1. Define the integral $2\pi\delta(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}$ by a limit process using $e^{-\varepsilon|\omega|}$, and show that the δ-function may be defined by the expression:

$$\delta(t) = \lim_{\varepsilon\to 0}\left[\frac{\varepsilon}{\pi(\varepsilon^2+t^2)}\right].$$

Ans:

$$\tilde{f}_{(\omega)} = e^{-\varepsilon|\omega|}, \quad f(t) = \int_{-\infty}^{\infty}\tilde{f}(\omega)\, e^{-i\omega t}\, d\omega = \int_{-\infty}^{\infty} e^{-\varepsilon|\omega|}\, e^{-i\omega t}\, d\omega$$

$$= \int_0^{\infty} d\omega\, e^{-(\varepsilon+it)\omega} + \int_{-\infty}^{0} d\omega\, e^{(\varepsilon-it)\omega} \qquad (\varepsilon = 0^+)$$

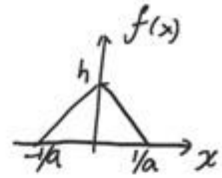
$$= -\frac{e^{-(\varepsilon+it)\omega}}{\varepsilon+it}\Big|_0^{\infty} + \frac{e^{(\varepsilon-it)\omega}}{\varepsilon-it}\Big|_{-\infty}^{0}$$

$$= \frac{1}{\varepsilon+it} + \frac{1}{\varepsilon-it}$$

$$= \frac{2\varepsilon}{\varepsilon^2+t^2}$$

$$\lim_{\varepsilon\to 0^+} f(t) = 2\pi\delta(t) = \lim_{\varepsilon\to 0^+}\frac{2\varepsilon}{\varepsilon^2+t^2} \Rightarrow \delta(t) = \lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi(\varepsilon^2+t^2)}.$$

2. Ans: $f(x) = \begin{cases} h(1-a|x|), & |x|<1/a, \text{ where } h \text{ is a constant;} \\ 0, & |x|>1/a. \end{cases}$



Find $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$

or $\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx} = \frac{1}{2\pi}\left(\int_0^{1/a} dx\cdot h(1-ax)\, e^{-ikx} + \int_{-1/a}^{0} dx\cdot h(1+ax)\, e^{-ikx}\right)$

$$= \frac{1}{2\pi}\left(\frac{h}{-ik}e^{-ikx}\Big|_0^{1/a} - ha\underbrace{\int_0^{1/a} dx\, x\, e^{-ikx}}_{-\frac{1}{ik}\left(x e^{-ikx}\big|_0^{1/a} - \int_0^{1/a} dx\, e^{-ikx}\right) = -\frac{1}{ik}\left(\frac{1}{a}e^{-ik/a} + \frac{1}{ik}e^{-ik/a} - \frac{1}{ik}\right)} + \frac{h}{ik}e^{-ikx}\Big|_{-1/a}^{0} + ha\int_{-1/a}^{0} dx\, x\, e^{-ikx}\right)$$

$$= -\frac{1}{2\pi}\cdot\frac{h}{ik}\left(e^{-ik/a} - e^{+ik/a}\right) + \frac{h}{2\pi ik}\left(e^{-ik/a} + \frac{a}{ik}e^{-ik/a} - \frac{a}{ik}\right)$$

$$- \frac{h}{2\pi ik}\left(e^{+ik/a} + \frac{a}{ik} - \frac{a}{ik}e^{+ik/a}\right)$$

$$= \frac{ha}{\pi k^2} - \frac{ha}{2\pi k^2}\big(\underbrace{e^{ik/a} + e^{-ik/a}}_{2\cos(\frac{k}{a})}\big) = \frac{ha}{\pi k^2}\left(1 - \cos\left(\frac{k}{a}\right)\right)$$

Exercise 4.3: A complex vector may be represented in terms of linear or circularly polarized basis vectors as

$$\mathbf{E} = \hat{\mathbf{e}}_1 E_1 + \hat{\mathbf{e}}_2 E_2 = \hat{\mathbf{e}}_+ E_+ + \hat{\mathbf{e}}_- E_- \tag{4.25}$$

Use the orthogonality relations to show that the equations that transform between these representations are

$$E_\pm = \frac{1}{\sqrt{2}}(E_1 \mp iE_2) \tag{4.26}$$

$$E_{1,2} = \frac{1}{\sqrt{2}}\left[(1,i)E_+ + (1,-i)E_-\right] \tag{4.27}$$

Solution: If we take the dot product of (4.25) with $\hat{\mathbf{e}}_\pm$ we get

$$\mathbf{E}\cdot\hat{\mathbf{e}}_\pm = E_\pm = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 E_1 + \hat{\mathbf{e}}_2 E_2)\cdot(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2) = \frac{1}{\sqrt{2}}(E_1 \mp iE_2) \tag{4.28}$$

where we have evaluated all the dot products between basis vectors according to the orthonormality relations. Similarly, if we take the dot product of (4.25) with $\hat{\mathbf{e}}_{1(2)}$ we get

$$\mathbf{E}\cdot\hat{\mathbf{e}}_{1(2)} = E_{1(2)} = \frac{1}{\sqrt{2}}\left[(\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2)E_+ + (\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2)E_-\right]\cdot\hat{\mathbf{e}}_{1(2)} = \frac{1(i)}{\sqrt{2}}(E_+ \pm E_-) \tag{4.29}$$

which is equivalent to the notation used in the problem, both meaning

$$E_1 = \frac{1}{\sqrt{2}}(E_+ + E_-) \tag{4.30}$$

$$E_2 = \frac{i}{\sqrt{2}}(E_+ - E_-) \tag{4.31}$$

4.

Solution: The real vector

$$\mathrm{Re}\left(\hat{\mathbf{e}}_{\pm}e^{-i\omega t}\right)=\frac{1}{\sqrt{2}}\left[\hat{\mathbf{e}}_1\cos(\omega t)\pm\hat{\mathbf{e}}_2\sin(\omega t)\right] \tag{4.36}$$

rotates counter-clockwise (or clockwise), so the sum (4.33) is as shown in Figure 4.2. It is evident from the construction in the figure that the maximum value of the sum of the two vectors is

$$E_{max}=\left|a_-+a_+\right| \tag{4.37}$$

and the minimum value is

$$E_{min}=\left|a_--a_+\right| \tag{4.38}$$

The eccentricity is the ratio of these,

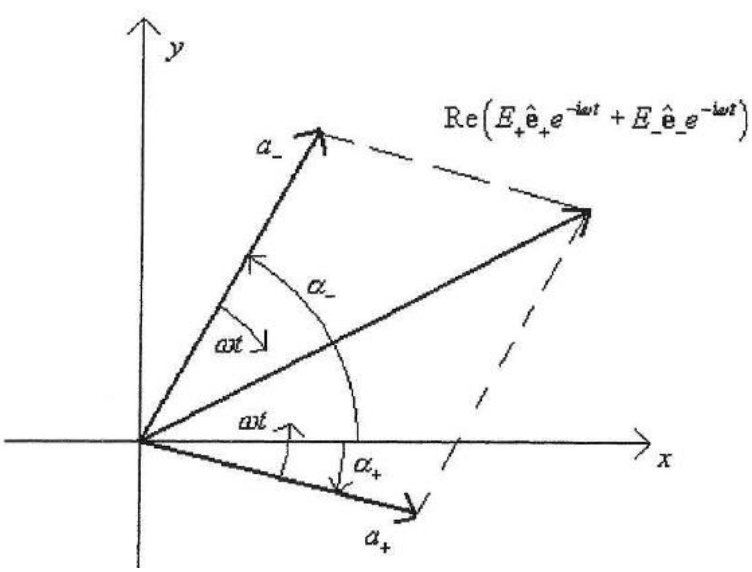
$$\chi=\frac{\left|a_-+a_+\right|}{\left|a_--a_+\right|} \tag{4.39}$$

The tilt corresponds to the angle at which the vectors are aligned to give the largest magnitude, which occurs when both vectors are rotated to the same angle

$$\theta=\omega t-\alpha_+=-\left(\omega t-\alpha_-\right) \tag{4.40}$$

or,

$$\theta=\frac{1}{2}(\alpha_--\alpha_+) \tag{4.41}$$



5.

Solution: The amplitude of a plane wave diffracted through multiple identical apertures in the far-field approximation is

$$a = ia_0 \mathbf{k}\cdot\mathbf{n}'\frac{e^{ikr}}{2\pi r}\int e^{i\Delta\mathbf{k}\cdot\mathbf{r}'}\sum_n f(\mathbf{r}'-\mathbf{r}_n)\,dS' \tag{9.118}$$

where $f(\mathbf{r})$ is the aperture function and $\mathbf{r}_n$ is the position vector of the n^{th} aperture. By factoring out an exponential we show that the contribution from each aperture differs only by a phase factor, giving

$$a = ia_0 \mathbf{k}\cdot\mathbf{n}'\frac{e^{ikr}}{2\pi r}\sum_n e^{i\Delta\mathbf{k}\cdot\mathbf{r}_n}\int e^{i\Delta\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r}_n)} f(\mathbf{r}'-\mathbf{r}_n)\,dS' = ia_0 \mathbf{k}\cdot\mathbf{n}'\frac{e^{ikr}}{r}\tilde{f}(\Delta\mathbf{k})\sum_n e^{i\Delta\mathbf{k}\cdot\mathbf{r}_n} \tag{9.119}$$

where $\tilde{f}(\Delta\mathbf{k})$ is the Fourier transform of the aperture function, as defined by (9.109).

(b) For a single slit, the Fourier transform of the aperture function is

$$\tilde{f}(\Delta\mathbf{k}) = \frac{1}{2\pi}\int_{-w/2}^{w/2} e^{i\Delta k_w x}dx\int_{-l/2}^{l/2} e^{i\Delta k_l y}dy \tag{9.120}$$

where w is the width of the slit, and l its length, with Δk_w and Δk_w the changes in the wave vector in the corresponding directions. Evaluating the integrals, we have

$$\tilde{f}(\Delta\mathbf{k}) = \frac{1}{2\pi}\left(\frac{e^{i\Delta k_w w/2} - e^{-i\Delta k_w w/2}}{i\Delta k_w}\right)\left(\frac{e^{i\Delta k_l l/2} - e^{-i\Delta k_l l/2}}{i\Delta k_l}\right) = \frac{wl}{2\pi}\frac{\sin\frac{1}{2}\Delta k_w w}{\frac{1}{2}\Delta k_w w}\frac{\sin\frac{1}{2}\Delta k_l l}{\frac{1}{2}\Delta k_l l} \tag{9.121}$$

(c) We can work out an expression for the sum of the phase factors in (9.119). We start with a finite series of powers of x, that is

$$x\sum_{n=1}^{N} x^n = x^2 + x^3 + \ldots x^{N+1} = \sum_{n=1}^{N} x^n + x^{N+1} - x \tag{9.122}$$

which we solve for the series to get

$$\sum_{n=1}^{N} x^n = \frac{x\left(x^N - 1\right)}{x-1} \tag{9.123}$$

For a phase factor $x = e^{i\theta}$, our result becomes

$$\sum_{n=1}^{N} e^{in\theta} = \frac{e^{i\theta}\left(e^{iN\theta}-1\right)}{e^{i\theta}-1} = \frac{e^{i\theta}e^{i\frac{1}{2}N\theta}}{e^{i\frac{1}{2}\theta}}\left(\frac{e^{i\frac{1}{2}N\theta} - e^{-i\frac{1}{2}N\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}}\right) = e^{i\frac{1}{2}(N+1)\theta}\frac{\sin\frac{1}{2}N\theta}{\sin\frac{1}{2}\theta} \tag{9.124}$$

In our case of N slits, this is

$$\sum_{n=1}^{N} e^{i\Delta\mathbf{k}\cdot\mathbf{r}_n} = \sum_{n=1}^{N} e^{ind\Delta k_w} = e^{i\frac{1}{2}(N+1)d\Delta k_w} \frac{\sin\frac{1}{2}Nd\Delta k_w}{\sin\frac{1}{2}d\Delta k_w} \tag{9.125}$$

(d) The peaks occur at the zeros of the denominator, when we have

$$\tfrac{1}{2}\Delta k_w d = m\pi \tag{9.126}$$

where m is an integer. The difference between the incident and transmitted wave numbers is

$$\Delta k_w = k\sin\theta - k\sin\theta_0 \tag{9.127}$$

so the peaks occur at

$$kd\left(\sin\theta - \sin\theta_0\right) = \frac{2\pi}{\lambda} d\left(\sin\theta - \sin\theta_0\right) = m2\pi \tag{9.128}$$

or

$$\sin\theta - \sin\theta_0 = \frac{m\lambda}{d} \tag{9.129}$$

(e) The first minimum on either side of each peak occurs at the nearest zero of the numerator (where the denominator is nonzero). These zeros occur at

$$\tfrac{1}{2}N\Delta k_w d = \left(Nm \pm 1\right)\pi \tag{9.130}$$

or

$$\Delta k_w = \frac{m2\pi}{d} + \delta\Delta k_w = \frac{\left(Nm+1\right)2\pi}{Nd} \tag{9.131}$$

where

$$\delta\Delta k_w = \frac{2\pi}{Nd} \tag{9.132}$$

is the change in wave numbers that puts a diffraction peak of the new pattern into the neighboring diffraction minimum of the old. For fixed angles, this change in the magnitude of the wave vector is

$$\delta\Delta k_w = \left(\sin\theta - \sin\theta_0\right)\delta k \tag{9.133}$$

For $N \gg 1$, the fractional change in wavelength is

$$\frac{\delta\lambda}{\lambda} = \frac{\delta k}{k} = \frac{\delta\Delta k_w}{k\left(\sin\theta - \sin\theta_0\right)} = \frac{2\pi}{Ndk}\frac{1}{\left(\sin\theta - \sin\theta_0\right)} \tag{9.134}$$

Substituting into the denominator from (9.128), we get

$$\frac{\delta\lambda}{\lambda} = \frac{1}{mN} \tag{9.135}$$

1.

