Quantum Software Development

EE-193 / CS-150 | Spring 2024 | Tufts University

Lecture 2: Visualizing Single-Qubit States

Tufts University School of Engineering

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Visualizing Single-Qubit States

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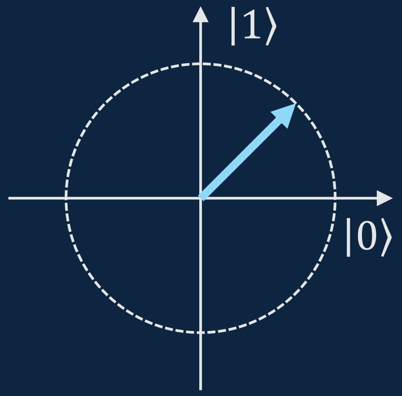
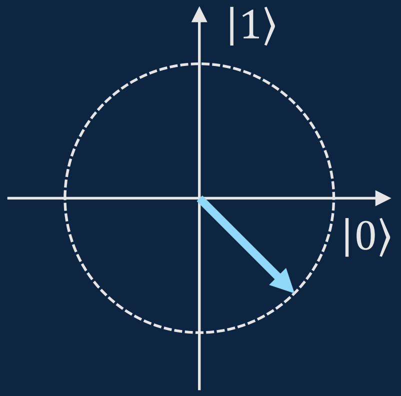
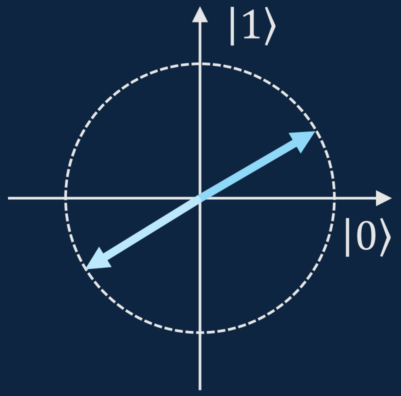
In quantum computing, “phase” refers to the complex argument of each superposition term.







$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} -a \\ -b \end{bmatrix}$$

Same magnitude, but different *relative* phase

***Global* phase differences are indistinguishable**

Which of these states are equivalent?

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2}}(e^{\frac{i\pi}{3}}|0\rangle + e^{\frac{i\pi}{3}}|1\rangle)$$

$$\frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$\frac{1}{\sqrt{2}}(-|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2}}(e^{\frac{i\pi}{3}}|0\rangle - e^{\frac{i\pi}{3}}|1\rangle)$$

By convention, the $|0\rangle$ part is always positive and real; all phase information is contained in the $|1\rangle$ part.

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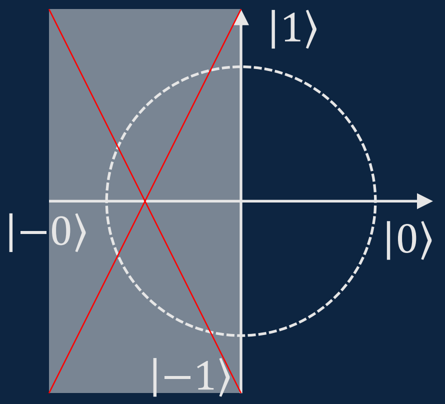
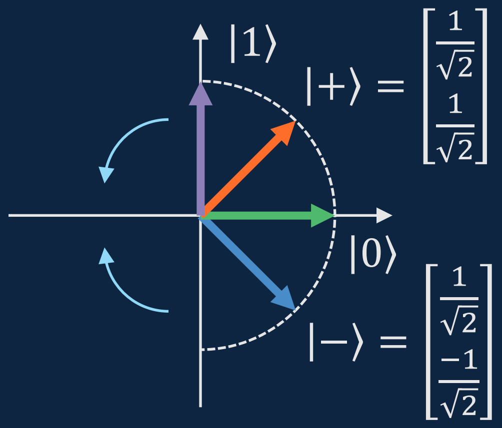
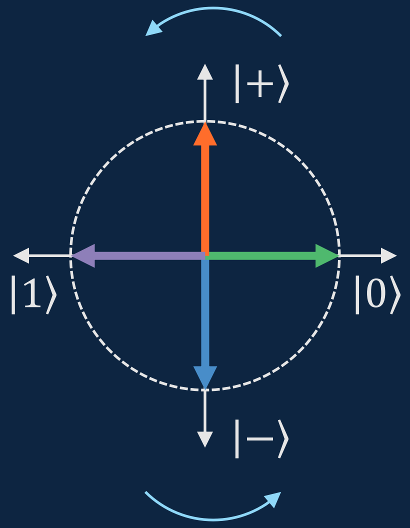
A mathematical transformation allows the set of unique, real-valued states to be represented on a circle.



