Quantum Software Development

Lecture 3: Working with Multiple Qubits, Quantum Control Logic

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Working With Multiple Qubits

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A multi-qubit state is composed by taking the tensor product of each constituent qubit's state.

$$|\psi_1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad |\psi_2\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$|\psi\rangle = |\psi_1, \psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$= \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a \cdot c \\ a \cdot d \\ b \cdot c \\ b \cdot d \end{bmatrix}$$

E.g., two $|0\rangle$ qubits together:

$$|00\rangle = |0\rangle \otimes |0\rangle$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} |00\rangle \text{ component} \\ |01\rangle \text{ component} \\ |10\rangle \text{ component} \\ |11\rangle \text{ component} \end{array}$$

An n-qubit quantum state can be expressed as a vector in $\mathbb{C}^{2^n}$, i.e., 2^n complex-valued elements.

Like with single qubits, Dirac notation shows the possible measurement outcomes for a multi-qubit system.

$$|Q\rangle = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

$$P =$$

$$|w|^2 \qquad |x|^2 \qquad |y|^2 \qquad |z|^2$$

$$|Q\rangle = |00\rangle \qquad |Q\rangle = |01\rangle \qquad |Q\rangle = |10\rangle \qquad |Q\rangle = |11\rangle$$

Multiple qubits together are called a register.

In Q#, a register is an array of qubits.

For example,

```
use register = Qubit[2];
```

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What is the complete state vector for each of these multi-qubit states?

$$|1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

$$|+\rangle \otimes |+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$|0\rangle \otimes |-\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$$

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Assume implicit identity gates when applying a single-qubit gate to a multi-qubit system.

$$|\psi\rangle = |\psi_1\psi_2\rangle = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

$$\boxed{X(\psi_1) \rightarrow ?}$$

$$X|\psi_1\rangle \otimes I|\psi_2\rangle = (X \otimes I)|\psi_1\psi_2\rangle = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)|\psi\rangle$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ w \\ x \end{bmatrix} = y|00\rangle + z|01\rangle + w|10\rangle + x|11\rangle$$

Matrix math is tedious. Use Dirac notation instead!

$$|\psi\rangle = |\psi_1\psi_2\rangle = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

$$\boxed{X(\psi_1) \rightarrow ?}$$

$$X|\psi_1\rangle \otimes |\psi_2\rangle = w(X|0\rangle \otimes |0\rangle) + x(X|0\rangle \otimes |1\rangle) + y(X|1\rangle \otimes |0\rangle) + z(X|1\rangle \otimes |1\rangle)$$

$$= w(|1\rangle \otimes |0\rangle) + x(|1\rangle \otimes |1\rangle) + y(|0\rangle \otimes |0\rangle) + z(|0\rangle \otimes |1\rangle)$$

$$= w|10\rangle + x|11\rangle + y|00\rangle + z|01\rangle$$

$$= y|00\rangle + z|01\rangle + w|10\rangle + x|11\rangle$$

Knowing how a gate transforms $|0\rangle$ and $|1\rangle$ is sufficient for Dirac notation calculations!

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When working with registers containing many qubits, it is typical to express their values in decimal.

$$|\psi\rangle = |\psi_1\psi_2\psi_3\psi_4\rangle = \frac{1}{2}(|0010\rangle + |0110\rangle + |1010\rangle + |1110\rangle)$$

$$= \frac{1}{2}(|2\rangle + |6\rangle + |10\rangle + |14\rangle)$$

Uniform Superposition:

$$|\psi\rangle = |+,+,\dots+\rangle = \frac{1}{\sqrt{2^n}}(|00\dots0\rangle + |00\dots1\rangle + \cdots + |11\dots1\rangle)$$

$$= \frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle + \cdots + |2^n - 1\rangle) = \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}|k\rangle$$

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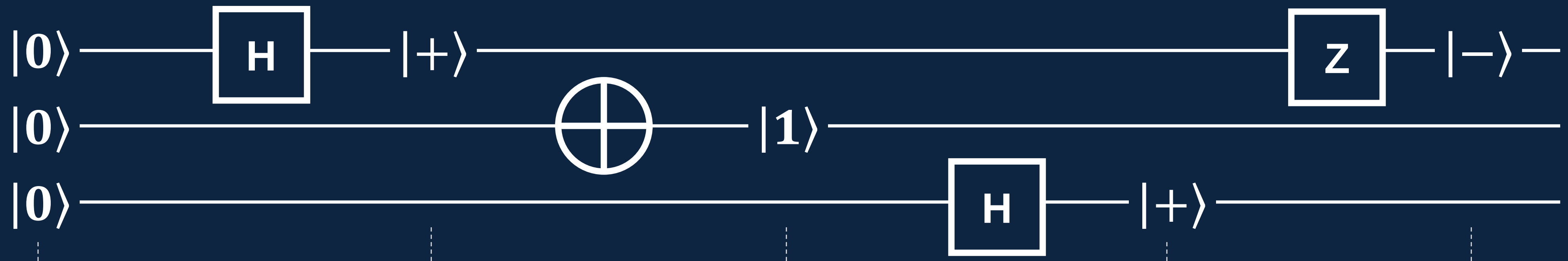
What is the state of the system after each gate?



