Quantum Software Development

Lecture 6: Quantum Interference
Midterm Review

February 21, 2024

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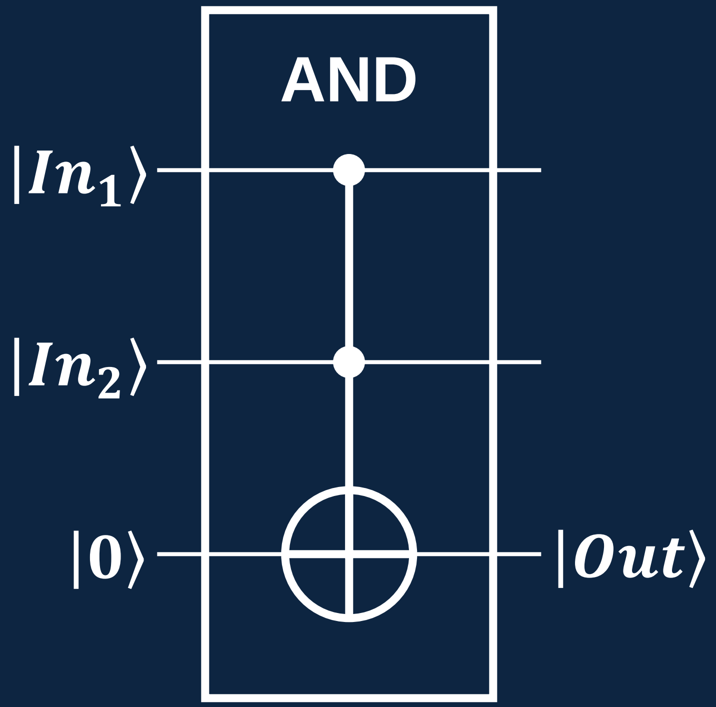
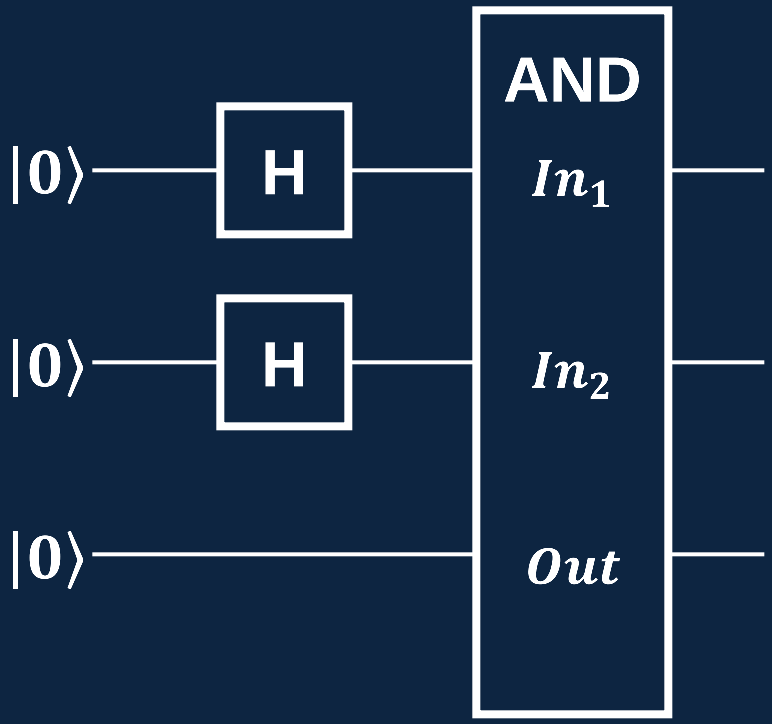
Quantum Interference

Superposition allows a computation to be performed for many input values at once. How is this useful?