Solution: From (4.129) in the text, the field modes excited by the moving electron have equations of motion given by

$$\frac{d^2 A_{\mathbf{n}p}}{dt^2}+\omega_{\mathbf{n}}^2 A_{\mathbf{n}p}=\frac{q}{\varepsilon_0 V}\mathbf{v}_0\cdot\hat{\mathbf{e}}_{\mathbf{n}p}^{*}e^{-i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}_0} \tag{4.131}$$

where $\mathbf{v}_0$ is the velocity of the electron, and

$$\omega_{\mathbf{n}}=k_{\mathbf{n}}c \tag{4.132}$$

is the frequency of the $\mathbf{n}^{th}$ mode. We consider the electron to be at rest for negative times, and to have velocity $\mathbf{v}_0=\boldsymbol{\beta}c$ for $t\geq 0$. The equations of motion for the field form a set of unforced, undamped harmonic oscillators for negative times, and for $t\geq 0$ they are

$$\frac{d^2 A_{\mathbf{n}p}}{dt^2}+\omega_{\mathbf{n}}^2 A_{\mathbf{n}p}=\frac{qc}{\varepsilon_0 V}\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_{\mathbf{n}p}^{*}e^{-i(\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta})\omega_{\mathbf{n}}t} \tag{4.133}$$

Multiplying by the integrating factor $\sin\left[\omega_{\mathbf{n}}(t-t')\right]$, and integrating by parts, we get

$$\begin{aligned}&\sin\left[\omega_{\mathbf{n}}(t-t')\right]\frac{dA_{\mathbf{n}p}}{dt'}\bigg|_0^t+\int_0^t dt'\frac{dA_{\mathbf{n}p}}{dt'}\omega_{\mathbf{n}}\cos\left[\omega_{\mathbf{n}}(t-t')\right]+\\ &\omega_{\mathbf{n}}\left\{A_{\mathbf{n}p}\cos\left[\omega_{\mathbf{n}}(t-t')\right]\Big|_0^t-\int_0^t dt'\frac{dA_{\mathbf{n}p}}{dt'}\cos\left[\omega_{\mathbf{n}}(t-t')\right]\right\}\\ &=\frac{qc}{\varepsilon_0 V}\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_{\mathbf{n}p}^{*}\int_0^t dt' e^{-i(\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta})\omega_{\mathbf{n}}t'}\sin\left[\omega_{\mathbf{n}}(t-t')\right]\end{aligned} \tag{4.134}$$

The integrals on the left-hand side cancel, and we can easily evaluate the boundary terms on the left-hand side because $\frac{dA_{\mathbf{n}p}}{dt}=A_{\mathbf{n}p}=0$ when $t=0$. We then have

$$A_{\mathbf{n}p}=\frac{qc}{\varepsilon_0 V\omega_{\mathbf{n}}}\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_{\mathbf{n}p}^{*}\int_0^t dt' e^{-i(\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta})\omega_{\mathbf{n}}t'}\sin\left[\omega_{\mathbf{n}}(t-t')\right] \tag{4.135}$$

By rewriting the sine function in terms of complex exponentials, we get

$$A_{\mathbf{n}p} = \frac{qc}{2i\varepsilon_0 V\omega_{\mathbf{n}}}\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\left[e^{i\omega_{\mathbf{n}}t}\int_0^t dt' e^{-i\left(1+\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}\right)\omega_{\mathbf{n}}t'} - e^{-i\omega_{\mathbf{n}}t}\int_0^t dt' e^{i\left(1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}\right)\omega_{\mathbf{n}}t'}\right] \tag{4.136}$$

At long times, we use the hint in the problem to evaluate this expression as $t\to\infty$, and we get

$$A_{\mathbf{n}p} = -\frac{qc}{2\varepsilon_0 V\omega_{\mathbf{n}}^2}\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\left(\frac{e^{i\omega_{\mathbf{n}}t}}{1+\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}} + \frac{e^{-i\omega_{\mathbf{n}}t}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}}\right) \tag{4.137}$$

But we are interested in the spectral fluence radiated in the direction $\hat{\mathbf{n}}$, so it is important to observe that in the term

$$\hat{\mathbf{e}}_{\mathbf{n}p}A_{\mathbf{n}p}(t)e^{i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}} = -\hat{\mathbf{e}}_{\mathbf{n}p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\frac{qc}{2\varepsilon_0 V\omega_{\mathbf{n}}^2}\left[\frac{e^{i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}+\omega_{\mathbf{n}}t)}}{1+\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}} + \frac{e^{i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}-\omega_{\mathbf{n}}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}}\right] \tag{4.138}$$

of the sum (4.121) in the text representing the vector potential field, the second term represents a wave traveling in the direction $\hat{\mathbf{k}}_{\mathbf{n}}$, but the first term represents a wave traveling in the opposite direction. We therefore rearrange the sum so that the term $\mathbf{n},p$ contains waves traveling in the direction $\hat{\mathbf{k}}_{\mathbf{n}}$. Recalling that $\mathbf{k}_{-\mathbf{n}} = -\mathbf{k}_{\mathbf{n}}$ (but $\omega_{\mathbf{n}} = \omega_{-\mathbf{n}}$) and $\hat{\mathbf{e}}_{-n} = \hat{\mathbf{e}}^*_{n}$, we get

$$\mathbf{A}(\mathbf{r},t) = -\sum_{\mathbf{n},p}\frac{qc}{2\varepsilon_0 V\omega_{\mathbf{n}}^2}\left[\hat{\mathbf{e}}_{\mathbf{n}p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\frac{e^{i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}-\omega_{\mathbf{n}}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}} + \hat{\mathbf{e}}^*_{\mathbf{n}p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_{\mathbf{n}p}\right)\frac{e^{-i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}-\omega_{\mathbf{n}}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}}\right] \tag{4.139}$$

Differentiating with respect to time, we find that the electric field is

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} = -\sum_{\mathbf{n},p}\frac{iqc}{2\varepsilon_0 V\omega_{\mathbf{n}}}\left[\hat{\mathbf{e}}_{\mathbf{n}p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\frac{e^{i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}-\omega_{\mathbf{n}}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}} - \hat{\mathbf{e}}^*_{\mathbf{n}p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_{\mathbf{n}p}\right)\frac{e^{-i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}-\omega_{\mathbf{n}}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}}\right] \tag{4.140}$$

Note that when we rearrange it this way, each term in the sum over $(\mathbf{n},p)$ is explicitly real. The energy in the electric field is

$$\mathcal{W}_E = \frac{\varepsilon_0}{2}\int_V dV\mathbf{E}\cdot\mathbf{E} \tag{4.141}$$

where the product $\mathbf{E}\cdot\mathbf{E}$ is a double sum over $(\mathbf{n},p)$ and $(\mathbf{n}',p')$ of terms of the form

$$\left[\hat{\mathbf{e}}_{\mathbf{n}p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\frac{e^{i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}-\omega_{\mathbf{n}}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}} - \hat{\mathbf{e}}^*_{\mathbf{n}p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\frac{e^{-i(\mathbf{k}_{\mathbf{n}}\cdot\mathbf{r}-\omega_{\mathbf{n}}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}}\cdot\boldsymbol{\beta}}\right]$$

$$\cdot\left[\hat{\mathbf{e}}_{\mathbf{n}'p'}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}'p'}\right)\frac{e^{i(\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{r}-\omega_{\mathbf{n}'}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}'}\cdot\boldsymbol{\beta}} - \hat{\mathbf{e}}^*_{\mathbf{n}'p}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}'p'}\right)\frac{e^{-i(\mathbf{k}_{\mathbf{n}'}\cdot\mathbf{r}-\omega_{\mathbf{n}'}t)}}{1-\hat{\mathbf{k}}_{\mathbf{n}'}\cdot\boldsymbol{\beta}}\right]$$

$$= \left(\hat{\mathbf{e}}_{\mathbf{n}p} \cdot \hat{\mathbf{e}}^*_{\mathbf{n}'p'}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}_{\mathbf{n}'p'}\right) \frac{e^{i(\mathbf{k_n}\cdot\mathbf{r}-\omega_\mathbf{n}t)}}{1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}} \frac{e^{-i(\mathbf{k_{n'}}\cdot\mathbf{r}-\omega_\mathbf{n'}t)}}{1-\hat{\mathbf{k}}_\mathbf{n'}\cdot\boldsymbol{\beta}}$$

$$+ \left(\hat{\mathbf{e}}^*_{\mathbf{n}p} \cdot \hat{\mathbf{e}}_{\mathbf{n}'p'}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}_{\mathbf{n}p}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}^*_{\mathbf{n}'p'}\right) \frac{e^{-i(\mathbf{k_n}\cdot\mathbf{r}-\omega_\mathbf{n}t)}}{1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}} \frac{e^{i(\mathbf{k_{n'}}\cdot\mathbf{r}-\omega_\mathbf{n'}t)}}{1-\hat{\mathbf{k}}_\mathbf{n'}\cdot\boldsymbol{\beta}}$$

$$+ \left(\hat{\mathbf{e}}_{\mathbf{n}p} \cdot \hat{\mathbf{e}}_{\mathbf{n}'p'}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}^*_{\mathbf{n}'p'}\right) \frac{e^{i(\mathbf{k_n}\cdot\mathbf{r}-\omega_\mathbf{n}t)}}{1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}} \frac{e^{i(\mathbf{k_{n'}}\cdot\mathbf{r}-\omega_\mathbf{n'}t)}}{1-\hat{\mathbf{k}}_\mathbf{n'}\cdot\boldsymbol{\beta}}$$

$$+ \left(\hat{\mathbf{e}}^*_{\mathbf{n}p} \cdot \hat{\mathbf{e}}^*_{\mathbf{n}'p'}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}_{\mathbf{n}p}\right)\left(\boldsymbol{\beta} \cdot \hat{\mathbf{e}}_{\mathbf{n}'p'}\right) \frac{e^{-i(\mathbf{k_n}\cdot\mathbf{r}-\omega_\mathbf{n}t)}}{1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}} \frac{e^{-i(\mathbf{k_{n'}}\cdot\mathbf{r}-\omega_\mathbf{n'}t)}}{1-\hat{\mathbf{k}}_\mathbf{n'}\cdot\boldsymbol{\beta}} \qquad (4.142)$$

When we integrate over the volume V, the first two terms on the right vanish unless $\mathbf{k_n} = \mathbf{k_{n'}}$, while the second two terms vanish unless $\mathbf{k_n} = -\mathbf{k_{n'}}$. Then, since $\hat{\mathbf{e}}_{\mathbf{n}p} \cdot \hat{\mathbf{e}}^*_{\mathbf{n}p'} = \delta_{pp'}$ and $\hat{\mathbf{e}}_{\mathbf{n}p} \cdot \hat{\mathbf{e}}_{-\mathbf{n}p'} = \delta_{pp'}$, the sums over $(\mathbf{n}', p')$ collapse, leaving

$$\mathcal{W}_E = \frac{q^2c^2}{4\varepsilon_0 V}\sum_{\mathbf{n},p}\left[\frac{\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_{\mathbf{n}p}\right)}{\omega_\mathbf{n}^2\left(1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}\right)^2} + \frac{\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right)\left(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{-\mathbf{n}p}\right)}{\omega_\mathbf{n}^2\left(1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}\right)\left(1-\hat{\mathbf{k}}_{-\mathbf{n}}\cdot\boldsymbol{\beta}\right)}\left(\frac{e^{i2\omega_\mathbf{n}t}+e^{-i2\omega_\mathbf{n}t}}{2}\right)\right] \qquad (4.143)$$

Averaged over a complete cycle, the second term in the brackets vanishes and we are left with

$$\langle \mathcal{W}_E \rangle = \frac{q^2c^2}{4\varepsilon_0 V}\sum_{\mathbf{n},p}\frac{\left|\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right|^2}{\omega_\mathbf{n}^2\left(1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}\right)^2} \qquad (4.144)$$

The energy in the magnetic field of a wave is the same, so the total energy radiated is

$$\langle \mathcal{W} \rangle = \frac{q^2c^2}{2\varepsilon_0 V}\sum_{\mathbf{n},p}\frac{\left|\boldsymbol{\beta}\cdot\hat{\mathbf{e}}^*_{\mathbf{n}p}\right|^2}{\omega_\mathbf{n}^2\left(1-\hat{\mathbf{k}}_\mathbf{n}\cdot\boldsymbol{\beta}\right)^2} \qquad (4.145)$$

But from (4.146) in the text we see that the sum over polarizations is

$$\sum_{\mathbf{n},p}\left|\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_{\mathbf{n}p}\right|^2 = \beta^2\left(\sin^2\theta\right) = \left|\boldsymbol{\beta}\times\hat{\mathbf{k}}\right|^2 \qquad (4.146)$$

where θ is the angle between the velocity $\boldsymbol{\beta}$ and the wave vector $\hat{\mathbf{k}}$. If we use the result in (4.145) and convert the sum into an integral using (4.118) in the text, we get

$$\langle w \rangle = \frac{q^2c^2}{16\pi^3\varepsilon_0}\int k^2 dk d\Omega \frac{\left|\boldsymbol{\beta}\times\hat{\mathbf{k}}\right|^2}{\omega^2\left(1-\hat{\mathbf{k}}\cdot\boldsymbol{\beta}\right)^2} = \frac{q^2}{16\pi^3\varepsilon_0 c}\int d\omega d\Omega \frac{\left|\hat{\mathbf{k}}\times\boldsymbol{\beta}\right|^2}{\left(1-\hat{\mathbf{k}}\cdot\boldsymbol{\beta}\right)^2} \tag{4.147}$$

We immediately identify the radiation emitted in the direction $\hat{\mathbf{n}}$ at the frequency ω as

$$\frac{d^2\langle w \rangle}{d\omega d\Omega} = \frac{q^2}{16\pi^3\varepsilon_0 c}\left|\frac{\hat{\mathbf{n}}\times\boldsymbol{\beta}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}\right|^2 \tag{4.148}$$

2.