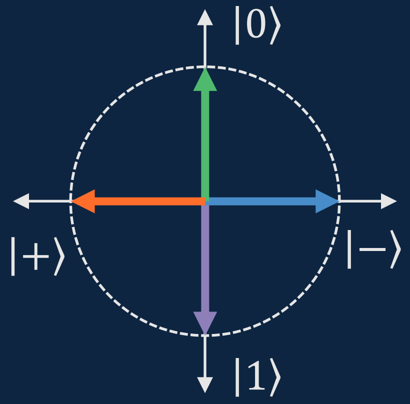
Delete the left half of the circle



“Stretch” it back into a circle



Rotate 90° counterclockwise



Next, add complex numbers back in…

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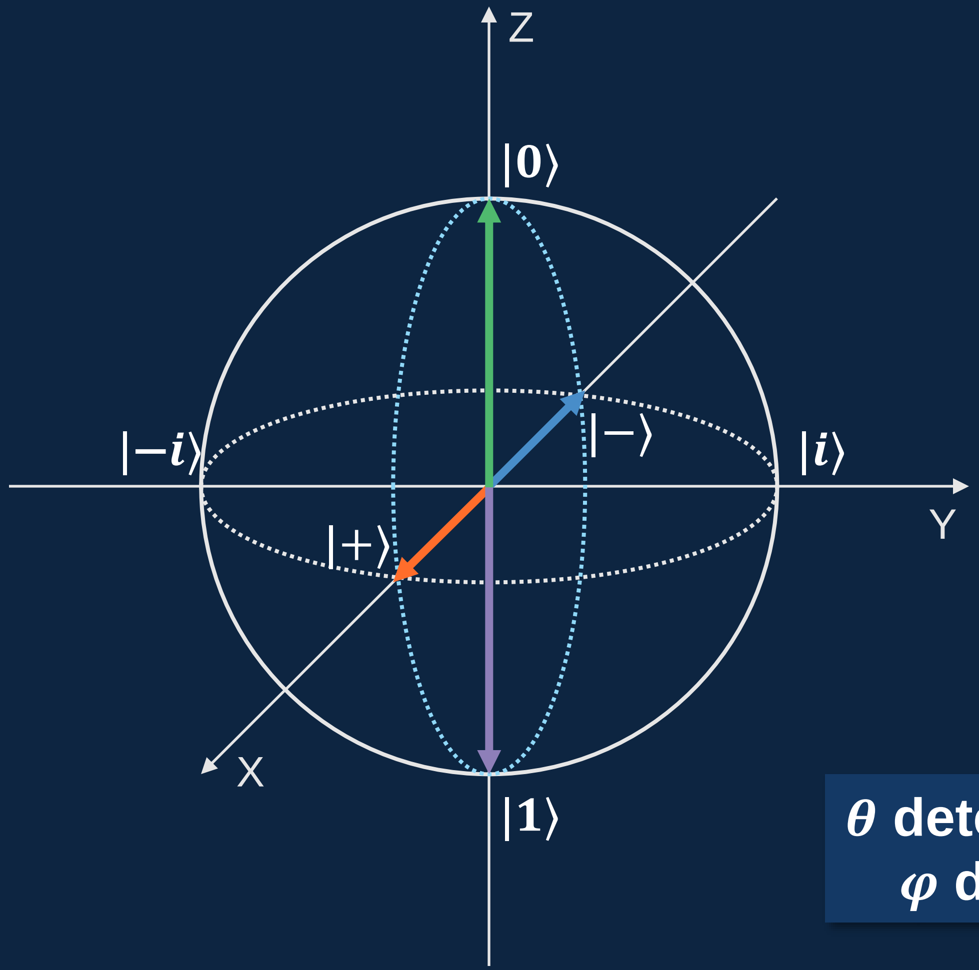
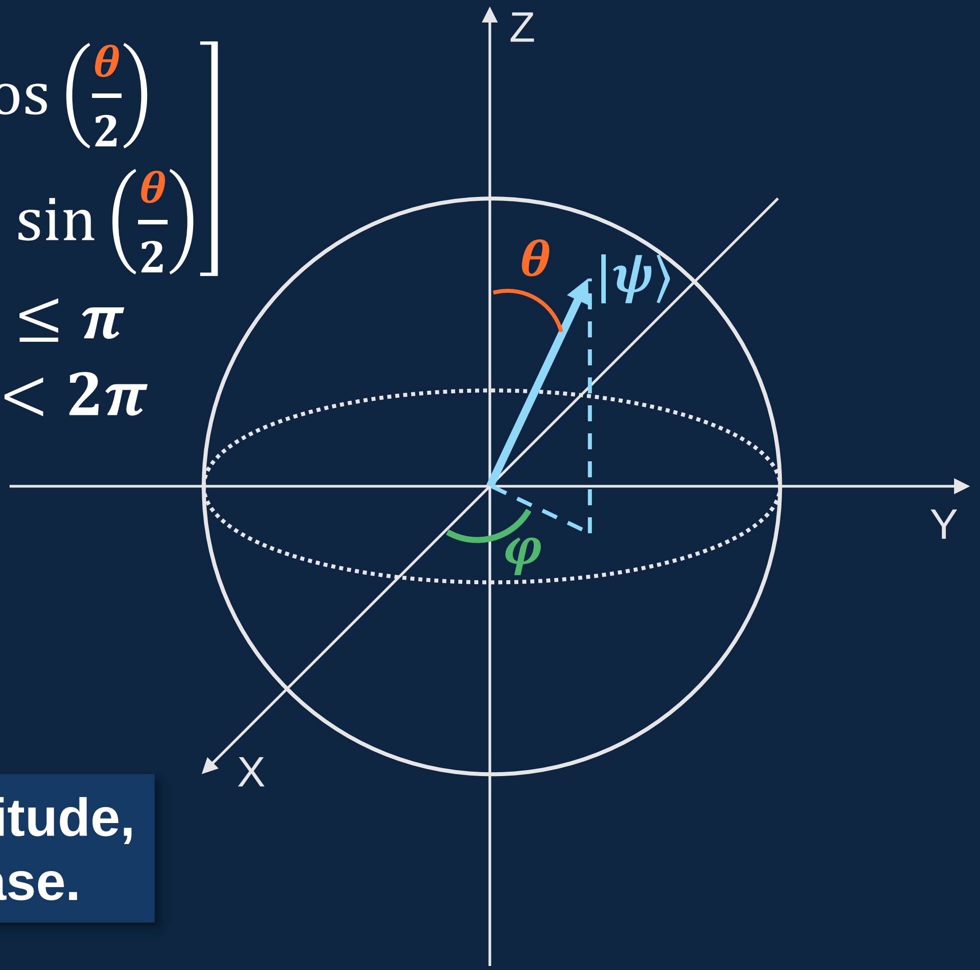
The Bloch Sphere is the standard way of visualizing a single-qubit state.