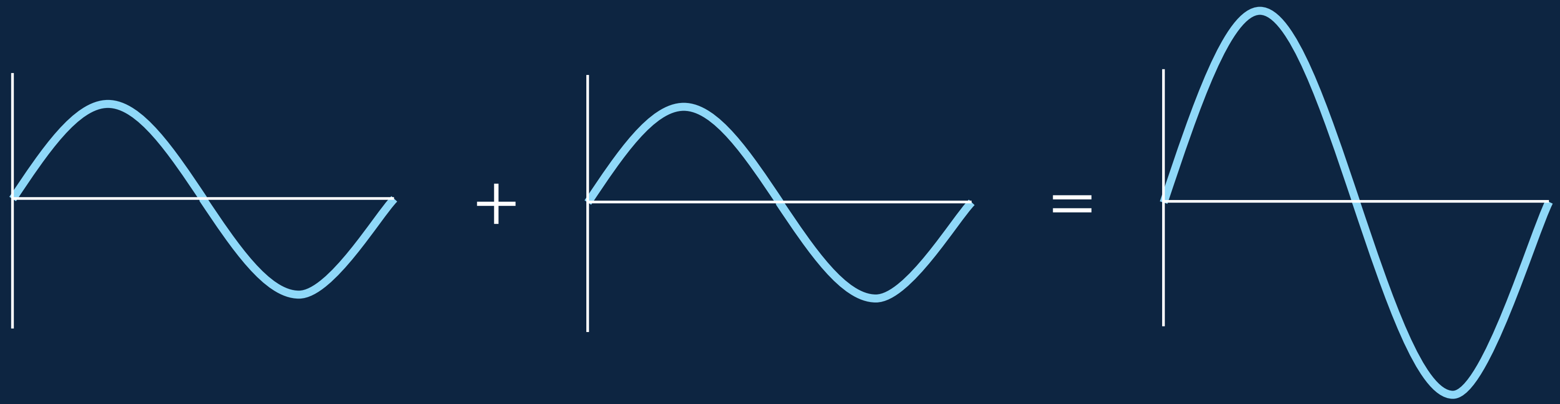
$$\frac{1}{2}\left(|00,0\rangle + |01,0\rangle + |10,0\rangle + |11,1\rangle\right)$$

Resulting state encodes the entire truth table!

But…only one of the terms can be measured at a time, randomly.

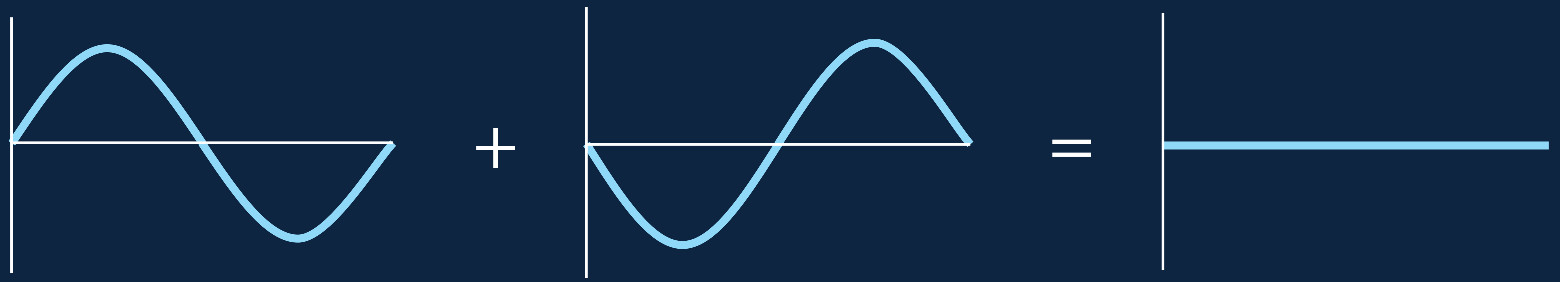
Signals with the same frequency interfere with each other based on their phase.

Constructive Interference (same phase)



Hearing Aid

Destructive Interference (opposite phase)



Noise-Cancelling Headphones

Superposition terms with the same value interfere with each other based on their phase.

$$H|+\rangle = H\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \rightarrow ?$$

$$H\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] = \frac{1}{\sqrt{2}}(H|0\rangle + H|1\rangle)$$

$$\rightarrow \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}(|0\rangle + |1\rangle + |0\rangle - |1\rangle)$$

$$= \frac{1}{2}(2 \cdot |0\rangle + 0 \cdot |1\rangle) = |0\rangle$$

Remember:

$$H|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Constructive interference on $|0\rangle$, destructive interference on $|1\rangle$

How does applying H to two qubits transform their state?