$$|000\rangle$$

$$\frac{1}{\sqrt{2}}(|000\rangle + |100\rangle)$$

$$\frac{1}{\sqrt{2}}(|010\rangle + |110\rangle)$$

$$\frac{1}{2}(|010\rangle + |011\rangle + |110\rangle + |111\rangle)$$

$$\frac{1}{2}(|010\rangle + |011\rangle - |110\rangle - |111\rangle)$$

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Quantum Control Logic

How would you translate this function into a quantum operation?

```
bool ToggleConditionally (bool a, bool b) {
    if (a) {
        b = !b;
    }
    return b;
}
```

```
operation ToggleConditionally (Qubit a, Qubit b) : Unit {
    if M(a) == One {
        X(b);
    }
}
```

Measurement destroys superposition!

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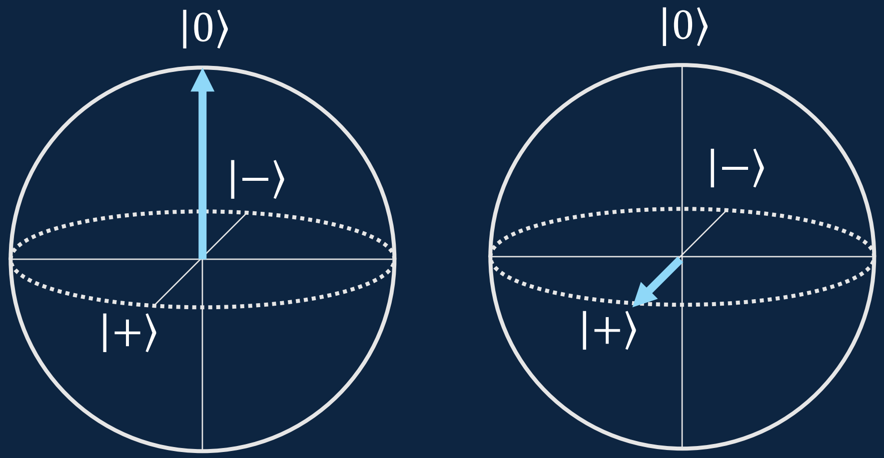
In some multi-qubit states, the state of each individual qubit cannot be expressed independently.

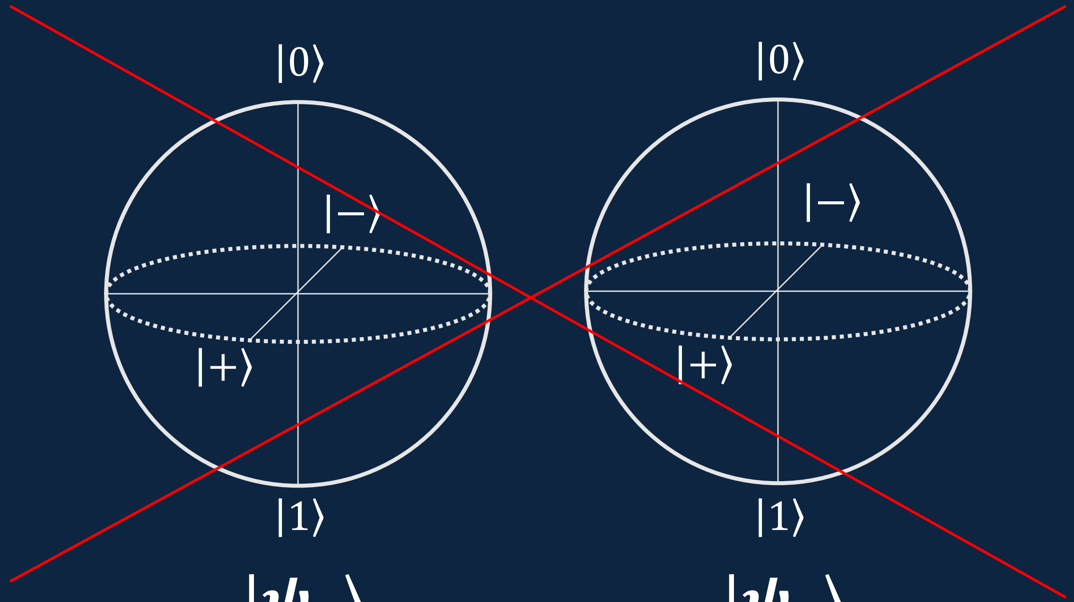
$$|\psi_1\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$
$$= |0\rangle \otimes |+\rangle$$



