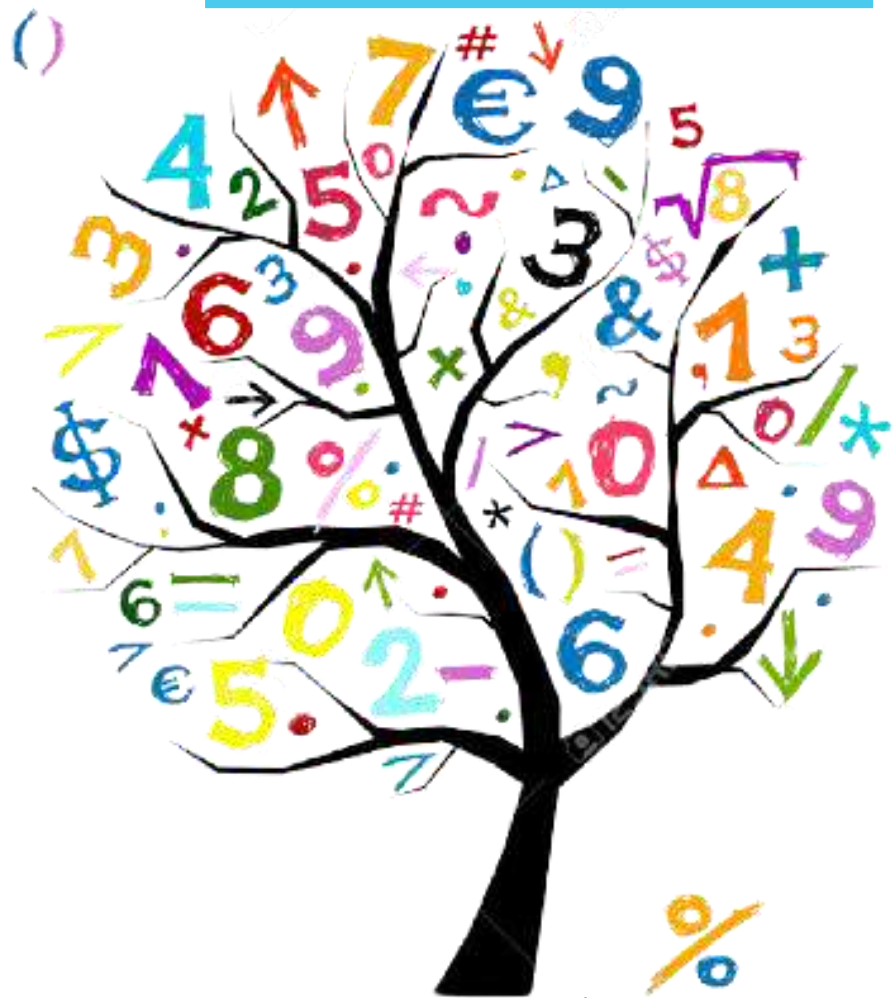


ENGINEERING MATHEMATICS-I




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ENGINEERING MATHEMATICS – I (3+0)

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Lesson 1

Rolle's Theorem, Lagrange's Mean Value Theorem , Cauchy's Mean Value Theorem

1.1 Introduction

In this lesson first we will state the Rolle's theorems, mean value theorems and study some of its applications.

Theorem 1. 1 [Rolle's Theorem]: Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . If $f(a) = f(b)$, then there exists at least one number c in (a, b) such that $f'(c) = 0$.

Proof: Assume $f(a) = f(b) = 0$. If $f(a) = f(b) = k$ and $k \neq 0$, then we consider $f(x) - k$ instead of $f(x)$. Since $f(x)$ is continuous on $[a, b]$ it attains

its bounds: Let M and m be both maximum and minimum of $f(x)$ on $[a, b]$. If

$M = m$, then $f(x) = m$ is throughout i.e., $f(x)$ is constant on

$[a, b] \Rightarrow f'(x) = 0$ for all x in $[a, b]$. Thus \exists at least one c such that $f'(c) = 0$.

Suppose $M \neq m$. If $f(x)$ varies on (a, b) then there are points where $f'(c) > 0$

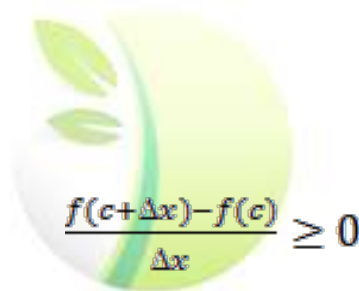
or points where $f'(c) < 0$. Without loss of generality assume $M > 0$ and the

function takes the maximum value at $x = c$, so that $f(c) = M$. It is to be noted that if $c = a$, $f(c) = f(a) = 0 = f(b)$, which is a contradiction. Now as $f(c)$ is the maximum value of the function, it follows that $f(c + \Delta x) - f(c) \leq 0$, both when $\Delta x > 0$ and $\Delta x < 0$.

