
Chapter 2: Random Variables

1 DEFINITION OF RANDOM VARIABLE

- When an experiment is performed, we are often interested in some numerical characteristic of the result.
- Here are some practical examples.
 - An experiment is to examine 100 electronic components, our interest is “the number of defectives”.
 - Flipping a coin 100 times, our interest could be the number of heads obtained, instead of the “H” and “T” sequence.
- This motivates us to assign a numerical value to a possible outcome of an experiment.

DEFINITION 1 (RANDOM VARIABLE)

*Let S be sample space for an experiment. A function X , which assigns a real number to every $s \in S$ is called a **random variable**.*

- So random variable X is a function from S to \mathbb{R} :

$$X : S \mapsto \mathbb{R}.$$

- For convenience, hereafter, we simplify “**random variable**” as “**RV**”.

Example 2.1

- Let $S = \{HH, HT, TH, TT\}$ be a sample space associated with the experiment of flipping two coins.
- Define the RV:

$X =$ number of heads obtained.

- Note that X is a **function** from S to \mathbb{R} , the set of real numbers:

$$X(HH) = 2, \quad X(HT) = X(TH) = 1, \quad X(TT) = 0.$$

The range of X is $R_X = \{0, 1, 2\}$.

L-example 2.1

- A coin is thrown until a “head” occurs.

$$S = \{H, TH, TTH, TTTH, TTTTH, \dots\}$$

- Let X = the number of “trials” required. We then have

$$X(H) = 1, X(TH) = 2, X(TTH) = 3, \dots, \text{ and so on.}$$

- $R_X = \{1, 2, 3, \dots, \}$

REMARK

- We use upper case letters X, Y, Z, X_1, X_2, \dots to denote **random variables**.
- We use lower case letters x, y, z, x_1, x_2 to denote their **observed values** in the experiment.
- The set $\{X = x\}$ is a subset of S , in the sense:

$$\{X = x\} = \{s \in S : X(s) = x\}.$$

- Likewise, the set $\{X \in A\}$, for A being a subset of \mathbb{R} , is also a subset of S :

$$\{s \in S : X(s) \in A\}.$$

- This gives $P(X = x)$ and $P(X \in A)$ based on probability defined on S :

$$P(X = x) = P(\{s \in S : X(s) = x\})$$

$$P(X \in A) = P(\{s \in S : X(s) \in A\})$$

Example 2.2

- Revisit Example 2.1; $S = \{HH, HT, TH, TT\}$ is the sample space of flipping two coins. X = number of heads obtained.
- Then $\{X = 0\} = \{TT\}$; $\{X = 1\} = \{HT, TH\}$; $\{X = 2\} = \{HH\}$; $\{X \geq 1\} = \{HT, TH, HH\}$.
- $P(X = 0) = P(TT) = 1/4$; $P(X = 1) = P(\{HT, TH\}) = 2/4$; $P(X = 2) = P(HH) = 1/4$; $P(X \geq 1) = P(\{HT, TH, HH\}) = 3/4$.

- We can summarize the probabilities of the RV X as a table:

x	0	1	2
$P(X = x)$	1/4	1/2	1/4

L-example 2.2

- When a pair of fair dice is rolled, what is the probability that a sum of 3 is obtained?

$$S = \left\{ (x_1, x_2) \mid x_1 = 1, 2, 3, 4, 5, 6; x_2 = 1, 2, 3, 4, 5, 6 \right\}.$$

- X = the sum of two dice. That is for any $(x_1, x_2) \in S$,

$$X((x_1, x_2)) = x_1 + x_2.$$

- The range of X is

$$R_X = \{2, 3, 4, \dots, 12\}.$$

- Since $\{X = 3\} = \{(1, 2), (2, 1)\}$, we have

$$P(X = 3) = P(\{(1, 2), (2, 1)\}) = 2/36.$$

- The probabilities of other possible values for X can be found similarly, and are tabulated below:

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2 PROBABILITY DISTRIBUTIONS

- We introduce the distribution of two types of RVs: **discrete** and **continuous**.
- In particular, denote by X the RV, and its range by R_X .
 - **Discrete**: the number of values in R_X is **finite** or **countable**; that is we can write $R_X = \{x_1, x_2, x_3, \dots\}$.
 - **Continuous**: R_X is an **interval** or a **collection of intervals**.

Discrete Probability Distributions

- For a discrete RV X , we can always write $R_X = \{x_1, x_2, x_3, \dots\}$.
- Each $x_i \in R_X$, there is a probability that X takes this value, i.e., $P(X = x_i)$.
- We can define a function $f(x) = P(X = x)$.
Note that $f(x_i) = P(X = x_i)$ for $x_i \in R_X$, and $f(x) = 0$ for $x \notin R_X$.
- $f(x)$ is called the **probability function, p.f.** (or **probability mass function, p.m.f.**) of X .
- The collection of pairs $(x_i, f(x_i)), i = 1, 2, 3, \dots$, is called the **probability distribution** of X .

The p.f. $f(x)$ of a discrete RV **must** satisfy:

- (1) $f(x_i) \geq 0$ for all $x_i \in R_X$;
- (2) $f(x) = 0$ for all $x \notin R_X$;
- (3) $\sum_{i=1}^{\infty} f(x_i) = 1$, or $\sum_{x_i \in R_X} f(x_i) = 1$.

