2 MODELING

2.1 Background

2.1.1 Model Convention

The rotary inverted pendulum model is shown in Figure 2.1. The rotary arm pivot is attached to the SRV02 system and is actuated. The arm has a length of L_r, a moment of inertia of J_r, and its angle, θ, increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive, i.e., $V_m > 0$.

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is $\frac{L_p}{2}$. The moment of inertia about its center of mass is J_p. The inverted pendulum angle, α, is zero when it is perfectly upright in the vertical position and increases positively when rotated CCW.

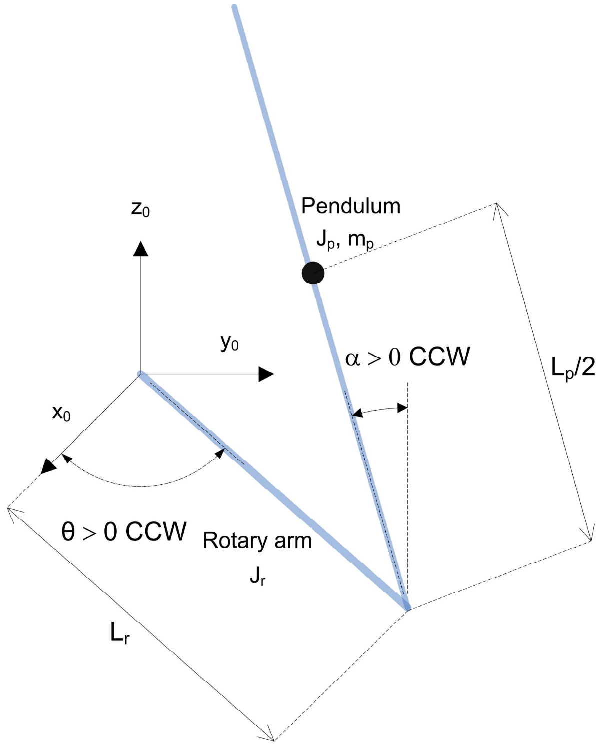


Figure 2.1: Rotary inverted pendulum conventions

2.1.2 Nonlinear Equations of Motion

Instead of using classical mechanics, the Lagrange method is used to find the equations of motion of the system. This systematic method is often used for more complicated systems such as robot manipulators with multiple joints.

More specifically, the equations that describe the motions of the rotary arm and the pendulum with respect to the servo motor voltage, i.e. the dynamics, will be obtained using the Euler-Lagrange equation:

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

The variables q_i are called *generalized coordinates*. For this system let

$$q(t)^\top = [\theta(t)\ \alpha(t)] \tag{2.1}$$

where, as shown in Figure 2.1, $\theta(t)$ is the rotary arm angle and $\alpha(t)$ is the inverted pendulum angle. The corresponding velocities are

$$\dot{q}(t)^\top = \left[\frac{\partial\theta(t)}{\partial t}\ \frac{\partial\alpha(t)}{\partial t}\right]$$

Note: The dot convention for the time derivative will be used throughout this document, i.e., $\dot{\theta} = \frac{d\theta}{dt}$. The time variable t will also be dropped from θ and α, i.e., $\theta = \theta(t)$ and $\alpha = \alpha(t)$.

With the generalized coordinates defined, the Euler-Lagrange equations for the rotary pendulum system are

$$\begin{aligned}\frac{\partial^2 L}{\partial t \partial\dot{\theta}} - \frac{\partial L}{\partial\theta} &= Q_1\\ \frac{\partial^2 L}{\partial t \partial\dot{\alpha}} - \frac{\partial L}{\partial\alpha} &= Q_2\end{aligned}$$

The Lagrangian of the system is described

$$L = T - V$$

where T is the total kinetic energy of the system and V is the total potential energy of the system. Thus the Lagrangian is the difference between a system's kinetic and potential energies.

The generalized forces Q_i are used to describe the non-conservative forces (e.g., friction) applied to a system with respect to the generalized coordinates. In this case, the generalized force acting on the rotary arm is

$$Q_1 = \tau - B_r\dot{\theta}$$

and acting on the pendulum is

$$Q_2 = -B_p\dot{\alpha}.$$

See [4] for a description of the corresponding SRV02 parameters (e.g. such as the back-emf constant, k_m). Our control variable is the input servo motor voltage, V_m. Opposing the applied torque is the viscous friction torque, or viscous damping, corresponding to the term B_r. Since the pendulum is not actuated, the only force acting on the link is the damping. The viscous damping coefficient of the pendulum is denoted by B_p.

The Euler-Lagrange equations is a systematic method of finding the equations of motion, i.e., EOMs, of a system. Once the kinetic and potential energy are obtained and the Lagrangian is found, then the task is to compute various derivatives to get the EOMs. After going through this process, the nonlinear equations of motion for the SRV02 rotary inverted pendulum are:

$$\begin{aligned}\left(m_pL_r^2 + \frac{1}{4}m_pL_p^2 - \frac{1}{4}m_pL_p^2\cos(\alpha)^2 + J_r\right)\ddot{\theta} - \left(\frac{1}{2}m_pL_pL_r\cos(\alpha)\right)\ddot{\alpha}&\\ +\left(\frac{1}{2}m_pL_p^2\sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2}m_pL_pL_r\sin(\alpha)\right)\dot{\alpha}^2 &= \tau - B_r\dot{\theta}\end{aligned} \tag{2.2}$$

$$\begin{aligned}-\frac{1}{2}m_pL_pL_r\cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} - \frac{1}{4}m_pL_p^2\cos(\alpha)\sin(\alpha)\dot{\theta}^2&\\ -\frac{1}{2}m_pL_pg\sin(\alpha) &= -B_p\dot{\alpha}.\end{aligned} \tag{2.3}$$

The torque applied at the base of the rotary arm (i.e., at the load gear) is generated by the servo motor as described by the equation

$$\tau = \frac{\eta_g K_g \eta_m k_t (V_m - K_g k_m \dot{\theta})}{R_m}. \tag{2.4}$$

See [4] for a description of the corresponding SRV02 parameters (e.g. such as the back-emf constant, k_m).

Both the equations match the typical form of an EOM for a single body:

$$J\ddot{x} + b\dot{x} + g(x) = \tau_1$$

where x is an angular position, J is the moment of inertia, b is the damping, $g(x)$ is the gravitational function, and τ_1 is the applied torque (scalar value).

For a generalized coordinate vector q, this can be generalized into the matrix form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{2.5}$$

where D is the inertial matrix, C is the damping matrix, $g(q)$ is the gravitational vector, and τ is the applied torque vector.

The nonlinear equations of motion given in 2.2 and 2.3 can be placed into this matrix format.

2.1.3 Linearizing

Here is an example of how to linearize a two-variable nonlinear function called $f(z)$. Variable z is defined

$$z^\top = [z_1 \; z_2]$$

and $f(z)$ is to be linearized about the operating point

$$z_0{}^\top = [a \; b]$$

The linearized function is

$$f_{lin} = f(z_0) + \left(\frac{\partial f(z)}{\partial z_1}\right)\bigg|_{z=z_0}(z_1 - a) + \left(\frac{\partial f(z)}{\partial z_2}\right)\bigg|_{z=z_0}(z_2 - b)$$

2.1.4 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \tag{2.6}$$

and

$$y = Cx + Du \tag{2.7}$$

where x is the state, u is the control input, A, B, C, and D are state-space matrices. For the rotary pendulum system, the state and output are defined

$$x^\top = [\theta \; \alpha \; \dot{\theta} \; \dot{\alpha}] \tag{2.8}$$

and

$$y^\top = [x_1 \; x_2]. \tag{2.9}$$

In the output equation, only the position of the servo and link angles are being measured. Based on this, the C and D matrices in the output equation are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.10}$$

and

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.11}$$

The velocities of the servo and pendulum angles can be computed in the digital controller, e.g., by taking the derivative and filtering the result though a high-pass filter.

2.2 Pre-Lab Questions

1. A-2 Linearize the first nonlinear inverted rotary pendulum equation, Equation 2.2. The initial conditions for all the variables are zero, i.e., $\theta_0 = 0$, $\alpha_0 = 0$, $\dot{\theta}_0 = 0$, $\dot{\alpha}_0 = 0$.

