Discrete Structures
for Computer Science

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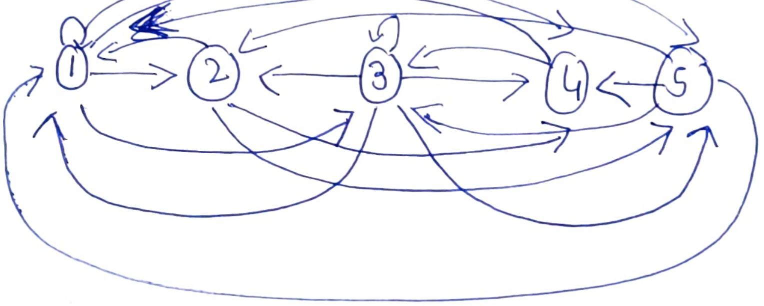
Assignment - II

1) Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs (1,1), (1,2), (1,3), (2,3), (2,4), (3,1), (3,4), (3,5), (4,2), (4,5), (5,1), (5,2) & (5,4).

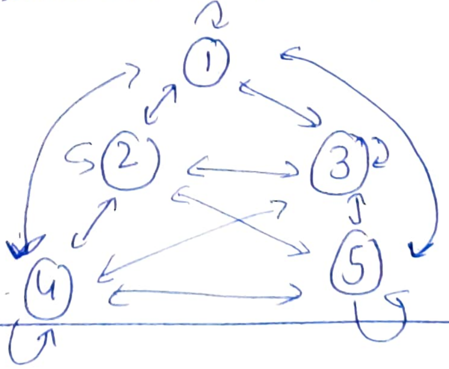
Find a) R^2

$R \circ R = R^2 = \{ (1,1)\ (1,3),\ (1,4),\ (1,5)\ (1,2),$
$(2,4),\ (2,5),\ (2,1)\ (3,1),\ (3,2),\ (3,3),$
$(3,4),\ (3,5),\ (4,1),\ (4,2),\ (4,3),\ (5,1),\ (5,2),\ (5,3),\ (5,4) \}$



b) $R^3 = R^2 \circ R$
$= \{ (1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (2,1)$
$(2,2)\ (2,3)\ (2,4)\ (2,5)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (4,1)\ (4,2)$
$(4,3),\ (4,4),\ (4,5),\ (5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5) \}$

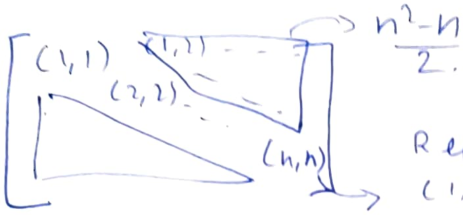
= All elements $|A \times A|$ relations formed.



c) $R^4 = R^3$ ($\because$ R^3 already includes all combination)

d) $R^5 = R^4 = R^3$

2) $\rightarrow$ no. of Relations = A relation such that it is reflexive & exactly one symmetric pair



Reflexive: (1,1) (2,2) ... (n,n) had only 1 choice

$\rightarrow$ one symmetric pair : if $\exists$ (x, y) then (y, x) will exists.

$\therefore$ x can be select in n ways & y can be select in $(n-1)$ ways

$\Rightarrow$ Total no. of relations $= n * (n-1)$

3) Partial orders: (a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\hookrightarrow$ ~~it is satisfies~~

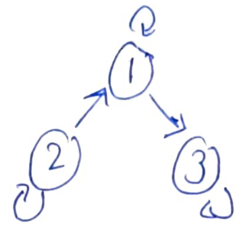
(i)

$\rightarrow$ To be POSET it should satisfy.

i) Reflexive ii) Anti-sym iii) Transitive

→ I have relation in such a way that elements are equal to 1.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \rightarrow R = \{(1,1)\ (1,3)\ (2,1)\ (2,2)\ (3,3)\}$$



Reflexive ✓

Anti symmetric ✓

In transitive $(1,3)$ & $(2,1)$

$(2,3)$ should form like it should be 1 but $(2,3)$ is 0.

[$(1,3) = 1 (A_{13})$; $(2,1) = 1 (A_{21})$ but $(2,3) = 0 (A_{23})$ it should be 1].

∴ it didn't satisfy transitive property.