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -1 \\ 1 \\ -1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 1 \\ -1 \\ -1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -1 \\ -1 \\ 1 \end{matrix}} \end{bmatrix}$$

$$H^{\otimes 2}|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}\boxed{(|00\rangle + |01\rangle + |10\rangle + |11\rangle)}$$

$$H^{\otimes 2}|01\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}\boxed{(|00\rangle - |01\rangle + |10\rangle - |11\rangle)}$$

$$H^{\otimes 2}|10\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}\boxed{(|00\rangle + |01\rangle - |10\rangle - |11\rangle)}$$

$$H^{\otimes 2}|11\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}\boxed{(|00\rangle - |01\rangle - |10\rangle + |11\rangle)}$$

The i^{th} row of the j^{th} column of the Hadamard transform is given by $(-1)^{i \cdot j}$, where $i \cdot j$ is the bitwise dot product.

$$H_{12,10} = (-1)^{1100 \cdot 1010} = (-1)^{1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0} = (-1)^{1} = -1$$

$$H^{\otimes 2} = \frac{1}{\sqrt{2^2}} \begin{bmatrix} (-1)^{00 \cdot 00} & (-1)^{00 \cdot 01} & (-1)^{00 \cdot 10} & (-1)^{00 \cdot 11} \\ (-1)^{01 \cdot 00} & (-1)^{01 \cdot 01} & (-1)^{01 \cdot 10} & (-1)^{01 \cdot 11} \\ (-1)^{10 \cdot 00} & (-1)^{10 \cdot 01} & (-1)^{10 \cdot 10} & (-1)^{10 \cdot 11} \\ (-1)^{11 \cdot 00} & (-1)^{11 \cdot 01} & (-1)^{11 \cdot 10} & (-1)^{11 \cdot 11} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H^{\otimes 2}|10\rangle = \frac{1}{\sqrt{2^2}} \left((-1)^{00 \cdot 10}|00\rangle + (-1)^{01 \cdot 10}|01\rangle + (-1)^{10 \cdot 10}|10\rangle + (-1)^{11 \cdot 10}|11\rangle \right)$$

$$= \frac{1}{2} \boxed{(|00\rangle + |01\rangle - |10\rangle - |11\rangle)}$$

When the input is a superposition of multiple values, the corresponding outputs add together.

$$H^{\otimes 2}\left(\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)\right) = \frac{1}{\sqrt{2}}\cdot\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$= \frac{1}{\sqrt{2}}\cdot\left(\frac{1}{2}(|00\rangle + |00\rangle) + \frac{1}{2}(|01\rangle - |01\rangle) + \frac{1}{2}(|10\rangle + |10\rangle) + \frac{1}{2}|11\rangle - |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}(1|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Check:

$$H^{\otimes 2}|0,+\rangle = H|0\rangle \otimes H|+\rangle = |+,0\rangle$$

$$H^{\otimes 2}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

What is the result of applying the Hadamard transform to each of these states?

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

$$\frac{1}{2}\left(\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}1+1+1-1\\1-1+1+1\\1+1-1+1\\1-1-1-1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2\\2\\2\\-2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{8}}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} + \frac{1}{\sqrt{8}}\begin{bmatrix}1\\-1\\-1\\1\\-1\\1\\1\\-1\end{bmatrix}\right) = \frac{1}{4}\begin{bmatrix}2\\0\\0\\2\\0\\2\\2\\0\end{bmatrix}$$