For any set $B \subset \mathbb{R}$, we have

$$P(X \in B) = \sum_{x_i \in B \cap R_X} f(x_i).$$

Example 2.3

- Revisit Examples 2.1 and 2.2. RV X is the number of heads when flipping two coins.
- The p.f. of X is given below

x	0	1	2
$f(x)$	1/4	1/2	1/4

- $f(x)$ satisfies (1) $f(x_i) \geq 0$ for $x_i = 0, 1$, or 2 ; (2) $f(x) = 0$ for other x ; (3) $f(0) + f(1) + f(2) = 1$.
- $B = [1, \infty)$; then $P(X \in B) = f(1) + f(2) = 3/4$.

L-example 2.3

Six lots of components are ready to be shipped by a certain supplier. The number of defective components in each lot is as follows:

Lot	1	2	3	4	5	6
# of defectives	0	2	0	1	2	0

- One of the lots is to be **randomly** selected and shipped to a customer.
- Let $X = \#$ of defectives in the shipped lot.
- Then $R_X = \{0, 1, 2\}$.

- The lots are selected randomly, so each has the same probability to be chosen.
- Let $f(x)$ be the p.f. of X .
- We have
 - $f(0) = P(X = 0) = P(\text{lot 1 or 3 or 6 is selected}) = 3/6.$
 - $f(1) = P(X = 1) = P(\text{lot 4 is selected}) = 1/6.$
 - $f(2) = P(X = 2) = P(\text{lot 2 or 5 is selected}) = 2/6.$

- The probability function of X can be summarized by

x	0	1	2
$f(x)$	$1/2$	$1/6$	$1/3$

- It satisfies all the properties of probability functions.
- If $B = \{0, 2\}$, $P(X \in B) = f(0) + f(2) = 1/2 + 1/3 = 5/6$.

L-example 2.4

(a) Find the constant c , such that

$$f(x) = cx, \quad \text{for } x = 1, 2, 3, 4,$$

and 0 otherwise, is a probability function of a random variable X .

(b) Compute $P(X \geq 3)$.

Solution:

(a) Based on the property $\sum_{i=1}^{\infty} f(x_i) = 1$, we have

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) = 1,$$

which is

$$c + 2c + 3c + 4c = 1.$$

Therefore $c = 1/10$.

(b) $P(X \geq 3) = f(3) + f(4) = 3/10 + 4/10 = 7/10$.

L-example 2.5

- Consider a group of five potential blood donors: A, B, C, D and E, of whom only A and B have type O+ blood.
- Five blood samples, one from each individual , will be typed in random order until an O+ individual is identified.

Solution:

- Let $Y = \#$ of typing needed to identify an O+ individual.
- Let O_i and O'_i be the events that an O+ and a non-O+ individual is typed in the i th typing

$$\begin{aligned}f(1) &= P(Y = 1) = P(O_1) = 2/5 = 0.4, \\f(2) &= P(Y = 2) = P(O'_1 \cap O_2) = P(O'_1)P(O_2|O'_1) \\&= \frac{3}{5} \cdot \frac{2}{4} = 0.3,\end{aligned}$$

$$\begin{aligned}
 f(3) &= P(O'_1)P(O'_2|O'_1)P(O_3|O'_1 \cap O'_2) \\
 &= \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = 0.2,
 \end{aligned}$$

$$\begin{aligned}
 f(4) &= P(Y = 4) \\
 &= P(O'_1)P(O'_2|O'_1)P(O'_3|O'_1 \cap O'_2)P(O_4|O'_1 \cap O'_2 \cap O'_3) \\
 &= \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} = 0.1,
 \end{aligned}$$

and $f(y) = 0$ if $y \neq 1, 2, 3, 4$.

- Then the probability function of Y is

y	1	2	3	4
$f(y)$	0.4	0.3	0.2	0.1

Continuous Probability Distributions

- For a continuous RV X , R_X is an interval or a collection of intervals.
- For any $x \in \mathbb{R}$, we must have $P(X = x) = 0$.
- The **probability function, p.f.**, (or **probability density function, p.d.f.**) is defined to quantify the probability that X is in a certain range.

The **p.d.f.** of a continuous RV X , denoted by $f(x)$, is a function that satisfies:

(1) $f(x) \geq 0$ for all $x \in R_X$; and $f(x) = 0$ for $x \notin R_X$.

(2) $\int_{R_X} f(x)dx = 1$.

(3) For any a and b such that $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

Note: (2) is equivalent to $\int_{-\infty}^{\infty} f(x)dx = 1$, since $f(x) = 0$ for $x \notin R_X$.

REMARK

- For any arbitrary specific value x_0 , we have

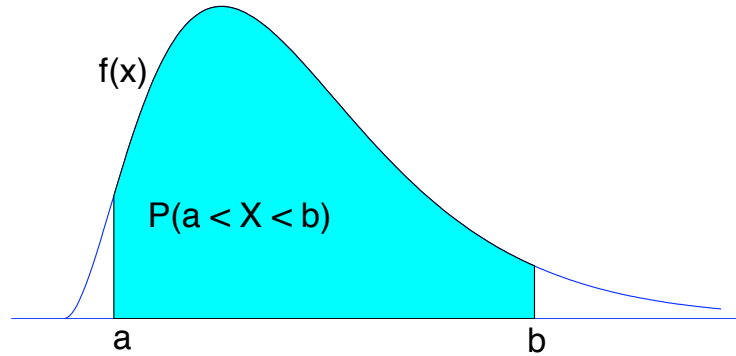
$$P(X = x_0) = \int_{x_0}^{x_0} f(x)dx = 0.$$

This gives an example of “ $P(A) = 0$, but A is not necessarily \emptyset .”

