2 PATHS AND CIRCUITS

This chapter serves two purposes. The first is to introduce additional concepts and terms in graph theory. These concepts, such as paths, circuits, and Euler graphs, deal mainly with the nature of connectivity in graphs. The degree of vertices, which is a local property of each vertex, will be shown to be related to the more global properties of the graph.

The second purpose is to illustrate with examples how to solve actual problems using graph theory. The celebrated Königsberg bridge problem, which was introduced in Chapter 1, will be solved. The solution of the seating arrangement problem, also introduced in Chapter 1, will be obtained by means of Hamiltonian circuits. A third problem, which involves stacking four multicolored cubes, will also be solved. These three unrelated problems will demonstrate the problem-solving power of graph theory. The reader may attempt to solve these problems without using graphs; the difficulty of such an approach will quickly convince him of the value of graph theory.

2-1. ISOMORPHISM

In geometry two figures are thought of as equivalent (and called congruent) if they have identical behavior in terms of geometric properties. Likewise, two graphs are thought of as equivalent (and called *isomorphic*) if they have identical behavior in terms of graph-theoretic properties. More precisely: Two graphs G and G' are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that edge e is incident on vertices v_1 and v_2 in G; then the corresponding edge e' in G' must be incident on the vertices v'_1 and v'_2 that correspond to

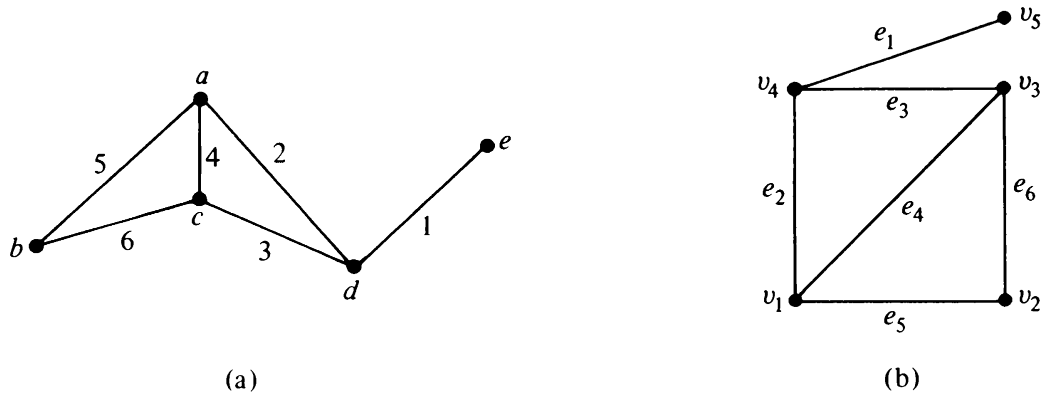


Fig. 2-1 Isomorphic graphs.

v_1 and v_2, respectively. For example, one can verify that the two graphs in Fig. 2-1 are isomorphic. The correspondence between the two graphs is as follows: The vertices *a*, *b*, *c*, *d*, and *e* correspond to v_1, v_2, v_3, v_4, and v_5, respectively. The edges 1, 2, 3, 4, 5, and 6 correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively.

Except for the labels (i.e., names) of their vertices and edges, isomorphic graphs are the same graph, perhaps drawn differently. As indicated in Chapter 1, a given graph can be drawn in many different ways. For example, Fig. 2-2 shows two different ways of drawing the same graph.

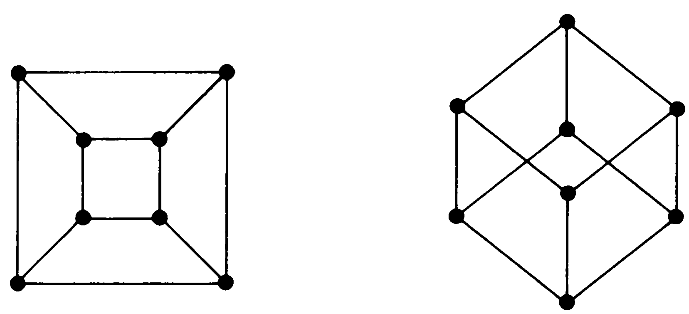
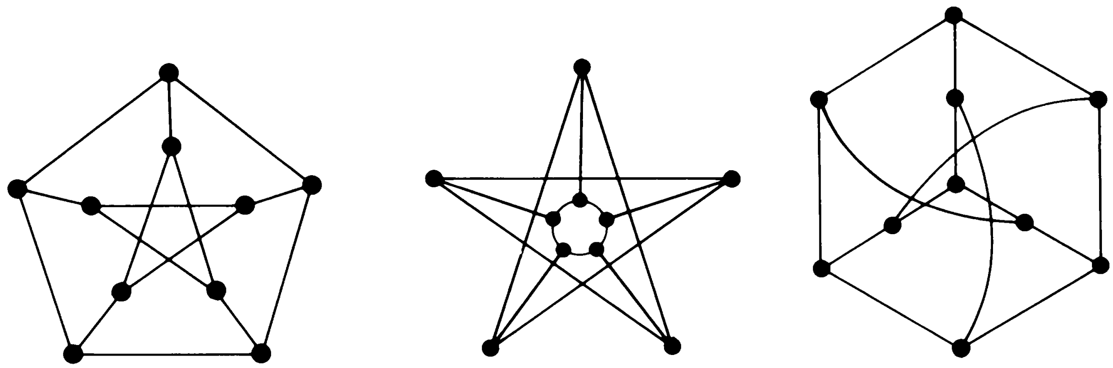


Fig. 2-2 Isomorphic graphs.