Answer 2.1

Outcome	**Solution**
A-2	Let variable z be

$$z^{\top} = [\theta, \alpha, \dot{\theta}, \dot{\alpha}, \ddot{\theta}, \ddot{\alpha}]$$

where $z_0{}^{\top} = [0\ 0\ 0\ 0\ 0\ 0]$. The left-hand side of Equation 2.2 is already linear. Set the right-hand side to function

$$f(z) = \left(m_p L_r^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 \cos(\alpha)^2 + J_r\right)\ddot{\theta} - \left(\frac{1}{2} m_p L_p L_r \cos(\alpha)\right)\ddot{\alpha} + \left(\frac{1}{2} m_p L_p^2 \sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2} m_p L_p L_r \sin(\alpha)\right)\dot{\alpha}^2.$$

From Section 2.1.3, the linearized function is in the form

$$f_{\text{lin}}(z) = f(z_0) + \left(\frac{\partial f(z)}{\partial \ddot{\theta}}\right)\Big|_{z=z_0}\ddot{\theta} + \left(\frac{\partial f(z)}{\partial \ddot{\alpha}}\right)\Big|_{z=z_0}\ddot{\alpha} + \left(\frac{\partial f(z)}{\partial \dot{\theta}}\right)\Big|_{z=z_0}\dot{\theta} + \left(\frac{\partial f(z)}{\partial \dot{\alpha}}\right)\Big|_{z=z_0}\dot{\alpha} + \left(\frac{\partial f(z)}{\partial \theta}\right)\Big|_{z=z_0}\theta + \left(\frac{\partial f(z)}{\partial \alpha}\right)\Big|_{z=z_0}\alpha \quad \text{(Ans.2.1)}$$

Linearizing $f(z)$ with respect to $\ddot{\theta}$ gives

$$\left(\frac{\partial f(z)}{\partial \ddot{\theta}}\right)\Big|_{z=z_0} = m_p L_r^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 \cos(0) + J_r = m_p L_r^2 + J_r$$

When linearizing $f(z)$ with respect to $\ddot{\alpha}$, we get

$$\left(\frac{\partial f(z)}{\partial \ddot{\alpha}}\right)\Big|_{z=z_0} = -\frac{1}{2} m_p L_p L_r \cos(0) = -\frac{1}{2} m_p L_p L_r$$

All the other terms are as follows:

$$\left(\frac{\partial f(z)}{\partial \dot{\theta}}\right)\Big|_{z=z_0} = 0, \left(\frac{\partial f(z)}{\partial \dot{\alpha}}\right)\Big|_{z=z_0} = 0,$$
$$\left(\frac{\partial f(z)}{\partial \theta}\right)\Big|_{z=z_0} = 0, \left(\frac{\partial f(z)}{\partial \alpha}\right)\Big|_{z=z_0} = 0, \text{ and } f(z_0) = 0$$

Evaluating Equation Ans.2.1 we obtain

$$f_{\text{lin}}(z) = \left(m_p L_r^2 + J_r\right)\ddot{\theta} - \frac{1}{2} m_p L_p L_r \ddot{\alpha}. \quad \text{(Ans.2.2)}$$

Incorportating this back into the original equation, we get the following linear equation of motion

$$\left(m_p L_r^2 + J_r\right)\ddot{\theta} - \frac{1}{2} m_p L_p L_r \ddot{\alpha} = \tau - B_r\dot{\theta}. \quad \text{(Ans.2.3)}$$

□□□

2. A-2 Linearize the second nonlinear inverted rotary pendulum equation, Equation 2.3, with initial conditions $\theta_0 = 0$, $\alpha_0 = 0$, $\dot{\theta}_0 = 0$, $\dot{\alpha}_0 = 0$.

Answer 2.2

Outcome	Solution
A-2	The same principles used for linearizing the first nonlinear EOM can be used for this. The left-hand side of Equation 2.3 is

$$
\begin{aligned}
f(z) &= -\frac{1}{2}m_pL_pL_r\cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} \\
&\quad -\frac{1}{4}m_pL_p^2\cos(\alpha)\sin(\alpha)\dot{\theta}^2 - \frac{1}{2}m_pL_pg\sin(\alpha)
\end{aligned}
$$

The linearization given in Equation Ans.2.1 is used for this equation. The solution to the $\ddot{\theta}$, $\ddot{\alpha}$, and α based derivatives are:

$$
\left.\left(\frac{\partial f(z)}{\partial \ddot{\theta}}\right)\right|_{z=z_0} = -\frac{1}{2}m_pL_pL_r,
$$

$$
\left.\left(\frac{\partial f(z)}{\partial \ddot{\alpha}}\right)\right|_{z=z_0} = J_p + \frac{1}{4}m_pL_p^2, \text{ and}
$$

$$
\left.\left(\frac{\partial f(z)}{\partial \alpha}\right)\right|_{z=z_0} = -\frac{1}{2}m_pL_pg.
$$

The other $\dot{\theta}$, $\dot{\alpha}$, and θ based derivetatives are zero and $f(z_0) = 0$. Evaluating the $f_{lin}(z)$ function, we obtain

$$
f_{lin}(z) = -\frac{1}{2}m_pL_pL_r\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} - \frac{1}{2}m_pL_pg\alpha,
$$

which is the linearized left-hand side of Equation 2.3. The second linear EOM is therefore

$$
-\frac{1}{2}m_pL_pL_r\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} - \frac{1}{2}m_pL_pg\alpha = -B_p\dot{\alpha}. \quad \text{(Ans.2.4)}
$$

□□□

3. A-2 Fit the two linear equations of motion found in the above exercises into the matrix form shown in Equation 2.5. Make sure the equation is in terms of θ and α (and its derivatives).

Answer 2.3

Outcome	Solution
A-2	For a two variable q, the matrix in in the form

$$
\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\ddot{q} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\dot{q} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}
$$

Given the generalized coordinate definion in Equation 2.1 and the linear equations Equation Ans.2.3 and Equation Ans.2.4, the matrix becomes

$$
\begin{bmatrix} m_pL_r^2 + J_r & -\frac{1}{2}m_pL_pL_r \\ -\frac{1}{2}m_pL_pL_r & J_p + \frac{1}{4}m_pL_p^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} B_r & 0 \\ 0 & B_p \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2}m_pL_pg\alpha \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}
$$
(Ans.2.5)

□□□

4. A-2 Solve for the acceleration terms in the equations of motion. You can either solve this using the two linear equations or using the matrix form. If you're doing it in the matrix form, recall that the inverse of a 2x2 matrix is

$$
A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad (2.12)
$$

where $\det(A) = ad - bc$.

In any case, you'll have two equations of the form: $\ddot{\theta} = g_1(\theta, \alpha, \dot{\theta}, \dot{\alpha})$ and $\ddot{\alpha} = g_2(\theta, \alpha, \dot{\theta}, \dot{\alpha})$. Make sure you collect the terms with respect to the θ, α, $\dot{\theta}$, and $\dot{\alpha}$ variables.

Answer 2.4

Outcome	Solution
A-2	Reorganize Equation Ans.2.5 to get

$$\begin{bmatrix} m_p L_r^2 + J_r & -\frac{1}{2} m_p L_p L_r \\ -\frac{1}{2} m_p L_p L_r & J_p + \frac{1}{4} m_p L_p^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} \tau - B_r \dot{\theta} \\ \frac{1}{2} m_p L_p g \alpha - B_p \dot{\alpha} \end{bmatrix}.$$

Using the hint from above, the inverse of the matrix is

$$\begin{bmatrix} m_p L_r^2 + J_r & -\frac{1}{2} m_p L_p L_r \\ -\frac{1}{2} m_p L_p L_r & J_p + \frac{1}{4} m_p L_p^2 \end{bmatrix}^{-1} = \frac{1}{J_T} \begin{bmatrix} J_p + \frac{1}{4} m_p L_p^2 & \frac{1}{2} m_p L_p L_r \\ \frac{1}{2} m_p L_p L_r & m_p L_r^2 + J_r \end{bmatrix}$$

where the determinant of the matrix equals

$$\begin{aligned} J_T &= (m_p L_r^2 + J_r)(J_p + \frac{1}{4} m_p L_p^2) - \frac{1}{4} m_p^2 L_p^2 L_r^2 \\ &= J_p m_p L_r^2 + J_r J_p + \frac{1}{4} J_r m_p L_p^2. \end{aligned}$$

Solving for the acceleration terms

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \frac{1}{J_T} \begin{bmatrix} J_p + \frac{1}{4} m_p L_p^2 & \frac{1}{2} m_p L_p L_r \\ \frac{1}{2} m_p L_p L_r & J_r + m_p L_r^2 \end{bmatrix} \begin{bmatrix} \tau - B_r \dot{\theta} \\ \frac{1}{2} m_p L_p g \alpha - B_p \dot{\alpha} \end{bmatrix}$$