Furthermore, we have

$$P(a < X < b) = P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b) = \int_a^b f(x) dx.$$

- They all represent the area under the graph of $f(x)$ between $x = a$ and $x = b$.



- To check that a function $f(x)$ is a p.d.f., it suffices to check (1) and (2), namely,

(1) $f(x) \geq 0$ for all $x \in R_X$; and $f(x) = 0$ for $x \notin R_X$.

(2) $\int_{R_X} f(x)dx = 1.$

Example 2.4 Let X be a continuous RV with p.d.f. given by

$$f(x) = \begin{cases} cx, & \text{for } 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c ;

(b) Find $P(X \leq 1/2)$.

Solution:

(a) Since

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 cxdx = c \cdot \frac{x^2}{2} \Big|_0^1 = c/2,$$

we set $c/2 = 1$, and result in $c = 2$.

(b)

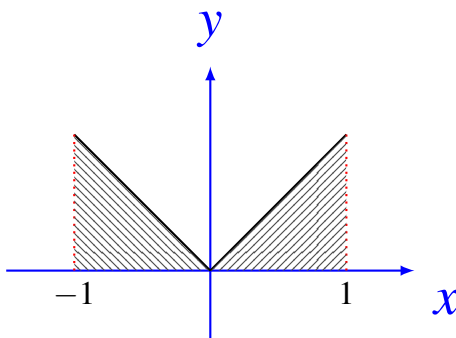
$$P(X \leq 1/2) = \int_{-\infty}^{1/2} f(x)dx = \int_0^{1/2} 2xdx = 1/4.$$

L-example 2.6 Let X be a random variable with probability function given by

$$f(x) = \begin{cases} c|x|, & |x| \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find c .

Solution: The area under the curve $|x|$, $|x| \leq 1$ is $2 \times (1 \times 1/2) = 1$.



Therefore $c \cdot 1 = 1$ results in $c = 1$.

L-example 2.7

- “Time headway” in traffic flow is the elapsed time between the time that one car finishes passing a fixed point and the instant that the next car begins to pass that point.
- Let X = the time headway for two randomly chosen consecutive cars on a highway during a period of heavy flow.

- The following p.d.f. for X was suggested:

$$f(x) = \begin{cases} 0.15e^{-0.15(x-0.5)}, & \text{for } x \geq 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Verify that $f(x)$ is a legitimate p.d.f. for the RV X .
- (b) Compute $P(X \leq 5)$.

Solution:

(a) To check that $f(x)$ is a p.d.f., we need only to verify (1) $f(x) \geq 0$ for any $x \in \mathbb{R}$; (2) $\int_{-\infty}^{\infty} f(x)dx = 1$. (1) is clearly satisfied, we prove (2):

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{0.5}^{\infty} 0.15e^{-0.15(x-0.5)}dx \\ &= 0.15e^{0.075} \int_{0.5}^{\infty} e^{-0.15x}dx \\ &= 0.15e^{0.075} \left(-\frac{1}{0.15}e^{-0.15x} \right) \Big|_{0.5}^{\infty} = 1.\end{aligned}$$

(b)

$$\begin{aligned}P(X \leq 5) &= \int_{-\infty}^5 f(x)dx = \int_{0.5}^5 0.15e^{-0.15(x-0.5)}dx \\&= 0.15e^{0.075} \left(-\frac{1}{0.15}e^{-0.15x} \right) \Big|_{0.5}^5 \\&= e^{0.075} (-e^{-0.75} + e^{-0.075}) = 0.4908.\end{aligned}$$

3 CUMULATIVE DISTRIBUTION FUNCTION

DEFINITION 2

For any RV X , we define its cumulative distribution function (c.d.f.) by

$$F(x) = P(X \leq x).$$

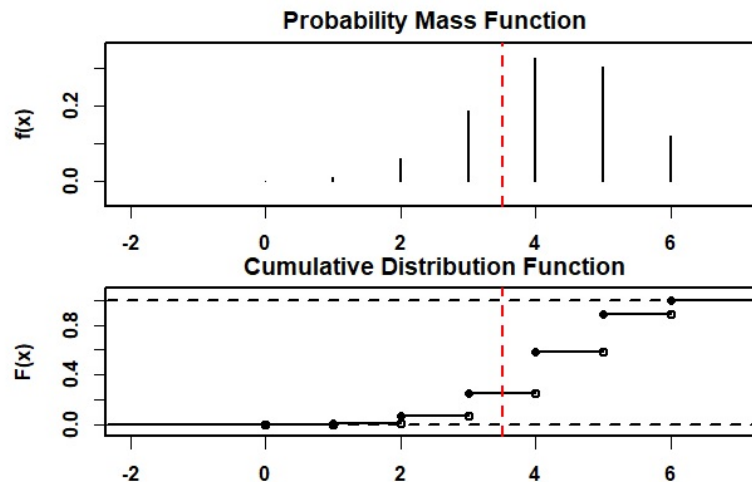
Note: This definition is applicable for X to be either a discrete or a continuous RV.

c.d.f. for Discrete RV

- If X is a **discrete RV**, we have

$$\begin{aligned} F(x) &= \sum_{t \in R_X; t \leq x} f(t) \\ &= \sum_{t \in R_X; t \leq x} P(X = t) \end{aligned}$$

- The c.d.f. of a discrete RV is a step function.



