Q2) LU Decomposition, Vector spaces and LT (5 marks)

i. Find the LU decomposition of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$ when it exists. For which real numbers a & b does it exist?

Soln.

$$A = LU$$

or, $$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} f & g & h \\ 0 & i & j \\ 0 & 0 & k \end{bmatrix}}_{U}$$

or, $$\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix} = \begin{bmatrix} f & g & h \\ cf & cg+i & ch+j \\ df & dg+ei & dh+ej+k \end{bmatrix}$$

on equating all elements (each) on both sides, we get

$\boxed{f=1}$ $\boxed{g=0}$ $\boxed{h=1}$

$cf = a$ or $c \cdot 1 = a$ or $\boxed{c=a}$

$df = b$ or $d \cdot 1 = b$ or $\boxed{d=b}$

$cg + i = a$ or $c \cdot 0 + i = a$ or $\boxed{i=a}$

$dg + ei = b$ or $d \cdot 0 + e \cdot a = b$ or $\boxed{e = b/a}$

$ch + j = a$ or $a \cdot 1 + j = a$ or $\boxed{j=0}$

$dh + ej + k = a$ or $b \cdot 1 + \frac{b}{a} \cdot 0 + k = a$ or $\boxed{k = a-b}$

$$\therefore \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & b/a & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix}}_{U}$$

matrix A (k-by-k) has LU factorization if

$$\text{Rank}(A_{11}) + k \geq \text{Rank}\begin{bmatrix} A_{11} & A_{12} \end{bmatrix} + \text{Rank}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$$

$$\text{Rank}(1) + 3 \geq \text{Rank}\begin{bmatrix} 1 & 0 \end{bmatrix} + \text{Rank}\begin{bmatrix} 1 \\ a \end{bmatrix}$$

i.e. all its leading principal minors are non-zero i.e. $\Delta_i \neq 0$, $i \in \{1, 2, \ldots$

i.e. $A_1 = 1$

leading submatrices are : $A_2 = \begin{bmatrix} 1 & 0 \\ a & a \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$

i.e. $|A_1| = 1 \neq 0$

$|A_2| = a - 0 \neq 0 \Rightarrow a \neq 0$

$|A_3| = 1(a^2 - ab) - 0 + 1(ab - ba) \neq 0$

or, $a^2 - ab \neq 0$

or $a(a-b) \neq 0$

or $\boxed{a \neq 0}$ or $\boxed{a \neq b}$

Thus $a, b \in R$ except $a \neq 0$ & $a \neq b$. Ans.

ii. Find the dimension of the vector space spanned by the vectors $\{[1,1,-2,0,1], [1,2,0,-4,1], [0,1,3,-3,2], [2,3,0,-2,0]\}$ and find a basis for the space. (1)

Soln:

The given vectors can be represented in matrix form as:

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 1 & 2 & 0 & -4 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 2 & 3 & 0 & -2 & 0 \end{bmatrix}$$

Applying Row transformations

$R_2 \rightarrow R_2 - R_1$

$R_4 \rightarrow R_4 - 2R_1$

$$\equiv \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 4 & -2 & -2 \end{bmatrix}$$

Applying again, $R_3 \rightarrow R_3 - R_2$

$R_4 \rightarrow R_4 - R_2$

$$\equiv \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix}$$

Finally applying $R_4 \rightarrow R_4 - 2R_3$, we get

$$\equiv \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}$$

From the above matrix, after row transformation it is clear that we have four linearly independent vectors. So, these 4 vectors will span the vector space of A.

Then, the basis are:

$$\{[1,1,-2,0,1],\ [0,1,2,-4,0],\ [0,0,1,1,2],\ [0,0,0,0,-6]\}$$

Also, dimension of the vector space

= number of linearly independent vectors / rank

= 4. Ans.

iii. Suppose that A is a matrix such that the complete solution to

$Ax = \begin{bmatrix}1\\4\\1\\1\end{bmatrix}_{4\times 1}$ is of the form:

$$x = \begin{bmatrix}0\\1\\1\end{bmatrix}_{3\times 1} + c\begin{bmatrix}0\\2\\1\end{bmatrix},\quad c \in R$$

(a) What can be said about the columns of matrix A? (0.5)

(b) Find the dimension of null space and rank of matrix A. (2)

Soln:

$$[A]_{m\times n}\begin{bmatrix}0\\1+2c\\1+c\end{bmatrix}_{3\times 1} = \begin{bmatrix}1\\4\\1\\1\end{bmatrix}_{4\times 1}$$

We know that matrix multiplication is possible only when no. of columns in A matrix = number of rows in B matrix (AB to exist)

$\therefore n = 3$ & $m = 4$

$\therefore$ Order of matrix $A = 4\times 3$ (4 rows & 3 columns)

$\Rightarrow$ matrix A has 3 columns. Ans.

The complete solution of $Ax = b$ will be all vectors that can be written as $x_p + x_n$, where x_p is one particular solution and x_n is a vector in the null space

Also we know that $\dim \text{Null}(A) + \dim \text{Col}(A) = n$ (col. A)

$\dim \text{Null}(A)$ → no. of free variables in row red. form of A

$\dim \text{Col}(A)$ → column corresp. to leading 1's in the row red. form of A

or, From Rank-Nullity theorem,

nullity (A) + rank (A) = columns (A)

$\because$ dim (null space) = number of special solution = 1

$\therefore$ 1 + rank(A) = 3

$\boxed{\dim(\text{null space}) = 1}$

or $\boxed{\text{Rank}(A) = 2}$