$\boxed{\text{∴ It is not POSET.}}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R = \{(1,1)\ (2,2)\ (3,3)\}$

→ $\boxed{\text{It is POSET}}$

Ref ✓

Anti-sym ✓

Transitive ✓

(c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ $R = \{(1,1)\ (1,3)\ (2,2)\ (2,3)\ (3,3)\ (3,4)\ (4,1)\ (4,2)\ (4,4)\}$

Ref ✓

Anti-sym ✓

Transi ✗ [$(1,3) \rightarrow 1$, $(3,4) \rightarrow 1$ but $(1,4)$ is 0]

$\boxed{\text{∴ It is not POSET}}$

4) (a) R & S are equivalence relations on A.
Prove / disprove $R \circ S$ is an equivalence relation

<u>Ans</u> $R \circ S$ is not an equivalence relation.

Here's why.
I will prove this by counter example.

suppose set $A = \{1, 2, 3\}$

→ equivalence relation R on A:

$$R = \{(1,1), (2,2) (3,3) (2,3) (3,2)\}$$

→ equivalence relation S on A:

$$S = \{(1,1), (2,2) (3,3) (1,3) (3,1)\}$$

~~⇒ ROS = { (1,1) (2,2) (3,3) }~~

$$R \circ S = \{ (1,2) \cancel{(1,3)} (2,3) \}$$

→ But For transitive property (1,3) element part is missing.

so ROS is not an equivalence relation

(b) Prove $R = \{(x, y) \mid x+y$ is an even integer $\}$ is an equivalence relation

(a) <u>Reflexive</u>: Let $(a,a) \in R$, $a \in A$.

$a + a = 2a$ is always even

∴ Reflexive

② symmetric. let $(a,b) \in R$ Then $(a+b) = 2K$

$\Rightarrow (b+a) = 2k$

$\therefore (b,a) \in R$.

$\therefore$ symmetric

③ Transitive: Let $(a,b) \in R$ & $(b,c) \in R$

$(a,c) \in R$

Then $(a+b) = 2k$

$(b+c) = 2l$

$a+c+2b = 2(k+l)$

$\therefore \boxed{a+c = 2(k+l-b) = 2u} \rightarrow (a,c) \in R$

$\therefore$ Hence transitive

$\therefore$ Given R is equivalence relation on $\mathbb{Z}$.

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5) $A = \{0,1,2,3\}$. Let $R = \{(0,1),(0,2)\ (1,1),(1,3),$
$(2,2),(3,0)\}$.

① Transitive closure of R.

$$t(R) = R \cup \{(a,c) \mid (a,b) \in R \wedge (b,c) \in R \wedge (a,c) \notin R\}$$

$$= R \cup \{(0,3),(1,0),(2,0),(3,2),(3,1)\}$$

$$= \{(0,1)(1,0)\ (0,2)(2,0)\ (0,3)(3,0)\ (1,1)\ (1,3)(2,2)(3,2)\ (3,1)\}$$

② symmetric closure of R.

$$S(R) = R \cup \{(b,a) \mid (a,b) \in R \wedge (b,a) \notin R\}$$

$$= R \cup \{(1,0),(2,0)\ (3,1)(0,3\}$$

$$= \{(0,1)\ (1,0)\ (0,2)\ (2,0)\ (1,1)\ (1,3)\ (3,1)\ (2,2)\ (3,0)\ (0,3)\}$$

(8) reflexive closure:

$$r(R) = R \cup \{(a,a) \mid (a,a) \notin R,\ a \in A\}$$

$$= R \cup \{(0,0)\ (3,3)\}$$

$$= \{(0,0)\ (1,1)\ (2,2)\ (3,3)\ (0,2)\ (1,3)\ (3,0)\}$$

6) (a) $(a,b) \in R$?

→ Total ordered pairs $= n \times n = n^2$

$(a,b) = 1$ choice

Remaining $= n^2 - 1$ → it has 2 choice
① it can be there
② it won't be there

$= 2^{n^2-1}$

$\boxed{\therefore \text{No. of relations} = 1 \cdot 2^{n^2-1}}$

(b) $(a,b) \notin R$?

→ Total ordered pairs $= n^2$

$(a,b) = 0$ choice

Remaining $= 2^{n^2-1}$

$\boxed{\therefore \text{no. of relations} = 2^{n^2-1}}$

7) Let R be a relation from a set A to a set B. The complementary relation $\bar{R}$ is the set of ordered pairs $\{(a,b) \mid (a,b) \notin R\}$

① show that the relation R on a set A is reflexive if & only if the inverse relation R^{-1} is reflexive.

Relation R is reflexive if $(a,a) \in R$ for every $a \in A$.