$$|\psi\rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \varphi < 2\pi$$

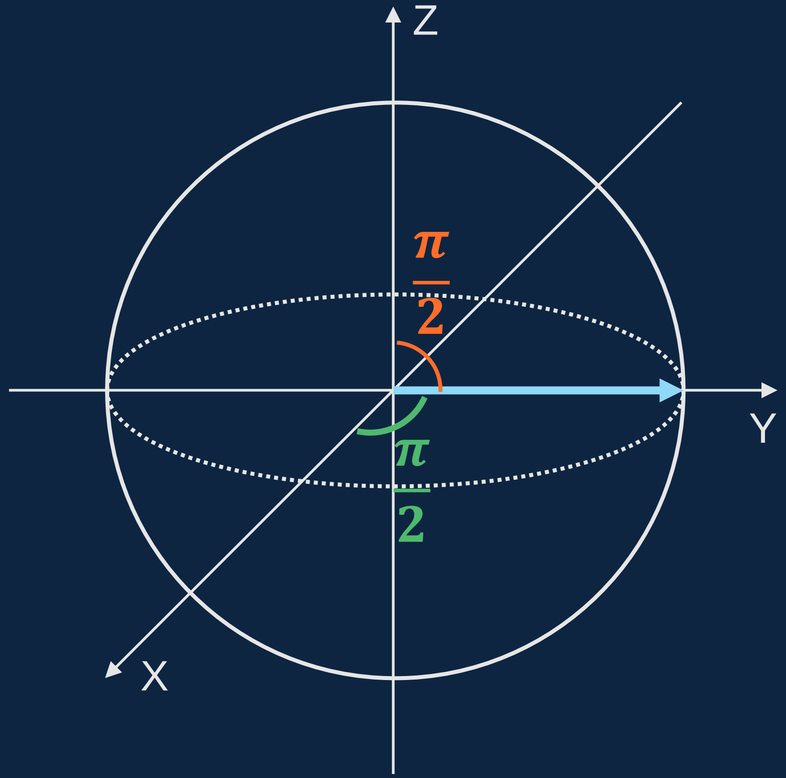


θ determines magnitude, φ determines phase.

What is the state of each qubit represented on the Bloch sphere?

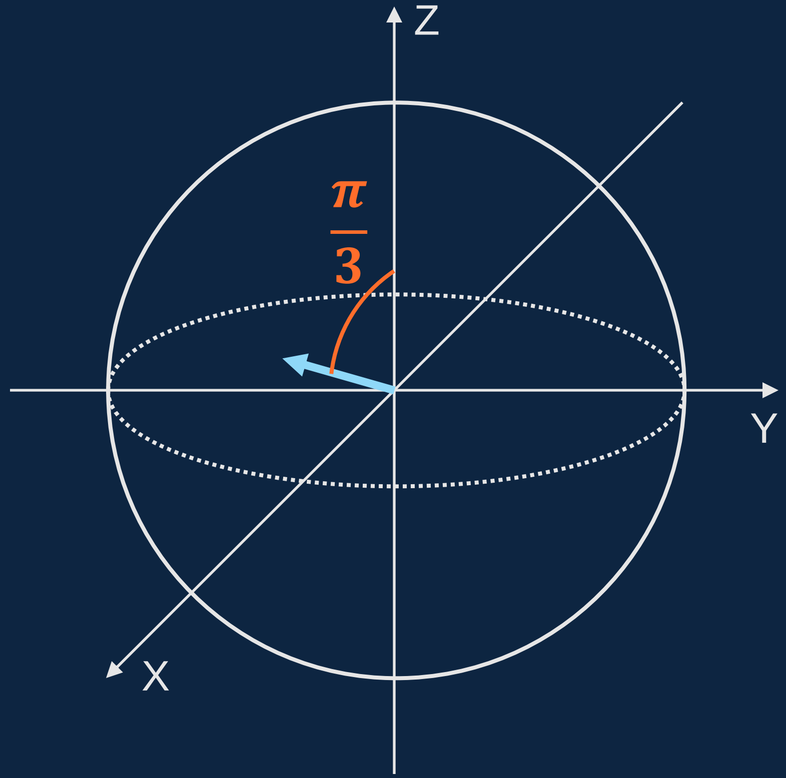
$$|\psi\rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$(\theta, \varphi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$



$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) \\ e^{\frac{i\pi}{2}}\sin\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$(\theta, \varphi) = \left(\frac{\pi}{3}, 0\right)$$

