Module 2 (Topology)

1 Metric topology :-

(i) Metric :- A metric on a set X is a f^{x^n}

$$d : X \times X \longrightarrow R$$

having following properties :-

ⓐ $d(x, y) \geq 0 \quad \forall\, x, y \in X$; equality holds iff $x = y$

ⓑ $d(x, y) = d(y, x) \quad \forall\, x, y \in X$

ⓒ $d(x, y) + d(y, z) \geq d(x, z) \quad \forall\, x, y, z \in X$
(Triangle inequality)

(ii) ϵ-Ball centered at x :-

given a metric d on X, the no. $d(x, y)$ is called the distance b/w x and y in the metric d.

given $\epsilon > 0$,

$$B(x, \epsilon) \text{ (or) } B_d(x, \epsilon) = \{ y \mid d(x, y) < \epsilon \}$$

consider the set of all pts y whose distance from x is less than ϵ.
It is called the ϵ-Ball centered at x.

(iii) Metric topology :- If d is a metric on set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is a Basis for a topology on X called metric topology induced by d.

Proof :- (a) 1^{st} condition for basis is trivial,

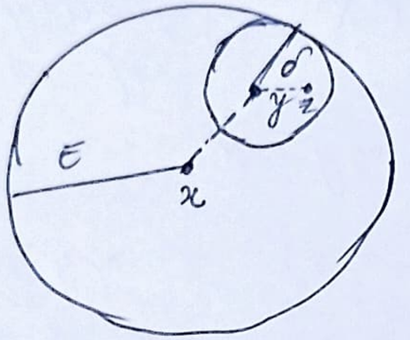
$\therefore x \in B(x, \epsilon)$ for any $\epsilon > 0$

(b) 2^{nd} cond^n of basis,

For checking 2^{nd} cond^n, we show that if $y \in B(x, \epsilon)$ then there is a basis element $B(y, \delta)$ centered at y s/t $B(y, \delta) \subset B(x, \epsilon)$.

$\Rightarrow$ Define δ as,

$\epsilon - d(x, y) = \delta$



Then, $B(y, \delta) \subset B(x, \epsilon)$

If $z \in B(y, \delta)$ then

$d(y, z) < \epsilon - d(x, y)$

$\Rightarrow d(x, y) + d(y, z) < \epsilon$ —— (i)

Now, from triangle inequality ; $d(x, z) \leq d(x, y) + d(y, z)$ —— (ii)

From (i) & (ii) ; $d(x, z) \leq d(x, y) + d(y, z) < \epsilon$

Now, let B_1 & B_2 be two basis elements & let $y \in B_1 \cap B_2$.

To show :- $B(y, \delta_1) \subset B_1$ & $B(y, \delta_2) \subset B_2$

(δ_1 and δ_2 are +ve no's)

Let δ be smaller of δ_1 and δ_2, we can conclude that $B(y, \delta) \subset B_1 \cap B_2$

(*) $\left\{ \begin{array}{l} B(y, \delta) \subset B(y, \delta_1) \\ B(y, \delta) \subset B(y, \delta_2) \end{array} \right. \Rightarrow B(y, \delta) \cap B(y, \delta) \subset B(y, \delta_1) \cap B(y, \delta_2)$

$\Rightarrow B(y, \delta) \subset B(y, \delta_1) \cap B(y, \delta_2) \subset B_1 \cap B_2 \}$

(iv) New defⁿ of metric topology :-

A set V is open in the metric topology induced by d iff for each $y \in V$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset V$.

Example 1) given set X, define $d : X \times X \to R$ s/T

$$d(x,y) = 1 \quad \text{if } x \neq y$$
$$d(x,y) = 0 \quad \text{if } x = y$$

Then (X, d) is called discrete metric space (or) trivial.

The topology induced by d is called discrete topology.

$d : R \times R \to R$

Example 2) The standard metric on $\mathbb{R}$ is defined by eqⁿ $d(x,y) = |x-y|$.

The topology induced by d is called order topology.

Q P/T $d(x,y) = 1$ if $x \neq y$
$d(x,y) = 0$ if $x = y$ is a metric

Ans) (i) $d(x,y)$ is either 0 or 1,

so, $d(x,y) \geq 0$

(ii) $d(x,y) = 0$ iff $x = y$ (trivial)

(iii) $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} = \begin{cases} 1 & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$

$= d(y,x)$

(iv) Triangle inequality

Let us consider pt $z \in X$. Then;

case 1) If $x \neq z$ and $y \neq z$;

$$d(x,z) + d(z,y) = (1+1) = 2 \geq d(x,y) = 1$$

case 2) If $x = z$ and $y \neq z$;

$$d(x,z) + d(z,y) = 0+1 = 1 \geq d(x,y) = 1$$

case 3) If $x \neq z$ and $y = z$;

$$d(x,z) + d(z,y) = 1+0 = 1 \geq d(x,y) = 1$$

case 4) If $x = z$ and $y = z$;

$$d(x,z) + d(z,y) = 0+0 = 0 \geq d(x,y) = 0$$

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(2) Metrizable :-

(i) If X is a topological space, X is said to be metrizable if there exists a metric d on set X that induces the topology X.