Inverse relation R^{-1} is set $\{ (b,a) \mid (a,b) \in R \}$

→ Assume R is reflexive → $(a,a) \in R$

By Inverse relation $(a,a) \in R^{-1}$

Then R^{-1} is reflexive as $(a,a) \in R^{-1} \; \forall (a \in A)$

→ Similarly, Assume R^{-1} is reflexive → $(a,a) \in R^{-1} \; \forall (a \in A)$

Then $(a,a) \in R$ from def^n of inverse relation

R is reflexive as $(a,a) \in R \; \forall a \in A$.

∴ R is reflexive iff R^{-1} is reflexive.

(2) Show that the relation R on a set A is reflexive if and only if the complementary relation $\bar{R}$ is irreflexive.

Relation R is said to be irreflexive if $\nexists a \in A$ such that $(a,a) \in R$.

→ If R is reflexive → $\bar{R}$ is irreflexive

$(a,a) \in R \; \forall (a \in A)$

by def^n of $\bar{R}$ $(a,a) \notin \bar{R}$ ∴ $(a,a) \in R \; \forall a \in A$

$\bar{R}$ is irreflexive.

→ Assume $\bar{R}$ is irreflexive

Then $\forall a \in A \; \{ (a,a) \notin \bar{R} \}$

It implies $(a,a) \in R \; \forall a \in A$

Hence R is reflexive

∴ R is reflexive iff $\bar{R}$ is irreflexive

8) Which of these are posets?

a) $(R, =)$?

① reflexive $\forall a \in A \ (a = a)$ is true

② Anti-symmetric $\forall a \in A \ ((a,b) \wedge (b,a)$, $a = b$

$(a = a) \in R$

③ Transitive; $(a,b) \wedge (b,c) \to (a,c)$

$a = b = c = a$

$\boxed{\therefore \text{It is POSET}}$

b) $(R, <)$? ① Reflexive won't satisfy as $\forall a \in A \ (a < a)$ is false

$\boxed{\therefore \text{It is not POSET}}$

c) $(R, \leq)$? ① Reflexive $\forall a \in A \ (a, a) \in R$ (a,a) True

② Anti-sym $(a \leq b) \wedge (b \leq a)$ ~~then~~ false

$\Rightarrow a = b$

③ Transitive $(a \leq b) \wedge (b \leq c) \to (a \leq c)$

True

$\boxed{\therefore \text{It is not POSET}}$

d) $(R, \neq)$ ① Reflexive won't satisfy as $\forall a \in A \ (a, a)$ where $a \neq a$ is false

$\boxed{\therefore \text{It is not POSET}}$

9) Prove / disprove. If R is an equivalence relation on A, then ROR is an equivalence relation on A.

Proof: Given R is Reflexive, sym & Transitive

① $\forall a \in A$, $(a,a) \in ROR$ $\therefore$ R is reflexive

ROR is composition with itself will always be reflexive

② Symmetric: If $(a,c) \in R$ & $(c,b) \in R$ & $\exists c \in A$

as R is symmetric $\rightarrow (a,b) \in ROR$

$(c,a) \in R$ & $(b,c) \in R$

$((b,a) \in ROR)$

$\therefore$ ROR is symmetric

③ Transitive: $\rightarrow$ If $(a,d) \in R$ & $(d,b) \in R$ & $\exists d \in A$

$\downarrow$

$(a,b) \in ROR$ > ①

$(a,b) \in R$

$\rightarrow$ if $(b,e) \in R$ & $(e,c) \in R$ & $\exists e \in A$

$\downarrow$

$(b,c) \in ROR$ > ②

$(b,c) \in R$

$\rightarrow$ From ① & ②

$(a,b) \in R$ & $(b,c) \in R \rightarrow (a,c) \in ROR$

$(a,c) \in R$

$\therefore$ ROR is transitive

$\therefore$ ROR is an equivalence relation on R.

$\therefore$ Hence proved.

10) Let R be the relation $\{(a,b) \mid a \neq b\}$ on the set of integers up to 15. ① Reflexive closure of R

$R = \{(0,1),(0,2), - (0,15), (1,0), (1,2) \ldots (14,15)\}$

$r(R) = R \cup \{(0,0),(1,1),(2,2), \ldots (15,15)\}$

② Transitive: R is already transitive $\boxed{t(R) = R}$

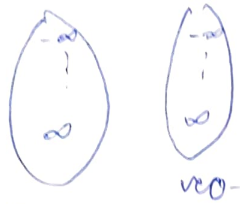
③ symmetric: R is already symmetric $\boxed{S(R) = R}$

Functions -

11) Find an example of a fun that is neither injective nor surjective.

Perfect example is $f: R \to R$ & $y = x^2$

$f(x) = x^2$.



co-domian

Domain

Range $\in [0, \infty)$

codomian $\in (-\infty, \infty)$

① $f(\pm 3) = 3^2 = 9$ (It's doesn't have unique image)

so not injective.

