1 Deriving the Power Flow Equations

The most simple expression for power is $P = V \cdot I$, voltage times current. This equation stays valid but needs to be extended since the standard transmission grid uses alternating currents (AC) to transfer power. If we imagine V and I being a sinusoidal signal which are not necessarily aligned, we see that the resulting power is not only depending on the amplitude of the voltage and current signals, but also on the phase-angele of the signal in each point in time. It is therefore helpful to describe electrical parameters in an AC system as a two dimensional value of amplitude and phase-angle. This can be achieved the easiest with complex numbers. Expressing the voltage exemplary as a complex number gives

$$V \coloneqq v(t) = \hat{V} e^{j\phi} = \hat{V}(\cos(\phi) + j\sin(\phi)) \tag{1}$$

where $\hat{V}$ is the amplitude of the voltage, ϕ the angle at a certain point in time and j the imaginary unit [1]. With the same logic all other parameters get an additional imaginary part:

$$S = P + jQ \qquad \text{Apparent Power} = \text{Active Power} + \text{Reactive Power} \tag{2}$$

$$Z = R + jX \qquad \text{Impedance} = \text{Resistance} + \text{Reactance} \tag{3}$$

$$Y = G + jB \qquad \text{Admittance} = \text{Conductance} + \text{Susceptance} \tag{4}$$

$$Y = \frac{1}{Z} = \frac{R}{R^2 + X^2} + j\frac{-X}{R^2 + X^2}. \tag{5}$$

With that knowledge, we reformulate the power equation from above towards $S_i = V_i I_i^*$ where I^* is the complex conjugate of I [2]. S_i represents the power flowing out of a node i and V_i is the voltage at this node. The current flowing out of a node i can be reformulated according to Ohm's law to:

$$I_i = \sum_{k=1}^{N} Y_{ik} V_k. \tag{6}$$

Where N is the amount of all nodes in the network and Y_{ik} the (negative[3]) admittance between node i and node k. If those two nodes are not connected, the admittance equals

[1] In electrical engineering it is common to use j instead of i for the imaginary unit in order to not get it mixed up with the symbol for the time dependent current $i(t)$.

[2] This is necessary because we need to calculate the phase-angle *difference* between the voltage and the current signal. Without the conjugation, the angles would add up.

[3] Because power flowing out of the node is defined postive.

zero. Thus, we get for the apparent power flowing out of i:

$$S_i = V_i \sum_{k=1}^{N} Y_{ik}^* V_k^*. \tag{7}$$

Using the definitions from equations (1) and (5), we get:

$$S_i = \sum_{k=1}^{N} V_i V_k (\cos(\phi_{ik}) + j \sin(\phi_{ik}))(G_{ik} - jB_{ik}) \tag{8}$$

With $\phi_{ik} = \phi_i - \phi_k$ and also splitting the real and the imaginary part of the power we get the **power flow equations:**

$$P_i = \sum_{k=1}^{N} V_i V_k (G_{ik} \cos(\phi_i - \phi_k) + B_{ik} \sin(\phi_i - \phi_k)) \tag{9}$$

$$Q_i = \sum_{k=1}^{N} V_i V_k (G_{ik} \sin(\phi_i - \phi_k) + B_{ik} \cos(\phi_i - \phi_k)). \tag{10}$$

2 Linear Load Flow and PTDF Matrix

When it comes to Load Flow modelling, it is essential to make assumptions in order to accomplish a transition from a non-linear AC-characteristic to the linear DC-Load Flow system. In AC-Load Flow, four variables (voltage angle, voltage magnitude, active and reactive power injections) need to be determined for the nodes in the network. These four variables as well as losses have a non-linear relationship which makes the formulation of the problem highly complex and hard to solve. In order to simplify the computation and to create a DC-Load Flow formulation, the following assumptions are made:

1. Neglecting reactive power flows
2. Neglecting active power losses. We assume that $R \ll jX$, thus the impedances are given as $Z \approx jX$ and $Y \approx jB$.
3. Voltage angles are small (small angle approximation),
4. The voltage profile is flat, meaning that the voltage amplitude is equal for all nodes (in per unit values)

These assumptions are often made in techno-economic studies since they create linear characteristics. Consequently, there are only two variables that need to be determined (voltage angle, active power injections) for each node. Based on these assumptions the active power flow through on (lossless) transmission line $l_{i,k}$ and in matrix format, with $\mathbf{B_d}$ being $l \times l$-diagonal matrix with the line susceptances on the diagonal:

$$P_{ik} = \frac{|V_i|\,|V_k|}{X_l} \sin(\phi_i - \phi_k) \tag{11}$$

$$P_l = B_l(\phi_i - \phi_k) \tag{12}$$

$$\mathbf{P_l} = \boldsymbol{B_d} \cdot \boldsymbol{A} \cdot \boldsymbol{\phi_k} \tag{13}$$

While the power injection at note n becomes:

$$P_n = \sum_k B_l(\phi_i - \phi_k) \tag{14}$$

$$\boldsymbol{P_n} = \boldsymbol{A^T} \cdot \boldsymbol{B_d} \cdot \boldsymbol{A} \cdot \boldsymbol{\phi_k} \tag{15}$$

The incidence matrix $\boldsymbol{A}$ is a $l \times n$, with $a_{ij} = 1$ if the line starts at i and $a_{ij} = -1$ if it ends at j, matrix that captures the topology of the grid. The PTDF (Power Transfer Distribution Factor) -matrix is defined as a node to line sensitivity, so basically $\frac{P_l}{P_n}$.

$$\boldsymbol{PTDF} = (\boldsymbol{B_d} \cdot \boldsymbol{A})(\boldsymbol{A^T} \cdot \boldsymbol{B_d} \cdot \boldsymbol{A})^{-1} \tag{16}$$

The equation for nodal balance is linearly dependent (Equation (15)) and therefore the matrix $\boldsymbol{A^T} \cdot \boldsymbol{B_d} \cdot \boldsymbol{A}$ is singular and there is no inverse. To overcome this a node has to be defined as a reference node (also called slack node). This node is removed from the power flow equations and sorted back in with the value zero. Therefore the PTDF matrix gives all node to line sensitivities, under the restriction that the slack node will balance all other grid injections.

Using the PTDF matrix we can calculate the flows on each line based on the injections at each node. The injection of the slack node will always balance all other injections. Note that the direction is relative to the signs in incidence matrix, or if nodes are defined as starting- or end node.

$$\boldsymbol{Flow} = \boldsymbol{PTDF} \cdot \boldsymbol{Inj} \tag{17}$$