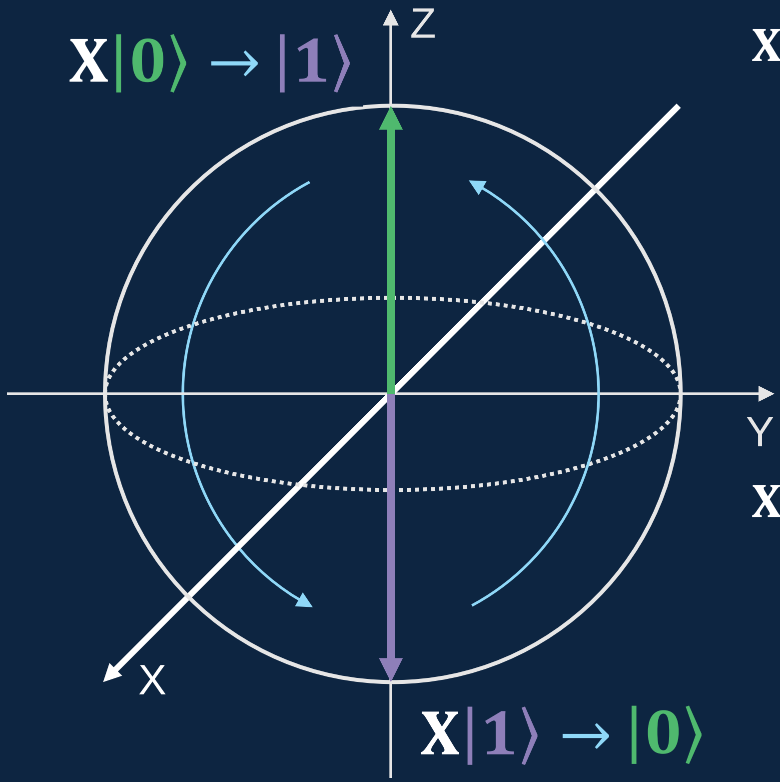


$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) \\ e^{0}\sin\left(\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

Single-qubit gates rotate qubits around the Bloch sphere.

X gate rotates 180° about X-axis

$\mathbf{X}|0\rangle \rightarrow |1\rangle$

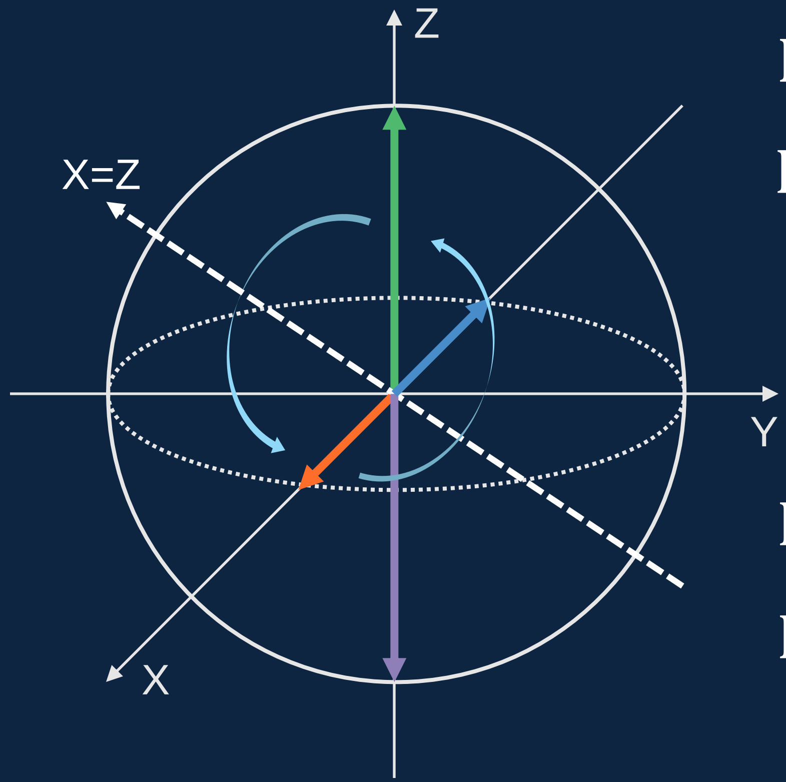


$\mathbf{X}|1\rangle \rightarrow |0\rangle$

$$\mathbf{X}|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv |1\rangle$$

$$\mathbf{X}|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 + 1 \cdot 1 \\ 1 \cdot 0 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv |0\rangle$$