② As squaring some elements will left over in co-domain (no - pre - images) so not surjective. [As Range $\neq$ codomain]

12) Define functions f, g & h as follows:

$f: R \to R; \forall x \in R, f(x) = x^2$.

$g: N \to N; \forall x \in N, g(x) = x^2$.

$h: A \to B; \forall x \in A, h(x) = x^2$

where $A = \{0, 1, 2, 3, 4\}$ &

$B = \{0, 1, 4, 9, 16\}$

which is one-one & onto ?

→ one-one : f is not one-one As $f(\pm 1) = 1$ (same images)

: h & g is one-one because as only natural numbers are given input

→ onto : f is not onto (Range $[0, \infty) \neq$ codomain R)

: g is not onto (R $\neq$ codomain (N) [$1 \to 4 \to 9$ For 3, 2 they won't have pre-images]

: h is onto $h(0) = 0$

$h(1) = 1$

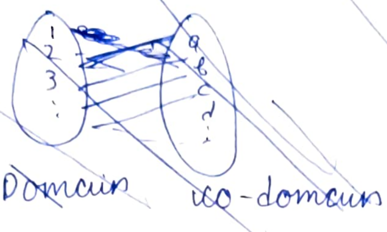
$h(2) = 4$

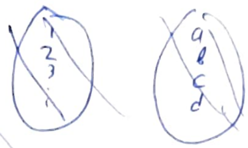
$h(3) = 9$ $h(4) = 16$

Range = codomain (B)

~~13) which is one-one?~~

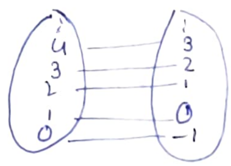
~~a) $f(n) = n-1$~~





13) which is one-one?

a) $f(n) = n-1$



$f(n)$ is 1-1 because every element in domain is mapped to distinct element in co-domain [unique images]

b) $f(n) = n^2+1$

For $f(-1) = 2$, $f(1) = 2$ [As in co-domain it ~~has~~ doesn't have unique images]

$f(n)$ is not 1-1 function.

c) $f(n) = n^3$ $f(n)$ is 1-1 [unique images]

14) (a) $f: R \to R$, $g: R \to R$ one-one then $f+g$ is also one-one

→ No, $f+g$ is not necessarily 1-1 function

Counter example: let $f(x) = x+1$ & $g(x) = -x$.

$\Rightarrow f(x)+g(x) = (f+g)(x) = 1$ [then all will connect to 1, no unique images]

$\therefore (f+g)(x)$ is not necessarily 1-1 function.

b) $f: R \to R$, $g: R \to R$ > onto, is $f+g$ also onto?

No, $(f+g)(x)$ is not necessarily onto function

As $f(x)$ & $g(x)$ are onto, then for any y in co-domain $\exists\, x_1, x_2$ such that

$$y = f(x_1) + g(x_1) \qquad \text{let } x = x_1 = x_2$$

$$y = f(x_2) + g(x_2)$$

$$(f+g)(x) = f(x) + g(x)$$
$$= f(x) + (y - f(x))$$
$$= y$$

$\therefore$ For any y in co-domain, we can find x in do-main, then [Range will not equal to co-domain]

$\therefore$ $(f+g)(x)$ is not necessarily onto function.

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15) Suppose that f is a function from A to B, where A & B are finite sets with $|A| = |B|$. Show that f is one-one & onto.

Ans given $|A| = |B|$ it implies both cardinality are equal.

(i) f is 1-1 (let's assume)

$$\forall (a, b) \in A \quad f(a) = f(b) \to a = b$$

Every element in domain is mapped to distinct element in B.

$\therefore$ Every element in co-domain have pre-image
$\therefore$ f is onto.

(2) f is onto (let's assume):

Every element in co-domain is mapped to atleast one element from domain. But $|A| = |B|$ so it will map to only one element in B. This ensures it will map to unique elements in co-domain. [unique images & pre-images]

$\therefore$ f is 1-1.

$\therefore$ From (1) & (2).

$\Rightarrow$ f is 1-1 iff it is onto

16) (a) T is 1-1?

Let's us assume m & n have same images $T(m) = T(n)$ it means that set of divisors (m) = set of divisors (n)

$\Rightarrow$ (largest divisor of m = largest divisor of n)

$\Rightarrow m = n$

$\Rightarrow$ T is 1-1 function.

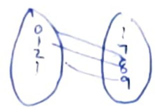
(b) T is onto?

Counter example: Let $\{1, 3, 5\} \in D$.