From the matrix muliplication, the first equation is

$$\ddot{\theta} = \frac{1}{J_T} \left(J_p + \frac{1}{4} m_p L_p^2 \right) (\tau - B_r \dot{\theta}) + \frac{1}{2 J_T} m_p L_p L_r \left(\frac{1}{2} m_p L_p g \alpha - B_p \dot{\alpha} \right).$$

Expanding the equation and collecting like terms gives us

$$\ddot{\theta} = \frac{1}{J_T} \left(-\left(J_p + \frac{1}{4} m_p L_p^2 \right) B_r \dot{\theta} - \frac{1}{2} m_p L_p L_r B_p \dot{\alpha} + \frac{1}{4} m_p^2 L_p^2 L_r g \alpha + \left(J_p + \frac{1}{4} m_p L_p^2 \right) \tau \right). \tag{Ans.2.6}$$

For the second equation, the matrix muliplication leads to

$$\ddot{\alpha} = \frac{1}{2 J_T} m_p L_p L_r (\tau - B_r \dot{\theta}) + \frac{1}{J_T} \left(J_r + m_p L_r^2 \right) \left(\frac{1}{2} m_p L_p g \alpha - B_p \dot{\alpha} \right).$$

By collecting like terms we obtain

$$\ddot{\alpha} = \frac{1}{J_T} \left(\frac{1}{2} m_p L_p L_r B_r \dot{\theta} - \left(J_r + m_p L_r^2 \right) B_p \dot{\alpha} + \frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2 \right) \alpha + \frac{1}{2} m_p L_p L_r \tau \right). \tag{Ans.2.7}$$

□□□

5. A-1, A-2 Find the linear state-space of the rotary inverted pendulum system. Make sure you give the A and B matrices (C and D have already been given in Section 2.1).

Answer 2.5

Outcome	Solution
A-1	From the defined state in Equation 2.8, it is given that $\dot{x}_1 = x_3$ and $\dot{x}_2 = x_4$. Substitute state x into the equations of motion found, where (as given in Equation 2.8) we have $\theta = x_1$, $\alpha = x_2$, $\dot{\theta} = x_3$, $\dot{\alpha} = x_4$. The A and B matrices for $\dot{x} = Ax + Bu$ can then be found.
A-2	Substituting x into Equation Ans.2.6 and Equation Ans.2.7 gives

$$\begin{aligned}\dot{x}_3 &= \frac{1}{J_T}\left(-\left(J_p + \frac{1}{4}m_p L_p^2\right) B_r x_3 - \frac{1}{2} m_p L_p L_r B_p x_4 \right.\\ &\left. +\frac{1}{4} m_p^2 L_p^2 L_r g x_2 + \left(J_p + \frac{1}{4} m_p L_p^2\right) u \right)\end{aligned}$$

and

$$\begin{aligned}\dot{x}_4 &= \frac{1}{J_T}\left(\frac{1}{2} m_p L_p L_r B_r x_3 - \left(J_r + m_p L_r^2\right) B_p x_4 \right.\\ &\left. +\frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2\right) x_2 + \frac{1}{2} m_p L_p L_r u \right).\end{aligned}$$

The A and B matrices in the $\dot{x} = Ax + Bu$ equation are

$$A = \frac{1}{J_T}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} m_p^2 L_p^2 L_r g & -\left(J_p + \frac{1}{4} m_p L_p^2\right) B_r & -\frac{1}{2} m_p L_p L_r B_p \\ 0 & \frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2\right) & \frac{1}{2} m_p L_p L_r B_r & -\left(J_r + m_p L_r^2\right) B_p \end{bmatrix}$$

and

$$B = \frac{1}{J_T}\begin{bmatrix} 0 \\ 0 \\ J_p + \frac{1}{4} m_p L_p^2 \\ \frac{1}{2} m_p L_p L_r \end{bmatrix}.$$

□□□

2.3 In-Lab Exercises

2.3.1 Simulation: Model Analysis

1. Run the *setup_rotpen_student.m* script. The SRV02 and pendulum model parameters are automaticaly loaded using the *config_srv02.m* and *config_sp.m* functions. It then calls the *ROPTEN_ABCD_eqns_student.m* script to load the model in the Matlab workspace.

2. B-5 Open the *ROTPEN_ABCD_eqns_student.m* script. The script should contain the following code:

```
% State Space Representation
A = eye(4,4);
B = [0;0;0;1];
C = eye(2,4);
D = zeros(2,1);

% Add actuator dynamics
B = Kg * kt * B / Rm;
A(3,3) = A(3,3) - Kg^2*kt*km/Rm*B(3);
A(4,3) = A(4,4) - Kg^2*kt*km/Rm*B(4);

system = ss(A,B,C,D);
```

The representative C and D matrices have already been included. You need to enter the state-space matrices A and B that you found in Section 2.2. The actuator dynamics have been added to convert your state-space matrices to be in terms of voltage. Recall that the input of the state-space model you found in Section 2.2 is the torque acting at the servo load gear (i.e., the pivot of the pendulum). However, *we do not control torque directly - we control the servo input voltage*. The above code uses the voltage-torque relationship given in Equation 2.4 in Section 2.1.2 to tranform torque to voltage.

Answer 2.6

Outcome	**Solution**
B-5	As given in the ROTPEN_ABCD_eqns.m script, the state-space model is entered as:

```
% State Space Representation
Jt = Jr*Jp + Mp*(Lp/2)^2*Jr + Jp*Mp*Lr^2;
A = [0 0 1 0;
     0 0 0 1;
   0 Mp^2*(Lp/2)^2*Lr*g/Jt -Dr*(Jp+Mp*(Lp/2)^2)/Jt -Mp*(Lp/2)*Lr*Dp/Jt;
   0 Mp*g*(Lp/2)*(Jr+Mp*Lr^2)/Jt -Mp*(Lp/2)*Lr*Dr/Jt -Dp*(Jr+Mp*Lr^2)/Jt];

B = [0; 0; (Jp+Mp*(Lp/2)^2)/Jt; Mp*(Lp/2)*Lr/Jt];
C = eye(2,4);
D = zeros(2,1);

% Add actuator dynamics
B = Kg * kt * B / Rm;
A(3,3) = A(3,3) - Kg^2*kt*kn/Ra*B(3);
A(4,3) = A(4,4) - Kg^2*kt*kn/Ra*B(4);
```

□□□

3. B-5, K-3 Run the *ROTPEN_ABCD_eqns_student.m* script to load the state-space matrices in the Matlab workspace. Show the numerical matrices that are displayed in the Matlab prompt.

Answer 2.7

Outcome	Solution
B-5	The model shown below should have been loaded if they used the correct model parameters (e.g., correct syntax) and ran the script properly. In Matlab, the model parameters are denoted as Mp, Lp, Lr, Jr, Dr, Jp, Dp, and g.
K-3	If the model was developed and entered properly, they should appear as:

```
A =

         0         0    1.0000         0
         0         0         0    1.0000
         0   81.4033  -10.2536   -0.9319
         0  122.0545  -10.3320   -1.3972

B =

         0
         0
   83.4659
  -80.3162


C =

     1     0     0     0
     0     1     0     0


D =

     0
     0
```

These matrices indicate that the model of the rotary pendulum has been successfully loaded.

□□□

4. K-1 Find the open-loop poles of the system.

Answer 2.8

Outcome	Solution
K-1	Using the MATLAB command *eig(A)*, we find that the open-loop poles of the system are -17.1, 8.34, -2.87, and 0.

□□□

Before ending this lab... To do the pre-lab questions in Section 3.3, you need the A and B matrices (numerical representation) and the open-loop poles. Make sure you record these.

2.3.2 Implementation: Calibration

Experimental Setup

The *q_rotpen_model_student* Simulink diagram shown in Figure 2.2 is used to confirm that the actual system hardware matches the modeling conventions. It is also a good check that the system is connected properly. The QUARC blocks are used to interface with encoders of the system. For more information about QUARC, see Reference [3]. This model outputs the rotary arm and pendulum link angles and can apply a voltage to the DC motor.