Hence,

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0$$

when $\Delta x > 0$



$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \geq 0$$

when $\Delta x < 0$. Since it is given that the derivative at $x = c$ exists, we get

$f'(c) \leq 0$ when $\Delta x > 0$ and $f'(c) \geq 0$ when $\Delta x < 0$. Combining the two

inequalities we have, $f'(c) = 0$.

Note: Rolle's theorem shows that b/w any two zero's of a function $f(a)$ there

exists at least one zero of $f'(x)$ i.e., $f(a) = f(b)$ clearly f is continuous on $[-1, 1]$

Example 1: Verify the Roll's theorem for $f(x) = x^2$ for all $x \in [-1, 1]$.

Solution:

(i) $f(1) = f(-1) = 1$, (ii) f is differentiable on $[-1,1]$, so all conditions of Roll's theorems are satisfying. Hence $f'(c) = 2c = 0$ implies $c = 0$ and $c \in (-1,1)$.

Example 2: $f(x) = 1 - |x|$ in $[-1,1]$.

Solution:

$f(-1) = f(1) = 0$, f is continuous. But $f(x)$ is not differentiable at $x = 0$.

Note that $f'(x) \neq 0$, for which $f(x)$ is differentiable. As $f'(x) = -1$, for $x > 0$

and $f'(x) = 1$, for $x < 0$.

Example 3: Show that the equation $3x^5 + 15x - 8 = 0$, has only one real root

Solution:

$f(x) = 3x^5 + 15x - 8$ is an odd degree polynomial, hence it has at least one real root as complex roots occurs in pair.

Suppose \exists two real roots x_1, x_2 such that $x_1 < x_2$, then on $[x_1, x_2]$, all properties of Roll's theorem satisfied, hence $\exists c \in (x_1, x_2)$, such that $f'(c) = 0$,

But $f'(x) = 15x^4 + 15 = 15(x^4 + 1) > 0$, for all x , a contradiction to

Rolle's theorem. Hence the equation has only one real root.

1.2. Mean Value Theorems

Theorem 1.2 [Lagrange's Mean Value Theorem]: If a function $f(x)$ is continuous on $[a, b]$, differentiable (a, b) , then there exists at least one point c ,

$a < c < b$ such that $f(b) - f(a) = f'(c)(b - a)$. Hence Lagrange's mean

value theorem can be written as

$$f(b) - f(a) = hf'(a + \theta h), \text{ where } h = b - a; 0 \leq \theta \leq 1.$$

Geometrical Representation: If all points of the arc AB there is a tangent line, then there is a point C between A and B at which the tangent is parallel to the chord connecting the points A and B .

1.2.1 Cauchy's Mean Value Theorem

Cauchy's mean value theorem, also known as the extended mean value theorem, is the more general form of the mean value theorem.

Theorem 1.2 [Cauchy's Mean Value Theorem]: It states that if functions f and g are both continuous on the closed interval $[a, b]$, and differentiable on the

open interval (a, b) and $g(a) \neq g(b)$ then there exists some $c \in (a, b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Note 1: Cauchy's mean value theorem can be used to prove L'Hospital's rule. The mean value theorem (Lagrange) is the special case of Cauchy's mean value theorem when $g(t) = t$.

Note 2: The proof of Cauchy's mean value theorem is based on the same idea as the proof of the mean value theorem

1.2.2 Another form of the statement: If $f(x)$ and $g(x)$ are derivable in

$[a, a+h]$ and $g'(x) \neq 0$ for any $x \in [a, a+h]$, then there exists at least one number $\theta \in (0,1)$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)} \quad (0 < \theta < 1)$$

Example 4: Write the Cauchy formula for the functions $f(x) = x^2$, $g(x) = x^3$ on $[1,2]$.

Solution:

Clearly f and g are continuous and diff. on $[1,2]$ $g'(x) = 3x^2 = 0$ iff

$x = 0, 0 \notin [1,2]$. $f'(x) = 2x$. Hence $g(1) \neq g(2)$

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

$$\text{i.e., } \frac{4-1}{8-1} = \frac{2c}{3c^2} \text{ implies } \frac{3}{7} = \frac{2}{3c}, \text{ so } c = \frac{14}{9}.$$

1.2.3 The Intermediate Value Theorem It states the following: If $y = f(x)$ is continuous on $[a, b]$, and N is a number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = N$.