H gate rotates 180° about the line X=Z



$\mathrm{H}|0\rangle \rightarrow |+\rangle$

$\mathrm{H}|+\rangle \rightarrow |0\rangle$

$\mathrm{H}|1\rangle \rightarrow |-\rangle$

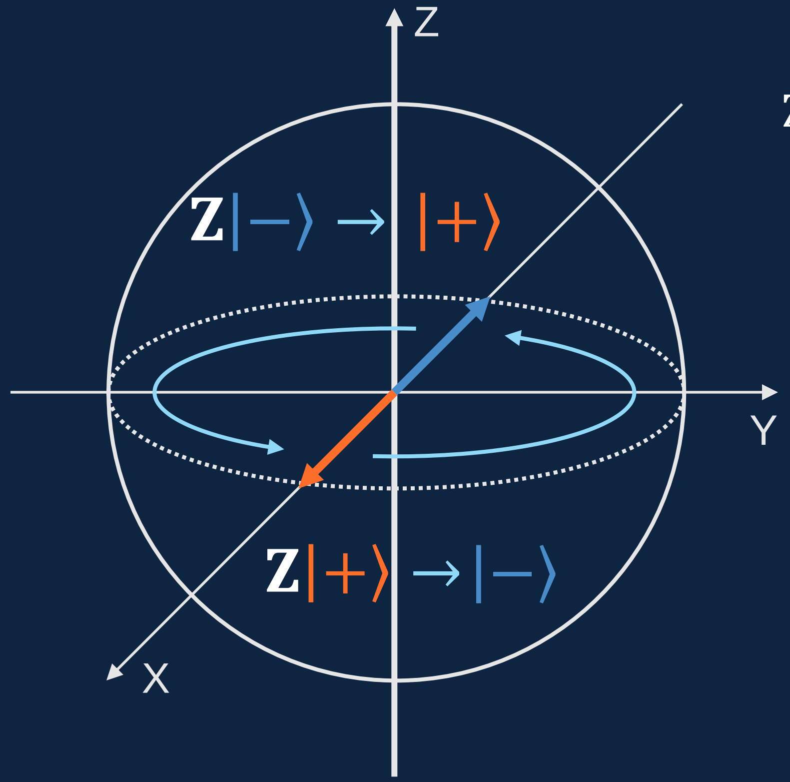
$\mathrm{H}|-\rangle \rightarrow |1\rangle$

The Z gate rotates 180° about the Z-axis; it flips the phase of the $|1\rangle$ term.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definition

Symbol



$$Z|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} \\ 0 \cdot \frac{1}{\sqrt{2}} - 1 \cdot \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \equiv |-\rangle$$

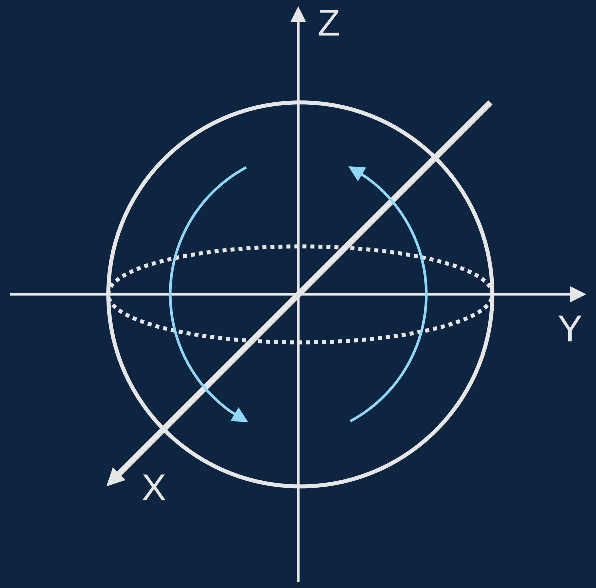
$$Z|-\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} \\ 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \equiv |+\rangle$$