But there is no +ve Z that has only 1, 3, 5 as divisors 15 is the Z' have $\{1, 3, 5, 15\}$ as integer but there is no Z' having only $\{1, 3, 5\}$ as divisors. $\boxed{\therefore \text{ T is not onto.}}$

17) check one-one ?

(a) $f(n) = n+7$



For $\forall n \in \mathbb{Z}$,
$f(n)$ has unique images
$\therefore f(n)$ is 1-1.

(b) $f(n) = 2n-3$

$f(n_1) = f(n_2)$

$2n_1 - 3 = 2n_2 - 3$

$\boxed{n_1 = n_2}$ → " for one-one to prove this property is required.

c) $f(n) = \lceil n/2 \rceil$

$f(2) = \lceil 1 \rceil = 1$
$f(1) = \lceil 1/2 \rceil = 1$ > ~~it~~ As both have same images
$\therefore$ It is not one-one.

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18) If X & Y are sets and $F: X \to Y$ is 1-1 & onto, then $F^{-1}: Y \to X$ is also 1-1 & onto.

Proof: Given $F: X \to Y$ is bijective $\therefore |X| = |Y|$

→ F is one-one if for every $x_1, x_2 \in X$
$F(x_1) = F(x_2) \Rightarrow \boxed{x_1 = x_2}$

→ F is onto if for every $y \in Y$,
$\exists x \in X \quad (F(x) = y)$

claim: To prove F^{-1} is 1-1

Let $F^{-1}(y_1) = F^{-1}(y_2)$

As $F^{-1}(y) \in X \quad [y \in y_1, y_2]$

$F(x_1) = F(x_2) \quad \therefore x_1 = x_2 = x$

$F(F^{-1}(y_1)) = F(F^{-1}(y_2))$

$(y_1 = y_2)$ & $\forall y \in Y \quad F(F^{-1}(y)) = y$

$\therefore F^{-1}$ is 1-1

claim: F^{-1} is onto

For every $x \in X$ $F(x) = y$ &

every $x \in X$; $\exists$ a unique y

such that $F^{-1}(y) = x$

$\therefore F^{-1}$ is onto.

19) Suppose $F: X \to Y$ is onto. Prove that for every subset $B \subseteq Y$, $F(F^{-1}(B)) = B$.

Proof: $F^{-1}(B) = \{x \in X \; (F(x) \in B)\}$ & $B \subseteq Y$

$F(F^{-1}(B)) = \{F(x) \mid x \in F^{-1}(B)\}$

As F is onto, every element in Y is image of same element x under $F \Rightarrow$

$F(x) = y$.

$F(F^{-1}(B)) = B$

claim 1: $F(F^{-1}(B)) \subseteq B$ $\to$ For any element b in $F(F^{-1}(b))$, $\exists$ element (a) in $F^{-1}(B)$ such that $F(a) = b \Rightarrow$ a is in $F^{-1}(B)$

$F(a)$ is in B

b is in B

$\Rightarrow \boxed{\therefore F(F^{-1}(B)) \subseteq B}$

claim 2: $B \subseteq F(F^{-1}(B))$:

For any element b in B, since F is onto, $\exists$ a in x $F(a) = b$, since a maps to b under F, a must b in $F^{-1}(B)$ because $F^{-1}(B)$ contains all elements in X that map to elements in B. b is in $F(F^{-1}(B))$ $\Rightarrow$ $B \subseteq F(F^{-1}(B))$

From claim 1 & claim 2

$\boxed{F(F^{-1}(B)) = B}$ for every subset $B \subseteq Y$,

20) examples of finite sets A & B with $|A|, |B| \geq 4$

(a) F is 1-1 but not onto

$A = \{1,2,3,4\}$ $B = \{1,4,9,16,25\}$

$[f : x^2 \mid x \in A]$

$f(1) = 1$
$f(2) = 4$
$f(3) = 9$
$f(4) = 16$

one-one not onto.

As $\{25\} \notin R$

(b) F is onto but not 1-1

$A = \{-1,0,1,2,3\}$ $B = \{1,0,4,9\}$

$[f : x2 \mid x \in A]$

$f(-1) = 1$
$f(0) = 1$
$f(1) = 1$

[For $f(-1)$ & $f(1) = 1$ same image, so not 1-1 but onto]

(c) F is onto & one-one

$A = \{0,1,2,3\}$ $B = \{0,1,4,9\}$

$[f : x^2 \mid x \in A]$

$f(0) = 0$
$f(1) = 1$
$f(2) = 4$
$f(3) = 9$

[It satisfies both having unique image & Range = ~~codomain~~ co-domain]