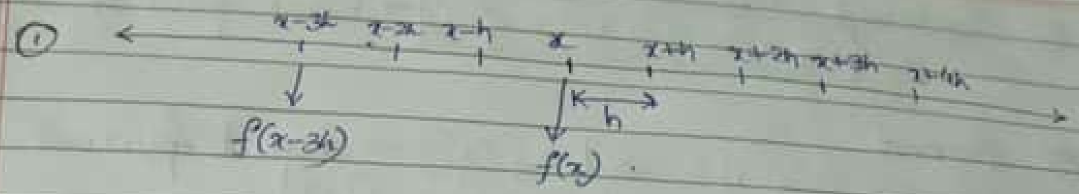
Unit - II

Interpolation

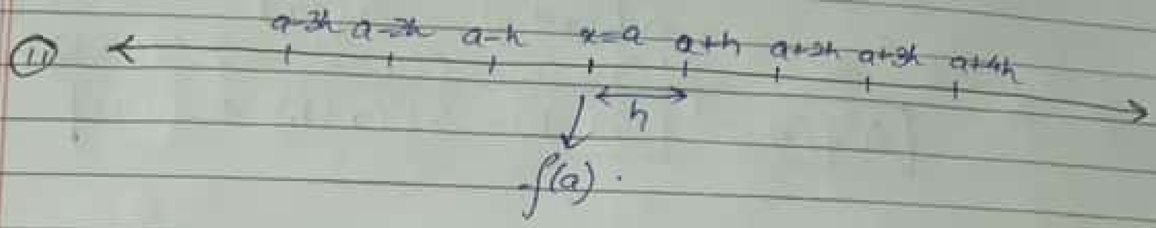
Date: 30 | 01 | 2020

Real Lines —.

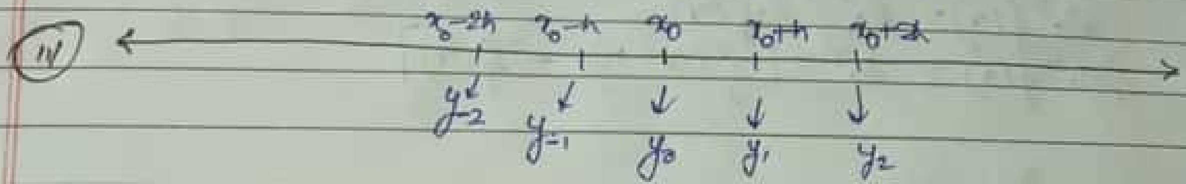
(i)



(ii)



(iii)



Interpolation - Finding the value within the range.
Extrapolation - Finding the value out of range.

Finite difference:

Let $y=f(x)$ be a function defined on $x_0, x_0+h+x_0+2h, \ldots, x_0+nh$ by $y_i = f(x_0+ih)$ for all $i = 0, 1, 2, \ldots$ so on. The values of independent variables x are arguments and corresponding functional values are known as entries.

The following three types of differences are commonly use in numerical analysis.

(i) Forward differences
(ii) Backward differences
(iii) Central differences

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(1) forward difference (Δ)

$$\boxed{\Delta f(x) = f(x+h) - f(x)} \rightarrow \text{First forward diff.}$$

(ii) Second

$$\Delta^2 f(x) = \Delta[\Delta f(x)] \rightarrow \text{Second forward diff.}$$
$$= \Delta[f(x+h) - f(x)]$$
$$= \Delta f(x+h) - \Delta f(x)$$
$$= \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\}$$
$$\boxed{\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x).}$$

$$\Delta^n f(x) = \Delta^{n-1} \Delta f(x)$$
$$\Rightarrow \Delta^n f(x) = \Delta^{n-1}[f(x+h) - f(x)]$$

$$\Delta^3 f(x) = \Delta[\Delta^2 f(x)]$$
$$= \Delta[f(x+2h) - 2f(x+h) + f(x)].$$
$$= [f(x+3h) - f(x+h) - 2f(x+2h) + 2f(x+h)$$
$$+ f(x+h) - f(x)]$$
$$\Delta^3 f(x) = f(x+3h) - 2f(x+2h) + 2f(x+h) - f(x).$$

Q. Construct a forward difference table and find $\Delta^4 f(1)$ if $f(1)=1$, $f(2)=3$, $f(3)=8$, $f(4)=15$, $f(5)=25$.

$\Rightarrow$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	1				
		2			
2	3		3		
		5		-1	
3	8		2		2
		7		1	
4	15		3		
		10			
5	25				

$$\boxed{\Delta^4 f(1) = 2}$$

Date - 5 Feb 2020.

