism

Let

- T_0 – the base theory (a set of sentences closed under entailment)
- α – a new assertion
- $\neg\alpha$ – its negation

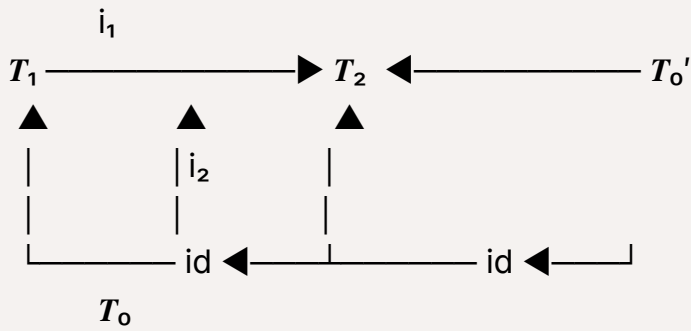


Both arrows are *injections* (monos) in \mathbf{Th} , the paraconsistent-theory category.

Together they realise the formal contradiction; nothing collapses because hom-sets are enriched over Belnap's 4-value lattice .

2 Push-out (synthesis): the universal "glue" object T_2

In \mathbf{Th} every span $T_1 \leftarrow T_0 \rightarrow T_0'$ has a colimit:



Concrete construction (your Prop. 4.2 idea).

Introduce a fresh symbol p that “re-threads” the conflict:

$$\begin{aligned}
 T_2 &:= T_0 \\
 &\cup \{ \alpha \rightarrow p, \\
 &\quad \neg \alpha \rightarrow \neg p, \\
 &\quad p \vee \neg p \} \quad (\text{minimal, adds just one atom})
 \end{aligned}$$

p is the **new emergent concept**; T_2 is initial among theories receiving both injections while preserving entailment relations.

Mathematically: for any theory S and maps $f_1: T_1 \rightarrow S$, $f_2: T_0' \rightarrow S$ that agree on T_0 , there exists a **unique** $h: T_2 \rightarrow S$ with $h \circ i_1 = f_1$ and $h \circ i_2 = f_2$.

3 Semantic realisation via a functor

$F: \mathbf{Th} \rightarrow \mathbf{Sem}$

Choose **Sem** to be a suitable model category (sets, Kripke frames, graphs...).

Below, I use **Set** of *two-world Kripke models* for clarity.

$$\begin{array}{ccc}
 F(T_0) & \leftarrow & F(T_1) \\
 \searrow & & \searrow \\
 & \text{colim} \longrightarrow & F(T_2) \quad (\text{preserves the push-out}) \\
 \nearrow & & \nearrow \\
 F(T_0') & \leftarrow &
 \end{array}$$

Minimal two-world model that realises T_2

World	p	α	$\neg\alpha$
w_1	T	T	F
w_2	F	F	T

- The link $p \vee \neg p$ ensures every world lands in exactly one context.
- No explosion: both worlds coexist; $\alpha \wedge \neg\alpha$ holds globally in four-valued sense, yet each world is classically consistent.

Walk-through with a tangible KB fragment

| Thesis:

| Antithesis:

① Injection

T_o = taxonomy facts ($\text{penguin}(x) \rightarrow \text{bird}(x)$).

Add α on one branch, $\neg\alpha$ on the other.

② Push-out

Introduce $\text{context}(\text{penguin}) \equiv p$.

Add rules

$\alpha \rightarrow p$	(flying rule applies only in p-world)
$\neg\alpha \rightarrow \neg p$	(non-flying rule in $\neg p$ -world)
$p \vee \neg p$	(each individual tagged)

Result: the KB now **contains a fresh attribute** "penguin-context" that reconciles both flight rules.