1.2.4 Applications of the Mean Value Theorem to Geometric properties of Functions.

Let f be a function which is continuous on a closed interval $[a, b]$ and assume f .

has a derivative at each point of the open interval (a, b) . Then we have

1. (i) If $f'(x) > 0$ for all $x \in (a, b)$, f is strictly increasing on $[a, b]$.
2. (ii) If $f'(x) < 0$ for all $x \in (a, b)$, f is strictly decreasing on $[a, b]$.
3. (iii) If $f'(x) = 0$ for all $x \in (a, b)$, f is constant.

Intermediate value Theorem for Derivatives: If $f'(x)$ exists for $a \leq x \leq b$, with $f'(a) \neq f'(b)$ then for any number d between $f'(a)$ and $f'(b)$ there is a number $a < c < b$ where $f'(c) = d$.

Application: If $f'(x)$ exists with $f'(x) \neq 0$, on any interval then f has a differentiable inverse, there.

Converse of Rolle's theorem : - (need not true).

Example 1.5 Let $f(x)$ be continuous on $[a, b]$ and differentiable (a, b) . If $\exists c \in (a, b)$ such that $f'(c) = 0$, does it follow that $f(a) = f(b)$?

Solution:

No: Take for example $f(x) = x^2$ on $[-1, 2]$, $f'(x) = 2x = 0$ implies $x = 0$.

But $f(-1) = 1$ and $f(2) = 4$.

Example 1.6 Show that $|\sin x - \sin y| \leq |x - y|$

Solution:

Let $f(t) = \sin t$ on $[y, x]$, By mean value theorem $\sin x - \sin y = f'(c)(x - y)$,

But $f'(t) = \cos t$, and $|\cos t| \leq 1$, for all t . Hence

$$|\sin x - \sin y| = |f'(c)(x - y)| \leq |x - y|.$$

Example 1.7 Show that $\tan^{-1}x_2 - \tan^{-1}x_1 < x_2 - x_1$, for all $x_2 > x_1$.

Solution:

Let $f(x) = \tan^{-1}x$ on $[x_1, x_2]$. By mean value theorem $\tan^{-1}x_2 - \tan^{-1}x_1 =$

$$f'(c)(x_2 - x_1) = \frac{1}{1+c^2}(x_2 - x_1) \text{ but } \frac{1}{1+c^2} < 1 \text{ for all } c. \text{ Hence the results.}$$

Questions: Answer the following question.

1. Verify the truth of Rolle's theorem for the functions

(a) $f(x) = x^2 - 3x + 2$ on $[1, 2]$

(b) $f(x) = (x-1)(x-2)(x-3)$ on $[1, 3]$

(c) $f(x) = \sin x$ on (a) $[0, \pi]$

2. The function $f(x) = 4x^3 + x^2 - 4x - 1$ has roots 1 and -1. Find the root of the derivative $f'(x)$ mentioned in Rolle's theorem.

3. Verify Lagrange's formula for the function $f(x) = 2x - x^2$ on $[0, 1]$.

4. Apply Lagrange theorem and prove the inequalities

(i) $e^x \geq 1 + x$ (ii) $\ln(1 + x) < x$ ($x > 0$)

(iii) $b^n - a^n < nb^{n-1}(b - a)$ for $(b > a)$

5. Using Cauchy's mean value theorem show that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Keywords: Rolle's Theorem, Lagrange's and Cauchy's mean value; L'Hospital's rule; Intermediate value.

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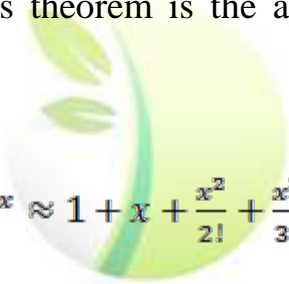
Lesson 2

Taylor's theorem / Taylor's expansion, Maclaurin's expansion

2.1 Introduction

In calculus, Taylor's theorem gives us a polynomial which approximates the function in terms of the derivatives of the function. Since the derivatives are usually easy to compute, there is no difficulty in computing these polynomials.

A simple example of Taylor's theorem is the approximation of the exponential function e^x near $x = 0$.



$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

The precise statement of the Taylor's theorem is as follows:

Theorem 2.1: If $n \geq 0$ is an integer and f is a function which is n times continuously differentiable on the closed interval $[a, x]$ and $n + 1$ times differentiable on the open interval (a, x) , then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f^{(2)}(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x)$$

Here, $n!$ denotes the factorial of n , and $R_n(x)$ is a remainder term, denoting the difference between the Taylor polynomial of degree n and the original function. The remainder term $R_n(x)$ depends on x and is small if x is close enough to a .

Several expressions are available for it. The Lagrange form is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1} = a + \theta(x - a)$$

where $0 < \theta < 1$.

If we put $a = 0$, Taylor's formula reduces to Maclaurin's formula.

where ξ lies between a and x .