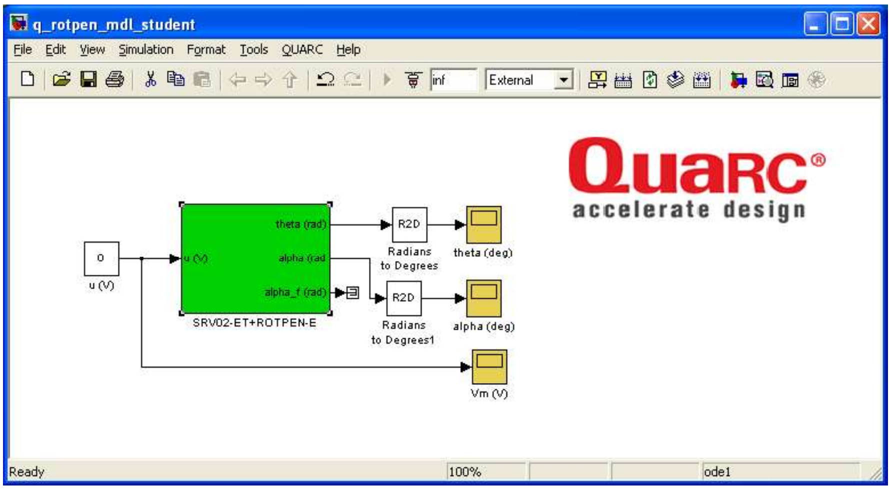


Figure 2.2: q_rotpen_mdl_student Simulink diagram used to confirm modeling conventions

IMPORTANT: Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, then you need to go to Section 5.3 to configure the lab files first.

1. Run the *setup_rotpen.m* script to load your Rotary Pendulum model.
2. In the *q_rotpen_model_student* Simulink diagram, go to QUARC | Build to build the QUARC controller.
3. Turn ON the power amplifier.
4. Go to QUARC | Start to run the controller.
5. B-9 Rotate the arm and the pendulum in the counter-clockwise direction and examine the direction of their response. Does the direction of these measurements agree with the modeling conventions given in Section 2.1.1? Explain why or why not.

   Answer 2.9

Outcome	**Solution**
B-9	Yes, the measurements agree with the model conventions. The rotary arm angle, θ, goes positive when it is rotated CCW and the pendulum angle, α, goes positive when it is rotated CCW.

   □□□

6. Go to the *SRV02-ET+ROTPEN-E* subsystem block, shown in Figure 2.3.
7. K-1, B-9 The Source block called *u (V)* in *q_rotpen_mdl_student* Simulink diagram is the control input. When you set *u (V)* to 1 V, the rotary arm must move according to the model conventions that were defined in Section 2.1.1. As shown in Figure 2.3, the *Direction Convention* Gain block is currently set to 0. Change this value such that the model conventions are adhered to. Plot the rotary arm response and the motor voltage in a Matlab figure when 1 V is applied.

   Note: When the controller stops, the last 10 seconds of data is automatically saved in the Matlab workspace to the variables *data_theta* and *data_Vm*. The time is stored in *data_alpha(:,1)* vector, the pendulum angle is stored the *data_alpha(:,2)* vector, and the control input is in the *data_Vm(:,2)* structure.

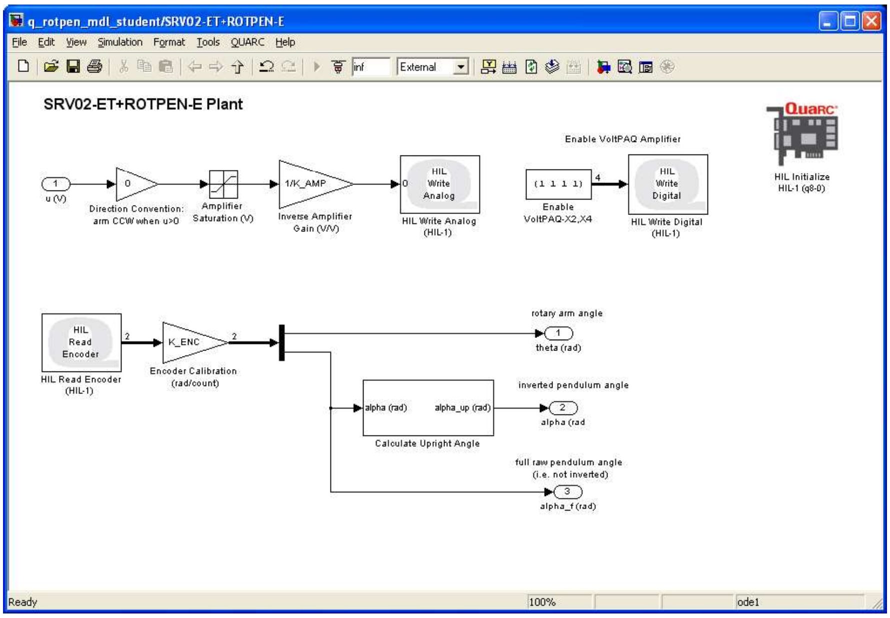


Figure 2.3: SRV02-ET+ROTPEN-E Subsystem - Student Version

Answer 2.10

Outcome **Solution**

B-9 In order for $\dot{\theta} > 0$ when $u > 0$, set the *Direction Convention* Gain block to -1. The -1 ensures that the arm increases in the positive direction when a positive voltage is applied.

K-1 The arm and voltage responses when 1 V is applied are shown in Figure Ans.2.1.

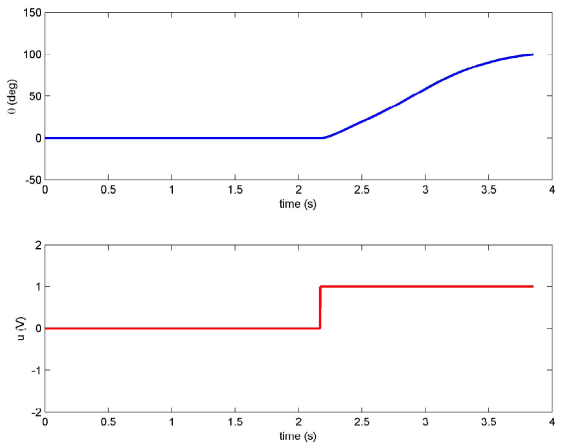


Figure Ans.2.1: Arm response when applying 1 V to control

□□□

8. Click on the STOP button to stop running the QUARC controller.
9. Shut off the power amplifier.

2.4 Results

B-6 Fill out Table 2 with your answers from your modeling lab results - both simulation and implementation.

Description	Symbol	Value	Units
State-Space Matrix	A	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 80.3 & -10.2 & -0.930 \\ 0 & 122 & -10.3 & -1.40 \end{bmatrix}$	
State-Space Matrix	B	$\begin{bmatrix} 0 \\ 0 \\ 83.2 \\ 80.1 \end{bmatrix}$	
State-Space Matrix	C	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	
State-Space Matrix	D	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	
Open-loop poles	OL	{-17.1, 8.34, -2.87, 0 }	

Table 2: Results

3 BALANCE CONTROL

3.1 Specifications

The control design and time-response requirements are:

Specification 1: Damping ratio: $\zeta = 0.7$.

Specification 2: Natural frequency: $\omega_n = 4$ rad/s.

Specification 3: Maximum pendulum angle deflection: $|\alpha| < 15$ deg.

Specification 4: Maximum control effort / voltage: $|V_m| < 10$ V.

The necessary closed-loop poles are found from specifications 1 and 2. The pendulum deflection and control effort requirements (i.e., specifications 3 and 4) are to be satisfied when the rotary arm is tracking a ± 20 degree angle square wave.

3.2 Background

In Section 2, we found a linear state-state space model that represents the inverted rotary pendulum system. This model is used to investigate the inverted pendulum stability properties in Section 3.2.1. In Section 3.2.2, the notion of controllabitliy is introduced. The procedure to transform matrices to their companion form is described in Section 3.2.3. Once in their companion form, it is easier to design a gain according to the pole-placement principles, which is discussed in Section 3.2.4. Lastly, Section 3.2.6 describes the state-feedback control used to balance the pendulum.

3.2.1 Stability

The stability of a system can be determined from its poles ([8]):

- Stable systems have poles only in the left-hand plane.
- Unstable systems have at least one pole in the right-hand plane and/or poles of multiplicity greater than 1 on the imaginary axis.
- Marginally stable systems have one pole on the imaginary axis and the other poles in the left-hand plane.

The poles are the roots of the system's characteristic equation. From the state-space, the characteristic equation of the system can be found using

$$\text{det}\,(sI - A) = 0$$

where *det()* is the determinant function, s is the Laplace operator, and I the identity matrix. These are the *eigenvalues* of the state-space matrix A.