- For any two numbers $a < b$, we have

$$P(a \leq X \leq b) = P(X \leq b) - P(X < a) = F(b) - F(a-),$$

where “ $a-$ ” represents the largest value in R_X , that is $< a$. More mathematically,

$$F(a-) = \lim_{x \uparrow a} F(x).$$

Example 2.5

- Revisit Examples 2.1 and 2.2. RV X is the number of heads of flipping two fair coins, it has the p.f.:

x	0	1	2
$f(x)$	1/4	1/2	1/4

- We have $F(0) = f(0) = 1/4$; $F(1) = f(0) + f(1) = 3/4$; $F(2) = f(0) + f(1) + f(2) = 1$.

- We therefore obtain the c.d.f.:

$$F(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \leq x < 1 \\ 3/4 & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

Example 2.6 Take the c.d.f. derived from Example 2.5:

$$F(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \leq x < 1 \\ 3/4 & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}.$$

Derive the corresponding p.f. (pretending that the c.d.f. is only information available for this distribution).

Solution:

- As $F(\cdot)$ only has four possible values, so the distribution is a discrete distribution.
- We obtain $R_X = \{0, 1, 2\}$, which are the jumping points of $F(\cdot)$. It is also the set so that $f(x)$ is non-zero.
- We have

$$f(0) = P(X = 0) = F(0) - F(0-) = 1/4 - 0 = 1/4;$$

$$f(1) = P(X = 1) = F(1) - F(1-) = 3/4 - 1/4 = 1/2;$$

$$f(2) = P(X = 2) = F(2) - F(2-) = 1 - 3/4 = 1/4.$$

L-example 2.8

- Let X = the number of days of sick leave taken by a randomly selected employee of a large company during a particular year.
- If the maximum number of allowable sick leave days per year is 14, possible values of X are $0, 1, 2, \dots, 14$.
- Suppose $F(0) = 0.58, F(1) = 0.72, F(2) = 0.76, F(3) = 0.81, F(4) = 0.88$, and $F(5) = 0.94$.

- We have

$$\begin{aligned}P(2 \leq X \leq 5) &= F(5) - F(2-) \\&= F(5) - F(1) = 0.94 - 0.72 = 0.22.\end{aligned}$$

- and

$$\begin{aligned}P(X = 3) &= F(3) - F(3-) = F(3) - F(2) \\&= 0.81 - 0.76 = 0.05.\end{aligned}$$

L-example 2.9 The p.f. for RV X is given by

$$f(x) = \begin{cases} (1-p)^{x-1}p, & \text{for } x = 1, 2, 3, \dots; \\ 0, & \text{otherwise,} \end{cases}$$

where $p \in (0, 1)$ is a fixed value. Find the c.d.f. for X .

Solution:

- For any $x = 1, 2, 3, \dots$, set $q = 1 - p$

$$\begin{aligned} F(x) &= P(X \leq x) = \sum_{t \leq x} f(t) = \sum_{t=1}^x (1-p)^{t-1} p \\ &= p (1 + q + q^2 + \dots + q^{x-1}) \\ &= p \cdot \frac{1 - q^x}{1 - q} = 1 - (1 - p)^x. \end{aligned}$$

- Question: What is the value of $F(x)$, when x is not a positive integer? For example, $x = 4.3$.

$$F(4.3) = 0?$$

L-example 2.10 Suppose that the c.d.f. for RV X is given by

$$F(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & \text{for } x \geq 1; \\ 0, & \text{for } x < 1, \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of x . For example, $\lfloor 3.6 \rfloor = 3$, $\lfloor 4 \rfloor = 4$, $\lfloor 4.7 \rfloor = 4$. Find its p.f. $f(x)$.

$$\begin{aligned} f(x) &= P(X = x) \\ &= F(x) - F(x-) \end{aligned}$$

Solution:


- $F(x)$ changes values only for $x = 1, 2, 3, \dots$; therefore it is a discrete distribution.
- $R_X = \{1, 2, 3, \dots\}$, i.e., the set of positive integers.
- for any $x \in R_X$,


$$\begin{aligned} f(x) &= F(x) - F(x-) = (1 - (1 - p)^x) - (1 - (1 - p)^{x-1}) \\ &= (1 - p)^{x-1}(1 - (1 - p)) = (1 - p)^{x-1}p, \end{aligned}$$

and $f(x) = 0$ otherwise.

L-example 2.11

- Many manufacturers have quality control programs that include inspection of incoming materials for defects.
- Suppose a computer manufacturer receives computer boards in lots of five.
- Two boards are selected from each lot for inspection.

Lots :  Pick 2

5 boards : 0 0 0 

(a) List all possible inspected boards for a lot.

(b) Suppose that boards 1 and 2 are the only defectives in a lot of five.

Define $X = \#$ of defective boards observed among an inspection.

Find the probability distribution of X .

$$f(x) = P(X=x)$$

(c) Let $F(x)$ be the c.d.f. of X . Derive $F(x)$.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1/2 & \text{for } 0 \leq x < 1 \end{cases}$$

$$= \begin{cases} 3/10 \\ 3/5 \\ 1 \\ 10 \end{cases}$$

$$5C_2 = 10$$

Solution:

(a) $\#(S) = \binom{5}{2} = 10$. The possible selections are

$$\left\{ \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \right\}.$$

(b) X may take values of 0, 1, and 2.

$$f(0) = P(X = 0) = P(\{\{3,4\}, \{3,5\}, \{4,5\}\}) = 3/10,$$

$$f(2) = P(X = 2) = P(\{\{1,2\}\}) = 1/10,$$

$$f(1) = P(X = 1) = 1 - [f(0) + f(2)] = 6/10,$$

and $f(x) = 0$ elsewhere.