Q: (a) Show that the total Thompson cross section of unpolarized light is given by $\sigma_T = \frac{8\pi}{3}r_c^2$, where $r_c = 2.82 \times 10^{-15}$ m is the classical particle radius.
A: The scattering cross section for a fix polarization $\hat{e_0}$ is given by:

$$\frac{d\sigma_{Thom}}{d\Omega} = (\frac{e^2}{4\pi\epsilon_0 mc^2})^2|\hat{k}\times\hat{e_0}|^2 = r_e^2|\hat{k}\times\hat{e_0}|^2 \tag{1}$$

To find the cross section of unpolarized light we assume a random mixture of polarization directions. Take two orthogonal polarization vectors $\hat{e_1}$ and $\hat{e_2}$.

$$\frac{d\sigma_{Thom}}{d\Omega} = \frac{1}{2}\sum_{m=1}^{2} r_c^2(1-|\hat{k}\cdot\hat{e_m}|^2) \tag{2}$$

we can define polarization vectors as follows: $\frac{d\sigma_\perp}{d\Omega} = r_c^2$ and $\frac{d\sigma_\parallel}{d\Omega} = r_c^2\cos^2\Theta$
Therefore we find:

$$\frac{d\sigma_{Thom}}{d\Omega} = \frac{1}{2}(\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}) = \frac{1}{2}r_c^2(1+\cos^2\Theta) \tag{3}$$

The total cross section is then

$$\sigma_{Thom} = \int d\Omega \frac{d\sigma_{Thom}}{d\Omega} = \frac{8\pi}{3}r_c^2 \tag{4}$$

Q: (b)Find the degree of polarization defined as:

$$\Pi(\Theta) = \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}} \tag{5}$$

A: we can define polarization vectors as follows: $\frac{d\sigma_\perp}{d\Omega} = r_c^2$ and $\frac{d\sigma_\parallel}{d\Omega} = r_c^2\cos^2\Theta$
and find:

$$\Pi(\Theta) = \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}} = \frac{\sin^2\Theta}{1+\cos^2\Theta} \tag{6}$$

Q: (c) Make a drawing of the angular dependence of the cross section and the degree of polarization.
A: see figure 1

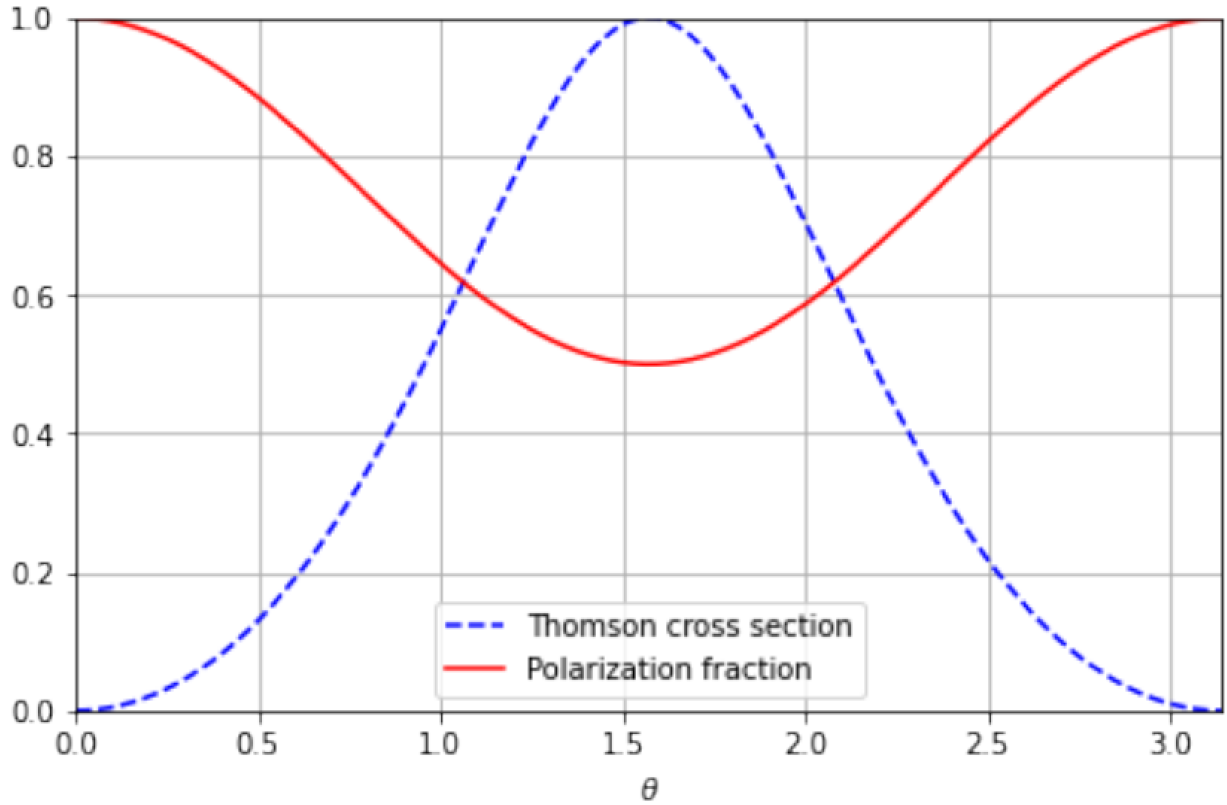


FIG. 1. ***Thomson cross section and polarization fraction***

3.

Solution: (a) We can count up the electrons using the differential with respect to momentum,

$$dN_e = 2d^3\mathbf{n} = \frac{2V}{(2\pi)^3\hbar^3}d^3\mathbf{p} \tag{4.234}$$

where the factor 2 reflects the two spin states (up and down) possible for electrons. We integrate the number of electrons up to the Fermi momentum,

$$N_e = \frac{2V}{(2\pi)^3\hbar^3}\int_0^{p_F} p^2 dp d\Omega = \frac{V}{3\pi^2\hbar^3}p_F{}^3 \tag{4.235}$$

which we solve for the Fermi momentum,

$$p_F = \hbar\left(\frac{3\pi^2}{V}\right)^{\frac{1}{3}} = \hbar\left(3\pi^2 n\right)^{\frac{1}{3}} \tag{4.236}$$

where n is the number density of electrons. The nonrelativistic Fermi energy is

$$\varepsilon_F = \frac{p_F{}^2}{2m_e} = \frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{\frac{2}{3}} \tag{4.237}$$

(b) Neutrons are also fermions, so the formula calculated for the number of electrons applies to neutrons as well,

$$N_n = \frac{V}{3\pi^2\hbar^3}p_F{}^3 \tag{4.238}$$

If we are interested in neutron stars, we must use the relativistic expression for the Fermi energy,

$$\varepsilon_F^2 = m^2c^4 + p_F{}^2c^2 = m^2c^4\left(1+\left(3\pi^2\frac{\hbar^3}{m^3c^3}n\right)^{\frac{2}{3}}\right) \tag{4.239}$$

In terms of the Compton wavelength, this energy is

$$\varepsilon_F = mc^2\sqrt{1+\left(\frac{3}{8\pi}\lambda_C{}^3 n\right)^{\frac{2}{3}}} \tag{4.240}$$

4.

(a) Q: What is the radiation flux F (energy per unit area per unit time) from the star at the planet's position?

S: The flux at the surface of the star is $\sim \sigma_B T^4$ (blackbody), where σ_B is the Stefan-Boltzmann constant. The luminosity,L, of the star is then $L = 4\pi R^2 \sigma_B T^4$. The flux at the planet's position is

$$F = \frac{L}{4\pi d^2} = \left(\frac{R}{d}\right)^2 \sigma_B T^4. \tag{7}$$

(b) Q: Assume the planet absorbs a fraction η of all the radiation reaching it, while the rest is reflected into space. What is the total energy L_p absorbed by the planet per unit time?

S:The cross-section,σ of the planet (with radius, r_p) to intercept and absorb the radiation is $\sigma = \pi r_p^2$. This is the geometric cross section and reasonable to use if $d \gg R$. So the total energy absorbed by the planet is

$$L_p = \eta F \pi r_p^2 = \eta \left(\frac{R}{d}\right)^2 \sigma_B T^4 \pi r_p^2. \tag{8}$$

(c) Q: Assume that there is no internal heating source in the planet and the total absorbed radiation reaches a new thermal equilibrium on the planet and is emitted back to space in the form of blackbody radiation. What is the temperature T_p on the surface of the planet?

A: We approximate the radiation from the planet as blackbody radiation with a luminosity of the planet given by $L_p = 4\pi r_p^2 k_B T_p^4$. By combining with the result in (b), we have

$$T_p = \left(\frac{\eta}{4}\right)^{\frac{1}{4}} \left(\frac{R}{d}\right)^{\frac{1}{2}} T. \tag{9}$$

(d) Q: If the temperature range for the habitable zone is $T_1 < T < T_2$, what would be the range of distance for a planet to be in the habitable zone? Express your results in terms of T, R, η,T_1, and T_2.

A: Substituting the result in (c) to $T_1 < T < T_2$, we obtain

$$\left(\frac{\eta}{4}\right)^{\frac{1}{2}} \left(\frac{T}{T_2}\right)^2 R < d < \left(\frac{\eta}{4}\right)^{\frac{1}{2}} \left(\frac{T}{T_1}\right)^2 R. \tag{10}$$

(e) Q: The Sun has a surface temperature of approximately $T = 5800$K and radius of $R = 6.96 \times 10^8$m. Let's take $T_1 \simeq 273$K, $T_2 \simeq 373$K, and $\eta = 0.7$. Compute the distance range of the habitable zone in Astronomical Units (AU). An Astronomical Unit is defined as the average distance between the Earth and the Sun (1 AU=1.496×10^{11}m). Is the Earth in the habitable zone? Interpret your result and comment what effects have been neglected, it might to think about the Earth and the Moon.

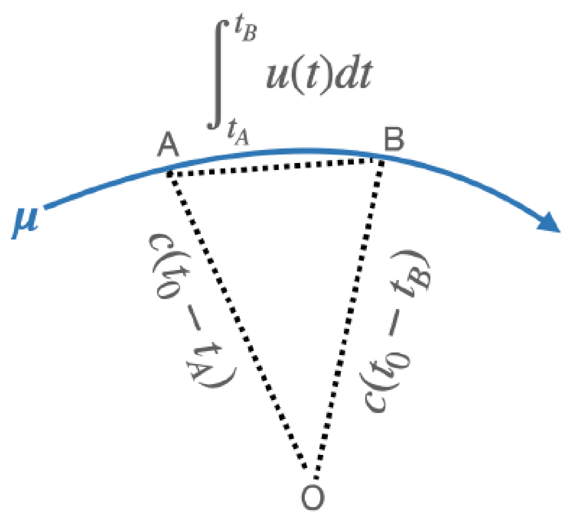
A: Substituting the numbers into the result in (d), we find that $0.47\text{AU} < d < 0.88\text{AU}$. Hence, the Earth is just outside of the range we derived, which is not too surprising given the order-of-magnitude nature of our calculation. The estimation neglects the effects caused by the planet's atmosphere. The difference between the Earth and the moon is the atmosphere. The greenhouse effect increases the temperature on the surface of the planet. The temperature on the moon is strongly depending on if it is facing the Sun or not. At night and day it is respectively $-183°C$ and $106°C$, so on average well below $0°C$.

1.