(ii) <u>Backward difference (∇)</u>

$$\boxed{\nabla f(x) = f(x) - f(x-h)} \rightarrow \text{first backward diff.}$$

$$\nabla^2 f(x) = \nabla[\nabla f(x)] \quad \rightarrow \text{Second backward diff.}$$
$$= \nabla[f(x) - f(x-h)]$$
$$= \nabla f(x) - \nabla f(x-h)$$
$$= [f(x) - f(x-h)] - [f(x-h) - f(x-2h)]$$
$$\boxed{\nabla^2 f(x) = f(x) - 2f(x-h) + f(x-2h)}$$

$$\nabla^n f(x) = \nabla^{n-1}[\nabla f(x)]$$
$$= \nabla^{n-1}[f(x) - f(x-h)].$$

Q. Construct a backward difference table.
$f(1)=4$, $f(2)=8$, $f(3)=12$, $f(4)=18$, $f(5)=36$,
(i) $\nabla^4 f(5)$
(ii) $\nabla^4 f(4)$
(iii) $\nabla^3 f(4)$
(iv) $\nabla^2 f(3)$.

⇒

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	4	4	$\boxed{0}$	2	$\boxed{8}$
2	8	4	$\boxed{2}$	10	
3	12	6	12		
4	18	18			
5	36				

$\nabla^4 f(5) = 8$
$\nabla^4 f(4) = 0$
$\nabla^3 f(4) = 2$.
$\nabla^2 f(3) = 0$.

Q. Construct a forward difference table.
$f(1)=4$ $f(2)=8$ $f(3)=12$, $f(4)=18$, $f(5)=36$
(i) $\Delta^4 f(1)$
(ii) $\Delta^4 f(2)$
(iii) $\Delta^3 f(1)$
(iv) $\Delta^2 f(3)$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	4	4	0	$\boxed{2}$	$\boxed{8}$
2	8	4	2	10	
3	12	6	$\boxed{12}$		
4	18	18			
5	36				

$\Delta^4 f(1) = 8$
$\Delta^4 f(2) = 0$
$\Delta^3 f(1) = 2$
$\Delta^2 f(3) = 12.$

(iii) Central Difference (δ)

$\frac{0+1}{2}$

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$	$\delta^4 f(x)$
0	y_0				
		$\delta y_{1/2}$			
1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
2	y_2		$\delta^2 y_2$		$\delta^4 y_2$.
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
4	y_4				

Different Types of Operators.

① Shifting Operator.

$$Ef(x) = f(x+h)$$

where E is known as shift operator.

A shifting operator, E is defined by —

$$E f(x) = f(x+h)$$
$$E^2 f(x) = E\left[E f(x+h)\right]$$
$$= E f(x+h)$$
$$E^2 f(x) = f(x+2h).$$

$$E^3 f(x) = f(x+3h)$$
$$E^n f(x) = f(x+nh).$$

Similarly, $E^{-1} f(x) = f(x-h).$

$$E f^{-2}(x) = f(x-2h)$$

$$E f^{-n}(x) = f(x-nh)$$

(ii) Averaging operator (μ)
The ~~ad~~ averaging operator (μ) is defined by

$$\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$$

OR

$$\mu f(x) = \frac{1}{2}\left[y_{x+h/2} + y_{x-h/2}\right]$$

(iii) Central Difference Operator (δ).
The central operator (δ) is defined by

$$\delta f(x) = \left[f(x+h/2) - f(x-h/2)\right]$$

Relation Between Operators.

(a) Relation between E and Δ $\quad E = 1+\Delta$

$$\Delta f(x) = f(x+h) - f(x)$$
$$E f(x) = f(x+h)$$

$$\Delta f(x) = E f(x) - f(x).$$
$$\Rightarrow \Delta = E - 1$$
$$\Rightarrow \boxed{E = \Delta + 1}$$

(b) Relation between ∇ and E^{-1}

$$\nabla f(x) = f(x) - f(x-h)$$
$$E^{-1} f(x) = f(x-h)$$

$(a+b)^2 = (a-b)^2 + 4ab$.

$$\Delta f(x) = f(x) - E^{-1} f(x)$$
$$\Rightarrow \Delta = 1 - E^{-1}$$
$$\Rightarrow \boxed{E^{-1} = 1 - \Delta}$$

(c) Relation between E and δ

$$\delta f(x) = f(x+h/2) - f(x-h/2)$$
$$\Rightarrow \delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$$
$$\Rightarrow \boxed{\delta = E^{1/2} - E^{-1/2}}$$

(d) Relation between μ and E.