$$= \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$$

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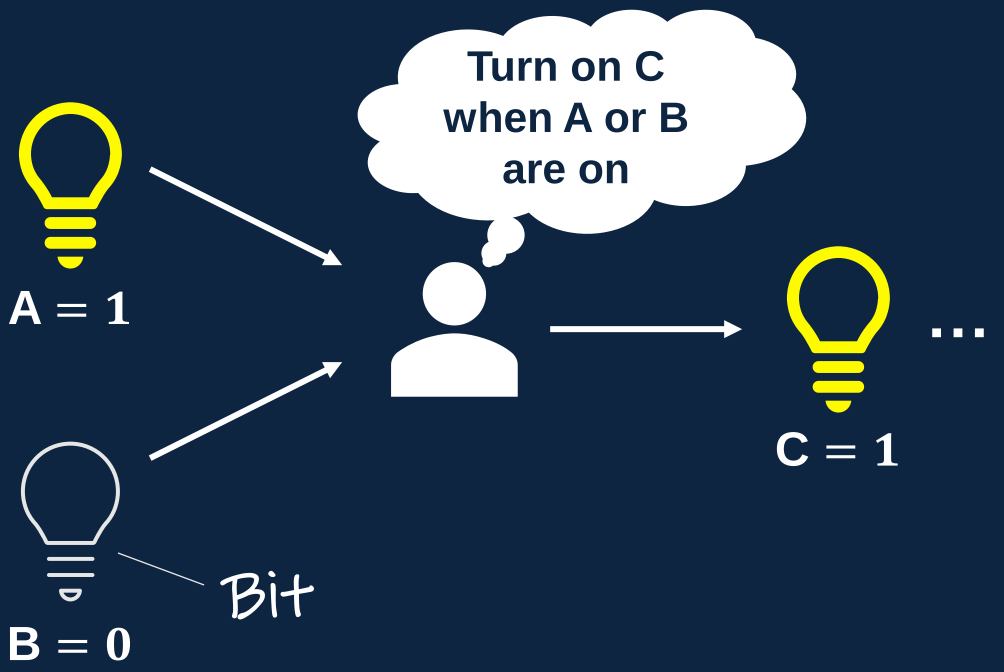
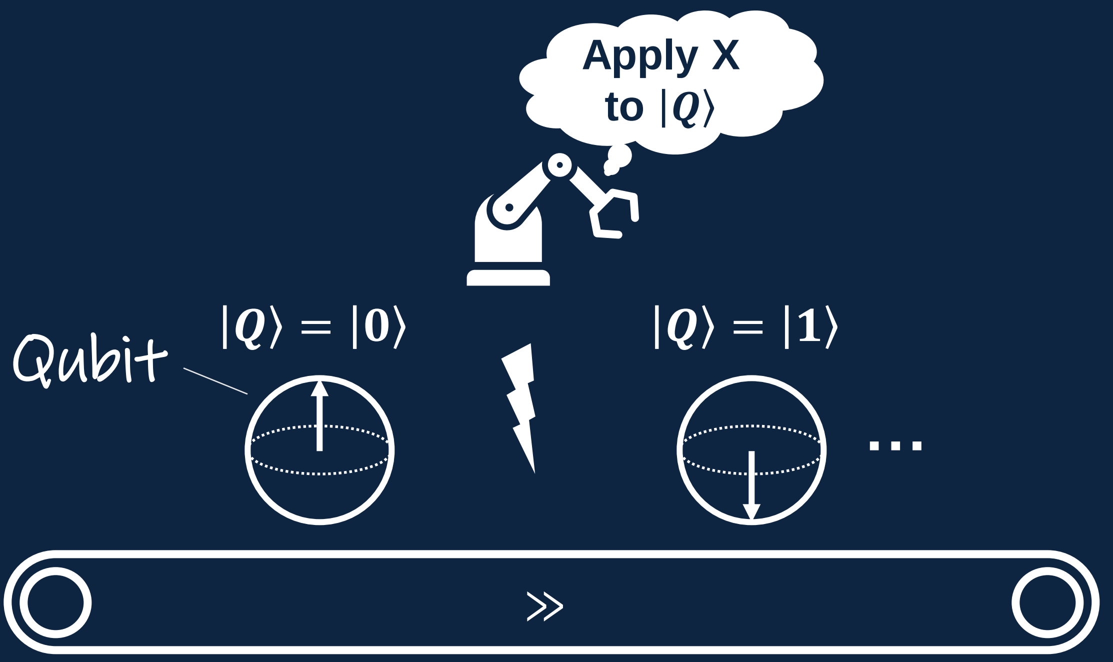
Midterm Review

Quantum computers process information using the principles of quantum mechanics, not digital logic.

