Exercícios resolvidos da Ficha 5

Exercício 1

Considere a semântica denotacional do seguinte programa

$$y := 1\,;\, \texttt{while}\ x \neq 1\ \texttt{do}\ \{y := y * x\,;\, x := x - 1\}$$

Aplique a função resultante a um estado s_0 tal que $s_0\,x = 3$, e indique o valor da variável y no estado de chegada.

Sugere-se que siga os seguintes passos:

1. Construa $\mathcal{S}_{\rm ds}[\![y := 1\,;\, \texttt{while}\ x \neq 1\ \texttt{do}\ \{y := y * x\,;\, x := x - 1\}]\!]$ identificando a funcional F envolvida.

2. Calcule as várias funções $F^n\ \bot$ usadas na definição de $\mathrm{FIX}\,F$ e apresente uma definição de $\mathrm{FIX}\,F$.

3. Tem agora todos os dado para calcular o valor de y no estado de chegada.

Resolução

1.

$$\begin{aligned}
&\mathcal{S}_{\rm ds}[\![y := 1\,;\, \texttt{while}\ x \neq 1\ \texttt{do}\ \{y := y * x\,;\, x := x - 1\}]\!]\, s_0\\
&= (\mathcal{S}_{\rm ds}[\![\texttt{while}\ x \neq 1\ \texttt{do}\ \{y := y * x\,;\, x := x - 1\}]\!] \circ \mathcal{S}_{\rm ds}[\![y := 1]\!])s_0\\
&= \mathcal{S}_{\rm ds}[\![\texttt{while}\ x \neq 1\ \texttt{do}\ \{y := y * x\,;\, x := x - 1\}]\!]\ (s_0[y \mapsto 1])\\
&= (\mathrm{FIX}\,F)(s_0[y \mapsto 1])
\end{aligned}$$

onde $\quad (F\ g) = \mathrm{cond}(\mathcal{B}[\![x \neq 1]\!]\,, g \circ \mathcal{S}_{\rm ds}[\![y := y * x\,;\, x := x - 1]\!]\,, \mathrm{id})$
ou seja,

$$\begin{aligned}
(F\ g)\,s \quad &= \begin{cases} g\,(\mathcal{S}_{\rm ds}[\![y := y * x\,;\, x := x - 1]\!]\,s) & \text{se}\ \ \mathcal{B}[\![x \neq 1]\!]\,s = \mathbf{tt}\\ \mathrm{id}\,s & \text{se}\ \ \mathcal{B}[\![x \neq 1]\!]\,s = \mathbf{ff}\end{cases}\\
&= \begin{cases} g\,((\mathcal{S}_{\rm ds}[\![x := x - 1]\!] \circ \mathcal{S}_{\rm ds}[\![y := y * x]\!])\,s) & \text{se}\ \ s\,x \neq 1\\ s & \text{se}\ \ s\,x = 1\end{cases}\\
&= \begin{cases} g\,((\mathcal{S}_{\rm ds}[\![x := x - 1]\!]\,(\mathcal{S}_{\rm ds}[\![y := y * x]\!]\ s)) & \text{se}\ \ s\,x \neq 1\\ s & \text{se}\ \ s\,x = 1\end{cases}\\
&= \begin{cases} g\,(\mathcal{S}_{\rm ds}[\![x := x - 1]\!]\,(s[y \mapsto (s\,y) \times (s\,x)])) & \text{se}\ \ s\,x \neq 1\\ s & \text{se}\ \ s\,x = 1\end{cases}\\
&= \begin{cases} g\,(s[y \mapsto (s\,y) \times (s\,x)][x \mapsto (s\,x) - 1]) & \text{se}\ \ s\,x \neq 1\\ s & \text{se}\ \ s\,x = 1\end{cases}
\end{aligned}$$

2. Relembre que $\perp$ é a função que, para todo o estado s, $\perp\ s = \underline{\text{undef}}$, e que

$$(F\ g)\, s = \begin{cases} g\,(s[y \mapsto (s\,y)\times(s\,x)][x \mapsto (s\,x)-1]) & \text{se}\ \ s\,x \neq 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