A metric space is a metrizable space X together with a specific metric d that gives the topology of X.

(ii) Definition :- Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number M such that

$$\Rightarrow d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A. If A is bounded and non-empty, the diameter of A is defined to be the number,

$$\text{diam } A = \sup \{ d(a_1, a_2) \mid a_1, a_2 \in A \}$$

Theorem 20.1> Let X be a metric space with metric d. Define $\bar{d} : X \times X \longrightarrow \mathbb{R}$ by the equation,

$$\bar{d}(x, y) = \min\{d(x, y), 1\}$$

The $\bar{d}$ is a metric that induces the same topology as d.

NOTE :- The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof :- (To prove that $\bar{d}$ is a metric)

(i) case 1> If $d(x,y) \geq 1$;

$$\bar{d}(x,y) = \min(d(x,y), 1) = 1 > 0$$

case 2> If $d(x,y) < 1$;

$$\bar{d}(x,y) = \min(d(x,y), 1) = d(x,y)$$

$$\Rightarrow \bar{d}(x,y) \geq 0 \qquad [\text{But, } d(x,y) > 0]$$

(ii) $\bar{d}(x, x) = \min(d(x, x), 1)$
$= \min(0, 1) = 0$

So, $\bar{d}(x, y) = 0$ iff $x = y$.

(iii) $\bar{d}(x,y) = \min(d(x,y), 1)$
$= \min(d(y,x), 1)$
$= \bar{d}(y, x)$

(iv) checking triangle inequality; $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$ (6)

Case 1) If either $d(x,y) \geq 1$ or $d(y,z) \geq 1$, then ~~eig~~ RHS of inequality is atleast 1 and LHS (by defn) is at most 1. Hence, inequality holds.

Case 2) If $d(x,y) < 1$ and $d(y,z) < 1$.

$$d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z) \quad \text{— (i)}$$

$$\therefore \bar{d}(x,z) \leq d(x,z) \quad \text{— (ii)}$$

From (i) & (ii); $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

Hence, triangle inequality holds for $\bar{d}$.

Finally, we S/T $\bar{d}$ generates same topology as d.

NOTE:- In any metric space, the collection of ϵ-balls with $\epsilon < 1$ forms a basis for metric topology, for every basis element containing x contains such an ϵ-centered at x. It follows that d and $\bar{d}$ induce same topology on X.

(8) Definition:-

given, $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$,

(i) norm of x, $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$

(ii) Euclidean metric d on $\mathbb{R}^n$, $d(x,y) = \|x-y\|$

$d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ $\quad = \left[(x_1-y_1)^2 + \cdots + (x_n-y_n)^2\right]^{1/2}$

(iii) square metric ρ, $\rho(x,y) = \max\{|x_1-y_1|, \ldots, |x_n-y_n|\}$

Q P/T ρ is a metric.

Proof :- given, $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$

$y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

(i) $\rho(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$

(ct) $\geq |x_i - y_i| > 0$ if $x_i \neq y_i$

$\rho(x, x) = \max\{|x_1 - x_1|, \ldots, |x_n - x_n|\}$
$= \max\{0, 0, \ldots, 0\}$
$= 0$

(ii) so, $\rho(x,y) = 0$ iff $x = y$

(iii) $\rho(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$
$= \max\{|y_1 - x_1|, \ldots, |y_n - x_n|\}$
$= \rho(y, x)$

(iv) From triangle inequality for $\mathbb{R}$; for each positive integer i,

$\Rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$

Then by defn of ρ,

$|x_i - z_i| \leq \rho(x,y) + \rho(y,z)$

and, $\rho(x, z) = \max\{|x_i - z_i|\} \leq \rho(x,y) + \rho(y,z)$

(7) **Lemma 20.2>** Let d and d' be two metrics on the set X; let τ and τ' be the topologies they induce, respectively. Then τ' is finer than τ iff for each x in X and each $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon)$$

Proof :- suppose that τ' is finer than τ. given the basis element $B_d(x, \epsilon)$ for τ, there is a basis element B' for topology τ' such that $x \in B' \subset B_d(x, \epsilon)$ [using the below ~~result~~ Lemma]

~~Result~~ Lemma

As we know that, if $\mathcal{B}$ and $\mathcal{B}'$ are bases for topologies τ and τ' respectively on X then the following are equivalent:-

(1) τ' is finer than τ'

(2) For each $x \in X$ and each Basis element $B \in \mathcal{B}$ containing x, there is a Basis element $B' \in \mathcal{B}'$ s/T $x \in B' \subset B$.