It is not always an easy task to determine whether or not two given graphs are isomorphic. For instance, the three graphs shown in Fig. 2-3 are all isomorphic, but just by looking at them you cannot tell that. It is left as an exercise for the reader to show, by redrawing and labeling the vertices and edges, that the three graphs in Fig. 2-3 are isomorphic (see Problem 2-3).

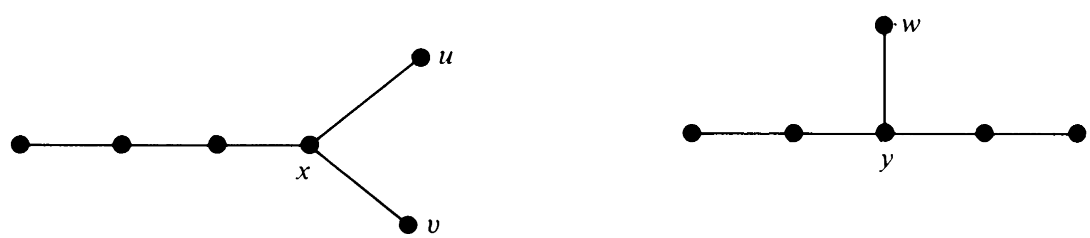
It is immediately apparent by the definition of isomorphism that two isomorphic graphs must have

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.



(a) (b) (c)

Fig. 2-3 Isomorphic graphs.



(a) (b)

Fig. 2-4 Two graphs that are not isomorphic.

However, these conditions are by no means sufficient. For instance, the two graphs shown in Fig. 2-4 satisfy all three conditions, yet they are not isomorphic. That the graphs in Figs. 2-4(a) and (b) are not isomorphic can be shown as follows: If the graph in Fig. 2-4(a) were to be isomorphic to the one in (b), vertex x must correspond to y, because there are no other vertices of degree three. Now in (b) there is only one pendant vertex, w, adjacent to y, while in (a) there are two pendant vertices, u and v, adjacent to x.

Finding a simple and efficient criterion for detection of isomorphism is still actively pursued and is an important unsolved problem in graph theory. In Chapter 11 we shall discuss various proposed algorithms and their programs for automatic detection of isomorphism by means of a computer. For now, we move to a different topic.

2-2. SUBGRAPHS

A graph g is said to be a *subgraph* of a graph G if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G. For instance, the graph in Fig. 2-5(b) is a subgraph of the one in Fig. 2-5(a). (Obviously, when considering a subgraph, the original graph must

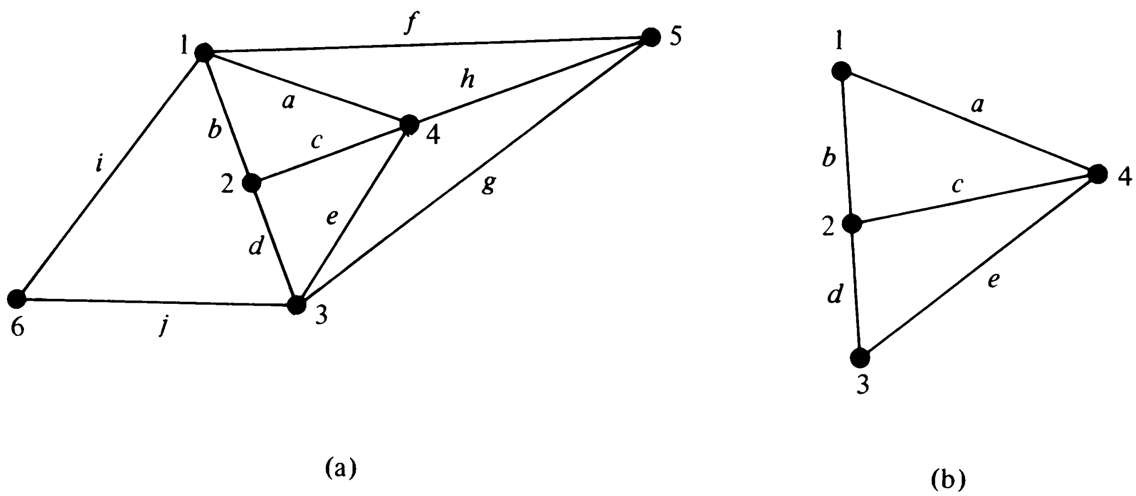


Fig. 2-5 Graph (a) and one of its subgraphs (b).

not be altered by identifying two distinct vertices, or by adding new edges or vertices.) The concept of subgraph is akin to the concept of subset in set theory. A subgraph can be thought of as being contained in (or a part of) another graph. The symbol from set theory, $g \subset G$, is used in stating "g is a subgraph of G."

The following observations can be made immediately:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.

Edge-Disjoint Subgraphs: Two (or more) subgraphs g_1 and g_2 of a graph G are said to be *edge disjoint* if g_1 and g_2 do not have any edges in common. For example, the two graphs in Figs. 2-7(a) and (b) are edge-disjoint subgraphs of the graph in Fig. 2-6. Note that although edge-disjoint graphs do not have any edge in common, they may have vertices in common. Subgraphs that do not even have vertices in common are said to be *vertex disjoint*. (Obviously, graphs that have no vertices in common cannot possibly have edges in common.)

2-3. A PUZZLE WITH MULTICOLORED CUBES

Now we shall take a brief pause to illustrate, with an example, how a problem can be solved by using graphs. Two steps are involved here: First, the physical problem is converted into a problem of graph theory. Second,