Para simplificar a apresentação, <u>vamos assumir que</u> $s' = s[y \mapsto (s\,y)\times(s\,x)][x \mapsto (s\,x)-1]$

$$(F^0\ \perp)\, s \quad = \quad \perp s \quad = \quad \underline{\text{undef}}$$

$$(F^1\ \perp)\, s \quad = \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$(F^2\ \perp)\, s \quad = \quad (F(F\ \perp))\, s \quad = \quad \begin{cases} (F\ \perp)\, s' & \text{se}\ \ s\,x \neq 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases} \quad = \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \wedge s'\,x \neq 1 \\ s' & \text{se}\ \ s\,x \neq 1 \wedge s'\,x = 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$= \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \wedge (s\,x)-1 \neq 1 \\ s[y \mapsto (s\,y)\times(s\,x)][x \mapsto (s\,x)-1] & \text{se}\ \ s\,x \neq 1 \wedge (s\,x)-1 = 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$= \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \wedge s\,x \neq 2 \\ s[y \mapsto (s\,y)\times 2][x \mapsto 1] & \text{se}\ \ s\,x = 2 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$(F^3\ \perp)\, s \quad = \quad (F(F^2\ \perp))\ s \quad = \quad \begin{cases} (F^2\ \perp)\, s' & \text{se}\ \ s\,x \neq 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$= \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \wedge s'\,x \neq 1 \wedge s'\,x \neq 2 \\ s'[y \mapsto (s'\,y)\times 2][x \mapsto 1] & \text{se}\ \ s\,x \neq 1 \wedge s'\,x = 2 \\ s' & \text{se}\ \ s\,x \neq 1 \wedge s'\,x = 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$= \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \wedge (s\,x)-1 \neq 1 \wedge (s\,x)-1 \neq 2 \\ s'[y \mapsto (s'\,y)\times 2][x \mapsto 1] & \text{se}\ \ s\,x \neq 1 \wedge (s\,x)-1 = 2 \\ s' & \text{se}\ \ s\,x \neq 1 \wedge (s\,x)-1 = 1 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$= \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \wedge s\,x \neq 2 \wedge s\,x \neq 3 \\ s[y \mapsto (s\,y)\times(s\,x)\times 2][x \mapsto 1] & \text{se}\ \ s\,x = 3 \\ s[y \mapsto (s\,y)\times(s\,x)][x \mapsto (s\,x)-1] & \text{se}\ \ s\,x = 2 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$= \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x \neq 1 \wedge s\,x \neq 2 \wedge s\,x \neq 3 \\ s[y \mapsto (s\,y)\times 3\times 2][x \mapsto 1] & \text{se}\ \ s\,x = 3 \\ s[y \mapsto (s\,y)\times 2][x \mapsto 1] & \text{se}\ \ s\,x = 2 \\ s & \text{se}\ \ s\,x = 1 \end{cases}$$

$$\vdots$$

$$(F^n\ \perp)\, s \quad = \quad \begin{cases} \underline{\text{undef}} & \text{se}\ \ s\,x < 1\ \vee\ s\,x > n \\ s[y \mapsto (s\,y)\times j\times\ldots\times 2][x \mapsto 1] & \text{se}\ \ s\,x = j\ \wedge\ 1 \leq j \leq n \end{cases}$$

Ou, escrito de outro modo,

$$(F^n \ \bot)\, s \ = \ \begin{cases} \underline{\text{undef}} & \text{se} \ \ s\,x < 1 \ \vee \ s\,x > n \\ s[\, y \mapsto (s\,y) \times \prod_{i=2}^{s\,x} i \,][x \mapsto 1] & \text{se} \ \ 1 \leq s\,x \leq n \end{cases}$$

Portanto,

$$(\mathrm{FIX}\, F)\, s = \begin{cases} \underline{\text{undef}} & \text{se} \ \ s\,x < 1 \\ s[y \mapsto (s\,y) \times (s\,x) \times \ldots \times 2][x \mapsto 1] & \text{se} \ \ s\,x \geq 1 \end{cases}$$

3. O estado s_0 é tal que $s_0\,x = 3$. Portanto, o valor da variável y no estado de chegada é

$$\begin{array}{rcl} (\mathcal{S}_{\mathrm{ds}}[\![y := 1\, ;\, \texttt{while}\ x \neq 1\ \texttt{do}\ \{ y := y * x\, ;\, x := x - 1 \}]\!]\, s_0)\, y & = & ((\mathrm{FIX}\, F)(s_0[y \mapsto 1]))\, y \\ & = & (s_0[y \mapsto 1 \times 3 \times 2][x \mapsto 1])\, y \\ & = & 1 \times 3 \times 2 \\ & = & 6 \end{array}$$