3.2.2 Controllability

If the control input u of a system can take each state variable, x_i where $i = 1 \ldots n$, from an initial state to a final state then the system is controllable, otherwise it is uncontrollable ([8]).

Rank Test The system is controllable if the rank of its controllability matrix

$$T = \left[B \; AB \; A^2B \ldots A^nB\right] \tag{3.1}$$

equals the number of states in the system,

$$\text{rank}(T) = n.$$

3.2.3 Companion Matrix

If (A, B) are controllable and B is $n \times 1$, then A is similar to a companion matrix ([1]). Let the characteristic equation of A be

$$s^n + a_n s^{n-1} + \ldots + a_1.$$

Then the companion matrices of A and B are

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix} \tag{3.2}$$

and

$$\tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.3}$$

Define

$$W = T\tilde{T}^{-1}$$

where T is the controllability matrix defined in Equation 3.1 and

$$\tilde{T} = [\tilde{B} \ \tilde{B}\tilde{A} \ldots \tilde{B}\tilde{A}^n].$$

Then

$$W^{-1}AW = \tilde{A}$$

and

$$W^{-1}B = \tilde{B}.$$

3.2.4 Pole Placement

If (A,B) are controllable, then pole placement can be used to design the controller. Given the control law $u = -Kx$, the state-space in Equation 2.6 becomes

$$\begin{aligned} \dot{x} &= Ax + B(-Kx) \\ &= (A - BK)x \end{aligned}$$

To illustate how to design gain K, consider the following system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -1 & -5 \end{bmatrix} \tag{3.4}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.5}$$

Note that A and B are already in the companion form. We want the closed-loop poles to be at $[-1 \ -2 \ -3]$. The *desired* characteristic equation is therefore

$$(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6 \tag{3.6}$$

For the gain $K = [k_1\ k_2\ k_3]$, apply control $u = -Kx$ and get

$$A - KB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3-k_1 & -1-k_2 & -5-k_3 \end{bmatrix}.$$

The characteristic equation of $A - KB$ is

$$s^3 + (k_3+5)s^2 + (k_2+1)s + (k_1-3) \tag{3.7}$$

Equating the coefficients between Equation 3.7 and the desired polynomial in Equation 3.6

$$\begin{aligned} k_1 - 3 &= 6 \\ k_2 + 1 &= 11 \\ k_3 + 5 &= 6 \end{aligned}$$

Solving for the gains, we find that a gain of $K = [9\ 10\ 1]$ is required to move the poles to their desired location.

We can generalize the procedure to design a gain K for a controllable (A,B) system as follows:

Step 1 Find the companion matrices $\tilde{A}$ and $\tilde{B}$. Compute $W = T\tilde{T}^{-1}$.

Step 2 Compute $\tilde{K}$ to assign the poles of $\tilde{A} - \tilde{B}\tilde{K}$ to the desired locations. Applying the control law $u = -Kx$ to the general system given in Equation 3.2,

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1-k_1 & -a_2-k_2 & \cdots & -a_{n-1}-k_{n-1} & -a_n-k_n \end{bmatrix} \tag{3.8}$$

Step 3 Find $K = \tilde{K}W^{-1}$ to get the feedback gain for the original system (A,B).

Remark: It is important to do the $\tilde{K} \to K$ conversion. Remember that (A,B) represents the actual system while the companion matrices $\tilde{A}$ and $\tilde{B}$ do not.

3.2.5 Desired Poles

The rotary inverted pendulum system has four poles. As depicted in Figure 3.1, poles p_1 and p_2 are the complex conjugate *dominant* poles and are chosen to satisfy the natural frequency, ω_n, and damping ratio, ζ, specifications given in Section 3.1. Let the conjugate poles be

$$p_1 = -\sigma + j\omega_d \tag{3.9}$$

and

$$p_2 = -\sigma - j\omega_d \tag{3.10}$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the *damped* natural frequency. The remaining closed-loop poles, p_3 and p_4, are placed along the real-axis to the left of the dominant poles, as shown in Figure 3.1.

3.2.6 Feedback Control

The feedback control loop that balances the rotary pendulum is illustrated in Figure 3.2. The reference state is defined

$$x_d = [\theta_d\ 0\ 0\ 0]$$

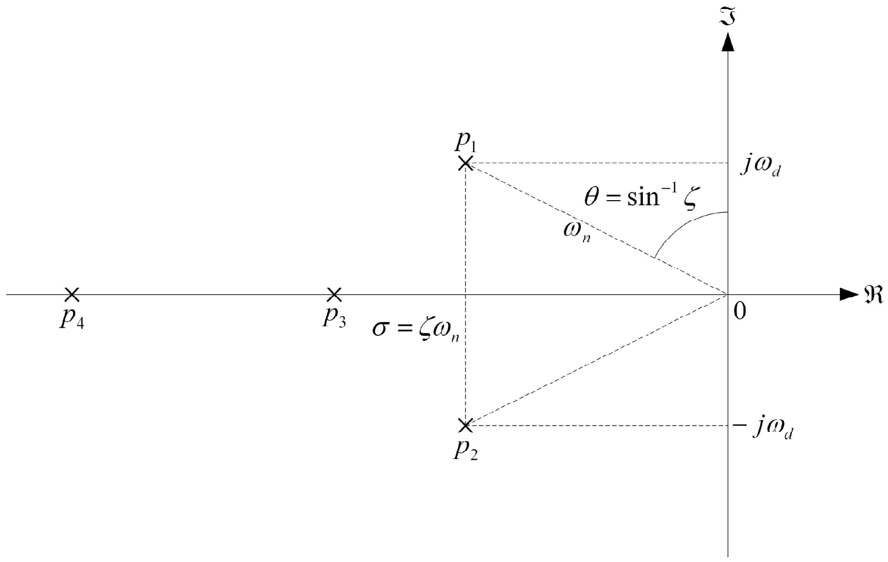


Figure 3.1: Desired closed-loop pole locations

where θ_d is the desired rotary arm angle. The controller is

$$u = K(x_d - x). \tag{3.11}$$

Note that if $x_d = 0$ then $u = -Kx$, which is the control used in the pole-placement algorithm.

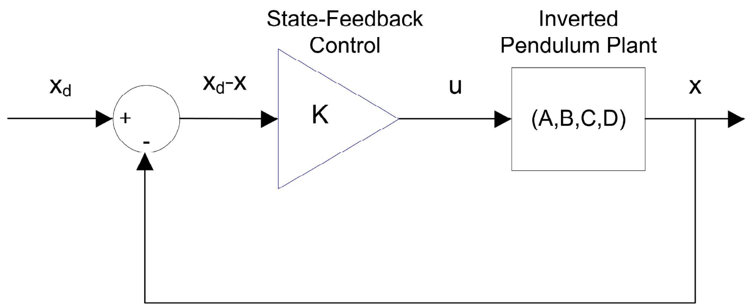


Figure 3.2: State-feedback control loop

When running this on the actual system, the pendulum begins in the hanging, downward position. We only want the balance control to be enabled when the pendulum is brought up around its upright vertical position. The controller is therefore

$$u = \begin{cases} K(x_d - x) & |x_2| < \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

where ϵ is the angle about which the controller should engage. For example if $\epsilon = 10$ degrees, then the control will begin when the pendulum is within ± 10 degrees of its upright position, i.e., when $|x_2| < 10$ degrees.

3.3 Pre-Lab Questions

1. A-1, A-3 Based on your analysis in Section 2.3, is the system stable, marginally stable, or unstable? Did you expect the stability of the inverted pendulum to be as what was determined?

Answer 3.1

Outcome	Solution
A-1	The open-loop poles determined in Section 2.3 are -17.1, 8.34, -2.87, and 0. Because one pole is in the right-hand plane, the system is unstable.
A-3	This makes sense, as an inverted pendulum does not stay inverted by itself - it falls down.

□□□

2. A-1, A-2 Using the open-loop poles, find the characteristic equation of A.

Answer 3.2

Outcome	Solution
A-1	As given in Section 3.2.1, the roots of the characteristic equation are the open-loop poles.
A-2	Given the open-loop poles, the open-loop polynomial equation is

$$s(s+17.1)(s-8.34)(s+2.87) = s^4+11.6s^3-117.3s^2-408.3s \quad \text{(Ans.3.1)}$$

□□□

3. A-2 Give the corresponding companion matrices $\tilde{A}$ and $\tilde{B}$. Do not compute the transformation matrix W (this will be done in the lab using QUARC®).

