CS301
Design and Analysis of Algorithms

Lecture: Growth of Function

About this lecture

- Introduce Asymptotic Notation
  - $\Theta(\ )$, $\Omega(\ )$, $O(\ )$, $o(\ )$, $\omega(\ )$

Dominating Term

Recall that for input size n,

- Insertion Sort 'srunning time is:

$An^2 + Bn + C$, (A,B,C are constants)

- Merge Sort 's running time is:

$Dn \log n + En + F$, (D,E,F are constants)

- To compare their running times for large n, we can just focus on the dominating term (the term that grows fastest when n increases)
  - An^2 vs $Dn \log n$

Dominating Term

- If we look more closely, the leading constants in the dominating term does not affect much in this comparison
  - We may as well compare n^2 vs $n \log n$ (instead of An^2 vs $Dn \log n$)
- As a result, we conclude that Merge Sort is better than Insertion Sort when n is sufficiently large

Asymptotic Efficiency

- The previous comparison studies the asymptotic efficiency of two algorithms
- If algorithm P is asymptotically faster than algorithm Q, P is often a better choice
- To aid (and simplify) our study in the asymptotic efficiency, we now introduce some useful asymptotic notation

Big-O notation

Definition: Given a function $g(n)$, we denote $(g(n))$ to be the set of functions

$$\{ f(n) \mid \text{there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq c\, g(n) \text{ for all } n \geq n_0 \}$$

Rough Meaning: $(g(n))$ includes all functions that are upper bounded by $g(n)$

Big-O notation (example)

- $4n$ ε $O(5n)$ [proof: c = 1, n ¸ 1]
- $4n$ ε $O(n)$ [proof: c = 4, n ¸ 1]
- $4n + 3$ ε $O(n)$ [proof: c = 5, n ¸ 3]
- n ε $O(0.001n^2)$ [proof: c = 1, n ¸ 100]
- $\log_e n$ ε $O(\log n)$ [proof: c = 1, n ¸ 1]
- $\log n$ ε $O(\log_e n)$ [proof: c = log e, n ¸ 1]

Remark: Usually, we will slightly abuse the notation, and write f(n) = O(g(n)) to mean f(n) ε O(g(n))

Big-Omega notation

Definition: Given a function g(n), we denote (g(n)) to be the set of functions

{ f(n) | there exists positive constants c and n_0 such that

$$0 \cdot c\, g(n) \cdot f(n)$$

for all n ¸ n_0 }

Rough Meaning: (g(n)) includes all functions that are lower bounded by g(n)

Big-O and Big-Omega

- Similar to Big-O, we will slightly abuse the notation, and write $f(n) = \Omega(g(n))$ to mean $f(n) \in \Omega(g(n))$

Relationship between Big-O and Big- :

$$f(n) = \Omega(g(n)) \Leftrightarrow g(n) = O(f(n))$$

Big- notation (example)

- $5n = \Omega(4n)$ [proof: $c = 1, n \geq 1$]
- $n = \Omega(4n)$ [proof: $c = 1/4, n \geq 1$]
- $4n + 3 = \Omega(n)$ [proof: $c = 1, n \geq 1$]
- $0.001n^2 = \Omega(n)$ [proof: $c = 1, n \geq 100$]
- $\log_e n = \Omega(\log n)$ [proof: $c = 1/\log e, n \geq 1$]
- $\log n = \Omega(\log_e n)$ [proof: $c = 1, n \geq 1$]

Θ notation (Big-O \ Big-Ω)

Definition: Given a function $g(n)$, we denote $\Theta(g(n))$ to be the set of functions

$$\{\, f(n) \mid \text{there exists positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1\, g(n) \le f(n) \le c_2\, g(n) \text{ for all } n \ge n_0 \,\}$$

Meaning: Those functions which can be both upper bounded and lower bounded by of $g(n)$

Big-O, Big- Ω , and Θ

- Similarly, we write $f(n) = \Theta(g(n))$ to mean

  $f(n) \in (g(n))$

Relationship between Big-O, Big-Ω ,
and Θ :

$$f(n) = \Theta(g(n))$$

$$\Leftrightarrow$$

$$f(n) = \Omega(g(n)) \text{ and } f(n) = O(g(n))$$

notation (example)

- $4n = \Theta(n)$ [$c_1 = 1, c_2 = 4, n \geq 1$]
- $4n + 3 = \Theta(n)$ [$c_1 = 1, c_2 = 5, n \geq 3$]
- $\log_e n = \Theta(\log n)$ [$c_1 = 1/\log e, c_2 = 1, n \geq 1$]
- Running Time of Insertion Sort = $\Theta(n^2)$
  - If not specified, running time refers to the worst-case running time
- Running Time of Merge Sort = $\Theta(n \log n)$

Little-o notation

Definition: Given a function g(n), we denote o(g(n)) to be the set of functions

{ f(n) | for any positive c, there exists positive constant n_0 such that

$$0 \leq f(n) < c\,g(n)$$

for all $n \geq n_0$ }

Note the similarities and differences with Big-O

Little-o (equivalent definition)

Definition: Given a function $g(n)$, $o(g(n))$ is the set of functions

$$\{ f(n) \mid \lim_{n\to\infty} \left(f(n)/g(n)\right) = 0 \}$$

Examples:

- $4n = o(n^2)$
- $n \log n = o(n^{1.000001})$
- $n \log n = o(n \log^2 n)$

Little-omega notation

Definition: Given a function $g(n)$, we denote $\omega(g(n))$ to be the set of functions

$\{ f(n) \mid$ for any positive c, there exists positive constant n_0 such that

$$0 \leq c\, g(n) < f(n)$$

for all $n \geq n_0 \}$

Note the similarities and differences with the Big-Omega definition

Little-omega (equivalent definition)

Definition: Given a function $g(n)$, $\omega(g(n))$ is the set of functions

$$\{\ f(n) \mid \lim_{n\to\infty}\left(g(n)/f(n)\right) = 0\ \}$$

Relationship between Little-o and Little-ω :

$$f(n) = \omega(g(n)) \Leftrightarrow g(n) = o(f(n))$$

To remember the notation:

O is like $\leq$: $f(n) = O(g(n))$ means $f(n) \leq cg(n)$

Ω is like $\leq$: $f(n) = \Omega(g(n))$ means $f(n) \geq cg(n)$

Θ is like $=$: $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = (f(n))$

o is like $<$: $f(n) = o(g(n))$ means $f(n) < cg(n)$

ω is like $>$: $f(n) = \omega(g(n))$ means $f(n) > cg(n)$