(c) It is sufficient to derive $F(0), F(1), F(2)$:

$$F(0) = P(X \leq 0) = f(0) = 0.3,$$

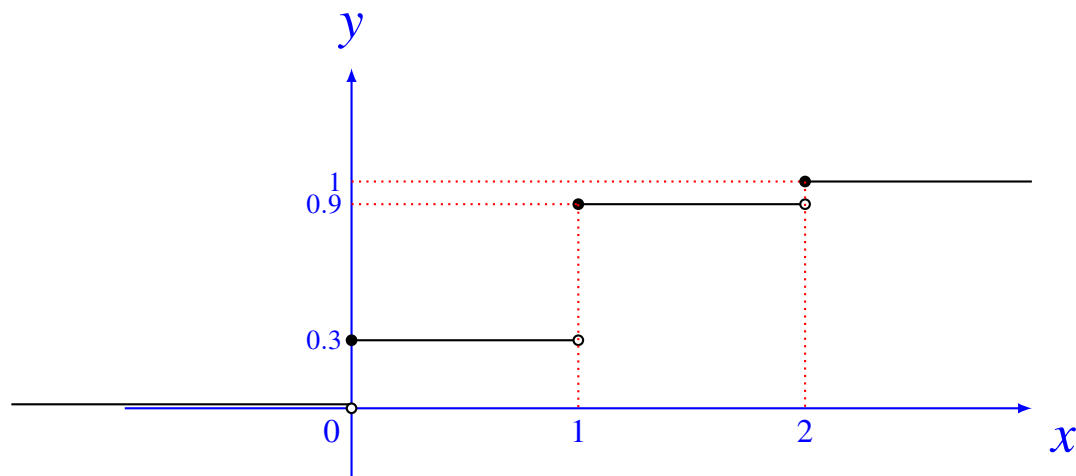
$$F(1) = P(X \leq 1) = f(0) + f(1) = 0.3 + 0.6 = 0.9$$

$$F(2) = P(X \leq 2) = f(0) + f(1) + f(2) = 1.$$

Therefore

$$F(x) = \begin{cases} 0, & x < 0, \\ 0.3, & 0 \leq x < 1, \\ 0.9, & 1 \leq x < 2, \\ 1, & 2 \leq x. \end{cases}$$

This c.d.f. can be drawn as a figure below:



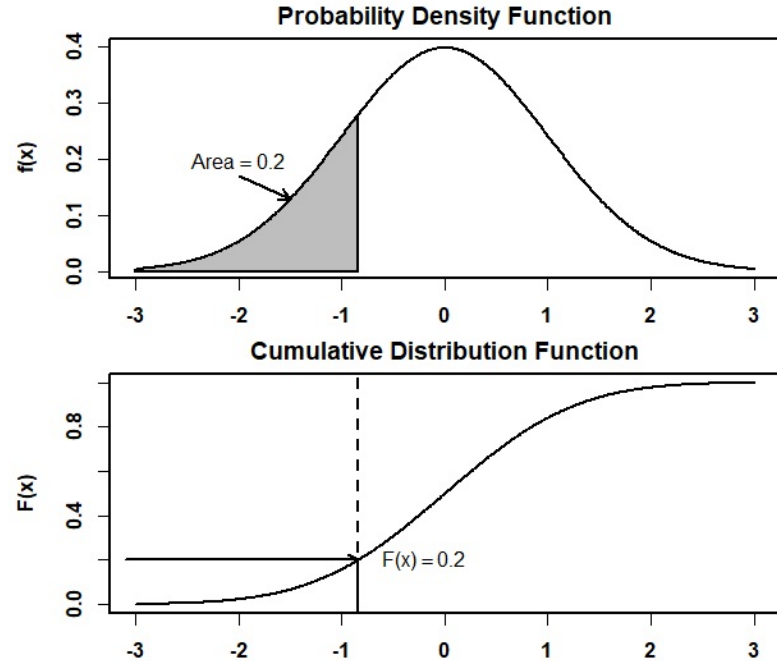
c.d.f. for Continuous RV

- If X is a continuous RV,

$$F(x) = \int_{-\infty}^x f(t)dt.$$

$$f(x) = \frac{dF(x)}{dx}.$$

- $P(a \leq X \leq b) = P(a < X < b) = F(b) - F(a).$



Example 2.7

- The p.d.f. of a RV X is given by

$$f(x) = \begin{cases} 2x & 0 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

- The c.d.f. of X is

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t) dt \\ &= \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases} \end{aligned}$$

Example 2.8 Take the c.d.f. derived from Example 2.7:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

Derive the corresponding p.f. (pretending that the c.d.f. is only information available for this distribution).

Solution:

- $F(x)$ is a c.d.f. for a continuous distribution, because when it is not equal to 0 and 1, it assumes different values in the interval $x \in [0, 1)$.
- $f(x) = 0$ when $x \notin [0, 1)$ because $\frac{d(0)}{dx} = \frac{d(1)}{dx} = 0$.
- $f(x) = \frac{d(x^2)}{dx} = 2x$ when $x \in [0, 1)$.

L-example 2.12

- Let X be the vibratory stress (psi) on a wind turbine blade at a particular wind tunnel.
- The following p.d.f. for X is proposed:

$$f(x) = \begin{cases} \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, & \text{for } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where θ is a given constant.

- Verify that $f(x)$ is a legitimate p.d.f., and find its c.d.f. $F(x)$.