Classical computation is a series of small decisions performed by digital logic gates.

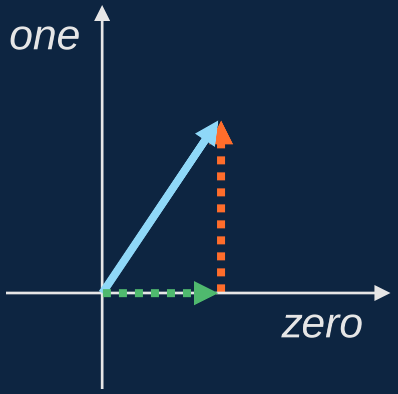
Quantum computation is a series of small transformations to a quantum state.





A quantum state is made of components that determine the probability of a given measurement outcome.

A qubit's state is the <u>superposition</u> (sum) of its $|0\rangle$ and $|1\rangle$ components.



Amplitudes

$$|Q\rangle = a|0\rangle + b|1\rangle$$

Classical analog

```
struct qubit {
    complex<double> zero;
    complex<double> one;
};
```

Measurement sets a qubit to $|0\rangle$ or $|1\rangle$ probabilistically.

$$\text{Prob}_{|0\rangle} = |a|^2 \quad |Q\rangle = |0\rangle$$

$$\text{Prob}_{|1\rangle} = |b|^2 \quad |Q\rangle = |1\rangle$$

$$\boxed{|a|^2 + |b|^2 = 1}$$

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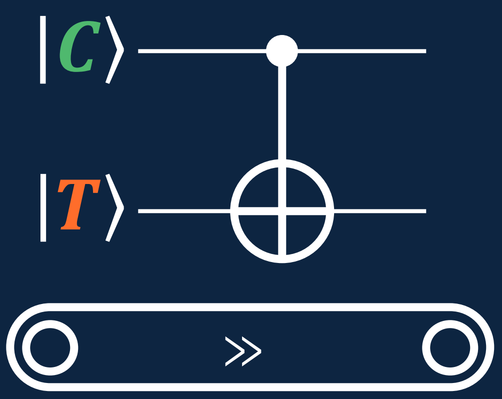
Quantum logic gates are applied independently to each component of the quantum state – without measurement.

The CNOT gate flips the value of the target qubit if the control qubit is a $|1\rangle$.




```
if (c) {
    t = !t;
}
```

Classical analog

$|C\rangle = |0\rangle \to$ do nothing

$|C\rangle = |1\rangle \to$ flip $|T\rangle$

Both branches can be computed at once; the result is an <u>entangled</u> two-qubit state.

$$|C\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \quad |T\rangle = |0\rangle$$

$$\mathrm{CNOT}|C, T\rangle$$

$$|T\rangle = |0\rangle \qquad |T\rangle = |1\rangle$$

$$\frac{1}{\sqrt{2}}|0, 0\rangle + \frac{1}{\sqrt{2}}|1, 1\rangle$$

C and T always have the same value when measured!

After a gate is applied, components corresponding to the same measurement outcome are combined.

Phase does not affect measurement but is relevant when applying a gate.

H gate

Positive phase

$$H|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$H|1\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Negative phase

Constructive and destructive interference can occur when combining like terms.

$$|Q\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$H|Q\rangle$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(2 \cdot \frac{1}{\sqrt{2}}|0\rangle + 0 \cdot \frac{1}{\sqrt{2}}|1\rangle\right) = |0\rangle$$

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Exam 1 Topics

Quantum Information

Qubit, amplitude, state vector, Dirac notation, superposition, measurement, phase

Quantum Logic Gates

Gate model of quantum computation, matrix definition, common single-qubit gates

Bloch Sphere

Visualization of single-qubit states, spherical coordinates, parametrized gates

Multi-Qubit States

Tensor product of state vectors, registers, applying gates to complex superpositions

Quantum Control Logic

Entanglement, controlled operations, phase kickback, common multi-qubit gates

Quantum Communication

Superdense coding, quantum key distribution

Quantum Error Correction

Bit-flip code, Shor code, Steane code

Quantum Interference

Hadamard transform

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