Quantum Software Development

Lecture 9: The Quantum Fourier Transform

March 27, 2024

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The Quantum Fourier Transform

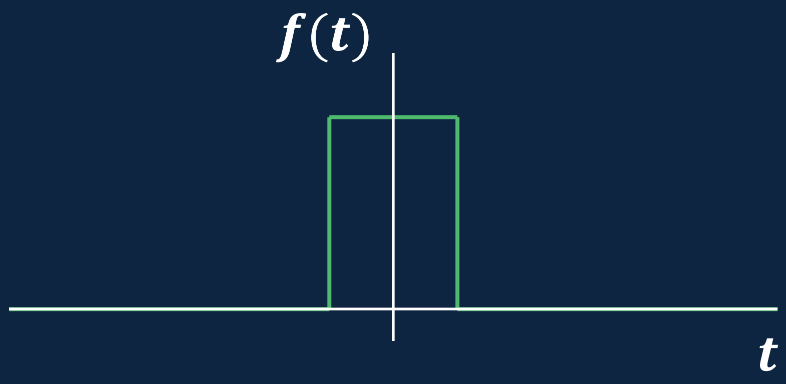
The information in a signal is often encoded in its frequency components.

The human ear senses the frequency components (pitches) in a sound.

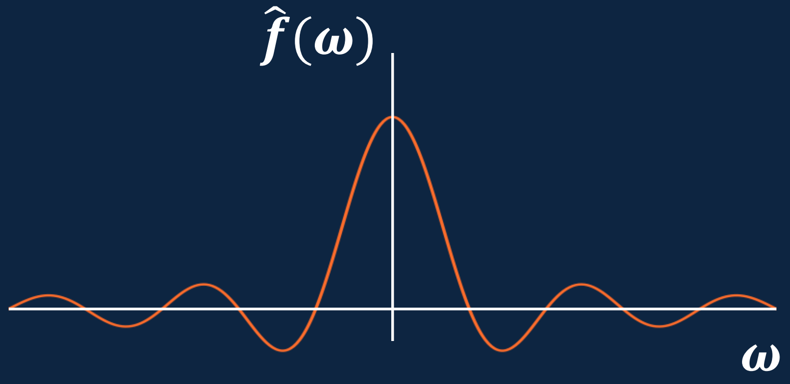
Any signal can be decomposed into a series of sine waves.

$$\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x) + \cdots$$

The Fourier transform maps a signal between the time and frequency domains.

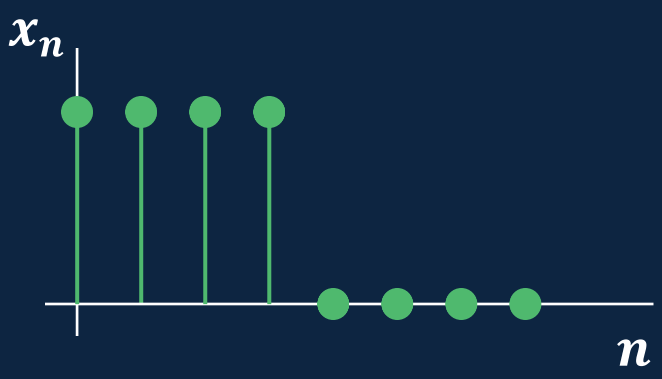


$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt$$



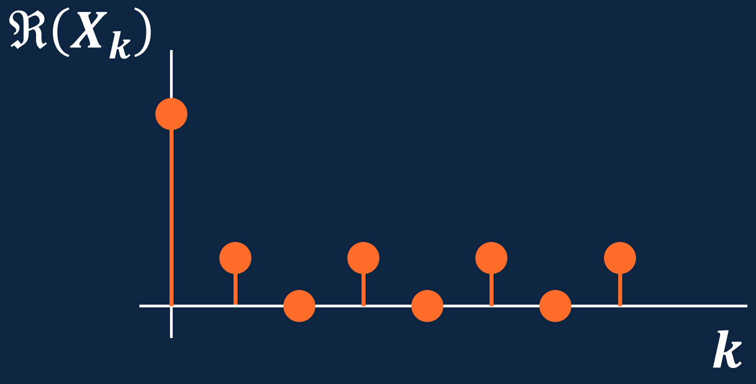
The discrete Fourier transform decomposes a signal with N samples into N frequency “bins”