Solution:

$$\left(\frac{-x^2}{2\theta^2} \right)' = -2x / 2\theta^2 = \frac{-x}{\theta^2}$$

- We first verify that $f(x)$ is a p.d.f.. It is obvious that $f(x) > 0$ for $x > 0$.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_0^{\infty} \frac{x}{\theta^2} e^{-x^2/(2\theta^2)} dx = - \int_0^{\infty} d \left(e^{-x^2/(2\theta^2)} \right) \\ &= - e^{-x^2/(2\theta^2)} \Big|_{x=0}^{\infty} \\ &= -0 - (-1) = 1. \end{aligned}$$

This verifies that $f(x)$ is a valid p.d.f.

$$\begin{aligned} & \quad \quad \quad (-x^2) \\ & \quad \quad \quad e \\ & = \frac{1}{e^{x^2}} \rightarrow 0 \end{aligned}$$

- For $x \leq 0$, it is clearly $F(x) = 0$. For $x > 0$,

$$\begin{aligned}
 F(x) &= P(X \leq x) = \int_{-\infty}^x f(t) dt = \int_0^x \frac{t}{\theta^2} e^{-t^2/(2\theta^2)} dt \\
 &= -e^{-t^2/(2\theta^2)} \Big|_{t=0}^x \\
 &= 1 - e^{-x^2/(2\theta^2)}.
 \end{aligned}$$

$$F(x) = \begin{cases} 1 - e^{-\frac{x^2}{2\theta^2}} \\ 0 \end{cases}$$

L-example 2.13 With the c.d.f. given in the last example:

$$F(x) = 1 - e^{-x^2/(2\theta^2)},$$

for $x \geq 0$ and $F(x) = 0$ otherwise. Derive its p.f.

uncountable range \rightarrow continuous distribution

- As $F(x)$ assumes different values in the interval $x \geq 0$, therefore we have continuous distribution. For any $x \geq 0$, we have

$$\begin{aligned} f(x) &= \frac{dF(x)}{dx} = \frac{d \left[1 - e^{-x^2/(2\theta^2)} \right]}{dx} \\ &= \frac{-d \left[e^{-x^2/(2\theta^2)} \right]}{dx} = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \end{aligned}$$

and $f(x) = 0$ for $x < 0$ since $d(F(x))/dx = d(0)/dx = 0$. This complies with the p.d.f. given in the last example.

REMARK

- No matter whether X is discrete or continuous, $F(x)$ is non-decreasing. In the sense that for any $x_1 < x_2$, $F(x_1) \leq F(x_2)$.
- p.f. and c.d.f. have a one-to-one correspondence. That is, with p.f. given, c.d.f. is uniquely determined; and vice versa.

- The ranges of $F(x)$ and $f(x)$ satisfy:
 - $0 \leq F(x) \leq 1$;
 - for discrete distribution, $0 \leq f(x) \leq 1$;
 - for continuous distribution, $f(x) \geq 0$, but **NO NEED** that $f(x) \leq 1$.

$$f(x) > 1 ?$$

Not a probability!

4 EXPECTATION AND VARIANCE OF A RV

- For a RV X , one natural practical question is: what is the **average value** of X , if the corresponding experiment is repeated many times.

For example, X is the number of heads for flipping a coin. We may want to know the average number of heads if we repeat to flip the coin “continuously”.

- Such an average, over a long run, is called the “**mean**” or “**expectation**” of X .

DEFINITION 3 (EXPECTATION OF DISCRETE RV)

Let X be a discrete RV with $R_X = \{x_1, x_2, x_3, \dots\}$ and p.f. $f(x)$. The “*expectation*” or “*mean*” of X is defined by

$$E(X) = \sum_{x_i \in R_X} x_i f(x_i).$$

By convention, we also denote $\mu_X = E(X)$.

DEFINITION 4 (EXPECTATION OF CONTINUOUS RV)

Let X be a continuous RV with p.f. $f(x)$. The “*expectation*” or “*mean*” of X is defined by

$$\mu_X = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{x \in R_X} xf(x)dx.$$

Note: The expected value is not necessarily a possible value of the random variable X .

Example 2.9 Suppose we toss a fair die and the upper face is recorded as X . We have $P(X = k) = 1/6$ for $k = 1, 2, 3, 4, 5, 6$, and

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3.5.$$

Example 2.10 The p.d.f. of weekly gravel sales X is

$$f(x) = \begin{cases} \frac{3}{2}(1 - x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then we have

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \frac{3}{2}(1 - x^2)dx \\ &= \frac{3}{2} \int_0^1 (x - x^3)dx = \frac{3}{2} \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 = 3/8. \end{aligned}$$

L-example 2.14

In a gambling game

- a man gains 5 if he gets all heads or all tails in tossing a fair coin 3 times;
- he pays 3 if either 1 or 2 heads show.

What is his expected gain?

Solution:

- Let X be the amount he can gain in the game.
- Then $X = 5$ or -3 with the following probabilities:

$$\begin{aligned}P(X = 5) &= P(\{HHH, TTT\}) = 1/8 + 1/8 = 1/4; \\P(X = -3) &= 1 - P(X = 5) = 3/4.\end{aligned}$$

- $E(X) = 5 \left(\frac{1}{4}\right) + (-3) \left(\frac{3}{4}\right) = -1.$
- This means he will lose 1 per toss, if he **continuously play the game for a long run.**

L-example 2.15

- Suppose “ X = the total number of hours (in units of 100 hours) that a family runs a vacuum cleaner over a period of one year”.
- The probability function of X is given by

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \leq x \leq 2, \\ 0, & \text{otherwise} \end{cases}$$

- Find the average number of hours per year that families run their vacuum cleaners.