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$
$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$
$$\therefore \mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-1/2}\right)$$
$$\Rightarrow \boxed{2\mu = E^{1/2} + E^{-1/2}}$$

★ (e) Relation between μ and δ.

$$\boxed{\mu^2 = 1 + \frac{\delta^2}{4}} \quad \text{or} \quad \boxed{\mu = \sqrt{1 + \frac{\delta^2}{4}}}$$

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x - h/2\right)\right]$$
$$\Rightarrow \mu f(x) = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) f(x)$$
$$\Rightarrow \mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right).$$

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$$\Rightarrow \mu^2 = \frac{1}{4}\left[E^{1/2} + E^{-1/2}\right]^2$$

$$\Rightarrow \mu^2 = \frac{1}{4}\left[(E^{1/2} - E^{-1/2})^2 + 4E^{1/2}E^{-1/2}\right]$$

$$\delta = (E^{1/2} - E^{-1/2})$$

$$\Rightarrow \mu^2 = \frac{1}{4}\left[\delta^2 + 4\right]$$

$$\Rightarrow \boxed{\mu^2 = \frac{\delta^2}{4} + 1}$$

* (f) Relation between μ, δ and E

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

$$\Rightarrow 2\mu = E^{1/2} + E^{-1/2} \quad \text{--- (i)}$$

$$\delta = E^{1/2} - E^{-1/2} \quad \text{--- (ii)}$$

Adding (i) and (ii), we get —

$$2\mu + \delta = 2E^{1/2}$$

$$\Rightarrow \boxed{E^{1/2} = \mu + \frac{\delta}{2}}$$

* (g) Relation between Δ, ∇, μ and δ

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] \quad \text{--- (i)}$$

$$\delta = (E^{1/2} - E^{-1/2}) \quad \text{--- (ii)}$$

Multiplying (i) & (ii) —

$$2\mu\delta = (E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$= (E - E^{-1})$$

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Calculation

$= [(\Delta+1) - (1-\nabla)]$

$= \Delta + \nabla$

$\Rightarrow 2\mu\delta = \Delta + \nabla$

$\Rightarrow \boxed{\mu\delta = \frac{1}{2}(\Delta + \nabla)}$

Q. Prove that $E\nabla = \nabla E = \Delta$.

$\Rightarrow$ $E\nabla f(x) = E[f(x) - f(x-h)]$

$= Ef(x) - Ef(x-h)$

$= Ef(x) - f(x)$

$\Rightarrow E\nabla = E - 1$ $\quad [\because E = 1+\Delta]$

$\Rightarrow E\nabla = \Delta$ —— (I)

$\nabla E f(x) = \nabla f(x+h)$

$= f(x+h) - f(x)$

$= \Delta f(x)$

$\Rightarrow \nabla E = \Delta$ —— (II)

from (I) and (II), we have —

$E\nabla = \nabla E = \Delta$.

Q. Prove that —

$(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$

$\Rightarrow$ $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2}$

$= (E^{1/2} + E^{-1/2})\, E^{1/2}$

$= E + E^{0}$

$= E + 1$

$= (1+\Delta) + 1$

$= 2 + \Delta$ Proved

$\Rightarrow (E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$.

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Q. Proved that —
$$E^{1/2} + E^{-1/2} = 2\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-1/2}.$$

$$\Rightarrow \quad 2\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-1/2}$$
$$= 2\left(1+\frac{\Delta}{2}\right)E^{-1/2}$$
$$= 2\left(\frac{2+\Delta}{2}\right)E^{-1/2}$$
$$= \{1+(\Delta+1)\}E^{-1/2}$$
$$= (1+E)E^{-1/2}$$
$$= E^{-1/2} + E^{1/2}$$
$$\Rightarrow E^{1/2} + E^{-1/2} = 2\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-1/2}$$ Proved

Date - 6 Feb 2020

— Interpolation —

Interpolation Formula for Equal Intervals.