Solution:
In vacuum it is not possible to have two values of retarded time corresponding to two different positions on the muon's trajectory. Figure I illustrates this. Let us first assume that there are two such points (A and B) on the trajectory (solid curve) of the electron. For an observer at point $\mathcal{O}$ at time t, the two points A and B correspond to retarded time t_A and t_B, respectively. The muon moves to A at t_A and to B at t_B along the trajectory. Its speed is denoted as $u(t)$. The distance traveled by the muon is the arc length $\widehat{AB} = \int_{t_A}^{t_B} u(t)dt$. Denote the distance between A and B as $\overline{AB}$, between A and $\mathcal{O}$ as $\overline{A\mathcal{O}}$, and between B and $\mathcal{O}$ as $\overline{BC}$. We have $\overline{A\mathcal{O}} = c(t-t_A)$ and $\overline{B\mathcal{O}} = c(t-t_B)$. Let us consider the triangle $AB\mathcal{O}$. We have

$$\overline{AB} + \overline{B\mathcal{O}} \leq \widehat{AB} + \overline{B\mathcal{O}} = \int_{t_A}^{t_B} u(t)dt + \overline{B\mathcal{O}} < \int_{t_A}^{t_B} cdt + c(t-t_B) = c(t-t_A) = \overline{A\mathcal{O}}, \tag{1}$$

where we have used $u(t) < c$. We reach the conclusion that $\overline{AB} + \overline{B\mathcal{O}} < \overline{A\mathcal{O}}$, i.e., the triangle $AB\mathcal{O}$ cannot exist. Therefore, our assumption of the existence of two values of retarded time for a field point at t is incorrect (in other words, to make the assumption work, we would need $u(t) \geq c$). In this case the triangle $AB\mathcal{O}$ can exist and hence for an observer at $\mathcal{O}$, two points on the track of the muon can have the same retarded time.



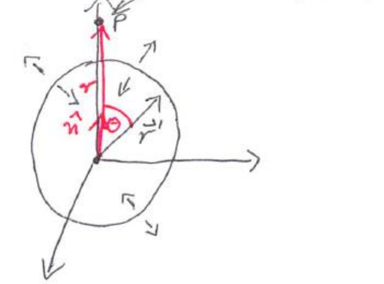
b) Instead of a vacuum assume the process described above takes place in a homogeneous medium with an index of refraction $n > 1$. Describe if the situation is different for this case and repeat (a).

Solution: We can make the same argument as under (a), but now the light propagates through the medium at the speed of light in the medium, given by c/n. The speed of muon travelling through the medium can be larger than c/n.

2.

Spherically symmetric distribution of charge oscillating radially

The current is given by



$$\vec{J}(\vec{r}', t_R) = \vec{r}' f(\vec{r}')\, e^{-i\omega t_R}$$

$$f(r') = f(r', t_R)$$

We can choose one observation point on the z-axis as the problem is spherically symmetric the solution we find for point p will apply for any other position

Calculate the vector potential $\vec{A}(\vec{r}, t)$

$$A(\vec{r}, t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \vec{J}(\vec{r}', t_R) \qquad t_R \approx t - \frac{r}{c} + \frac{\vec{r}'\cdot\hat{n}}{c}$$

$$\Rightarrow A(\vec{r}, t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \vec{r}' f(r')\, e^{-i\omega t} e^{+i\omega\frac{r}{c}} e^{-i\omega\frac{\vec{r}'\cdot\hat{n}}{c}}$$

$$\boxed{\begin{array}{l}\text{note: } \frac{\omega}{c} = k \\ \vec{r}'\cdot\hat{n} = r'\cos\theta\end{array}}$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \vec{r}' f(r')\, e^{-i\omega t} e^{ikr} e^{-ikr'\cos\theta}$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} e^{-i\omega t}\int d^3r'\, \vec{r}' f(r')\, e^{-ikr'\cos\theta}$$

Now look at the components of the vector potential $\vec{A}$

$$A_x = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}\int_0^\infty dr' r'^2 \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\phi' \sin\theta' \cos\phi' \, r' f(r') e^{-ikr'\cos\theta}$$

$$A_y = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}\int_0^\infty dr' r'^2 \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\phi' \sin\theta' \sin\phi' \, r' f(r') e^{-ikr'\cos\theta}$$

$$A_z = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}\int_0^\infty dr' r'^2 \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\phi' \cos\theta' \, r' f(r') e^{-ikr'\cos\theta}$$

Now choose the observation point on the z axis

$$\Rightarrow \hat{n} = \hat{z}$$

$$\Rightarrow \cos\theta' = \cos\theta$$

$$\Rightarrow \quad A_x = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}\int_0^\infty dr' r'^3 \int_0^{2\pi} d\phi' \cos\phi' \underbrace{\int_{-1}^{1} d\cos\theta' \sin\theta' e^{-ikr'\cos\theta'}}_{=0 \text{ (odd function)}}$$

$$A_y = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}\int_0^\infty dr' r'^3 \int_0^{2\pi} d\phi \sin\phi' \underbrace{\int_{-1}^{1} d\cos\theta' \sin\theta' e^{-ikr'\cos\theta'}}_{=0}$$

$\Rightarrow \vec{A}$ only has a component in the z-direction (radial direction)

The fields are $\quad \vec{B} = \nabla\times\vec{A} \sim \vec{A}\times\hat{n} = \underline{\underline{0}}$ ☒

$$|\vec{E}| = c|\vec{B}| = \underline{\underline{0}}$$ ☒

3.

Solution: The Fourier transform of the Maxwell equations is given by (10.5)-(10.8) in the text. If we start with the identity

$$\nabla\times\left(\nabla\times\tilde{\mathbf{E}}\right)=\nabla\left(\nabla\cdot\tilde{\mathbf{E}}\right)-\nabla^2\tilde{\mathbf{E}} \tag{10.25}$$

and substitute (10.5) and (10.7) for $\nabla\cdot\tilde{\mathbf{E}}$ and $\nabla\times\tilde{\mathbf{E}}$, we get

$$\nabla\times\left(i\omega\tilde{\mathbf{B}}\right)=\frac{1}{\varepsilon_0}\nabla\tilde{\rho}-\nabla^2\tilde{\mathbf{E}} \tag{10.26}$$

Substituting (10.8) for $\nabla\times\tilde{\mathbf{B}}$ we arrive at the Helmholtz equation

$$\nabla^2\tilde{\mathbf{E}}+k^2\tilde{\mathbf{E}}=\frac{1}{\varepsilon_0}\nabla\tilde{\rho}-i\mu_0\omega\tilde{\mathbf{J}} \tag{10.27}$$

where

$$\omega=kc \tag{10.28}$$

In terms of the Green function (10.22) in the text,

$$\tilde{G}=\frac{1}{4\pi R}e^{ikR} \tag{10.29}$$

the field at the observation point $\mathbf{r}_0$ is

$$\tilde{\mathbf{E}}(\mathbf{r}_0,\omega)=\int_{-\infty}^{\infty}d^3\mathbf{r}'\left[i\mu_0\omega\tilde{\mathbf{J}}(\mathbf{r}')-\frac{1}{\varepsilon_0}\nabla'\tilde{\rho}(\mathbf{r}')\right]\left(\frac{1}{4\pi R}e^{i\omega R/c}\right) \tag{10.30}$$

where we have used (10.28) to eliminate the wave number k in the exponent. We can eliminate the gradient in the integrand with an integration by parts. Using (10.24) in the text, we see that

$$\nabla'\left(\frac{1}{4\pi R}e^{i\omega R/c}\right)=\left(\frac{i\omega}{c}-\frac{1}{R}\right)\frac{e^{i\omega R/c}}{4\pi R}\nabla' R=-\left(\frac{i\omega}{c}-\frac{1}{R}\right)\frac{e^{i\omega R/c}}{R}\hat{\mathbf{R}} \tag{10.31}$$

Carrying out the integration by parts with the assumption that the sources vanish at infinity, we find that the field is

$$\tilde{\mathbf{E}}(\mathbf{r}_0,\omega)=\mu_0\int_{-\infty}^{\infty}d^3\mathbf{r}'\left[i\omega\tilde{\mathbf{J}}(\mathbf{r}')-\tilde{\rho}(\mathbf{r}')c\left(i\omega-\frac{c}{R}\right)\hat{\mathbf{R}}\right]\left(\frac{1}{4\pi R}e^{i\omega R/c}\right) \tag{10.32}$$

Substituting the explicit expressions for the Fourier transforms $\tilde{\mathbf{J}}$ and $\tilde{\rho}$, we get

$$\tilde{\mathbf{E}}=\frac{\mu_0}{\left(32\pi^3\right)^{1/2}}\int_{-\infty}^{\infty}d^3\mathbf{r}'\int_{-\infty}^{\infty}dt'\left[\frac{i\omega}{R}\mathbf{J}(\mathbf{r}',t')-\rho(\mathbf{r}',t')c\left(\frac{i\omega}{R}-\frac{c}{R^2}\right)\hat{\mathbf{R}}\right]e^{i\omega(t'+R/c)} \tag{10.33}$$

For the point source defined by (10.43) and (10.44) in the text, we can carry out the integral over all space trivially and get

$$\tilde{\mathbf{E}} = \frac{\mu_0 c q}{\left(32\pi^3\right)^{1/2}} \int_{-\infty}^{\infty} dt' \left[\frac{i\omega}{R_0}\left(\boldsymbol{\beta} - \hat{\mathbf{R}}_0\right) + \frac{c}{R_0^2}\hat{\mathbf{R}}_0 \right] e^{i\omega(t' + R_0/c)} \tag{10.34}$$

where

$$\mathbf{R}_0(t') = \mathbf{r}_0 - \mathbf{r}(t') \tag{10.35}$$

To get rid of the factor $i\omega$, we integrate the first term by parts with respect to t'. To do this we note that

$$\frac{dR_0^2}{dt'} = 2R_0 \frac{dR_0}{dt'} = \frac{d\mathbf{R}_0 \cdot \mathbf{R}_0}{dt'} = -2c\mathbf{R}_0 \cdot \boldsymbol{\beta} \tag{10.36}$$

so

$$\frac{d}{dt'}\left(t' + \frac{R_0}{c} \right) = 1 - \hat{\mathbf{R}}_0 \cdot \boldsymbol{\beta} \tag{10.37}$$

Assuming that the contributions from infinity can be ignored, the integral by parts gives us

$$\tilde{\mathbf{E}} = \frac{\mu_0 c q}{\left(32\pi^3\right)^{1/2}} \int_{-\infty}^{\infty} dt' \left\{ \frac{d}{dt'}\left[\frac{\hat{\mathbf{R}}_0 - \boldsymbol{\beta}}{R_0\left(1 - \hat{\mathbf{R}}_0 \cdot \boldsymbol{\beta}\right)} \right] + \frac{c}{R_0^2}\hat{\mathbf{R}}_0 \right\} e^{i\omega(t' + R_0/c)} \tag{10.38}$$

To invert the transform we multiply by $e^{-i\omega t_0}/\sqrt{2\pi}$ and integrate over ω to get

$$\mathbf{E}(\mathbf{r}_0, t_0) = \frac{q}{8\pi^2\varepsilon_0} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt' \left\{ \frac{d}{dt'}\left[\frac{\hat{\mathbf{R}}_0 - \boldsymbol{\beta}}{cR_0\left(1 - \hat{\mathbf{R}}_0 \cdot \boldsymbol{\beta}\right)} \right] + \frac{1}{R_0^2}\hat{\mathbf{R}}_0 \right\} e^{i\omega(t' - t_0 + R_0/c)} \tag{10.39}$$

The integral over ω produces a δ-function, so that

$$\mathbf{E}(\mathbf{r}_0, t_0) = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} dt' \left\{ \frac{d}{dt'}\left[\frac{\hat{\mathbf{R}}_0 - \boldsymbol{\beta}}{cR_0\left(1 - \hat{\mathbf{R}}_0 \cdot \boldsymbol{\beta}\right)} \right] + \frac{1}{R_0^2}\hat{\mathbf{R}}_0 \right\} \delta\left(t' - t_0 + R_0/c\right) \tag{10.40}$$

To carry out the integral over t' we use (10.38) in the text. The only contribution comes from the retarded time $t' = t_0 - R_0/c$, and the result is

$$\mathbf{E}(\mathbf{r}_0, t_0) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{1 - \hat{\mathbf{R}}_0 \cdot \boldsymbol{\beta}} \frac{d}{dt'}\left[\frac{\hat{\mathbf{R}}_0 - \boldsymbol{\beta}}{cR_0\left(1 - \hat{\mathbf{R}}_0 \cdot \boldsymbol{\beta}\right)} \right] + \frac{\hat{\mathbf{R}}_0}{R_0^2\left(1 - \hat{\mathbf{R}}_0 \cdot \boldsymbol{\beta}\right)} \right\}_{\text{retarded}} \tag{10.41}$$

To compute the derivative we use (10.36) and (10.40) in the text. Then,

$$\frac{d}{dt'}\left[\frac{\hat{\mathbf{R}}_0-\boldsymbol{\beta}}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}\right]=\frac{1}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}\left(\frac{d\hat{\mathbf{R}}_0}{dt'}-\dot{\boldsymbol{\beta}}\right)-\frac{\hat{\mathbf{R}}_0-\boldsymbol{\beta}}{cR_0^2\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}\frac{dR_0}{dt'}$$

$$+\frac{\hat{\mathbf{R}}_0-\boldsymbol{\beta}}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}\left(\hat{\mathbf{R}}_0\cdot\dot{\boldsymbol{\beta}}+\boldsymbol{\beta}\cdot\frac{d\hat{\mathbf{R}}_0}{dt'}\right)$$

$$=\frac{1}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}\left\{\frac{c}{R_0}\left[\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right]-\dot{\boldsymbol{\beta}}\right\}+\frac{\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}{R_0^2\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}$$

$$+\frac{\hat{\mathbf{R}}_0-\boldsymbol{\beta}}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}\left\{\hat{\mathbf{R}}_0\cdot\dot{\boldsymbol{\beta}}+\frac{c}{R_0}\left[\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2-\beta^2\right]\right\} \tag{10.42}$$