Answer 3.3

Outcome	Solution
A-2	The open-loop characteristic equation has the form $s^4 + a_4s^3 + a_3s^2 + a_2s+a_1$. Fitting the coefficients into the general companion matrix format given in Equation 3.2 and Equation 3.3:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 408.3 & 117.3 & -11.6 \end{bmatrix} \quad \text{(Ans.3.2)}$$

and

$$\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{(Ans.3.3)}$$

□□□

4. A-1, A-2 Find the location of the two dominant poles, p_1 and p_2, based on the specifications given in Section 3.1. Place the other poles at $p_3 = -30$ and $p_4 = -40$. Finally, give the desired characteristic equation.

Answer 3.4

Outcome	Solution
A-1	Using the pole locations in Equation 3.9 and Equation 3.10 and the damping ratio and natural frequency given in Section 3.1.
A-2	The components in equations 3.9 and 3.10 are

$$\sigma = \zeta\omega_n = 2.80$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.86$$

The desired location of the closed-loop poles is $-2.80 \pm j2.86$, -30, and -40. The characteristic polynomial is

$$(s+2.80-j2.86)(s+2.80+j2.86)(s+30)(s+40) = s^4+75.6s^3+1608s^2+7840s+19200 \qquad \text{(Ans.3.4)}$$

□□□

5. A-1, A-2 When applying the control $u = -\tilde{K}x$ to the companion form, it changes $(\tilde{A}, \tilde{B})$ to $(\tilde{A}-\tilde{B}\tilde{K}, \tilde{B})$. Find the gain $\tilde{K}$ that assigns the poles to their new desired location.

Answer 3.5

Outcome	Solution
A-1	Applying the control the companion matrix becomes

$$\tilde{A}-\tilde{B}\tilde{K} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & 408.3-k_2 & 117.3-k_3 & -11.6-k_4 \end{bmatrix}.$$

The characteristic equation for this system is

$$s^4+(11.6+k_4)s^3+(k_3-117.3)s^2+(k_2-408.3)s+k_1.$$

A-2 Equating these coefficents of this characteristic polynomial with the desired in Equation Ans.3.4 gives

$$11.6+k_4 = 75.6$$
$$k_3-117.3 = 1608$$
$$k_2-408.3 = 7840$$
$$k_1 = 19200$$

Solving for the gains k_i

$$\tilde{K} = [19200\ 8286\ 1725\ 64] \qquad \text{(Ans.3.5)}$$

□□□

3.4 In-Lab Exercises

3.4.1 Control Design

Note: Finding the control gain manually as dictated in Section 3 can be time consuming. The instructor may elect to have the student find the control gain through the standard *acker* Matlab command instead, which is in the last exercise.

1. Run the *setup_rotpen_student.m* script to load the rotary pendulum the model you found in pervious modeling lab.

2. B-4, B-7 Using Matlab commands, determine if the system is controllable. Explain why.

Answer 3.6

Outcome	**Solution**
B-4	Find the controllability matrix using the *ctrb* command and use *rank* to find the rank of that matrix. The resuling command are: `T = ctrb(A,B);` `rank(T)`
B-7	The system is controllable because the rank of its controllability matrix equals the number of states, i.e., $rank(T) = 4 = n$.

□□□

3. B-5, K-3 Open the *d_pole_placement_student.m* script. As shown below, the companion matrices $\tilde{A}$ and $\tilde{B}$ for the model are automatically found (denoted as Ac and Bc in Matlab).

```
% Characteristic equation: s^4 + a_4*s^3 + a_3*s^2 + a_2*s + a_1
a = poly(A);
%
% Companion matrices (Ac, Bc)
Ac = [  0 1 0 0;
        0 0 1 0;
        0 0 0 1;
        -a(5) -a(4) -a(3) -a(2)];
%
Bc = [0; 0; 0; 1];
% Controllability
T = 0;
% Controllability of companion matrices
Tc = 0;
% Transformation matrices
W = 0;
```

In order to find the gain K, we need to find the transformation matrix $W = T\tilde{T}^{-1}$ (note: $\tilde{T}$ is denoted as T_c in Matlab). Modify the *d_pole_placement_student.m* script to calculate the controllability matrix T, the companion controllabilty matrix Tc, the inverse of Tc, and W. Show your completed script and the resulting T, Tc, Tc^{-1}, and W matrices.

Answer 3.7

Outcome	Solution
B-5	If the experimental procedure is followed correctly, the following results should have been obtained.
K-3	This code can be used to find transformation matrix W:

```
% Controllability
T = ctrb(A,B);
% Controllability of companion matrices
Tc = ctrb(Ac,Bc);
% Transformation matrices
W = T*inv(Tc);
```

See the d_pole_placement.m script for the full solution. The resulting matrices are:

$$T = \begin{bmatrix} 0 & 83.5 & -930.7 & 16989 \\ 0 & 80.3 & -974.6 & 20780 \\ 83.5 & -930.7 & 16989 & -272898 \\ 80.3 & -974.6 & 20780 & -323519 \end{bmatrix}$$

$$\tilde{T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -11.7 \\ 0 & 1 & -11.7 & 253.1 \\ 1 & -11.7 & 253.1 & -3906 \end{bmatrix}$$

$$\tilde{T}^{-1} = \begin{bmatrix} -410.4 & -117.4 & 11.7 & 1 \\ -117.4 & 11.7 & 1 & 0 \\ 11.65 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} -3649 & 41.8 & 83.5 & 0 \\ 0 & -38.8 & 80.3 & 0 \\ 0 & -3649 & 41.8 & 83.5 \\ 0 & 0 & -38.8 & 80.3 \end{bmatrix}$$

□□□

4. Enter the companion gain, $\tilde{K}$, you found in the pre-lab as *Kc* in *d_pole_placement_student.m* and modify it to find gain K using the transformation detailed in Section 3. Run the script again to calculate the feedback gain K and record its value in Table 3.

Answer 3.8

As given in the pole-placement procedure in Section 3.2.4, the control gain is found using the equation $K = \tilde{K}W^{-1}$. Recall that gain $\tilde{K}$ was found in Equation Ans.3.5 in Section 3.3. Using this, we can find gain K in Matlab using the commands:

```
Kc = [ 19200 8286 1725 64 ];
K = Kc*inv(W)
```

This generates the balance control gain

$$K = [-5.26\ 30.0\ -2.65\ 3.55]. \qquad \text{(Ans.3.6)}$$

□□□

5. K-1, B-9 Evaluate the closed-loop poles of the system, i.e., the eigenvalues of $A - BK$. Record the closed-loop poles of the system when using the gain K calculated above. Have the poles been placed to their desired locations? If not, then go back and re-investigate your control design until you find a gain that positions the poles to the required location.

Answer 3.9

Outcome	Solution
K-1	Using the Matlab command *eig(A-B*K)* to find the poles of the closed-loop system we have:

```
>> eig(A-B*K)

ans =

 -40.0000
 -30.0000
  -2.8000 + 2.8566i
  -2.8000 - 2.8566i
```

B-9	The closed-loop poles are at $-2.8 \pm j2.86$, -30, and -40, equivalent to the location of the desired poles.

□□□

6. K-1 In the previous exercises, gain K was found manually through matrix operations. All that work can instead be done using a pre-defined *Compensator Design* Matlab command. Find gain K using a Matlab pole-placement command and verify that the gain is the same as generated before.

Answer 3.10

Outcome	Solution
K-1	The gain can be found using the 'acker' or 'place' commands as follows:

```
% Control Specifications
zeta = 0.7;
wn = 4;
% Location of dominant poles along real-axis
sigma = zeta*wn;
% Location of dominant poles along img axis (damped natural freqency)
wd = wn*sqrt(1-zeta^2);
% Desired poles (-30 and -40 are given)
DP = [-sigma+j*wd, -sigma-j*wd, -30, -40];
% Find control gain using Matlab pole-placement command
K = acker(A,B,DP);
```

Students may also enter the desired poles directly. See d_balance.m for the full code. The gain generated is

$$K = [-5.28\ 30.14\ \ -2.65\ 3.55]$$

which is the same as found manually in Equation Ans.3.6.

□□□

3.4.2 Simulating the Balance Control

Experiment Setup

The *s_rotpen_bal* Simulink diagram shown in Figure 3.3 is used to simulate the closed-loop response of the Rotary Pendulum using the state-feedback control described in Section 3 with the control gain K found in Section 3.4.1.