Notes

- In fact, the mean value theorem is used to prove Taylor's theorem with the Lagrange remainder term.
- The Taylor series of a real function $f(x)$ that is infinitely differentiable in a

neighborhood of a real number a , is the power series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

- In general, a function need not be equal to its Taylor series, since it is possible that the Taylor series does not converge, or that it converges to a different function.
- However, for some functions $f(x)$, one can show that the remainder term

$R_n(x)$ approaches zero as n approaches ∞ . Those functions can be expressed as a Taylor series in a neighbourhood of the point a and are called analytic.

Example 2.1 Show that $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

Solution:

Here $f(x) = \sinh x$, $f'(x) = \cosh x$, So

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$$

$$f(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$R_n(x) = \frac{h^n}{n!}f^{(n)}(a + \theta h). \text{ But for } a = 0 \text{ and } h = x$$

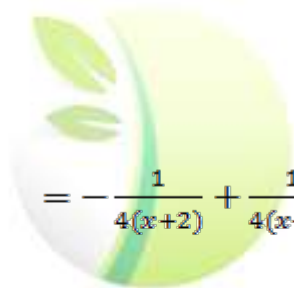
$$|R_n(x)| = \left| \frac{x^n}{n!}f^{(n)}(\theta x) \right|$$

$$\lim_{n \rightarrow \infty} |R_n| = \lim_{n \rightarrow \infty} \left| \frac{x^n}{n!} \right| |\cosh(\theta x)| = 0$$

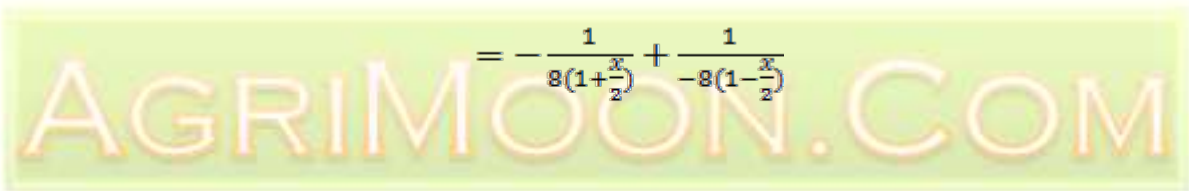
Example 2.2 . Find the Taylor series expansion of $\frac{1}{x^2-4}$

Solution: $f(x) = \frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)}$

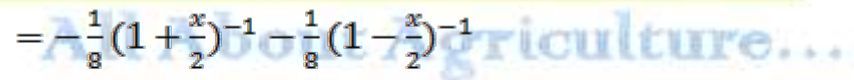
$$= \frac{A}{x+2} + \frac{B}{x-2}$$



$$= -\frac{1}{4(x+2)} + \frac{1}{4(x-2)}$$



$$= -\frac{1}{8(1+\frac{x}{2})} + \frac{1}{-8(1-\frac{x}{2})}$$



$$= -\frac{1}{8}(1+\frac{x}{2})^{-1} - \frac{1}{8}(1-\frac{x}{2})^{-1}$$

for $|\frac{x}{2}| < 1$, we have

$$= -\frac{1}{8} \left[1 - \frac{x}{2} + \left(-\frac{x}{2}\right)^2 + \left(-\frac{x}{2}\right)^3 \dots \right]$$

$$- \frac{1}{8} \left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 \dots \right]$$

$$= -\frac{1}{8} \left[2 + \left(\frac{x}{2}\right)^2 + \dots \right]$$

Example 2.3 : Find $f^{(100)}(0)$ if $f(x) = e^{x^2}$

Ans: $f^{(100)}(0) = \frac{100!}{50!}$.

Questions: Answer the following questions.

1. Expand in power of $x-2$ of the polynomial $x^4 - 5x^3 + 5x^2 + x + 2$.
2. Expand in power of $x+1$ of the polynomial $x^5 + 2x^4 - x^2 + x + 1$.
3. Write Taylor's formula for the function $y = \sqrt{x}$ when $a = 1, n = 3$.
4. Write the Maclaurin formula for the function $y = \sqrt{1+x}$ when $n = 2$.
5. Using the results of above problem, estimate the error of the approximate equation $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ when $x = 0.2$.
6. Write down the Taylor's expansion for the function $f(x) = \sin x$ about the point $a = \frac{\pi}{4}$ with $n = 4$.
7. Applying Taylor's theorem with remainder prove that $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0$.
8. Applying Maclaurin's theorem with remainder expand
 - (i) $\ln(1+x)$
 - (ii) $(1+x)^m$.

Keywords: Taylor's Formula, Taylor's Series, Maclaurin Formula and Series.

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