Sorting terms of order $1/R_0$ from terms of order $1/R_0^2$, we get

$$\frac{d}{dt'}\left[\frac{\hat{\mathbf{R}}_0-\boldsymbol{\beta}}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}\right]=\frac{1}{R_0^2}\left\{\frac{\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)\left(2\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)-\boldsymbol{\beta}}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}+\frac{\left[\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2-\beta^2\right]\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}\right\}$$

$$+\frac{1}{cR_0}\left[\frac{\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}_0\cdot\dot{\boldsymbol{\beta}}\right)}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}-\frac{\dot{\boldsymbol{\beta}}}{1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}}\right] \tag{10.43}$$

To simplify the $1/R_0$ term, we note that

$$\frac{\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}_0\cdot\dot{\boldsymbol{\beta}}\right)}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}-\frac{\dot{\boldsymbol{\beta}}}{1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}}=\frac{\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}_0\cdot\dot{\boldsymbol{\beta}}\right)-\left(\hat{\mathbf{R}}_0\cdot\hat{\mathbf{R}}_0-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)\dot{\boldsymbol{\beta}}}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}$$

$$=\frac{\hat{\mathbf{R}}_0\times\left[\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2} \tag{10.44}$$

when we use a vector identity for the triple cross product. To simplify the $1/R_0^2$ term, we combine it with the last term in (10.41) and get

$$\frac{\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)\left(2\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}+\frac{\left[\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2-\beta^2\right]\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}+\hat{\mathbf{R}}_0$$

$$=\frac{\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)\left(2\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)-\boldsymbol{\beta}\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}$$

$$+\frac{\left[\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2-\beta^2\right]\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)+\left[1-2\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}+\left(\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2\right]\hat{\mathbf{R}}_0}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2}$$

$$=\frac{\left(1-\beta^2\right)\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)}{\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^2} \tag{10.45}$$

If we substitute (10.44) and (10.45) back into (10.41), we arrive at the result

$$\mathbf{E}\left(\mathbf{r}_0,t_0\right)=\frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{R}}_0-\boldsymbol{\beta}}{\gamma^2R_0^2\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3}+\frac{\hat{\mathbf{R}}_0\times\left[\left(\hat{\mathbf{R}}_0-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3}\right]_{retarded}. \tag{10.46}$$

where $\gamma^2=1/\left(1-\beta^2\right)$.

4.

Exercise 10.3: By comparing (10.46) in the text to (10.41) in the text, show that for a point charge the electric and magnetic fields are related by

$$\mathbf{B} = \left[\frac{\hat{\mathbf{R}}_0 \times \mathbf{E}}{c}\right]_{retarded} \tag{10.47}$$

Solution: The Lienard-Wiechert form of the electric field, (10.46) in the text, is

$$\mathbf{E}(\mathbf{r}_0, t_0) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{\hat{\mathbf{R}}_0 - \boldsymbol{\beta}}{\gamma^2 R_0^2\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3} + \frac{\hat{\mathbf{R}}_0 \times \left[\left(\hat{\mathbf{R}}_0 - \boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{cR_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3}\right\}_{retarded} \tag{10.48}$$

Dividing by c and taking the cross product with the unit vector $\hat{\mathbf{R}}_0$, this is

$$\left[\frac{\hat{\mathbf{R}}_0 \times \mathbf{E}}{c}\right]_{retarded} = \frac{q}{4\pi\varepsilon_0 c^2}\left[\frac{c\hat{\mathbf{R}}_0 \times \left(\hat{\mathbf{R}}_0 - \boldsymbol{\beta}\right)}{\gamma^2 R_0^2\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3} + \frac{\hat{\mathbf{R}}_0 \times \left\{\hat{\mathbf{R}}_0 \times \left[\left(\hat{\mathbf{R}}_0 - \boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}}{R_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3}\right]_{retarded} \tag{10.49}$$

We change the prefactor to be in terms of the permeability and note that $\hat{\mathbf{R}}_0 \times \hat{\mathbf{R}}_0 = 0$ in the first term to get

$$\left[\frac{\hat{\mathbf{R}}_0 \times \mathbf{E}}{c}\right]_{retarded} = \frac{\mu_0 q}{4\pi}\left[\frac{-c\hat{\mathbf{R}}_0 \times \boldsymbol{\beta}}{\gamma^2 R_0^2\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3} + \frac{\hat{\mathbf{R}}_0 \times \left[\hat{\mathbf{R}}_0 \times \left(\hat{\mathbf{R}}_0 \times \dot{\boldsymbol{\beta}}\right) - \hat{\mathbf{R}}_0 \times \left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)\right]}{R_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3}\right]_{retarded} \tag{10.50}$$

From a vector identity, we get

$$\hat{\mathbf{R}}_0 \times \left(\hat{\mathbf{R}}_0 \times \dot{\boldsymbol{\beta}}\right) = \left(\hat{\mathbf{R}}_0 \cdot \dot{\boldsymbol{\beta}}\right)\hat{\mathbf{R}}_0 - \left(\hat{\mathbf{R}}_0 \cdot \hat{\mathbf{R}}_0\right)\dot{\boldsymbol{\beta}} = \left(\hat{\mathbf{R}}_0 \cdot \dot{\boldsymbol{\beta}}\right)\hat{\mathbf{R}}_0 - \dot{\boldsymbol{\beta}} \tag{10.51}$$

Substituting into (10.50) and using $\hat{\mathbf{R}}_0 \times \hat{\mathbf{R}}_0 = 0$ to eliminate one of the resulting terms, we get

$$\left[\frac{\hat{\mathbf{R}}_0 \times \mathbf{E}}{c}\right]_{retarded} = -\frac{\mu_0 q}{4\pi}\left\{\frac{c\hat{\mathbf{R}}_0 \times \boldsymbol{\beta}}{\gamma^2 R_0^2\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3} + \frac{\hat{\mathbf{R}}_0 \times \left[\dot{\boldsymbol{\beta}} + \hat{\mathbf{R}}_0 \times \left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)\right]}{R_0\left(1-\hat{\mathbf{R}}_0\cdot\boldsymbol{\beta}\right)^3}\right\}_{retarded} \tag{10.52}$$

Comparing this result to (10.41) in the text, we see that we have demonstrated (10.47).

1.

Exercise 6.1: Consider a volume V bounded by the surface S filled with a polarization $\mathbf{P}(\mathbf{r}')$ that depends on the position $\mathbf{r}'$. Assume that there are no free charges. As shown in Chapter 3, the potential of a point dipole at $\mathbf{r}'$ is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{R}}{R^3} \tag{6.1}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, so we can find the potential outside V from the integral

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\mathbf{P}\cdot\mathbf{R}}{R^3}\,dV \tag{6.2}$$

(a) Show that

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{R}}{R^3} \tag{6.3}$$

(b) Use this result together with the divergence theorem to show that the potential may be expressed in the form

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b}{R}\,dV + \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{R}\,dS \tag{6.4}$$

where the bound volume and surface charge densities are respectively

$$\rho_b(\mathbf{r}') = -\nabla'\cdot\mathbf{P}(\mathbf{r}') \tag{6.5}$$

$$\sigma_b(\mathbf{r}') = \mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}} \tag{6.6}$$

and $\hat{\mathbf{n}}$ is a unit vector outward normal to the surface S.

(c) The formula (6.4) looks like Coulomb's law. Does it apply inside V? Explain your answer.

(d) Consider a cylinder whose aspect ratio (ratio of the length L to the diameter D) is of the order of unity, filled with a uniform polarization parallel to its axis. Indicate the position and sign of the bound charge density. Sketch the equipotential surfaces. Sketch the lines of force of the electric field $\mathbf{E} = -\nabla\Phi$ inside and outside the cylinder.

(e) Sketch the lines of force of the displacement $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ inside and outside the cylinder. What is the approximate value of $\mathbf{D}$ at the very center of the cylinder for a very long cylinder ($L/D >> 1$), and for a very short cylinder ($L/D >> 1$)? Explain your answers.

Solution: (a) The length R is given by

$$\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{6.7}$$

where x, y, and z are the components of $\mathbf{r}$ according to

$$\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \tag{6.8}$$

The components of $\mathbf{r}'$ are defined similarly. The partial derivatives of $\frac{1}{R}$ with respect to the primed components are then

$$\frac{\partial}{\partial x'}\left(\frac{1}{R}\right) = -\frac{1}{2}\frac{-2(x-x')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} = \frac{(x-x')}{R^3} \tag{6.9}$$

and the analogous results in the other two directions. Using the definition of the gradient,

$$\nabla'\left(\frac{1}{R}\right) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x'} + \hat{\mathbf{y}}\frac{\partial}{\partial y'} + \hat{\mathbf{z}}\frac{\partial}{\partial z'}\right)\frac{1}{R} = \frac{(x-x')\hat{\mathbf{x}}+(y-y')\hat{\mathbf{y}}+(z-z')\hat{\mathbf{z}}}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} = \frac{\mathbf{r}-\mathbf{r}'}{R^3} = \frac{\mathbf{R}}{R^3} \tag{6.10}$$

(b) Using the vector identity,

$$\nabla\cdot(\phi\mathbf{A}) = \phi(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\phi) \tag{6.11}$$

we can write

$$\nabla'\cdot\left(\frac{\mathbf{P}}{R}\right) = \frac{\nabla'\cdot\mathbf{P}}{R} + \mathbf{P}\cdot\nabla'\left(\frac{1}{R}\right) = \frac{\nabla'\cdot\mathbf{P}}{R} + \frac{\mathbf{P}\cdot\mathbf{R}}{R^3} \tag{6.12}$$

This equation allows us to substitute for the integrand in(6.2)

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\mathbf{P}\cdot\mathbf{R}}{R^3}dV = \frac{1}{4\pi\varepsilon_0}\int_V\left(-\frac{(\nabla'\cdot\mathbf{P})}{R} + \nabla'\cdot\left(\frac{\mathbf{P}}{R}\right)\right)dV \tag{6.13}$$

Using the divergence theorem on the last term, we get

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\int_V \frac{(\nabla'\cdot\mathbf{P})}{R}dV + \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}\cdot\hat{\mathbf{n}}}{R}dS \tag{6.14}$$

or, using the definitions in the problem,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b}{R}dV + \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{R}dS \tag{6.15}$$

(c) The formula (6.4) applies both inside and outside the volume V since at any point the field due to the dipoles is correctly given by (6.1).

(d) The cylinder described is sketched in Figure 6.1. The positive bound surface charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} > 0$ is indicated by + signs and the negative bound surface charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} < 0$ is indicated by – signs. Equipotentials are indicated with dotted lines and the field lines of the electric field are drawn as solid lines.

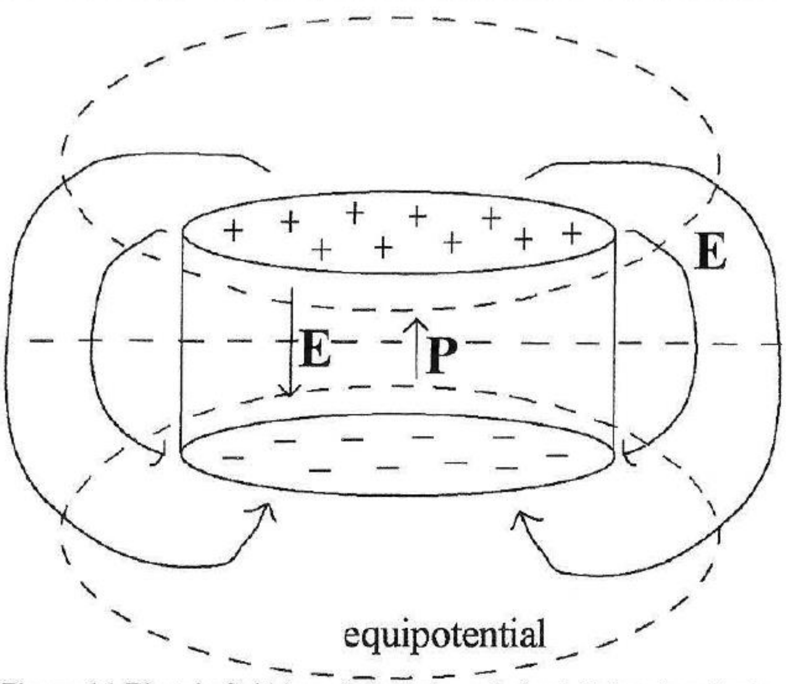


Figure 6.1 Electric field in and around a polarized dielectric cylinder

(e) The electric displacement in and around the dielectric is sketched in Figure 6.2. The displacement does not change sign as we cross the circular faces of the cylinder which contain the surface charge. The displacement does change sign as move across the surface along the length of the cylinder. Across this surface, however, the electric field does not change sign.