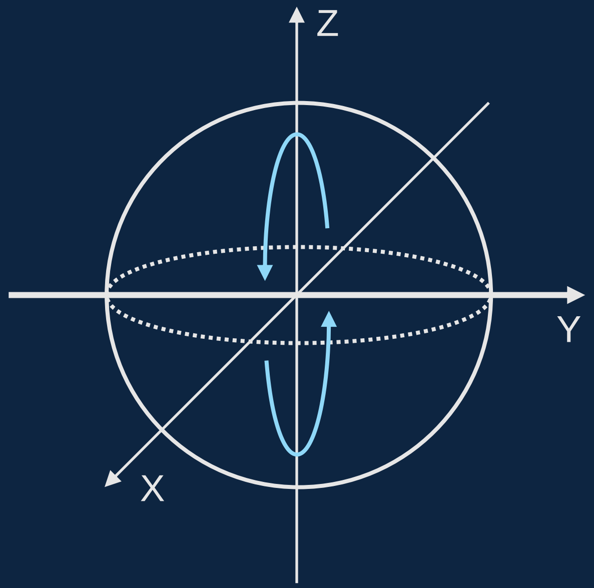
Parametrized gates allow for arbitrary rotations.

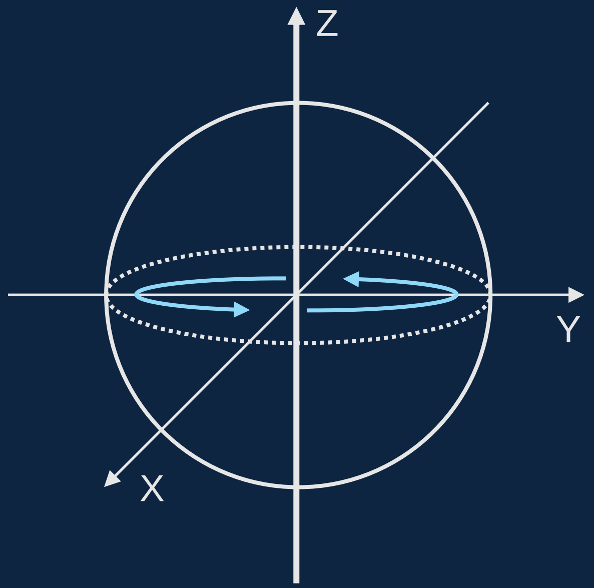
$$\mathrm{R_X} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\mathrm{R_Y} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\mathrm{R}_{\varphi} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$$







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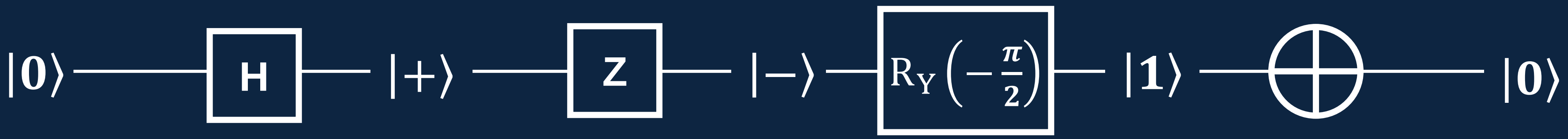
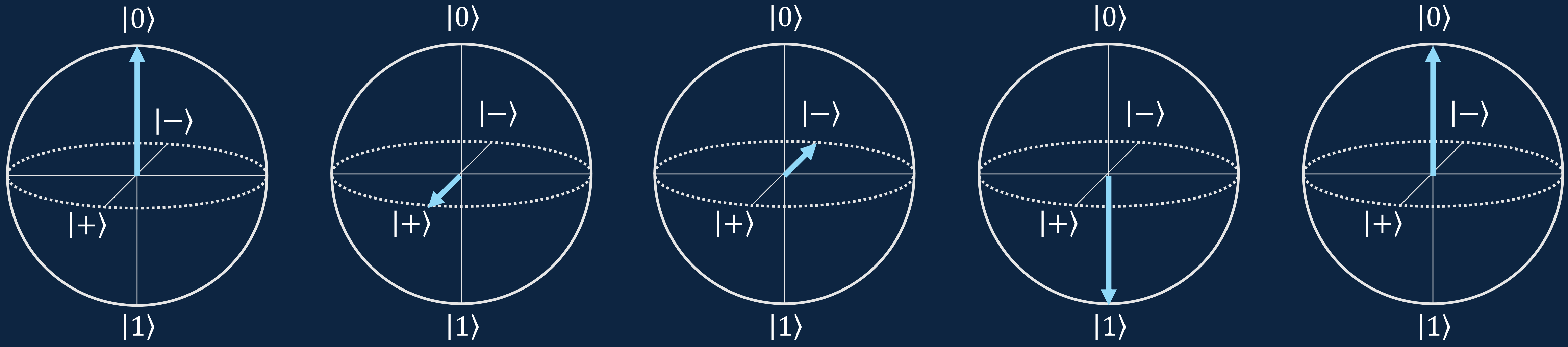
What is the state of the qubit after each gate?