$|\psi_1\rangle$ $|\psi_2\rangle$

$$|\psi_1\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

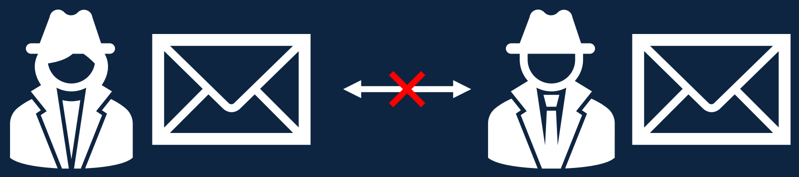
ψ_1 and ψ_2 are <u>entangled</u>.



$|\psi_1\rangle$ $|\psi_2\rangle$

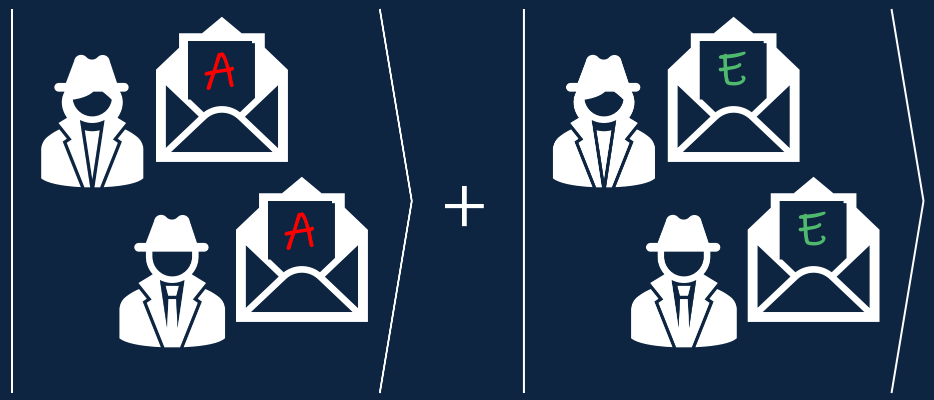
An imperfect classical analogy: Secret Orders

Two spies carry a secret message with orders to either execute or abort a mission. They cannot communicate with each other.



Since the messages are identical, only one of the spies needs to read it for the outcome of the mission to be determined.

A message sent with a superposition of “execute” and “abort” means the actions of the spies are entangled.



For which of these states are qubits entangled?

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

The two qubits will always be different when measured

$$\frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$$

$$= |0\rangle \otimes |-\rangle$$

No entanglement

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$= |+\rangle \otimes |+\rangle$$

No entanglement

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

The three qubits will always be the same when measured

$$\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$$

If the first qubit is a $|0\rangle$, the other two will be the same as each other; if it's a $|1\rangle$, the other two will be different from each other

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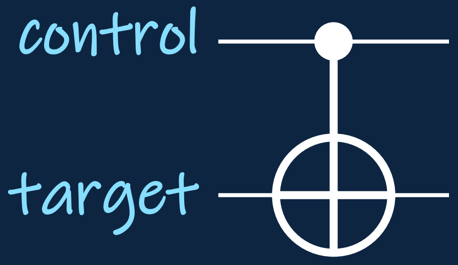
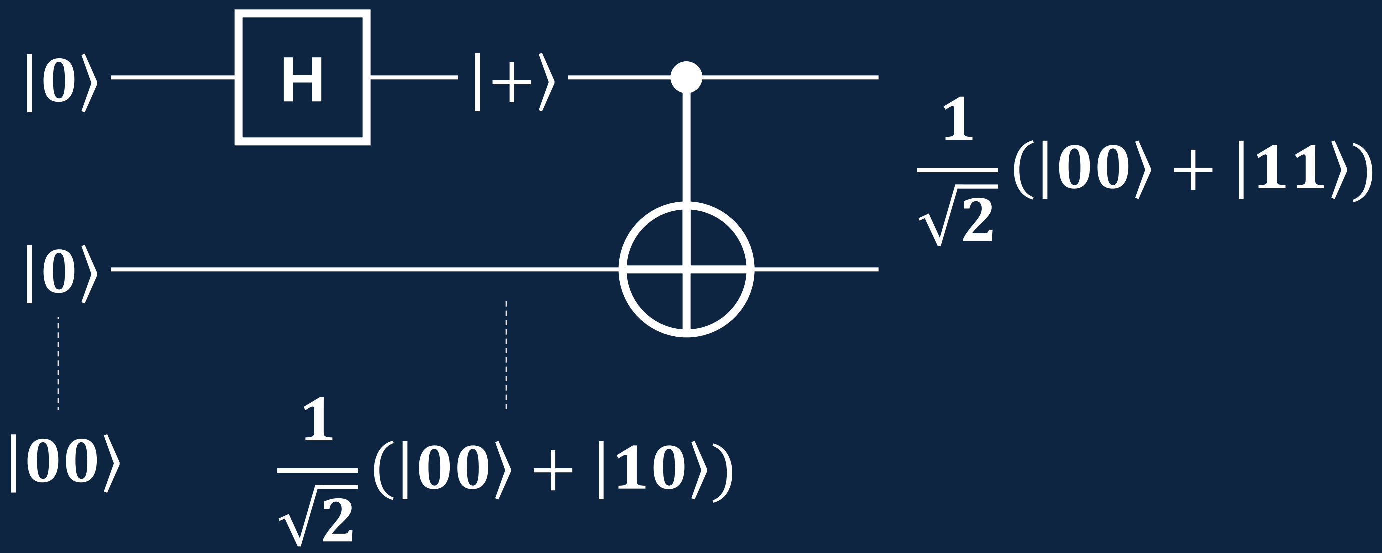
Multi-qubit gates can produce entangled states.