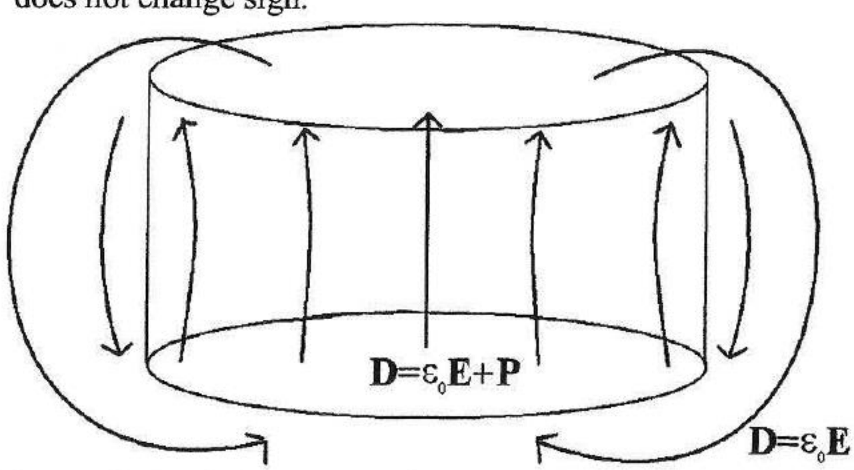


Figure 6.2 Electric displacement in and around a polarized dielectric cylinder

Two cylinders with different aspect ratios are pictured in Figure 6.3. In the long, skinny cylinder, we have two small areas of bound surface charge separated by a long distance. The electric field is small everywhere except near the ends, and in the center we have $\mathbf{D} \approx \mathbf{P}$. In the short, fat cylinder, we have large areas of surface charge close together. The electric field is strong inside dielectric and small outside the dielectric. In fact, using a small Gaussian cylinder that straddles the bound surface charge on one end of the cylinder we see that each end of the cylinder provides an electric field with magnitude given by

$$E = \frac{\sigma_b}{2\varepsilon_0} \tag{6.16}$$

Outside the ends of the cylinder these fields cancel, but inside they are in the same direction (pointing away from the positive end) and the magnitude of the field inside is

$$E = \frac{\sigma_b}{\varepsilon_0} \tag{6.17}$$

From (6.6), we see that the value of the uniform polarization inside the dielectric must be $\mathbf{P} = \sigma_b \hat{\mathbf{n}}$. This polarization always points toward the positive end, so the magnitude of the displacement at the center of the cylinder is zero.

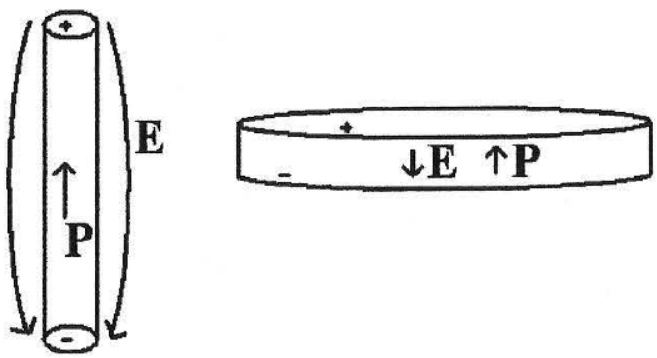


Figure 6.3 Two dielectric cylinders with differing aspect ratios

2.

Solution: From the text, we see that the electric field outside a uniformly polarized dielectric sphere of radius R is that of a point dipole at the origin,

$$\mathbf{E}_{out} = \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}})-\mathbf{p}}{4\pi\varepsilon_0 r^3} \tag{6.53}$$

where the dipole moment of the sphere is the total polarization,

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P} \tag{6.54}$$

Also from the text, the field inside the sphere has a magnitude directly related to the polarization, but points in the opposite direction,

$$\mathbf{E}_{in} = -\frac{\mathbf{P}}{3\varepsilon_0} \tag{6.55}$$

When we put this sphere in a uniform electric field, the field inside becomes

$$\mathbf{E}_{in} = \mathbf{E}_0 - \frac{\mathbf{P}}{3\varepsilon_0} \tag{6.56}$$

but the polarization itself depends on the field inside the sphere,

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}_{in} = \varepsilon_0\chi_e\mathbf{E}_0 - \frac{\chi_e\mathbf{P}}{3} \tag{6.57}$$

and we just solve for the polarization,

$$\mathbf{P} = \frac{3\varepsilon_0\chi_e\mathbf{E}_0}{3+\chi_e} \tag{6.58}$$

We plug this back in to (6.56) to get the field inside the sphere,

$$\mathbf{E}_{in} = \frac{\mathbf{E}_0}{1+\dfrac{\chi_e}{3}} \tag{6.59}$$

and into (6.53) plus the external field to get the electric field outside the sphere

$$\mathbf{E}_{out} = \mathbf{E}_0 + \frac{\chi_e}{3+\chi_e}\frac{R^3}{r^3}\left[3\hat{\mathbf{r}}(\mathbf{E}_0\cdot\hat{\mathbf{r}})-\mathbf{E}_0\right] \tag{6.60}$$

3.

Solution: (a) Outside a uniformly polarized dielectric sphere, the electric field is

$$\mathbf{E}_{out} = \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}})-\mathbf{p}}{4\pi\varepsilon_0 r^3} \tag{6.74}$$

With the appropriate changes, this becomes the magnetic field outside a sphere filled with uniform magnetization $\mathbf{M}$. Outside the sphere, this is proportional to the magnetic induction,

$$\mathbf{H}_{out} = \frac{\mathbf{B}_{out}}{\mu_0} = \frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}})-\mathbf{m}}{4\pi r^3} \tag{6.75}$$

where

$$\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M} \tag{6.76}$$

Inside the dielectric sphere,

$$\mathbf{E}_{in} = -\frac{\mathbf{P}}{3\varepsilon_0} \tag{6.77}$$

so, by analogy, inside the sphere of uniform magnetization,

$$\mathbf{H}_{in} = \frac{\mathbf{B}_{in}}{\mu_0} - \mathbf{M} = -\frac{\mathbf{M}}{3} \tag{6.78}$$

Solving for the magnetic induction, then, we have

$$\mathbf{B}_{in} = \frac{2}{3}\mu_0\mathbf{M} \tag{6.79}$$

The field lines for the magnetized sphere are shown in Figure 6.8.

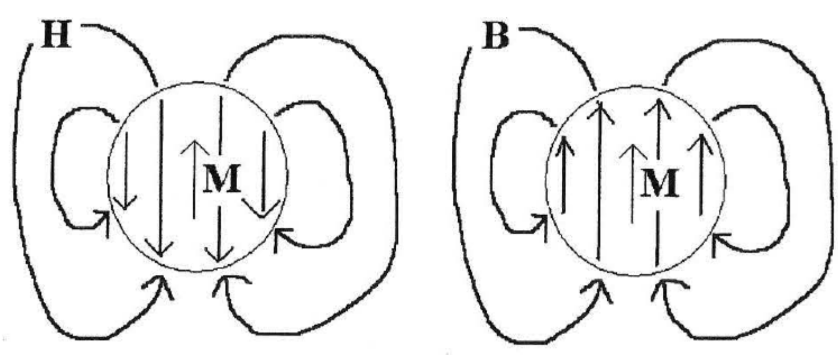


Figure 6.8 Field lines for the magnetized sphere

(b) For a spherical cavity inside a uniformly polarized dielectric material, the electric field is opposite that of the field inside a sphere of the material,

$$\mathbf{E}_{cav} = \frac{\mathbf{P}}{3\varepsilon_0} \tag{6.80}$$

The magnetic field (again, directly related to the induction) is then

$$\mathbf{H}_{cav} = \frac{\mathbf{B}_{cav}}{\mu_0} = \frac{\mathbf{M}}{3} \tag{6.81}$$

4.

Problem 4: EM Waves in conducting materials

no free charges $\rho_f = 0$

$$\Rightarrow \nabla\cdot\vec{B} = \nabla\cdot\vec{H} = 0$$

$$\Rightarrow \nabla\cdot\vec{E} = \nabla\cdot\vec{D} = 0$$

$$\nabla\cdot\vec{E} = \nabla\,\vec{E}_0\, e^{ikz-\chi z-i\omega t} = \vec{E}_0\cdot i(k+i\chi)\hat{e}_z\, e^{ikz-\chi z-i\omega t} = 0$$

$$\nabla\cdot\vec{H} = \nabla\,\vec{H}_0\, e^{ikz-\chi z-i\omega t} = \vec{H}_0\, i(k+i\chi)\hat{e}_z\, e^{ikz-\chi z-i\omega t} = 0$$

$\Rightarrow$ both $\vec{H}_0$ and $\vec{E}_0$ lie in the (x,y) plane

Faraday's law $\quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t}$

$$\Rightarrow i(k+i\chi)\hat{e}_z\times\vec{E} = -\mu_0(-i\omega)\vec{H} = +i\mu_0\omega\vec{H}$$

$\boxed{\hat{e}_z \text{ is unit vector in } z\text{-direction}}$

$$\Rightarrow \vec{H} = \frac{(k+i\chi)}{\mu_0\omega}\,\hat{e}_z\times\vec{E}$$

$$\Rightarrow \vec{H}_0 = \frac{(k+i\chi)}{\mu_0\omega}\,\hat{e}_z\times\vec{E}_0 \tag{1}$$

Maxwell-Ampère Law: $\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \underbrace{\sigma\vec{E}}_{\vec{J}} + \underbrace{\varepsilon\frac{\partial \vec{E}}{\partial t}}_{\vec{D}=\varepsilon\vec{E}}$

$$\Rightarrow \quad i(k+i\chi)\hat{e}_z\times\vec{H}_0 = \sigma\vec{E}_0 + (-i\omega)\varepsilon\vec{E}_0 = -i\vec{E}_0(\omega\varepsilon + i\sigma)$$

$$\Rightarrow \quad \vec{E}_0 = \frac{-(k+i\chi)}{(\omega\varepsilon+i\sigma)}\hat{e}_z\times\vec{H}_0 \qquad (2)$$

Make sure equation (1)+(2) are consistent

$$\vec{E}_0 = -\frac{(k+i\chi)}{(\omega\varepsilon+i\sigma)}\hat{e}_z\times\frac{(k+i\chi)}{\mu_0\omega}(\hat{e}_z\times\vec{E}_0) = -\frac{(k+i\chi)^2}{\mu_0\omega(\omega\varepsilon+i\sigma)}\underbrace{(\hat{e}_z\times(\hat{e}_z\times\vec{E}_0))}_{\hat{e}_z\underbrace{(\hat{e}_z\cdot\vec{E}_0)}_{=0} - \vec{E}_0\underbrace{(\hat{e}_z\cdot\hat{e}_z)}_{=1}}$$

$$\Rightarrow \quad (k+i\chi)^2 = \mu_0\omega(\omega\varepsilon+i\sigma) = \mu_0\varepsilon_0\omega^2\underbrace{\left(\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}\right)}_{n(\omega)^2}$$

Define a complex index of refraction $n(\omega) = \sqrt{\varepsilon_r + \frac{i\sigma}{\varepsilon_0\omega}}$

Relative permittivity: $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$

$$(k+i\chi)^2 = \mu_0\varepsilon\,\omega^2 n^2(\omega) = \frac{\omega^2}{c^2}n^2(\omega) \qquad c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\Longrightarrow \quad n(\omega) = \frac{(k+i\chi)}{\omega}c$$

a) $\Rightarrow \underline{\underline{k = \frac{\omega}{c} \mathrm{Re}\{n(\omega)\}}}$ $\quad$ $\underline{\underline{\chi = \frac{\omega}{c} \mathrm{Im}\{n(\omega)\}}}$

b) Use eq. (1) $\vec{H}_0 = -\frac{(k+i\chi)}{\mu_0 \omega} \hat{e}_z \times \vec{E}_0 = -\frac{\omega}{c} n(\omega) \frac{1}{\mu_0 \omega} \hat{e}_z \times \vec{E}_0$

b) $$\vec{H}_0 = -\frac{n(\omega)}{c\mu_0} \hat{e}_z \times \vec{E}_0 \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$= -n(\omega)\sqrt{\frac{\epsilon_0}{\mu_0}}\, \hat{e}_z \times \vec{E}_0 \qquad \hat{e}_z \times (\hat{e}_z \times \vec{E}_0) = -\vec{E}_0$$

$$\Rightarrow \vec{H}_0 = \frac{n(\omega)}{Z_0} \hat{e}_z \times \vec{E}_0$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega \text{ vacuum impedance}$$

$\Rightarrow$ The electric field amplitude relates to the Magnetic field as

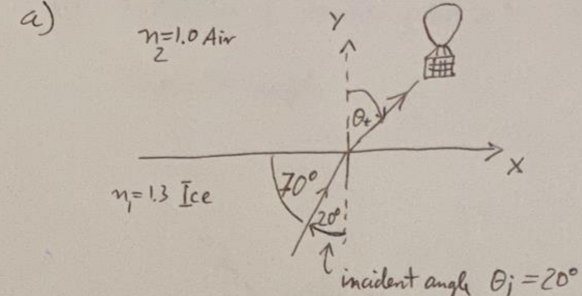
$$\underline{\underline{|\vec{H}_0| = \frac{n(\omega)}{Z_0} |\vec{E}_0|}} \quad \text{or} \quad \underline{\underline{|\vec{H}_0| = n(\omega)\sqrt{\frac{\epsilon_0}{\mu_0}}\, \vec{E}_0}}$$

$$\underline{\underline{|\vec{E}_0| = \frac{Z_0}{n(\omega)} |\vec{H}_0|}} \quad \text{or} \quad \underline{\underline{|\vec{E}_0| = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{n(\omega)} |\vec{H}_0|}}$$

Homework Solutions

Problem I: ANITA Experimental signatures

a)



