STAT 675 Homework # 2

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5.1 Compute a Monte Carlo estimate of $\int_0^{\pi/3} sint tdt$

```
set.seed(36)
m <- 10000
x <- runif(m, min = 0, max = pi/3)
theta.hat <- mean(sin(x)) * (pi/3)
theta <- cos(0) - cos(pi/3)  #could just set this to .5 but shwowing the calc to be clear
theta.hat - theta

## [1] 0.001851

rm(theta.hat, theta, x)
```

Not impressively accurate with 10,000 samples.

5.3 Monte Carlo estimate of $\int_0^{0.5} e^{-x}dx$ with a uniform and exponential. Since the integrand is a pdf we can estimate directly by taking sample, Y, from this pdf and applying the indicator function $I(Y \leq x)$.

```
x <- runif(m, min = 0, max = 0.5)
y <- rexp(m) <= 0.5
theta.hat <- 0.5 * mean(exp(-x))
theta.star <- mean(y)
var.theta.hat <- (0.25/m) * var(exp(-x))
var.theta.star <- (theta.star * (1 - theta.star))/m
r <- matrix(c(theta.hat, theta.star, var.theta.hat, var.theta.star), 2, 2)
rownames(r) <- c("theta.hat", "theta.star")
colnames(r) <- c("Theta", "Var.Theta")
tbl1 <- xtable(r, caption = "The estimates and estimated variances from two estimates of theta.")
digits(tbl1) <- 10
print(tbl1)
```

	Theta	Var.Theta
theta.hat	0.3940505503	0.0000003238
theta.star	0.3926000000	0.0000238465

Table 1: The estimates and estimated variances from two estimates of theta.

```r
rm(x, y, theta.hat, theta.star, var.theta.hat, var.theta.star, r, tbl1)
```

Since the inegrand is close to .5 the variance of the binomial is maximized and the indicator estimate has a large variance.

5.4 Function for cdf of Beta(3,3) $= \int_0^y 30x^2(1-x)^2dx$.

```r
betacdf <- function(x, m) {
    theta <- rep(0, length(x))
    for (i in 1:length(x)) {
        u <- rbeta(m, 3, 3) <= x[i]
        theta[i] <- mean(u)  #(30*u^2*(1-u)^2)*x
    }
    return(theta)
}
x <- seq(0, 1, 0.1)
r <- matrix(c(betacdf(x = x, m = 1e+06), pbeta(x, 3, 3)), 2, 11, byrow = TRUE)
colnames(r) <- x
rownames(r) <- c("My function", "R function")
tbl <- xtable(r, caption = "The estimates from a simple mote-carlo estimate and the R function.")
digits(tbl) <- 4
print(tbl)
```

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
My function	0.0000	0.0085	0.0579	0.1622	0.3179	0.4994	0.6833	0.8367	0.9423	0.9914	1.0000
R function	0.0000	0.0086	0.0579	0.1631	0.3174	0.5000	0.6826	0.8369	0.9421	0.9914	1.0000

Table 2: The estimates from a simple mote-carlo estimate and the R function.

```r
rm(x, r, tbl)
```

My function does a pretty good job.

5.6 Improvement in antithetic estimation of $\int_0^1 e^x dx$.

$$Var\left(\frac{e^u+e^{1-u}}{2}\right) = \frac{1}{4}\left[Var(e^u)+Var(e^{1-u})+2Cov(e^u,e^{1-u})\right]$$
$$= \frac{1}{4}\left[2Var(e^u)+2\left(Ee^ue^{1-u}-Ee^uEe^{1-u}\right)\right]$$
$$Note: Ee^ue^{1-u} = Ee = e$$
$$Also: Ee^{1-u} = \int_0^1 e^{1-u}du = -e^{1-u}\Big|_0^1 = -e^0+e^1 = e-1 = Ee^u$$

Therefore:

$$Var\left(\frac{e^u+e^{1-u}}{2}\right) = \frac{1}{4}\left[2Var(e^u)+2\left(e-(e-1)^2\right)\right]$$
$$= \frac{1}{2}\left[\left(\frac{e^2-1}{2}-(e-1)^2\right)+\left(e-(e-1)^2\right)\right]$$

This is computed below

```r
print(v.anti <- 0.5 * (((exp(1)^2 - 1)/2) - (exp(1) - 1)^2 + (exp(1) - ((exp(1) -
    1)^2))))

## [1] 0.003912
```

From the book the variance of the simple Monte Carlo estimate is 0.1210178 so the variance reduction is:

```r
100 * ((0.1210178 - v.anti)/0.1210178)

## [1] 96.77
```

5.7

```r
anti <- function(m) {
    #function to comput the antithetic estimate
    u <- runif(m/2)
    v <- 1 - u
    theta <- mean((exp(u) + exp(v))/2)
    v <- var((exp(u) + exp(v))/2)
    return(c(theta, v))
}
theta.ant <- anti(m = m)
simple <- function(m) {
    #function to comput the simple estimate
    u <- runif(m/2)
    v <- runif(m/2)
    theta <- mean((exp(u) + exp(v))/2)
    v <- var((exp(u) + exp(v))/2)
    return(c(theta, v))
}
theta.simp <- simple(m = m)
theta.ant[1]  #estimate

## [1] 1.718

theta.simp[1]

## [1] 1.719

theta.ant[2]

## [1] 0.00394

theta.simp[2]

## [1] 0.1202


reduction <- 100 * ((theta.simp[2] - theta.ant[2])/theta.simp[2])  #reduction in the variance of th
reduction

## [1] 96.72
```

5.8 Show that $\rho(X, X') = -1$.

$$\rho(X, X') = \frac{Cov(X, X')}{\sqrt{Var(X)Var(x')}} = \frac{Cov(aU, a(1-U))}{sqrtVar(aU)Var(a(1-U))}$$

$$= \frac{E\left[aU(a(1-U))\right] - E(aU)E(a(1-U))}{\sqrt{a^2Var(U)a^2Var(1-U)}} = \frac{a^2E(U-U^2) - aEU(a-aEU)}{a^2Var(U)}$$

$$= \frac{a^2EU - a^2EU^2 - \left[a^2EU - a^2(EU)^2\right]}{a^2Var(U)} = \frac{EU - EU^2 - EU - (EU)^2}{Var(U)} = \frac{-EU^2 + (EU^2)}{Var(U)}$$

$$= \frac{-1(Var(U))}{Var(U)} = -1$$

The result would hold for any random variable with exisitng first and second moments.

5.14 Obtain a Monte Carlo estimate of $\int_1^{\inf} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx$ by importance sampling.
I will use the pareto as my weighting distribution because it is the only distribution I could find in the back of Casella and Berger that could have the same support.

```
require(VGAM)
p <- rpareto(m, 1, 1)
g <- function(x) ((x^2)/sqrt(2 * pi)) * exp(-x^2/2)
fg <- g(p)/dpareto(p, 1, 1)
theta.hat <- mean(fg)
theta.hat

## [1] 0.4036
```