Modern Variable Selection

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Outline

- What are “Modern” methods? How do they differ?
- Penalization Methods (regularizations)
- Model Selection / Averaging

Why Modern Variable Selection

Traditional (generally p-value based) methods have a number of shortcomings.

- Traditional Selection (backward, forward, and best subset selection):
  - Suffer from α inflation
  - Do not address problem of correlated predictors
  - Difficult to decide the appropriate number of predictors
  - Do not have very good out-of-sample performance
  - Cannot handle a large number of predictors
  - See: Breiman (1996) **Heuristics of instability and stabilization in model selection.** *Annals of Statistics* 24 (6)
  - Also: *The Elements of Statistical Learning* by Hastie, Tibshirani, and Friedman

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*There is some current debate about the validity of variable importance from model averaging

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However,

- Still don't know the correct number of predictors
- Penalization methods introduce a new (unknown) parameter λ

*Only penalization methods

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  - LARS (Efron et al. 2004)
  - Elastic net (Zou and Hastie 2005)

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  - LASSO (Tibshirani 1996)
  - LARS (Efron et al. 2004)
  - Elastic net (Zou and Hastie 2005)
- Things I don't know about (among others)
  - SCAD
  - Non-negative garotte
  - COSSO

LASSO

LASSO works through the application of a penalty parameter (λ) to the fitted coefficients (β).
The l_1 regularization:

$$\min_{\beta} \mathit{Loss}\left(\beta; \mathbf{y}, \mathbf{X}\right) + \lambda J\left(\beta\right)$$

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- J is the penalty function wich (for l_1) takes the form:

$$J\left(\beta\right) = \sum_{j=1}^{d} |\beta_j|$$

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    - Grouped LASSO
    - Elastic net

Elastic Net

Modification of the l_1 regularization to a quadratic form:

$$\min_{\beta} Loss\left(\beta; \mathbf{y}, \mathbf{X}\right) + (1-\alpha)||\beta||^2 + \alpha||\beta||^2$$

with $\alpha = \frac{\lambda_2}{\lambda_2+\lambda_1}$

- Encourages grouping effect on related variables (not guaranteed, but not necessary to specify the groups)
- Can select more variables than rows in the data ($k > N$)
- Stabilizes the regularization path

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- The choice of regularization will influence how quickly coefficients are truncated to 0
- For Ridge regression this will never happen!
- LASSO and Elastic net both produce "sparse" results.

Sparsity

From: *Elements of Statistical Learning* Hastie, Tibshirani, and Friedman

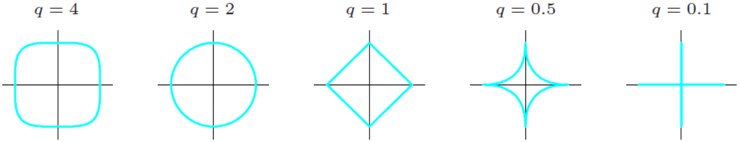

FIGURE 3.12. *Contours of constant value of* $\sum_j |\beta_j|^q$ *for given values of* q.

To get 0's the function must not be differentiable at the axes

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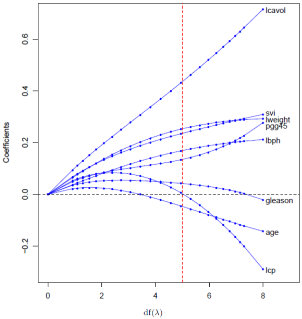


FIGURE 3.8. *Profiles of ridge coefficients for the prostate cancer example, as the tuning parameter λ is varied. Coefficients are plotted versus $\mathrm{df}(\lambda)$, the effective degrees of freedom. A vertical line is drawn at $\mathrm{df} = 5.0$, the value chosen by cross-validation.*

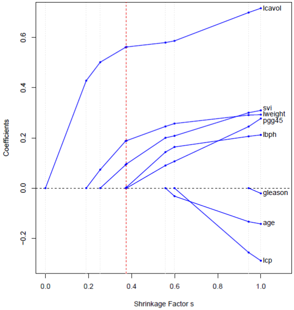
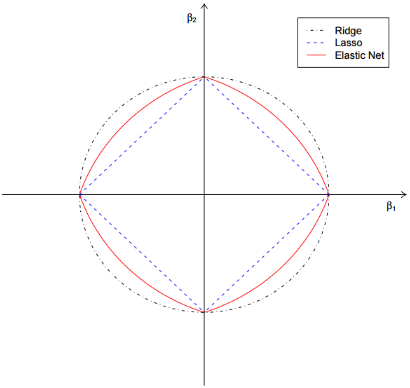