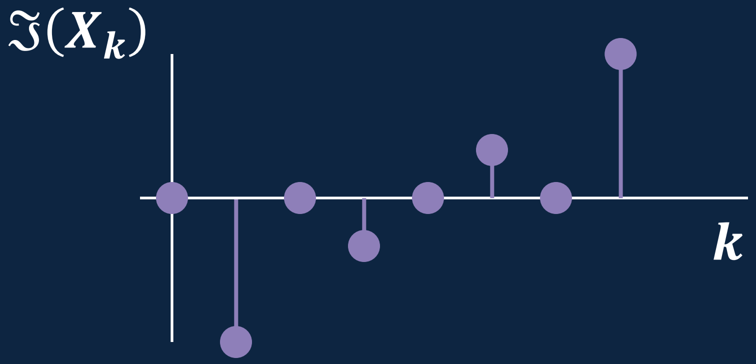
$N = 8$ samples



$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn}$$

X_k corresponds to a frequency of $\frac{k}{N}$

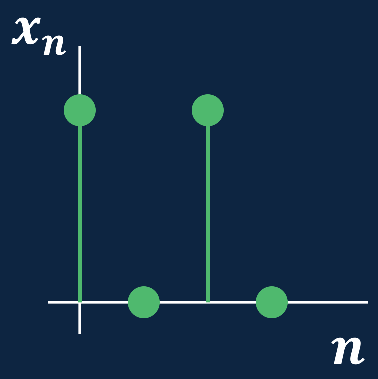




What is the DFT of this discrete signal?

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn}$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



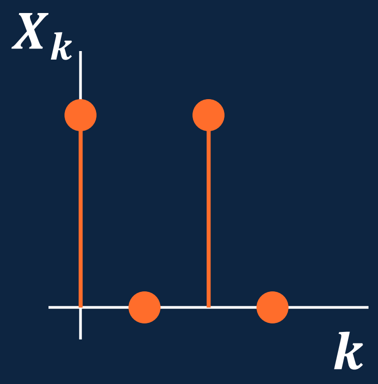
$$X_0 = \sum_{n=0}^{3} x_n e^{0} = 1 + 1 = 2$$

$$X_1 = \sum_{n=0}^{3} x_n e^{-\frac{\pi i}{2}n} = e^{0} + e^{-\pi i} = 1 - 1 = 0$$

$$X_2 = \sum_{n=0}^{3} x_n e^{-\pi i n} = e^{0} + e^{-2\pi i} = 1 + 1 = 2$$

$$X_3 = \sum_{n=0}^{3} x_n e^{-\frac{3\pi i}{2}n} = e^{0} + e^{-3\pi i} = 1 - 1 = 0$$

$$X_k = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$



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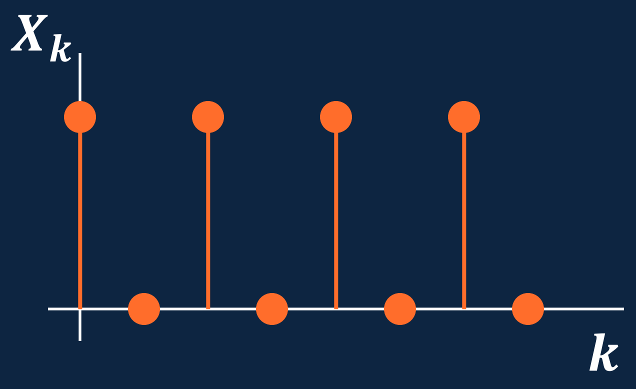
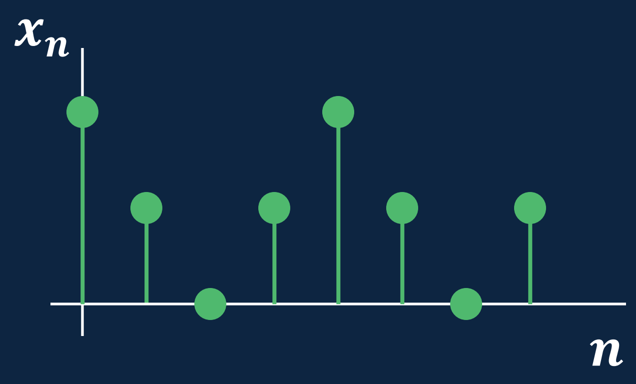
The inverse DFT “reconstructs” a time-domain signal from its frequency components.