Solution: The question is asking $100 \times E(X)$.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot x dx + \int_1^2 x(2-x)dx \\ &= \left(\frac{x^3}{3}\right)\bigg|_0^1 + \left(x^2 - \frac{x^3}{3}\right)\bigg|_1^2 \\ &= \left(\frac{1}{3} - 0\right) + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = 1. \end{aligned}$$

We conclude that on average, families run their vacuum cleaners 100 hours per year.

Properties of Expectation

(1) Let X be a random variable, and let a and b be any real numbers,

$$E(aX + b) = aE(X) + b. \quad (\text{Linearity})$$

(2) Let X and Y be two random variables, we have

$$E(X + Y) = E(X) + E(Y).$$

(3) Let $g(\cdot)$ be an arbitrary function.

- If X is a **discrete** RV with p.m.f. $f(x)$ and range R_X ,

$$E[g(X)] = \sum_{x \in R_X} g(x)f(x).$$

- If X is a **continuous** RV with p.d.f. $f(x)$ and range R_X ,

$$E[g(X)] = \int_{R_X} g(x)f(x)dx.$$

L-example 2.16 Let X be a random variable, and let a and b be any real numbers. Show that

$$E(aX + b) = aE(X) + b.$$

Solution:

- When X is a discrete random variable with p.f. $f(x)$,

$$\begin{aligned} E(aX + b) &= \sum_{x \in R_X} (ax + b)f(x) \\ &= \sum_{x \in R_X} axf(x) + \sum_{x \in R_X} bf(x) \\ &= a \left(\sum_{x \in R_X} xf(x) \right) + b \left(\sum_{x \in R_X} f(x) \right) = aE(X) + b. \end{aligned}$$

- When X is a continuous random variable with p.f. $f(x)$,

$$\begin{aligned} E(aX + b) &= \int_{-\infty}^{\infty} (ax + b)f(x)dx \\ &= \int_{-\infty}^{\infty} (ax)f(x)dx + \int_{-\infty}^{\infty} bf(x)dx \\ &= a \int_{-\infty}^{\infty} xf(x)dx + b \int_{-\infty}^{\infty} f(x)dx = aE(X) + b \end{aligned}$$

Note that based on properties (1) and (2), we have for constants a_1, a_2, \dots, a_k and RVs X_1, X_2, \dots, X_k ,

$$E(a_1X_1 + a_2X_2 + \dots + a_kX_k) = a_1E(X_1) + a_2E(X_2) + \dots + a_kE(X_k).$$

Variance

Let $g(x) = (x - \mu_X)^2$, this gives the definition of the variance for X .

DEFINITION 5 (VARIANCE)

*Let X be a RV. The **variance** of X is defined by*

$$\sigma_X^2 = V(X) = E(X - \mu_X)^2.$$

$$E \left([X - \mu_X]^2 \right)$$

REMARK

- The definition is applicable no matter whether X is discrete or continuous.
- If X is a **discrete** RV with p.m.f. $f(x)$ and range R_X ,

$$V(X) = \sum_{x \in R_X} (x - \mu_X)^2 f(x).$$

- If X is a **continuous** RV with p.d.f. $f(x)$,

$$V(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) dx.$$

Proof?

- For any X , $V(X) \geq 0$, and “=” holds if and only $P(X = E(X)) = 1$, or more intuitively, X is a **constant**.
- Let a and b be any real numbers, then $V(aX + b) = a^2 V(X)$.
- The variance can also be computed by an alternative formula:

$$V(X) = E(X^2) - [E(X)]^2.$$

- The positive square root of the variance is defined as the “**standard deviation**” of X :

$$\sigma_X = \sqrt{V(X)}.$$

Example 2.11 Let the p.f. of a RV X be given by

x	-1	0	1	2
$f(x)$	$1/8$	$2/8$	$1/8$	$4/8$

Find $E(X)$ and $V(X)$.

Solution:

$$\begin{aligned} E(X) &= \sum_{x \in R_X} xf(x) \\ &= (-1) \left(\frac{1}{8} \right) + 0 \left(\frac{2}{8} \right) + 1 \left(\frac{1}{8} \right) + 2 \left(\frac{4}{8} \right) = 1. \end{aligned}$$

$$\begin{aligned}
 V(X) &= \sum_{x \in R_X} [x - E(X)]^2 f(x) = \sum_{x \in R_X} [x - 1]^2 f(x) \\
 &= (-1 - 1)^2 \left(\frac{1}{8}\right) + (0 - 1)^2 \left(\frac{2}{8}\right) \\
 &\quad + (1 - 1)^2 \left(\frac{1}{8}\right) + (2 - 1)^2 \left(\frac{4}{8}\right) = \frac{5}{4}.
 \end{aligned}$$

Example 2.12 Denote by X the amount of time that a book on reserve at the library is checked out by a randomly selected student. Suppose X has the probability density function

$$f(x) = \begin{cases} x/2, & 0 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute $E(X)$, $V(X)$, and σ_X .