- Newton- Gregory Forward Interpolation Formula

Let $y=f(x)$ be the function which takes values $y_0, y_1, y_2, \ldots, y_n$ for values $x_0, x_1, x_2, \ldots, x_n$, of argument x. These arguments are equally with interval h, then Newton- Gregory Forward Formula is given by —

$$y_n(x_0+uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

+ ....

where $u = \dfrac{x - x_0}{h}$

Q. The population of a town is given below. Estimate the population for the year $\boxed{1896}$. $\to x$

Year (x)	x_0 1893	x_1 1903	x_2 1913	x_3 1923	x_4 1933
Population (y) (in thousands)	48	69	83	91	98

$\Rightarrow$ $u = \dfrac{x - x_0}{h} = \dfrac{1896 - 1893}{10} = 0.3$.

	x		y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	1893	y_0	$\boxed{48}$				
				$\boxed{21}$			
x_1	1903	y_1	69		$\boxed{-7}$		
				14		$\boxed{-1}$	
x_2	1913	y_2	83		-6		$\boxed{4}$
				8		5	
x_3	1923	y_3	91		-1		
				7			
x_4	1933	y_4	98				

$$y_n(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= 48 + 0.3 \times 21 + \frac{0.3(0.3-1)}{2!}(-7) + \frac{0.3(0.3-1)(0.3-2)}{3!} \times 1 + \frac{0.3(0.3-1)(0.3-2)(0.3-3)}{4!} \times 4$$

$$= 48 + 6.3 + 0.735 + 0.0595 - 0.1606$$

$$= \cancel{55.2} \; \underline{54.9339} \text{ (approx)}$$

Q. A cubic polynomial passes through the points $(0,-1)$, $(1,1)$, $(2,1)$ & $(3,-2)$. Then find the polynomial.

$\Rightarrow$ $u = \dfrac{x - x_0}{h} = \dfrac{x-0}{1} = x$

$\Rightarrow u = x$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
0	-1			
		2		
1	1		-2	
		0		-1
2	1		-3	
		-3		
3	-2			

$$y_n(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= -1 + x(2) + \frac{x(x-1)}{2!} \times (-2) + \frac{x(x-1)(x-2)}{3!}(-1)$$

$$= -1 + 2x - x(x-1) - \frac{x(x-1)(x-2)}{6}$$

$$= -1 + 2x - x^2 + x - \frac{x}{6}(x^2 - 3x + 2)$$

$$= -1 + 2x - x^2 + x - \frac{x^3}{6} + \frac{x^2}{2} - \frac{x}{3}.$$

$$= -\frac{x^3}{6} - \frac{x^2}{2} + \frac{8x}{3} - 1$$

Q. If $p_0 = 1$, $p_1 = 0$, $p_2 = 5$, $p_3 = 22$, $p_4 = 57$, then find $p_{0.5}$.

$\Rightarrow$ $x = 0.5$

$$u = \frac{x - x_0}{h} = \frac{0.5 - 0}{1} = \frac{0.5}{1} = 0.5$$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
0	1				
		-1			
1	0		6		
		5		6	
2	5		12		0
		17		6	
3	22		18		
		35			
4	57				

$$\begin{array}{r} 57 \\ -22 \\ \hline 35 \end{array}$$

$$y_{0.5} = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= 1 + 0.5(-1) + \frac{(0.5)(0.5-1)6}{2!} + \frac{0.5(0.5-1)(0.5-2)6}{3!} + \frac{0.5(0.5-1)(0.5-2)(0.5-3)\times 0}{4!}$$

$$= 1 - 0.5 - 0.75 + 0.375$$

$$= 0.125 \text{ (approx)}.$$

Date - 12 Feb 2020

- Newton - Gregory Backward Interpolation Formula.

Let the function be $y = f(x)$ for $x = x_0, x_1, x_2, \ldots x_n$, so on corresponding value of $y = y_0, y_1, y_2, \ldots, y_n$, so on, then Newton-Gregory Backward Interpolation Formula is given by -

$$f(x_n + uh) = y_n + \nabla y_n u + \frac{u(u+1)}{2!}\Delta^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots$$

where $u = \dfrac{x - x_n}{h}$

Q. For $f(x)$; $x = 7.5$, given —

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

$$u = \frac{x - x_n}{h} = 7.5 - 8 = -0.5$$

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$	$\nabla^6 f(x)$	$\nabla^7 f(x)$
1	1							
2	8	7						
3	27	19	12					
4	64	37	18	6				
5	125	61	24	6	0			
6	216	91	30	6	0	0		
7	343	127	36	6	0	0	0	
8	512	169	42	6	0	0	0	0

$512 - 343 = 169$

$-16+1$

$$y_{7.5} = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_n + \ldots$$

$$= 512 + (-0.5)\times 169 + \frac{(-0.5)(-0.5+1)}{2!}\times 42 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}\times 6 + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}\times 0$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.875 \text{ (approx)}.$$