$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N}kn}$$

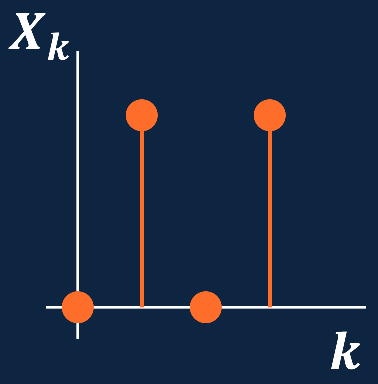


The DFT is an invertible, linear transformation.

What is the IDFT of this discrete signal?

$$\boxed{x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N}kn}}$$

$$X = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$



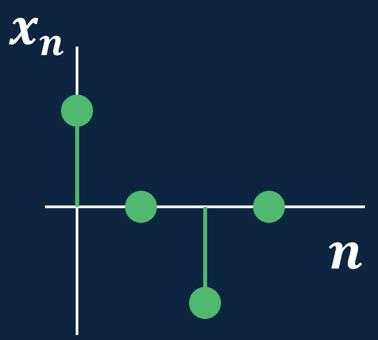
$$x_0 = \frac{1}{4}\sum_{k=0}^{3} X_k e^{0} = \frac{1}{2}(1+1) = 1$$

$$x_1 = \frac{1}{4}\sum_{k=0}^{3} X_k e^{\frac{\pi i}{2}k} = \frac{1}{2}\left(e^{\frac{\pi i}{2}} + e^{\frac{3\pi i}{2}}\right) = \frac{1}{2}(i - i) = 0$$

$$x_2 = \frac{1}{4}\sum_{k=0}^{3} X_k e^{\pi i k} = \frac{1}{2}\left(e^{\pi i} + e^{3\pi i}\right) = \frac{1}{2}(-1-1) = -1$$

$$x_3 = \frac{1}{4}\sum_{k=0}^{3} X_k e^{\frac{3\pi i}{2}k} = \frac{1}{2}\left(e^{\frac{3\pi i}{2}} + e^{\frac{9\pi i}{2}}\right) = \frac{1}{2}(-i + i) = 0$$

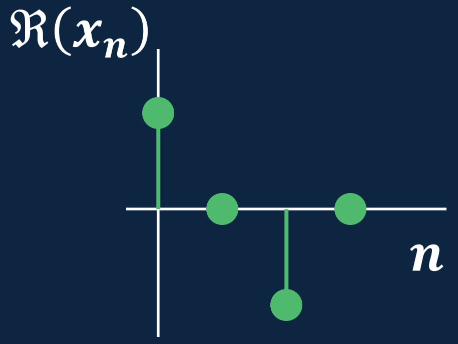
$$x = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$



How might a discrete signal be encoded into a quantum state?

$$x_n = e^{\frac{\pi i}{2}n}$$

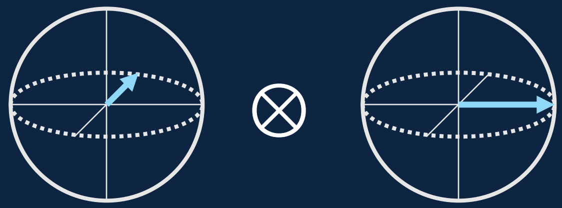
$$x = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$$