$\sin 20° \approx 0.34$

$\sin \theta_t \approx 0.34 \cdot \frac{n_1}{n_2} = 0.34 \cdot 1.3 \Rightarrow \underline{\theta_t \approx 26°}$

Note: θ_i needs to be 20° otherwise ~~at~~ 70° we would have total internal reflection

Total reflection angle

$\frac{n_2}{n_1} = \sin\theta_i \Rightarrow \frac{1}{1.3} = \sin\theta_i \Rightarrow \underline{\theta_i = 50°}$ critical angle

Snell's law $n_1 \sin\theta_i = n_2 \sin\theta_t$

Now consider two linear polarization directions

1) In x-y plane V-Pol or p-polarization

2) ~~Along~~ z direction H-Pol or s-polarization

Brau (7.234)

$$T_s = 1 - R_s = 1 - \left(\frac{\sqrt{n_i^2 - n_i^2 \sin^2\theta_i} - \sqrt{n_t^2 - n_i^2 \sin^2\theta_i}}{\sqrt{n_i^2 - n_i^2 \sin^2\theta_i} + \sqrt{n_t^2 - n_i^2 \sin^2\theta_i}} \right)^2$$

with $n_i = 1.3$ $n_t = n_{air} = 1.0$ $\theta_i = 20°$

$\Rightarrow R_s \approx 0.0237$ $\Rightarrow \underline{T_s \approx 0.976}$

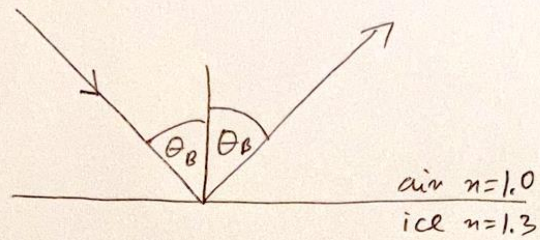
$$T_p = 1 - R_p = 1 - \left(\frac{n_t^2 \sqrt{n_i^2 - n_i^2 \sin^2\theta_i} - n_i^2 \sqrt{n_t^2 - n_i^2 \sin^2\theta_i}}{n_t^2 \sqrt{n_i^2 - n_i^2 \sin^2\theta_i} + n_i^2 \sqrt{n_t^2 - n_i^2 \sin^2\theta_i}} \right)^2$$

$\Rightarrow R_p \approx 0.0114$ $\Rightarrow \underline{T_p \approx 0.989}$

Conclusions: Fairly mixed polarization but p-polarization slightly larger

Problem 1: ANITA Experimental Signatures

b) Reflection of random polarization signal on the ground at Brewster angle



$$\tan\theta_B = \frac{n_{ice}}{n_{air}} = \frac{1.3}{1.0}$$

$$\Rightarrow \theta_B = 52.4^\circ$$

At Brewster angle θ_B reflectance for p-polarized light vanishes

$\Rightarrow$ reflected signal is purely s-polarized ∎

c) How to discriminate neutrino signals from antropogenic backgrounds
The following criteria can be used:

I) Polarization direction of the signal

II) Duration of the detected signal, neutrinos are single events represented as short pulses, while antropogenic signals are more continuous.

III) The location of the signal is used to map it back where it originated. The point of origin is compared to a map of Antarctica and known locations of researchers or adventurers.

2. Ans: For an Ohmic material, we confine ourselves to the linear response of the matter to the electric field. And we define a time-dependent conductivity function $\sigma(\tau)$, where τ is the time delay. This means that the current density $\vec{J}(\vec{r},t)$ may depend on the electric field at any time earlier than t but it cannot depend on the electric field at any time later than t:

$$\vec{J}(\vec{r},t) = \int_{-\infty}^{t} \vec{E}(\vec{r},t')\sigma(t-t')\,dt'. \qquad (1)$$

In pratice, $\sigma(\tau) \to 0$ as $\tau \to \infty$ because the electric field in the distant past has a negligible effect on the present-time current density. It's most convenient to extend the upper limit of integration in (1) to infinity and build causality directly into the conductivity. In other words, we constrain the conductivity so

$$\sigma(\tau<0) = 0,$$

and write

$$\vec{J}(\vec{r},t) = \int_{-\infty}^{\infty} dt'\,\sigma(t-t')\,\vec{E}(\vec{r},t'). \qquad (2)$$

The right side of (2) is a convolution integral. This motivates us to define the Fourier transform pairs

$$\tilde{\sigma}(\omega) = \int_{-\infty}^{\infty} dt\,\sigma(t)e^{i\omega t} \quad \text{and} \quad \sigma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\tilde{\sigma}(\omega)e^{-i\omega t}$$

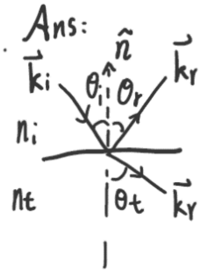
and write Fourier representations for the current density and electric field:

$$\vec{J}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\tilde{\vec{J}}(\vec{r},\omega)\,e^{-i\omega t} \quad \text{and} \quad \vec{E}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\tilde{\vec{E}}(\vec{r},\omega)\,e^{-i\omega t}.$$

These definitions allow us to apply the convolution theorem to the Fourier transform of (2) to conclude that

$$\tilde{\vec{J}}(\vec{r},\omega) = \tilde{\sigma}(\omega)\,\tilde{\vec{E}}(\vec{r},\omega).$$

3. Ans:



First we calculate the Poynting vector of the EM plane wave:

$$\vec{S}_a = \vec{E}_a \times \vec{H}_a = \vec{E}_a \times \left(\frac{\vec{k}_a \times \vec{E}_a}{\mu_a \omega} \right) = \frac{\vec{E}_a^{\,2}}{\mu_a \omega} \vec{k}_a, \qquad (1)$$

where the subscript a takes on the values $a = i, r, t$ for the incident, reflected, and transmitted waves, respectively, and μ_a is the corresponding permeability, usually $\mu_a \approx 1$.

And since $|\vec{k}_a| = n_a \frac{\omega}{c}$, we can simplify (1):

$$\vec{S}_a = \frac{\vec{E}_a^{\,2} n_a}{\mu_a c} \hat{k}_a .$$

Consider a incident plane (shown in the plot above) with a normal direction $\hat{n}$, and the light beam should be projected to the surface:

$$|\vec{S}_a \cdot \hat{n}| = \frac{\vec{E}_a^{\,2} n_a}{\mu_a c} |\hat{k}_a \cdot \hat{n}| = \frac{\vec{E}_a^{\,2} n_a}{\mu_a c} \cos\theta_a .$$

Given the incident angle θ_i, we can relate the reflected angle θ_r and refraction angle θ_t to θ_i:

$$\begin{cases} \theta_r = \theta_i , \\ n_i \sin\theta_i = n_t \sin\theta_t . \end{cases}$$

$$\Rightarrow \quad \cos\theta_r = \cos\theta_i , \quad \cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i} .$$

And given the magnitude square of the incident electric field $\vec{E}_i^{\,2}$, we can relate the magnitude square of the reflected and transmitted electric fields, $\vec{E}_r^{\,2}, \vec{E}_t^{\,2}$ to it:

$$\vec{E}_r^{\,2} = R \vec{E}_i^{\,2} , \quad \vec{E}_t^{\,2} = T \vec{E}_i^{\,2} ,$$

where R, T are dependent on θ_i and n_a $(a = i, r, t)$.

$$\Rightarrow \quad |\vec{S}_i \cdot \hat{n}| = \frac{\vec{E}_i^{\,2} n_i}{\mu_i c} \cos\theta_i , \quad |\vec{S}_r \cdot \hat{n}| = \frac{R \vec{E}_i^{\,2} n_r}{\mu_r c} \cos\theta_i , \quad |\vec{S}_t \cdot \hat{n}| = \frac{T \vec{E}_i^{\,2} n_t}{\mu_t c} \sqrt{1 - \left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i} .$$

4.

Solution: From (7.239) in the text, the phase shift of s-polarized light upon total internal reflection is

$$\Delta\phi_s = -2\arctan\left(\frac{\mu_i}{\mu_t}\sqrt{\frac{n_i^2\sin^2\theta_i - n_t^2}{n_i^2 - n_i^2\sin^2\theta_i}}\right) \approx -2\arctan\left(\frac{1}{n_i}\sqrt{\frac{n_i^2\sin^2\theta_i - 1}{1-\sin^2\theta_i}}\right) \tag{7.242}$$

where we have rewritten the result under the assumption that $\mu_i \approx \mu_t$, generally true for dielectrics, and with the substitution of unity for the index of refraction in air. The phase shift for p-polarized waves is

$$\Delta\phi_p = -2\arctan\left(\frac{\varepsilon_i}{\varepsilon_t}\sqrt{\frac{n_i^2\sin^2\theta_i - n_t^2}{n_i^2 - n_i^2\sin^2\theta_i}}\right) \approx -2\arctan\left(n_i\sqrt{\frac{n_i^2\sin^2\theta_i - 1}{1-\sin^2\theta_i}}\right) \tag{7.243}$$

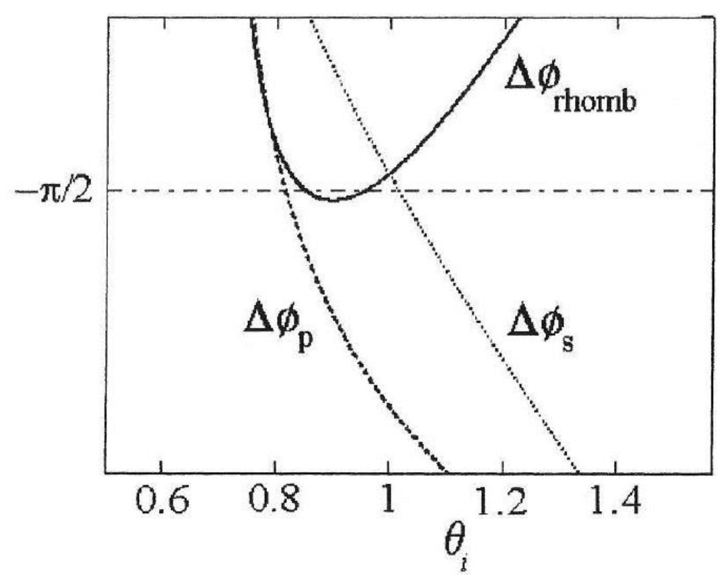
where we have used unity for n_t and eliminated the permittivities using the formula

$$n_i^2 = \frac{\mu_i\varepsilon_i}{\mu_0\varepsilon_0} \approx \frac{\varepsilon_i}{\varepsilon_t} \tag{7.244}$$

Changing linearly polarized light to circularly polarized light means a relative phase shift after two reflections of $\pi/2$ between the s-polarized and p-polarized components. We can define the relative shift through the Fresnel rhomb as

$$\Delta\phi = 2\Delta\phi_p - 2\Delta\phi_s \tag{7.245}$$

If we solve this graphically, as shown in Figure 7.4, we see that the relative phase shift equals $\frac{\pi}{2}$ for two incident angles, near .85 and .95 radians.



5.

Solution: As discussed in the text, the equations derived for the ratio between the transmitted electric field and the incident electric field require the index of refraction on the incident side to be real. They do not, however, require the index on the transmitted side to be real. We can start with (7.255) in the text,

$$\frac{e_r}{e_i} = \frac{\mu_t\sqrt{n_i^2 - n_i^2 \sin^2\theta_i} - \mu_i\sqrt{n_t^2 - n_i^2 \sin^2\theta_i}}{\mu_t\sqrt{n_i^2 - n_i^2 \sin^2\theta_i} + \mu_i\sqrt{n_t^2 - n_i^2 \sin^2\theta_i}} \tag{7.247}$$

which, at normal incidence $(\theta_i = 0)$, becomes

$$\frac{e_r}{e_i} = \frac{\mu_t n_i - \mu_i n_t}{\mu_t n_i + \mu_i n_t} \approx \frac{n_i - n_t}{n_i + n_t} \tag{7.248}$$

where we have made the further approximation $\mu_i = \mu_t = \mu_0$. Substituting our complex index of refraction on the transmitted side, we have

$$\frac{e_r}{e_i} = \frac{n_i - n'_t - in''_t}{n_i + n'_t + in''_t} \tag{7.249}$$

The complex conjugation is now important when we square to get the reflection coefficient,

$$R = \frac{e_r e_r^*}{e_i e_i^*} = \left(\frac{n_i - n'_t + in''_t}{n_i + n'_t + in''_t}\right)\left(\frac{n_i - n'_t - in''_t}{n_i + n'_t - in''_t}\right) = \frac{(n_i - n'_t)^2 + n''^2_t}{(n_i + n'_t)^2 + n''^2_t} \tag{7.250}$$

This reduces to (7.236) in the text when the losses vanish $(n_t'' = 0)$.

1.