Solution:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x \cdot x/2 dx = \frac{x^3}{6} \Big|_0^2 = 4/3.$$

We use $V(X) = E(X^2) - [E(X)]^2$ to compute $V(X)$,

$$E(X^2) = \int_0^2 x^2 \cdot x/2 dx = \int_0^2 x^3/2 dx = \frac{x^4}{8} \Big|_0^2 = 2.$$

$$V(X) = E(X^2) - [E(X)]^2 = 2 - (4/3)^2 = 2/9.$$

$$\sigma_X = \sqrt{V(X)} = \sqrt{2}/3.$$

L-example 2.17 Revisit Example 2.11. Let the p.f. of a RV X be given by

x	-1	0	1	2
$f(x)$	$1/8$	$2/8$	$1/8$	$4/8$

- (a) Compute $V(X)$ with the alternative formula.
- (b) Define $Y = X^2 + 2$. Compute $E(Y)$ and $V(Y)$.

Solution:

(a) We shall use the formula $V(X) = E(X^2) - [E(X)]^2$ to compute the variance. We can use $E(X) = 1$.

$$\begin{aligned} E(X^2) &= \sum_{x \in R_X} x^2 f(x) \\ &= (-1)^2 \left(\frac{1}{8}\right) + 0^2 \left(\frac{2}{8}\right) + 1^2 \left(\frac{1}{8}\right) + 2^2 \left(\frac{4}{8}\right) = 9/4. \\ V(X) &= E(X^2) - [E(X)]^2 = 9/4 - 1^2 = 5/4. \end{aligned}$$

(b) $E(Y) = E(X^2) + 2 = 9/4 + 2 = 17/4$. We use $V(Y) = E(Y^2) - [E(Y)]^2$ to compute the variance.

$$\begin{aligned} E(Y^2) &= E[(X^2 + 2)^2] = E(X^4 + 4X^2 + 4) \\ &= E(X^4) + 4(9/4) + 4 = E(X^4) + 13 \\ &= (-1)^4 \left(\frac{1}{8}\right) + 0^4 \left(\frac{2}{8}\right) + 1^4 \left(\frac{1}{8}\right) + 2^4 \left(\frac{4}{8}\right) + 13 \\ &= 85/4; \end{aligned}$$

Therefore

$$V(Y) = E(Y^2) - [E(Y)]^2 = 85/4 - (17/4)^2 = 51/16.$$

L-example 2.18 Show the property of variance:

$$V(X) = E(X^2) - [E(X)]^2.$$

Solution:

$$\begin{aligned} V(X) &= E[(X - \mu_X)^2] \\ &= E(X^2 - 2X\mu_X + \mu_X^2) \\ &= E(X^2) - E(2X\mu_X) + E(\mu_X^2) \\ &= E(X^2) - 2\mu_X E(X) + \mu_X^2 \\ &= E(X^2) - 2\mu_X^2 + \mu_X^2 = E(X^2) - \mu_X^2, \end{aligned}$$

since $\mu_X = E(X)$ is a constant.

L-example 2.19 Show the property of the variance: $V(aX + b) = a^2V(X)$, where a and b are constants.

Solution: Note that this property is equivalent to the following two properties

(a) $V(aX) = a^2V(X)$, and

(b) $V(X + b) = V(X)$.

Therefore, we only need to show (a) and (b). For (a)

$$\begin{aligned} V(aX) &= E[(aX)^2] - [E(aX)]^2 = E(a^2X^2) - [aE(X)]^2 \\ &= a^2E(X^2) - a^2[E(X)]^2 = a^2V(X). \end{aligned}$$

For (b),

$$\begin{aligned} V(X+b) &= E[(X+b)^2] - [E(X+b)]^2 \\ &= E(X^2 + 2Xb + b^2) - [E(X) + b]^2 \\ &= E(X^2) + 2bE(X) + b^2 - \{[E(X)]^2 + 2bE(X) + b^2\} \\ &= E(X^2) - [E(X)]^2 = V(X). \end{aligned}$$

L-example 2.20 Suppose that RV X has p.f.

$$f(x) = \begin{cases} \frac{x}{225}, & 0 < x < 15 \\ \frac{30-x}{225}, & 15 \leq x \leq 30 \\ 0, & \text{otherwise} \end{cases}$$

Compute $E(X)$ and $V(X)$.

Solution:

$$\begin{aligned} E(X) &= \int_0^{15} x \left(\frac{x}{225} \right) dx + \int_{15}^{30} x \left(\frac{30-x}{225} \right) dx \\ &= \frac{1}{225} \left\{ \left(\frac{x^3}{3} \right) \Big|_0^{15} + \left(15x^2 - \frac{x^3}{3} \right) \Big|_{15}^{30} \right\} \\ &= \frac{1}{225} \left\{ \frac{15^3}{3} + \left(15(30)^2 - \frac{30^3}{3} - 15(15)^2 + \frac{15^3}{3} \right) \right\} = 15. \end{aligned}$$

$$\int_0^4 \left(\frac{1}{8} x \right) x^2 dx = \int_0^4 \frac{1}{8} x^3 dx = \frac{x^4}{32} \Big|_0^4 = \frac{16}{32} = \frac{1}{2}$$

$$\begin{aligned} E(X^2) &= \int_0^{15} x^2 \left(\frac{x}{225} \right) dx + \int_{15}^{30} x^2 \left(\frac{30-x}{225} \right) dx \\ &= \frac{1}{225} \left\{ \left(\frac{x^4}{4} \right) \Big|_0^{15} + \left(10x^3 - \frac{x^4}{4} \right) \Big|_{15}^{30} \right\} = \frac{525}{2} = 262.5. \end{aligned}$$

Therefore

$$V(X) = E(X^2) - [E(X)]^2 = 262.5 - 15^2 = 37.5.$$