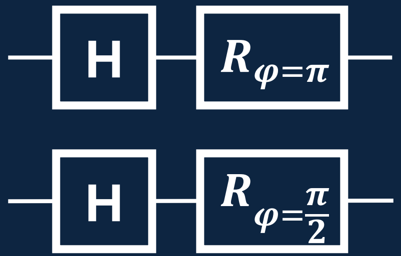


Encode the samples as the amplitudes of the state vector.

$$|x\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \frac{1}{2}(|00\rangle + i|01\rangle - |10\rangle - i|11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |-\rangle \otimes |i\rangle$$



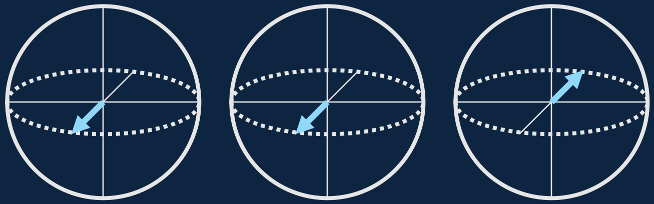


A discrete signal with frequency $f = \frac{1}{2^k}$ can be constructed using phase rotations.

$\boxed{\text{Example: } N = 8}$

$$f = \frac{1}{2} \qquad x_n = e^{\pi i n}$$

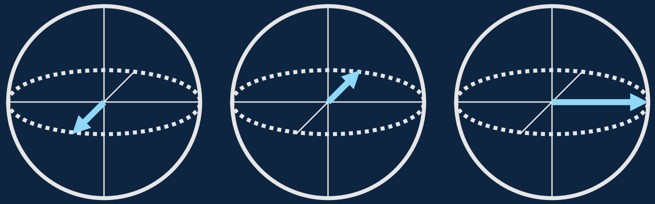
$$|x\rangle = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = |+\rangle \otimes |+\rangle \otimes |-\rangle$$



$\boxed{R_{\varphi=\pi}}$

$$f = \frac{1}{4} \qquad x_n = e^{\frac{\pi i}{2} n}$$

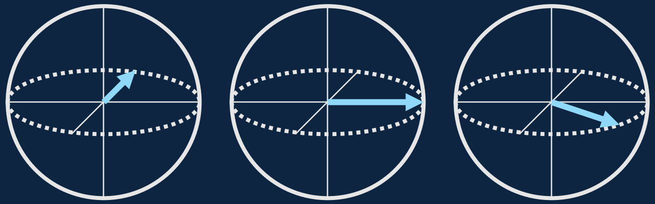
$$|x\rangle = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \\ 1 \\ i \\ -1 \\ -i \end{bmatrix} = |+\rangle \otimes |-\rangle \otimes |i\rangle$$



$\boxed{R_{\varphi=\pi}}$ $\boxed{R_{\varphi=\frac{\pi}{2}}}$

$$f = \frac{1}{8} \qquad x_n = e^{\frac{\pi i}{4} n}$$

$$|x\rangle = |-\rangle \otimes |i\rangle \otimes \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{e^{\frac{\pi i}{4}}}{\sqrt{2}} |1\rangle \right)$$

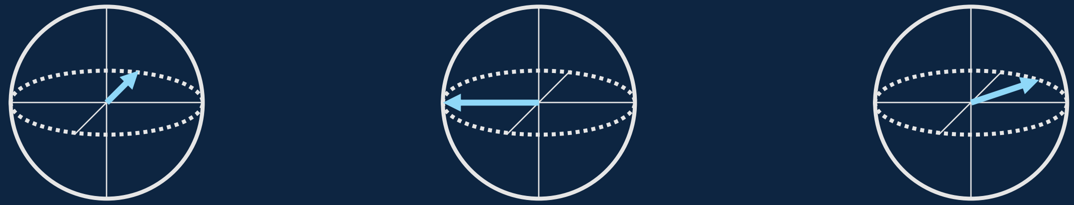


$\boxed{R_{\varphi=\pi}}$ $\boxed{R_{\varphi=\frac{\pi}{2}}}$ $\boxed{R_{\varphi=\frac{\pi}{4}}}$