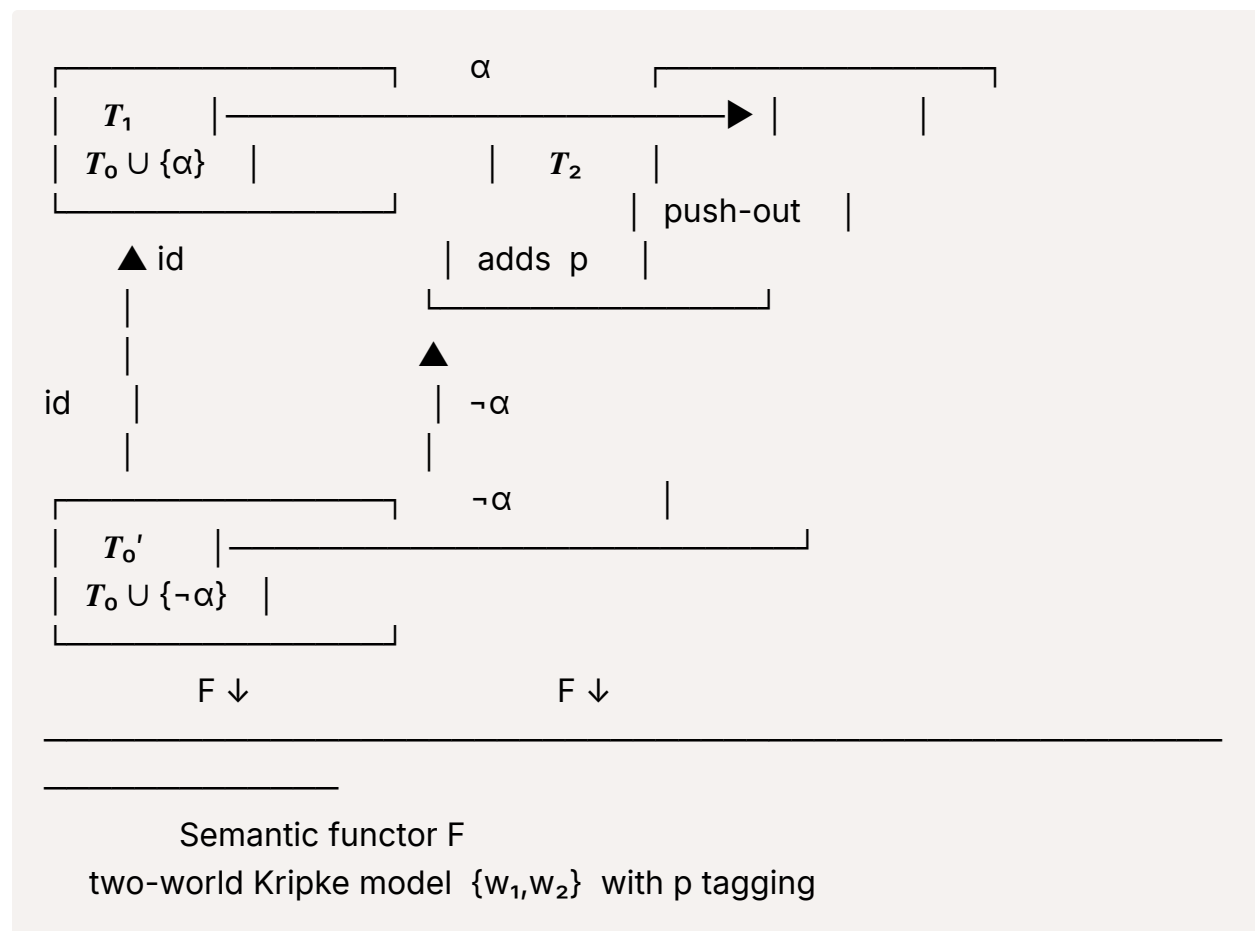
③ Semantic image

A data store instantiates two labelled sub-graphs or two rows in a truth-table.
Query engines can now answer:

- "Do birds fly (generically)?" – Yes in p world.
- "Do penguins fly?" – No in $\neg p$ world.

Explainability: system says *"Conflict resolved by creating a context dimension 'penguin'."*

Visual summary (one slide-style)



Key play-out: contradiction **injects**, system **pushes-out** to minimal extension, functor **realises** it as a concrete, queryable structure.

That's the life-cycle your framework formalises—and why we can mirror it in option portfolios, KB repair, or neural concept-splitting.