The *Signal Generator* block generates a 0.1 Hz square wave (with amplitude of 1). The *Amplitude (deg)* gain block is used to change the desired rotary arm position. The state-feedback gain K is set in the *Control Gain* gain block and is read from the Matlab workspace. The Simulink *State-Space* block reads the A, B, C, and D state-space matrices that are loaded in the Matlab workspace. The *Find State X* block contains high-pass filters to find the velocity of the rotary arm and pendulum.

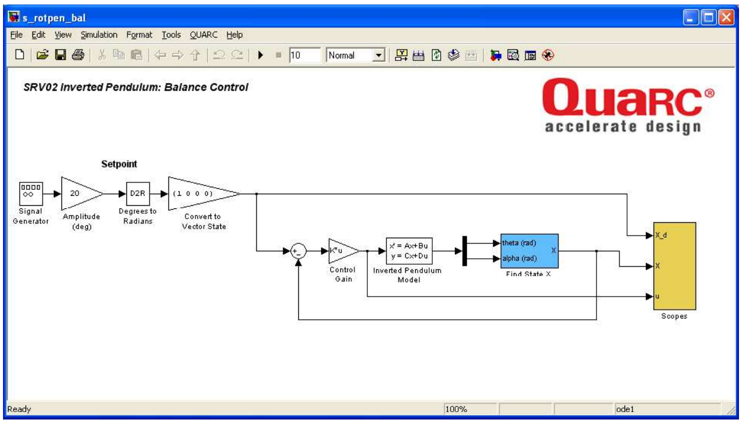


Figure 3.3: s_rotpen_bal Simulink diagram used to simulate the state-feedback control

IMPORTANT: Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, go to Section 5.4 to configure the lab files first. **Make sure the model you found in Section 2.3 is entered in ROTPEN_ABCD_eqns_student.m.**

1. Run *setup_rotpen.m*. Ensure the gain K you found in Section 3.4.1 is loaded.
2. B-5, K-3 Run the *s_rotpen_bal.mdl*. The response in the scopes shown in Figure 3.4 were generated using an arbitrary feedback control gain. Plot the simulated response of rotary arm, pendulum, and motor input voltage obtained using your obtained gain K in a Matlab figure and attach it to your report.

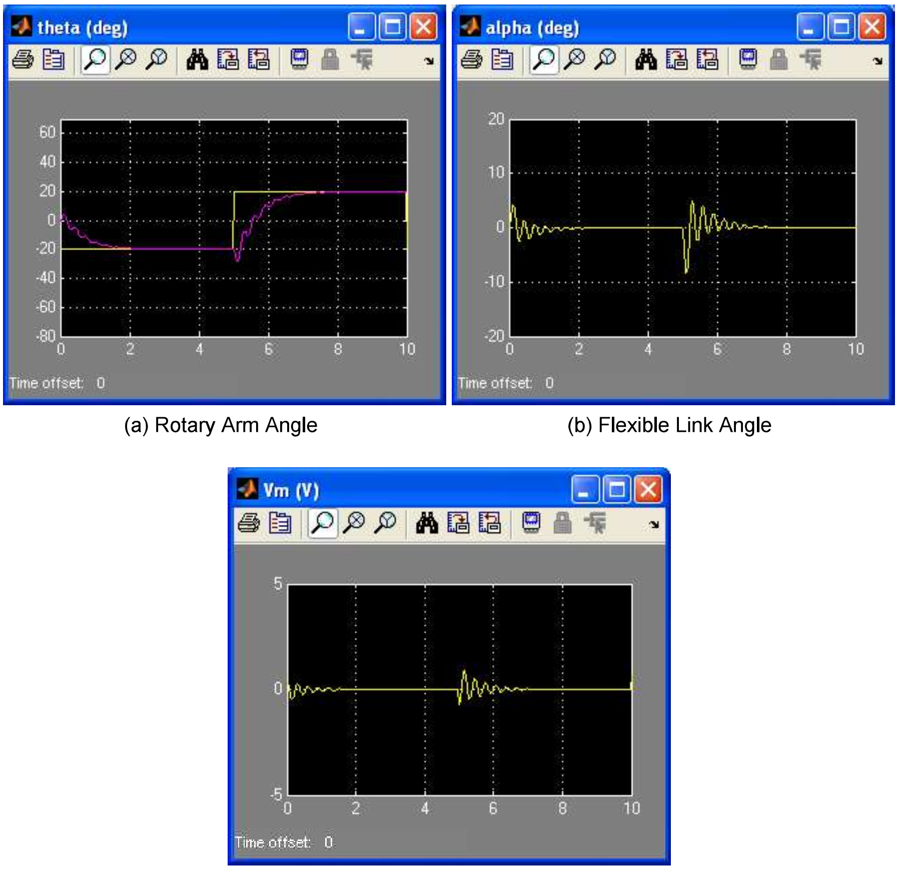
   Note: When the simulation stops, the last 10 seconds of data is automatically saved in the Matlab workspace to the variables *data_theta*, *data_alpha*, and *data_Vm*. The time is stored in the *data_theta(:,1)* vector, the desired and measured rotary arm angles are saved in the *data_theta(;,2)* and *data_theta(;,3)* arrays, the pendulum angle is stored the *data_alpha(:,2)* vector, and the control input is in the *data_Vm(:,2)* structure.

   Answer 3.11

Outcome	Solution
B-5	If the model and gain we're properly loaded, then the simulation should run and obtain a response similarly as shown in Figure Ans.3.1.
K-3	The typical closed-loop response is shown in Figure Ans.3.1. Use the Matlab *plot* command, you can generate a figure similarly as shown in Figure Ans.3.1. Run the *plot_rotpen_bal.m* script after running the s_rotpen_bal.mdl with the gain found in Section 3.4.1 to plot this response.

   □□□

3. K-1, B-9 Measure the pendulum deflection and voltage used. Are the specifications given in Section 3.1 satisfied?



(a) Rotary Arm Angle

(b) Flexible Link Angle

(c) Voltage

Figure 3.4: Balance Control Simulation using default gain

Answer 3.12

Outcome	Solution
K-1	As shown in Figure Ans.3.1, the maximum pendulum angle and voltage are

$$|V_m|_{max} = 3.7 \text{ V}$$
$$|\alpha|_{max} = 8.3 \text{ deg.}$$

B-9	The pendulum angle stays within ± 15 degrees and the input voltage is kept below ± 10 V. The natural frequency and damping ratio have already been satisfied in the control design. Therefore all the specifications have been met.

□□□

4. Close the Simulink diagram when you are done.

3.4.3 Implementing the Balance Controller

In this section, ths state-feedback control that was designed and simulated in the previous sections is run on the actual SRV02 Rotary Pendulum device.

Experiment Setup

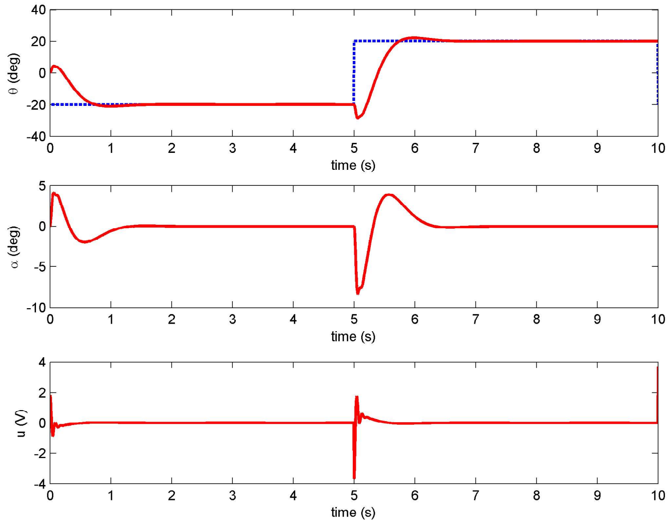


Figure Ans.3.1: Simulated closed-loop balance control response

The *q_rotpen_bal_student* Simulink diagram shown in Figure 3.5 is used to run the state-feedback control on the Quanser Rotary Pendulum system. The *SRV02-ET+ROTPEN-E* subsystem contains QUARC blocks that interface with the DC motor and sensors of the system. The feedback developed in Section 3.4.1 is implemented using a Simulink *Gain* block.