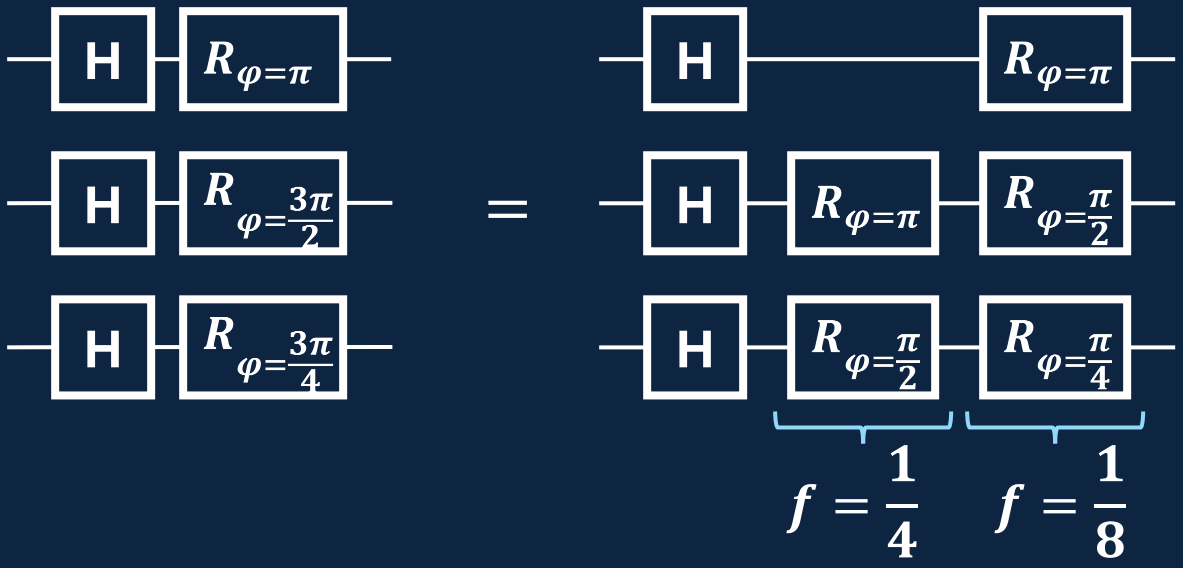
How could an 8-point signal with frequency $f = \frac{3}{8}$ be constructed?

$$x_n = e^{\frac{3\pi i}{4} n}$$

$\lvert x_n \rangle$	Amplitude
$\lvert 000 \rangle$	$e^{\frac{0\pi i}{4}}$
$\lvert 001 \rangle$	$e^{\frac{3\pi i}{4}}$
$\lvert 010 \rangle$	$e^{\frac{6\pi i}{4}}$
$\lvert 011 \rangle$	$e^{\frac{9\pi i}{4}} = e^{\frac{1\pi i}{4}}$
$\lvert 100 \rangle$	$e^{\frac{4\pi i}{4}}$
$\lvert 101 \rangle$	$e^{\frac{7\pi i}{4}}$
$\lvert 110 \rangle$	$e^{\frac{10\pi i}{4}} = e^{\frac{2\pi i}{4}}$
$\lvert 111 \rangle$	$e^{\frac{5\pi i}{4}}$



$$\frac{1}{\sqrt{8}}\left(\lvert 0\rangle + e^{\pi i}\lvert 1\rangle\right) \otimes \left(\lvert 0\rangle + e^{\frac{3\pi i}{2}}\lvert 1\rangle\right) \otimes \left(\lvert 0\rangle + e^{\frac{3\pi i}{4}}\lvert 1\rangle\right)$$