CNOT (CX) gate applies X to target when control is $|1\rangle$

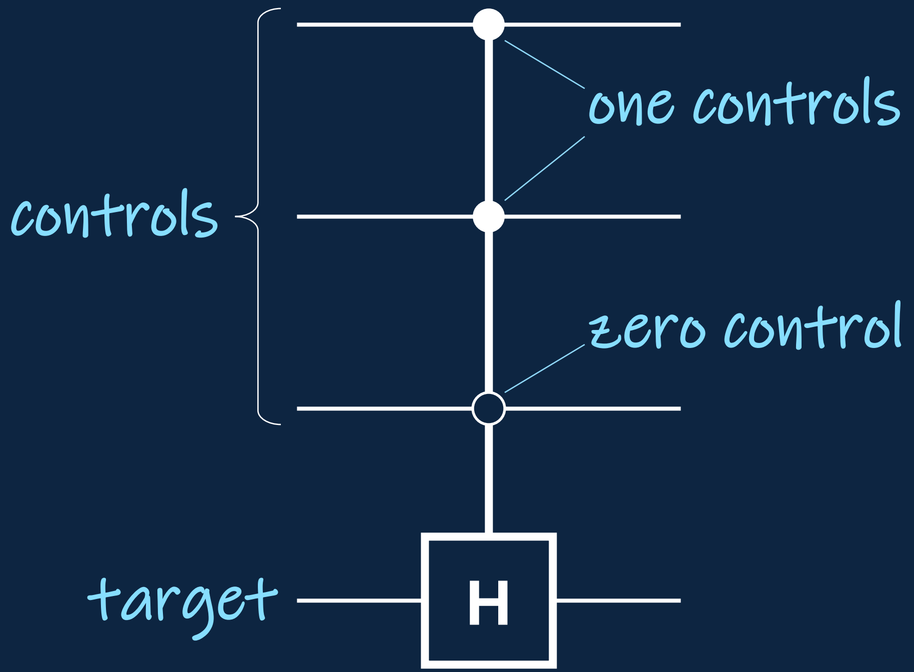


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



The control logic is encoded into the two-qubit state – without measurement!

Any gate can be controlled on any number of qubits; the gate is applied when all controls are as specified.



“Apply H to target when controls are $|110\rangle$”

$$\frac{1}{\sqrt{2}}|0100\rangle + \frac{1}{\sqrt{2}}|1100\rangle$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}|0100\rangle + \frac{1}{2}|1100\rangle + \frac{1}{2}|1101\rangle$$

In Q#, gates are one-controlled like this:

```
Controlled H(controls, target);
```

There is no built-in zero control.

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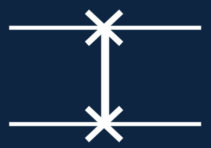
The SWAP gate swaps the amplitudes of two qubits.

$$\mathrm{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|\psi\rangle = |\psi_1\psi_2\rangle = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$



OR

$$\mathrm{SWAP}|\psi\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ y \\ x \\ z \end{bmatrix} = w|00\rangle + y|01\rangle + x|10\rangle + z|11\rangle$$

$$|\psi_2\psi_1\rangle = w|00\rangle + x|10\rangle + y|01\rangle + z|11\rangle = w|00\rangle + y|01\rangle + x|10\rangle + z|11\rangle$$

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