FIGURE 3.10. *Profiles of lasso coefficients, as the tuning parameter t is varied. Coefficients are plotted versus $s = t/\sum_1^p |\hat\beta_j|$. A vertical line is drawn at $s = 0.36$, the value chosen by cross-validation. Compare Figure 3.8 on page 65; the lasso profiles hit zero, while those for ridge do not. The profiles are piece-wise linear, and so are computed only at the points displayed; see Section 3.4.4 for details.*

Sparsity

From: Regularization and Variable Selection via the Elastic Net (Zou and Hastie 2005)



Elastic net regularization represents a compromise between LASSO and Ridge.

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- Can be estimated from the data but this is generally not done.
- Select *lambda* based on prediction error using cross-validation

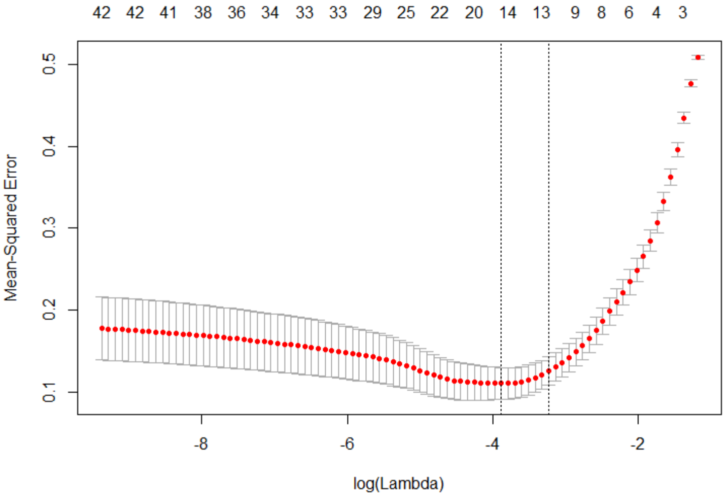
Cross-validation

K-fold Cross-validation steps: For each value of λ

1. Divide the data into two parts
2. "Train" the model on the first data set
3. Predict the outcome for the data in the second and calculate the error
4. Repeat k times for each value of λ

Cross-validation

Find the λ with the smallest prediction error and variance.



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- Results may be very specific to the original data.
- Interpretation of shrunken coefficients.

LASSO Implementation

SAS:
PROC GLMSELECT SEED=123;
PARTITION= ...; MODEL= .../ SELECTION=LASSO ;
(or SELECTION=LAR)
RUN;

YouTube Tutorial (click here)

R:
Package “glmnet” (created by Trevor Hastie and Junyang Qian)

cv.glmnet(x=predictors, y=response, family="gaussian", alpha=1, nlambda=100, nfolds=k)

glmnet(x=predictors, y=response, family="gaussian", alpha=1, lambda=best.lambda)

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- Select the top ranked model **or** Average the models to incorporate model selection uncertainty.

Information Criteria

All the information criteria follow the same general formula:

$$.IC = \text{-log(likelihood)} + \text{complexity penalty}$$

- $AIC = 2k - 2ln(L)$
- $BIC = -2ln(L) + k \cdot ln(n)$
- $KIC = -2\text{penalized log-likelihood} + C\left(\hat{\Sigma}_{\hat{\theta}}\right)*$

*I won't subject you to this function!

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This is called multi-model inference.

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These both follow the same general formula but have different philosophical motivations and interpretations

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- Adjust standard errors for model uncertainty

Multi-model Inference vs Regularization

- Must have plausable models for model averaging (already know what variables are important)
- Regularization methods can handle very high dimensions ($k > n$)
- Model averaging incorporates the uncertainty in the final model
- Both methods shrink coefficients and improve prediction.

Questions?

Regularization:
Stanford open course on statistical learning (you will learn R at the same time) by Tevor Hastie, R. Tibshirani, and others.

Model Averaging:
Burnham and Anderson. 2004. Multimodel Inference. *Socio. Meth. and Research* 33(2)