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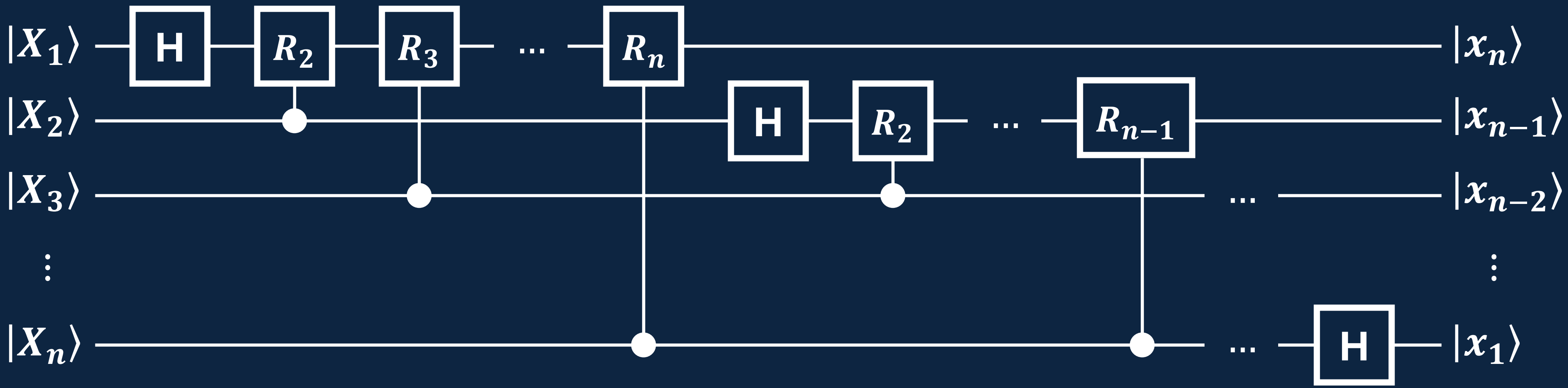
The quantum Fourier transform constructs a discrete signal of any frequency.



$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{bmatrix}$$

$$QFT|X\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\frac{2\pi i}{N} X n} |n\rangle$$

Output is in reverse order!

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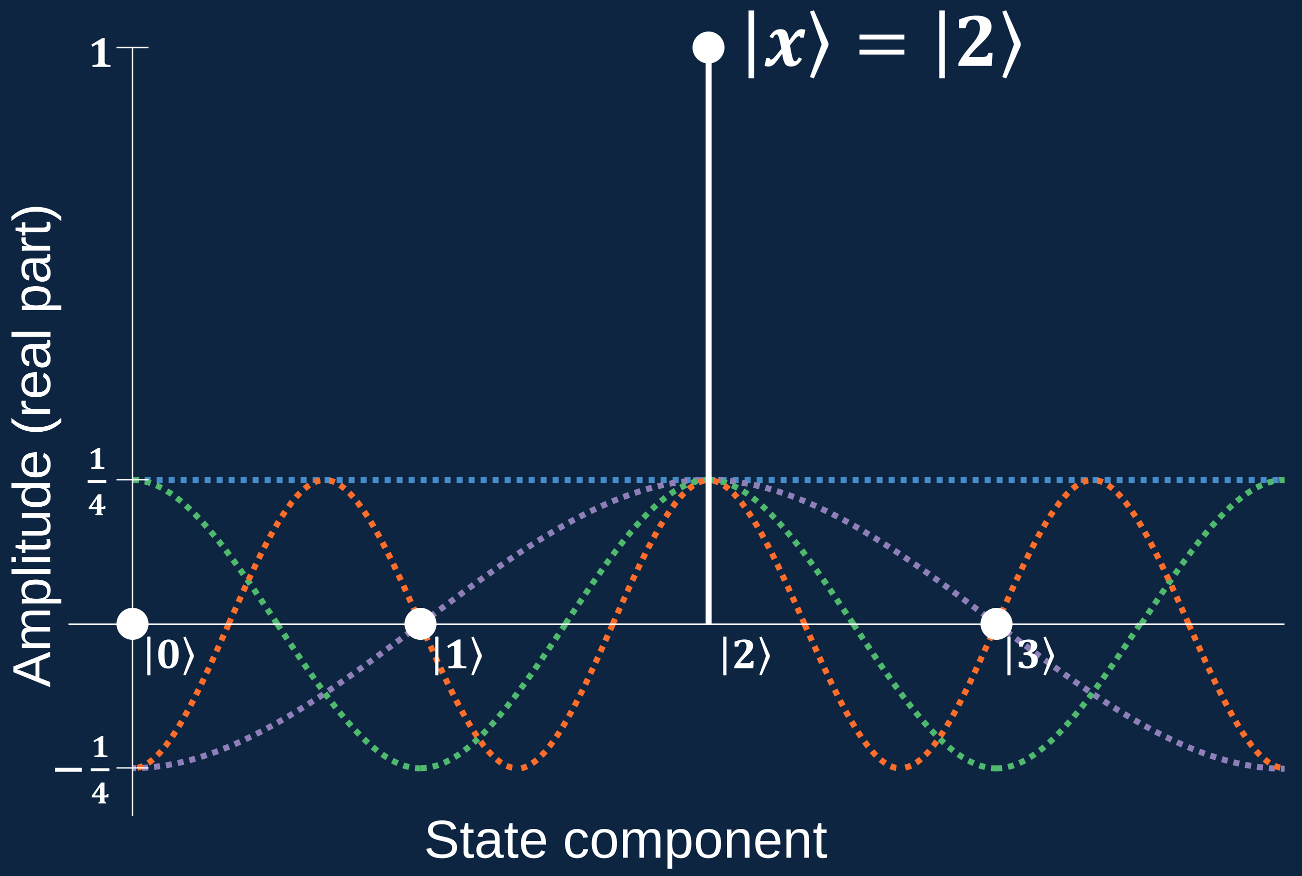
The QFT acts like the IDFT when a superposition of frequencies are input, due to interference.

