MFDS ASSIGNMENT - 1

GROUP NUMBER - 38

Solution for Q2.

(i) Find the LU decomposition of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$ when it exists. For which real numbers a and b does it exist.

Solution:

Suppose an $n \times n$ matrix A can be reduced to its row echelon form U without any row interchanges, that is, by using only the elementary row operations $R_i \to R_i - m_{ij}R_j$ for $j = 1, 2, \dots n-1$ and $i = j+1, j+2, \dots n$.

Define an $n \times n$ matrix $L := [l_{ij}]$ as follows.

$$l_{ij} := \begin{cases} m_{ij} & \text{if } i > j \\ 1 & \text{if } i = j \\ 0 & \text{if } i < j \end{cases}$$

By the above theorem, we need to Row Reduce the A matrix, without interchanging the Rows.

Thus, $A = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$

$$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{bmatrix} \begin{matrix} R_2 \to R_2 - (a)R_1 \\ R_3 \to R_3 - (b)R_1 \end{matrix}$$

$$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix} R_3 \to R_3 - \left(\frac{b}{a}\right) R_2$$

The last step $R_3 \to R_3 - \left(\frac{b}{a}\right)R_2$ is possible only when $a \neq 0$. Because we need the pivot element to be non-zero.

From the elementary Row operations, we have

$R_2 \to R_2 - (a)R_1$	Here from the theorem
$R_3 \to R_3 - (b)R_1$	$m_{21} = a$
$R_3 \to R_3 - \left(\frac{b}{a}\right)R_2$	$m_{31} = b$
	$m_{32} = \frac{b}{a}$

From this we can form the L matrix as follows

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & b/a & 1 \end{bmatrix}$$

Finally, we have,

$$\boxed{U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix}}$$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & b/a & 1 \end{bmatrix}}$$

$$\cancel{L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a/b & 1 \end{bmatrix}}$$

$A = LU$, will be possible only when $a \neq 0$.

So, a can be any real number other than zero.

and b can be any real number.

(ii) Find the dimension of the vectors space spanned by the vectors {[1,1,-2,0,1], [1,2,0,-4,1], [0,1,3,-3,2], [2, 3, 0,-2,0]} and find a basis for the space.

Solution:

Here we will form a matrix from the given ~~ma~~ vectors and let us name it as A.

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 1 & 2 & 0 & -4 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 2 & 3 & 0 & -2 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 4 & -2 & -2 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ \\ R_4 \to R_4 - 2R_1 \end{array}$$

$$\cong \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix} \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$\cong \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix} R_4 \to R_4 - 2R_3$$

Here, we have 4 linearly independent vectors.

So, these 4 vectors will span the vector space of A.

$\boxed{\text{Thus, the basis are } \{[1, 1, -2, 0, 1], [0, 1, 2, -4, 0], [0, 0, 1, 1, 2], [0, 0, 0, 0, -6]\}}$

Dimension of the vector space ~~3~~ ~~4~~ is the number of linearly independent ~~Dim~~ vectors.

$\boxed{\text{Dimension of the vector space} = 4}$

Q2) (iii) Suppose that A is a matrix such that the complete solution to $Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ is of the form:

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \; c \in R.$$

(a) What can be said about the columns of matrix A?

(b) Find the dimension of null space and rank of A.

Solution:

From the question $Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ we can infer

that $A_{(4\times m)} \times x_{(m\times 1)} = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}_{4\times 1}$.

Thus the dimension of A is $(4\times m)$ and x is $(m\times 1)$

From the complete solution $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \; c \in R$

we can infer that the dimension of x is (3×1)

So, finally we can say that

dimension of $x = (m\times 1) = (3\times 1)$. Thus $m = 3$.

So the dimension of $A = (4\times m) = (4\times 3)$.

$\boxed{\text{Matrix A has 3 Columns.}}$

The Complete Solution for A is

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}.$$

Complete Solution is a combination of particular Solution and special Solution.

$x = x_p + x_s$, where x_p is particular solution
x_s is special Solution
x is Complete Solution.

The number of special Solution will give us the dimension of null space.

Thus $\boxed{\dim(\text{null space}) = 1}$

By Rank-Nullity Theorem, we have.

$$\text{Rank}(A) + \text{Nullity}(A) = \text{Columns}(A)$$

We know that, $\text{Nullity}(A) = 1$ and ~~Re~~ $\text{Columns}(A) = 3$

So, $\boxed{\text{Rank}(A) = 2}$