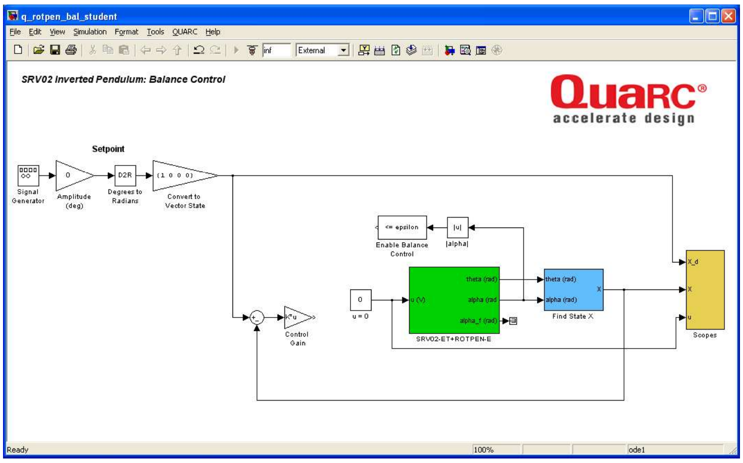


Figure 3.5: q_rotpen_bal_student Simulink diagram can be used to run balance controller

IMPORTANT: Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, then go to Section 5.5 to configure the lab files first.

1. Run the *setup_rotpen.m* script.
2. Make sure the gain K you found in Section 3.4.1 is loaded.
3. Open the q_rotpen_bal_student Simulink diagram.
4. Turn ON the power amplifier.
5. Go to QUARC | Build to build the controller.

6. Go to QUARC | Start to run the controller.

   Ensure the modifications you made in the Modeling Laboratory (Section 2.3) have been applied. Verify that the model conventions still hold (e.g., motor turns in expected way when a positive voltage is supplied).

7. K-3 As shown in Figure 3.5, the Simulink diagram is incomplete. Add the blocks from the Simulink library to implement the balance control. When implementing the control, keep in mind the following: unlike in the simulation, where the pendulum is already upright, the pendulum begins in the hanging down position. Thus when the controller starts, the inverted pendulum angle reads ± 180 and it goes up to zero when brought to the upright position. You will need to add a switch logic to implement Equation 3.12.

Answer 3.13

Outcome	Solution
K-3	The complete Simulink diagram is shown in Figure Ans.3.2. The *Multi-port Switch* from the *Simulink* | *Signal Routing* category is used to implement the switching control.

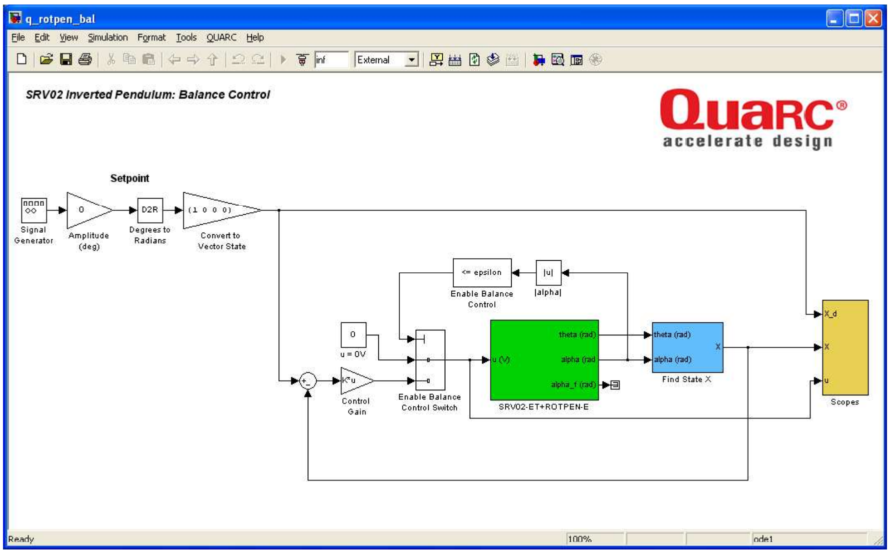


Figure Ans.3.2: q_rotpen_bal Simulink diagram used to implement balance control

□□□

8. B-5 Ensure the pendulum is in the hanging down position and is motionless. Go to QUARC | Build and QUARC | Run to start the QUARC controller. Once it is running, manually bring up the pendulum to its upright vertical position. You should feel the voltage kick-in when it is within the range where the balance control engages. Once it is balanced, introduce the ± 20 degree rotary arm command by setting the *Amplitude (deg)* gain in the Simulink diagram to 20.

Answer 3.14

Outcome	Solution
B-5	If the experimental procedure was followed correctly, they should be able to balance the pendulum and potentially generate a figure similar to Figure Ans.3.3.

□□□

9. K-3 The response should look similar to your simulation. Once you have obtained a response, click on the STOP button to stop the controller. Be careful, as the pendulum will fall down when the controller is stopped. Similarly as in the simulation, the response data will be saved to the workspace. Use this to plot the rotary arm, pendulum, and control input responses in a Matlab figure (see Section 3.4.2 for more information on plotting).

Answer 3.15

Outcome K-3

Solution Use the Matlab 'plot' command, you can generate a figure similarly as shown in Figure Ans.3.3. Run the 'plot_rotpen_bal.m' script after running the 'q_rotpen_bal.mdl' with the gain found in Equation Ans.3.6 to plot this response. Alternatively, this plot can be generated using the data stored in the *data_rotpen_bal_theta.mat*, *data_rotpen_bal_alpha.mat*, and *data_rotpen_bal_Vm.mat* files.

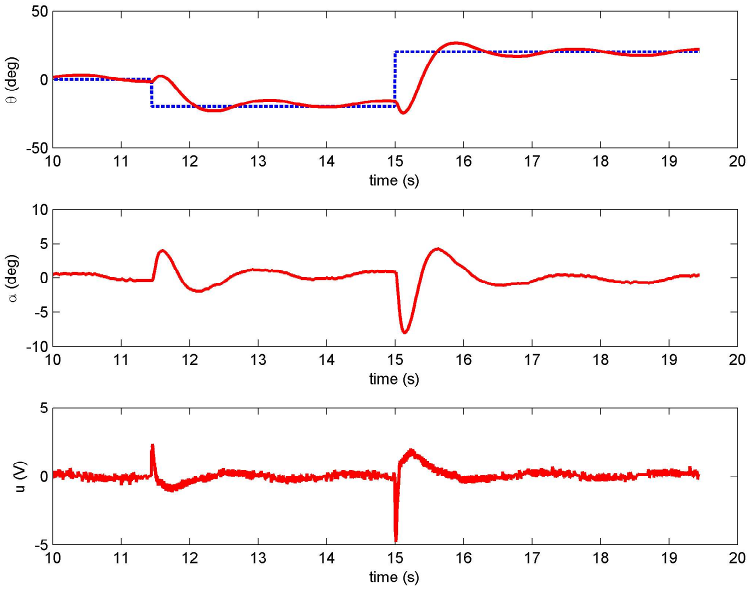


Figure Ans.3.3: Inverted Pendulum Response

□□□

10. K-1, B-9 Measure the pendulum deflection and voltage used. Are the specifications given in Section 3.1 satisfied for the implementation?

Answer 3.16

Outcome K-1

Solution The pendulum deflection and voltage measured in Figure Ans.3.3 are

$$|\alpha|_{max} = 8.0 \text{ deg}$$
$$|V_m|_{max} = 4.8 \text{ V}$$

B-9 As with the simulated response, the pendulum angle stays within ± 15 degrees and the motor voltage is kept between ± 10 V. The specifications are satisfied.

□□□

11. Shut off the power amplifier.

3.5 Results

B-6 Fill out Table 3 with your answers from your control lab results - both simulation and implementation.

Description	Symbol	Value	Units
Pre Lab Questions			
Desired poles	DP	{ -2.80±2.86, -30, -40 }	
Companion Gain	$\breve{K}$	[19200 -8286 1725 64]	
Simulation: Control Design			
Transformation Matrix	W	$\begin{bmatrix} -3649 & 41.8 & 83.5 & 0 \\ 0 & -38.8 & 80.3 & 0 \\ 0 & -3649 & 41.8 & 83.5 \\ 0 & 0 & -38.8 & 80.3 \end{bmatrix}$	
Control Gain	K	[-5.26 30.0 -2.65 3.55]	
Closed-loop poles	CLP	{ -2.86±2.86, -30, -40 }	
Simulation: Closed-Loop System			
Maximum deflection	$\lvert\alpha\rvert_{max}$	8.3	deg
Maximum voltage	$\lvert V_m\rvert_{max}$	3.7	V
Implementation			
Control Gain	K	[-5.26 30.0 -2.65 3.55]	
Maximum deflection	$\lvert\alpha\rvert_{max}$	8.0	deg
Maximum voltage	$\lvert V_m\rvert_{max}$	4.8	V

Table 3: Results