Frequency components

$$|X\rangle = \frac{1}{2}\left(|0\rangle - |1\rangle + |2\rangle - |3\rangle\right)$$

$\mathrm{QFT}|X\rangle$



The QFT is $O\left((\log N)^2\right)$, while the FFT is $O(N \log N)$!

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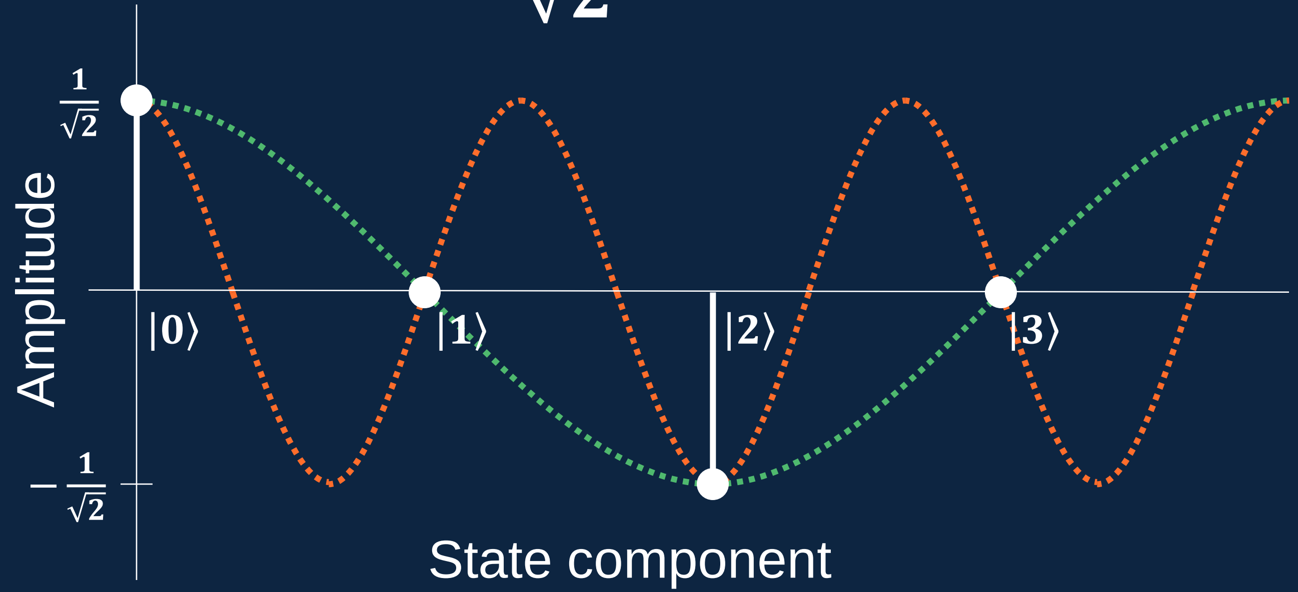
The inverse (adjoint) QFT decomposes a discrete signal into its frequency components, like the DFT.