So, within B' we can find a ball $B_{d'}(x, \delta)$ centered at x.

conversely, suppose the $\delta - \epsilon$ condition holds. Given a Basis element B for τ containing x, we can find within B a ball $B_d(x, \epsilon)$ centered at x. ~~Then~~ By the given condition, there is a δ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$. Then by above Lemma, τ' is finer than τ.

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(5) **Theorem 20.3>** The topologies on $\mathbb{R}^n$ by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

(6) Uniform metric :-

Consider,

$$\bar{d}(x, y) = \min\{\underset{d(x,y)}{|x-y|}, 1\}$$

$\hookrightarrow$ standard bounded metric on $\mathbb{R}$

given an index set J, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of R^J, ~~let us~~ define a metric $\bar{\rho}$ on R^J by the equation,

$$\Rightarrow \bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. $\bar{\rho}$ is called uniform metric on R^J and the topology induced by it is called uniform topology.

(7) **Theorem 20.5>**

Let $\bar{d}(a, b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define

$$D(x, y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

8 **Theorem 21.1** Let $f: X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, $\exists s$ $\delta > 0$ such that

$$d_X(x, y) \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

9 **Lemma 21.2 (The sequence Lemma):–**

Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x then $x \in \bar{A}$; the converse holds if X is materizable.

10 **Theorem 21.3** Let $f: X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \longrightarrow x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is materializable.

Proof:– Assume that f is continuous. Given, $x_n \longrightarrow x$, we want to show that $f(x_n) \longrightarrow f(x)$

Let V be a nbd of $f(x)$. Then $f^{-1}(V)$ is a nbd of x, and so there is an N such that $x_n \in f^{-1}(V)$ for $n \geq N$. Then $f(x_n) \in V$ for $n \geq N$.

To prove the converse, assume that the convergent sequence condition is satisfied. Let A be a subset of X; we show that $f(\bar{A}) \subset f(A)$, If $x \in \bar{A}$, then there is a sequence x_n of points of A converging to x (using sequence lemma)

By assumption, the sequence $f(x_n)$ converges to $f(x)$. Since $f(x_n) \in f(A)$, the sequence lemma implies that $f(x) \in \overline{f(A)}$.

Hence, $f(\overline{A}) \subset \overline{f(A)}$

(III) **Uniform limit theorem :-**

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f, then f is continuous.

Proof :- Let V be open in Y; let x_0 be a point of $f^{-1}(V)$. We want to find a nbd U of x_0 such that $f(U) \subset V$.

Let $y_0 = f(x_0)$. First choose ϵ so that the ϵ-ball $B(y_0, \epsilon)$ is contained in V. Then, using uniform convergence, choose N so that for all $n \geq N$ and all $x \in X$.

$$d(f_n(x), f(x)) < \epsilon/3$$

Finally, using continuity of f_N, choose a nbd U of x_0 such that f_N carries U into the $\epsilon/3$ ball in Y centered at $f_N(x_0)$.

We claim that f carries U into $B(y_0, \epsilon)$ and hence into V. For this, if $x \in U$, then;

$\Rightarrow$ $d(f(x), f_N(x)) < \epsilon/3$ (by choice of N)

$\Rightarrow$ $d(f_N(x), f_N(x_0)) < \epsilon/3$ (by choice of U)

$\Rightarrow$ $d(f_N(x_0), f(x_0)) < \epsilon/3$ (by choice of N)

Adding and using triangle inequality,

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 < \epsilon$$

$$\Rightarrow d(f(x), f(x_0)) < \epsilon.$$

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(12) Connectedness and compactness :—

(i) Intermediate value theorem :—
If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and if r is a real number between $f(a)$ and $f(b)$, then $\exists$s an element $c \in [a, b]$ such that $f(c) = r$.

(ii) Maximum value theorem :— If $f : [a, b] \to \mathbb{R}$ is continuous, then $\exists$s an element $c \in [a, b]$ such that $f(x) \leq f(c)$ for every $x \in [a, b]$

(iii) uniform continuity theorem :—
If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for every pair of numbers x_1, x_2 of $[a, b]$ for which $|x_1 - x_2| < \delta$.

(iv) The property of the space [a, b] on which the intermediate value theorem depends is the property called connectedness.

(v) The property on which maximum value thm & uniform continuity thm depends is the property called compactness.