Common Single-Qubit Gates

I (Identity)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Does nothing

H (Hadamard)

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Rotates 180° about X=Z

Z (Phase-flip)

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotates 180° about Z-axis

R_x

$$\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Rotates θ radians about X-axis

R_z

$$\begin{bmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{bmatrix}$$

Rotates θ radians about Z-axis **with** global phase applied

X (Not)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Rotates 180° about X-axis

Y

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Rotates 180° about Y-axis

S = $\sqrt{Z}$

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Rotates 90° about Z-axis

T = $\sqrt{S}$

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$$

Rotates 45° about Z-axis

R_y

$$\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Rotates θ radians about Y-axis

R_φ

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$$

Rotates φ radians about Z-axis **without** global phase applied

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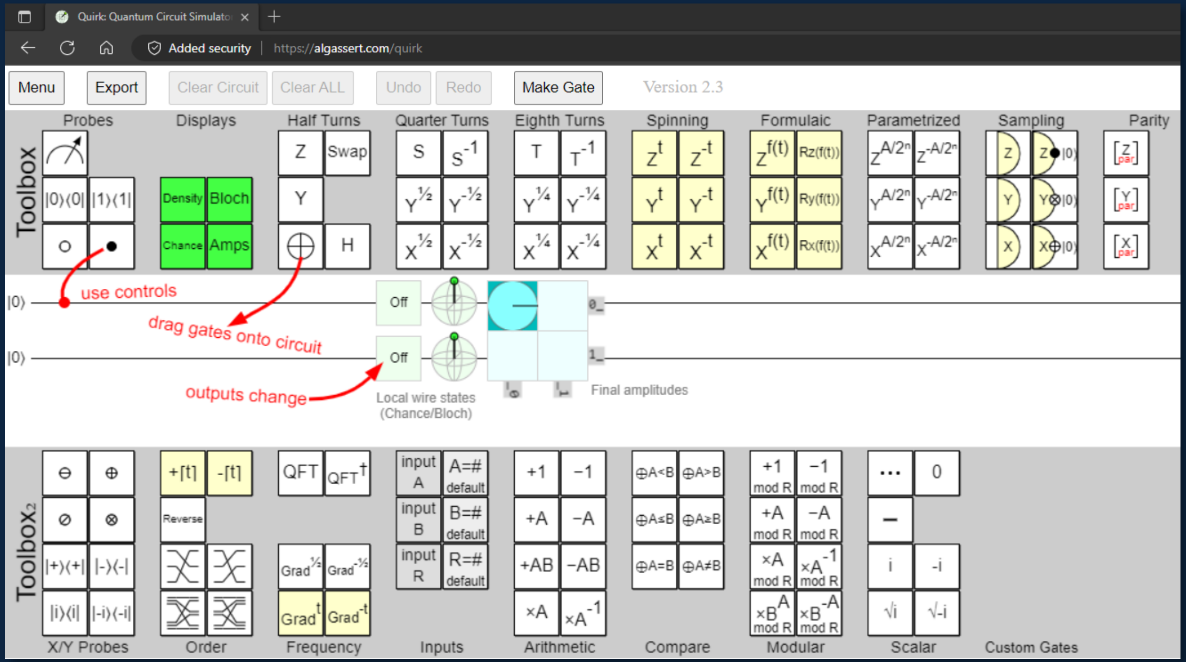
Try single-qubit gates in Quirk.

- Go to https://algassert.com/quirk
- Apply gates to the first qubit.
- How does each gate affect the Bloch sphere representation of the qubit's state?
- How could you prepare an arbitrary single-qubit state using the "formulaic" gates?



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