Solution: From (10.185) in the text, the general expression for the spectral angular fluence from transition radiation is

$$\frac{d^2\mathcal{W}}{d\omega d\Omega}=\frac{q^2}{16\pi^3\varepsilon_0 c}\left|\frac{\hat{\mathbf{n}}\times\boldsymbol{\beta}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}-\frac{\hat{\mathbf{n}}\times\boldsymbol{\beta}'}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}'}\right|^2 \tag{10.172}$$

where $\boldsymbol{\beta}$ is the velocity of the particle, $\boldsymbol{\beta}'$ is the velocity of its image charge counterpart, and $\hat{\mathbf{n}}$ is a unit vector in the observation direction. In the nonrelativistic limit, the denominators inside the absolute value are approximated by unity, and we have

$$\frac{d^2\mathcal{W}}{d\omega d\Omega}=\frac{q^2}{16\pi^3\varepsilon_0 c}\left|\hat{\mathbf{n}}\times(\boldsymbol{\beta}-\boldsymbol{\beta}')\right|^2 \tag{10.173}$$

As shown in Figure 10.2, the charge and image charge trajectories have angles with respect to the surface normal that are supplementary, so that

$$\boldsymbol{\beta}-\boldsymbol{\beta}'=-2\left(\boldsymbol{\beta}\cdot\hat{\mathbf{N}}\right)\hat{\mathbf{N}} \tag{10.174}$$

where $\hat{\mathbf{N}}$ is the surface unit normal. Making this substitution into the spectral angular fluence gives

$$\frac{d^2\mathcal{W}}{d\omega d\Omega}=\frac{q^2}{4\pi^3\varepsilon_0 c}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{N}}\right)^2\left|\hat{\mathbf{n}}\times\hat{\mathbf{N}}\right|^2=\frac{q^2}{4\pi^3\varepsilon_0 c}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{N}}\right)^2\sin^2\theta \tag{10.175}$$

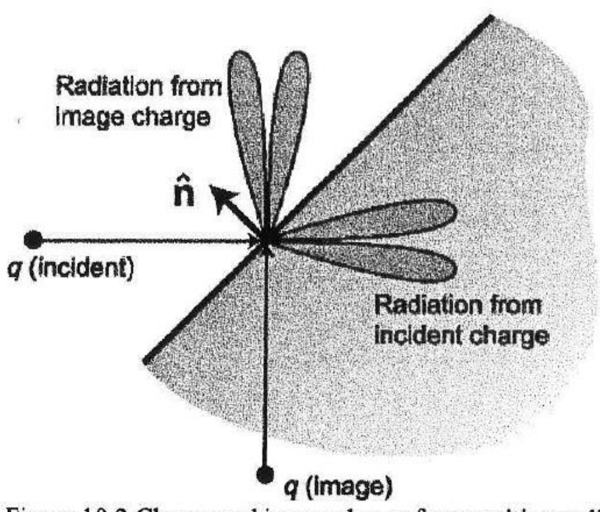


Figure 10.2 Charge and image charge for transition radiation

To get the total spectral fluence, we integrate over all observation directions, getting

$$\frac{d\mathcal{W}}{d\omega}=\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta d\theta\frac{d^2\mathcal{W}}{d\omega d\Omega}=\frac{q^2}{2\pi^2\varepsilon_0 c}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{N}}\right)^2\int_0^{\pi/2}d\theta\sin^3\theta \tag{10.176}$$

Evaluating the remaining integral, we have

$$\frac{d\mathcal{W}}{d\omega}=\frac{q^2}{3\pi^2\varepsilon_0 c}\left(\boldsymbol{\beta}\cdot\hat{\mathbf{N}}\right)^2 \tag{10.177}$$

This result is valid in the nonrelativistic limit. In the ultrarelativistic limit, we make a different approximation. In this high-energy limit, the radiation from the incident and image charges is strongly concentrated in the forward direction. We can then ignore any interference between the two radiation patterns. The total radiation into the vacuum is equal to the radiation from the image charge integrated over 4π steradians. With these conditions, the formula for the total spectral fluence becomes

$$\frac{d\mathcal{W}}{d\omega}=\frac{q^2}{16\pi^3\varepsilon_0 c}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\left|\frac{\hat{\mathbf{n}}\times\boldsymbol{\beta}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}\right|^2=\frac{q^2}{8\pi^2\varepsilon_0 c}\int_0^{\pi}\sin\theta d\theta\left|\frac{\hat{\mathbf{n}}\times\boldsymbol{\beta}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}\right|^2 \tag{10.178}$$

From the suggested vector identity, we get

$$\left|\hat{\mathbf{n}}\times\boldsymbol{\beta}\right|^2=\left(\hat{\mathbf{n}}\times\boldsymbol{\beta}\right)\cdot\left(\hat{\mathbf{n}}\times\boldsymbol{\beta}\right)=\left(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\right)\left(\boldsymbol{\beta}\cdot\boldsymbol{\beta}\right)-\left(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)\left(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right) \tag{10.179}$$

or, looking ahead to what we have to integrate, we rearrange to

$$\left|\hat{\mathbf{n}}\times\boldsymbol{\beta}\right|^2=\beta^2-\left(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^2=-\left(1-\beta^2\right)+1-\left(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^2=1-\left(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^2-\frac{1}{\gamma^2} \tag{10.180}$$

Putting this result into (10.178), we have

$$\frac{d\mathcal{W}}{d\omega}=\frac{q^2}{8\pi^2\varepsilon_0 c}\int_0^{\pi}\sin\theta d\theta\left[\frac{1+\left(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}-\frac{1}{\gamma^2\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^2}\right] \tag{10.181}$$

Changing integration variables to $x=\cos\theta$, we get

$$\frac{d\mathcal{W}}{d\omega}=-\frac{q^2}{8\pi^2\varepsilon_0 c}\int_1^{-1}dx\left[\frac{1+\beta x}{1-\beta x}-\frac{1}{\gamma^2\left(1-\beta x\right)^2}\right] \tag{10.182}$$

Expressing the first term in the integrand with partial fractions, this is

$$\frac{d\mathcal{W}}{d\omega}=-\frac{q^2}{8\pi^2\varepsilon_0 c}\int_1^{-1}dx\left[-1+\frac{2}{1-\beta x}-\frac{1}{\gamma^2\left(1-\beta x\right)^2}\right] \tag{10.183}$$

This form suggests another change of integration variable to $y=1-\beta x$, giving

$$\frac{d\mathcal{W}}{d\omega}=\frac{q^2}{8\pi^2\varepsilon_0 \beta c}\int_{1-\beta}^{1+\beta}dy\left[-1+\frac{2}{y}-\frac{1}{\gamma^2 y^2}\right] \tag{10.184}$$

Now, we can perform the integration to get

$$\frac{d\mathcal{W}}{d\omega}=\frac{q^2}{8\pi^2\varepsilon_0 c}\left[-2+\frac{2}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{2}{\beta}\right] \tag{10.185}$$

In the ultrarelativistic limit, we make the two approximations

$$1-\beta\approx 1/2\gamma^2 \tag{10.186}$$

and, where factors of the velocity are multiplied or added,

$$\beta\approx 1 \tag{10.187}$$

With these approximations, our result becomes

$$\frac{d\mathcal{W}}{d\omega}=\frac{q^2}{4\pi^2\varepsilon_0 c}\left[-2+\ln\left(4\gamma^2\right)\right]=\frac{q^2}{2\pi^2\varepsilon_0 c}\left[-1+\ln(2)+\ln(\gamma)\right] \tag{10.188}$$

The first two terms are constant – an artifact of the way the integral was handled without including overlap and integrating over 4π steradians. For $\gamma\gg 1$ the $\ln(\gamma)$ term is dominant, so our total spectral fluence in the ultrarelativistic limit is

$$\frac{d\mathcal{W}}{d\omega}=\frac{q^2}{2\pi^2\varepsilon_0 c}\ln(\gamma) \tag{10.189}$$

2.

(1) (a) The threshold energy for muons and protons:

Chrenkov radiation is observed when a particle travels through a medium at a velocity greater than the speed of light in that medium. Here we consider a water Chrenkov detector:

Speed of light in the water:

$$v = \frac{c}{n}.$$

$\Rightarrow$ We require the velocity of muons and protons:

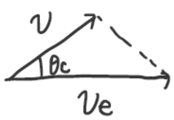
$$v_\mu,\ v_p \geqslant v.$$

Threshold energy of muons:

$$E_\mu = \sqrt{m_\mu^2c^4+(\gamma m_\mu v_\mu)^2c^2} \geqslant m_\mu c\sqrt{c^2+\frac{\left(\frac{c}{n}\right)^2}{1-\frac{1}{n^2}}} = m_\mu c^2\sqrt{1+\frac{1/n^2}{1-1/n^2}},\quad \gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{1}{n^2}}}.$$

Similarly, $E_p \geqslant m_pc^2\sqrt{1+\frac{1/n^2}{1-1/n^2}}$ Substitute m_μ, m_p, we find $E_\mu \geqslant 160.26\,\text{MeV}$, $E_p \geqslant 1.42\,\text{GeV}$.

(b)



$$\cos\theta_c = \frac{v}{v_e} = \frac{c/n}{c(p_e/E_e)},\quad \text{where } p_e = \sqrt{E_e^2-m_e^2c^4}\,/c.$$

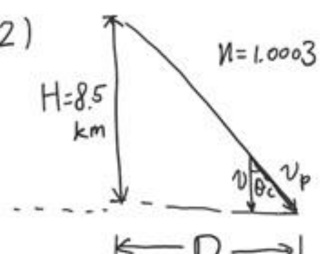
Substitute $m_ec^2=0.511\,\text{MeV}$, $E_e=2\,\text{GeV}$, we find $\theta_c \approx 41.25°$.

(c) Find $v_K = c\frac{p_k}{E_k} = c\frac{\sqrt{E_k^2-m_k^2c^4}}{E_k}$, compare v_K with $\frac{c}{n}$.

$\Rightarrow$ $\frac{c/n}{v_K} \approx 1.765 > 1$, so there is no Cherenkov light.

(d) No, since neutron is an electrically neutral particle, and assume neutron decay doesn't happen in the water Cherenkov detector (otherwise there will be high-energy electrons produced).

(2)



$$\cos\theta_c = \frac{v}{v_p} = \frac{c/n}{c\sqrt{1-\frac{m_p^2c^4}{E_p^2}}} = \frac{1}{n}\frac{1}{\sqrt{\frac{\gamma^2-1}{\gamma^2}}} = \frac{1}{n\beta},\quad D = H\tan\theta_c.$$

The maximum distance is

$$D_{max} = H\tan\theta_{c,max} = H\tan\theta_c\big|_{\beta=1} \approx 0.21\,\text{km}.$$

$$v_p \geqslant v \Rightarrow c\sqrt{1-\frac{m_p^2c^4}{E_p^2}} \geqslant \frac{c}{n} \Rightarrow \frac{m_p^2c^4}{E_p^2} \leqslant 1-\left(\frac{1}{n}\right)^2$$

$$E_p^2 \geqslant \frac{m_p^2c^4}{1-\left(\frac{1}{n}\right)^2} \equiv E_{p,min}^2$$

The threshold energy: $E_{p,min} = \frac{m_pc^2}{\sqrt{1-\frac{1}{n^2}}} \approx 38.3\,\text{GeV}.$

3.

Solution: From (10.387) in the text, the spectral angular fluence in this case is

$$\frac{d^2\mathcal{W}}{d\Omega d\omega}=\frac{\mu q^2 v^2 \sin^2\theta}{16\pi^3 v_\phi}\omega^2\tau^2\left|\frac{\sin\left[\frac{1}{2}\omega\tau\left(1-\frac{v}{v_\phi}\cos\theta\right)\right]}{\frac{1}{2}\omega\tau\left(1-\frac{v}{v_\phi}\cos\theta\right)}\right|^2 \tag{10.331}$$

The radiation at frequency ω peaks at the angle where

$$\cos\theta_C=\frac{v_\phi(\omega)}{v}=\frac{1}{n(\omega)\beta} \tag{10.332}$$

For nearby angles, we have

$$1-\frac{v}{v_\phi}\cos\theta=1-\frac{\cos\theta}{\cos\theta_C}=\frac{\cos\theta_C-\cos\theta}{\cos\theta_C} \tag{10.333}$$

But we can Taylor expand the cosine function as

$$\cos\theta\approx\cos\theta_C+\frac{d\cos\theta}{d\theta}(\theta-\theta_C)=\cos\theta_C-\sin\theta_C(\theta-\theta_C) \tag{10.334}$$

so

$$1-\frac{v}{v_\phi}\cos\theta=\frac{\cos\theta_C-\cos\theta_C+(\theta-\theta_C)\sin\theta_C}{\cos\theta_C}=(\theta-\theta_C)\tan\theta_C \tag{10.335}$$

The angular spectral fluence is then

$$\frac{d^2\mathcal{W}}{d\Omega d\omega}\approx\frac{\mu q^2 v^2\sin^2\theta_C}{16\pi^3 v_\phi}\omega^2\tau^2\left|\frac{\sin\left[\frac{1}{2}\omega\tau(\theta-\theta_C)\tan\theta_C\right]}{\frac{1}{2}\omega\tau(\theta-\theta_C)\tan\theta_C}\right|^2 \tag{10.336}$$

We can then make the substitution

$$\frac{v^2\sin^2\theta_C}{v_\phi}=\frac{v^2\tan^2\theta_C\cos^2\theta_C}{v\cos\theta_C}=v_\phi\tan^2\theta_C \tag{10.337}$$

to get

$$\frac{d^2\mathcal{W}}{d\Omega d\omega}\approx\frac{\mu q^2 v_\phi\tan^2\theta_C}{16\pi^3}\omega^2\tau^2\left|\frac{\sin\left[\frac{1}{2}\omega\tau(\theta-\theta_C)\tan\theta_C\right]}{\frac{1}{2}\omega\tau(\theta-\theta_C)\tan\theta_C}\right|^2 \tag{10.338}$$