$$|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |2\rangle)$$



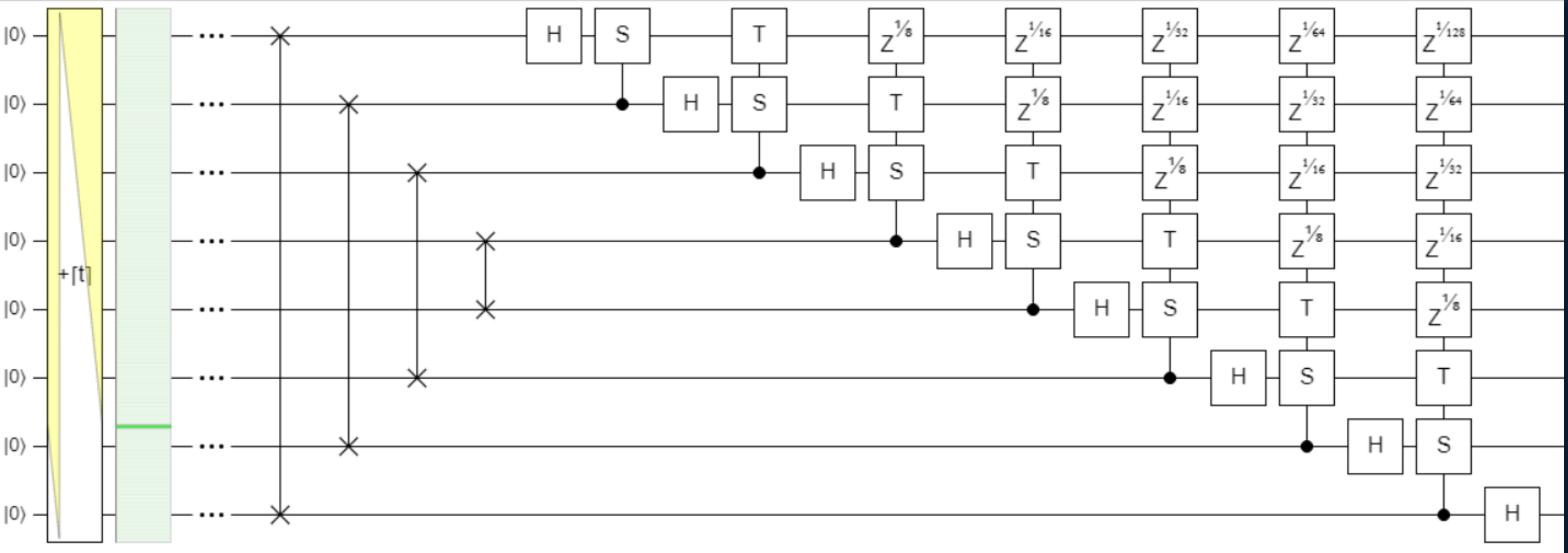
$$\mathrm{QFT}^\dagger|x\rangle$$

$$|X\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$$

$$QFT^\dagger|x\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{-\frac{2\pi i}{N}xk}|k\rangle$$

Try the QFT in Quirk.

- Go to https://algassert.com/quirk.
- Click on the Quantum Fourier Transform example circuit.
- How does the output pattern change with the input?



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