

Lecture 11 – Probability

DSC 10, Fall 2023

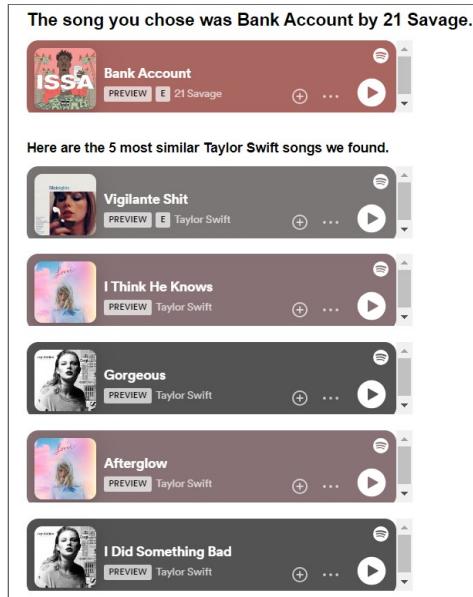
Announcements

- Quiz 2 is on **Wednesday in discussion section.**
 - The quiz covers Lectures 5 through 10.
 - Practice by solving problems from old exams at practice.dsc10.com.
- Lab 3 is due on **Thursday at 11:59PM**.
- Homework 3 is due on **Saturday at 11:59PM**.

hi!

Midterm Project released, due Saturday, 11/4 at 11:59PM

- In the project, you'll explore Taylor Swift's music and lyrics and implement some fun tools. You'll make a song recommender that suggests the Taylor Swift songs that are similar to your favorite song. Here's a sneak peek:



- **Start early!** You should be halfway done with the project by the end of this weekend.

Last time: for -loops

- Almost every for -loop in DSC 10 will use the **accumulator pattern**.
 - This means we initialize a variable, and repeatedly add on to it within a loop.
 - The variable could be an integer, an array, or even a string (as in Homework 3, Question 4: Triton Tweets).
 - Analogy: Start with a blank piece of paper and write something on it each time you run an experiment.
- Do **not** use for -loops to perform mathematical operations on every element of an array or Series.
 - Instead, use DataFrame manipulations and built-in array or Series methods.
- Helpful video : **For Loops (and when not to use them) in DSC 10.**

hi!

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes .

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- [Computational and Inferential Thinking, Chapter 9.5.](#)
- [Theory Meets Data, Chapters 1 and 2.](#)
- [Khan Academy's unit on Probability.](#)

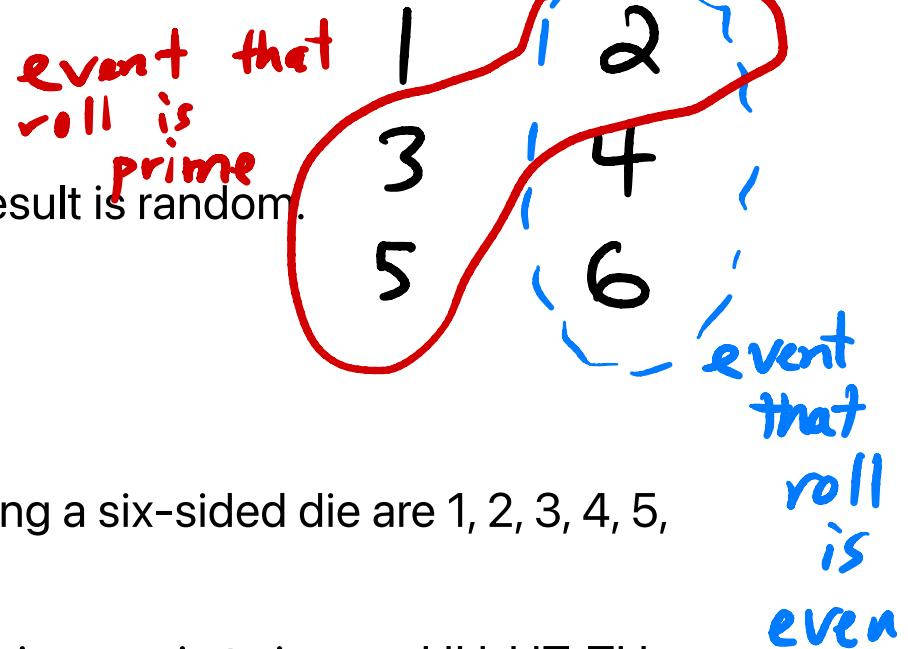
Probability theory

- Some things in life seem random.
 - e.g., flipping a coin or rolling a die .
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Example : Die

Terminology

- **Experiment:** A process or action whose result is random.
 - e.g., rolling a die.
 - e.g., flipping a coin twice.
- **Outcome:** The result of an experiment.
 - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
 - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event:** A set of outcomes.
 - e.g., the event that the die lands on an even number is the set of outcomes {2, 4, 6}.
 - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
 - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.



Terminology

- **Probability:** A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
 - 0: The event never happens.
 - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

$P(\text{roll is prime})$

$P(\text{roll } \geq 2)$

Equally-likely outcomes

WRONG! OH 1H 3H ← not equally likely.

ZH

- If all outcomes in event A are equally likely, then the probability of A is

$$P(A) = \frac{\# \text{ of outcomes satisfying } A}{\text{total } \# \text{ of outcomes}}$$

- Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

8 possible outcomes

$$(2 \cdot 2 \cdot 2)$$

HHH

THT

HHT

THT

HTH

TTH

HTT

TTT

probability = $\frac{3}{8}$

Concept Check  – Answer at cc.dsc10.com

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

Outcomes : 6

RG
RB
GR
GB
BR
BG

$$P(GR) = \frac{1}{6}$$

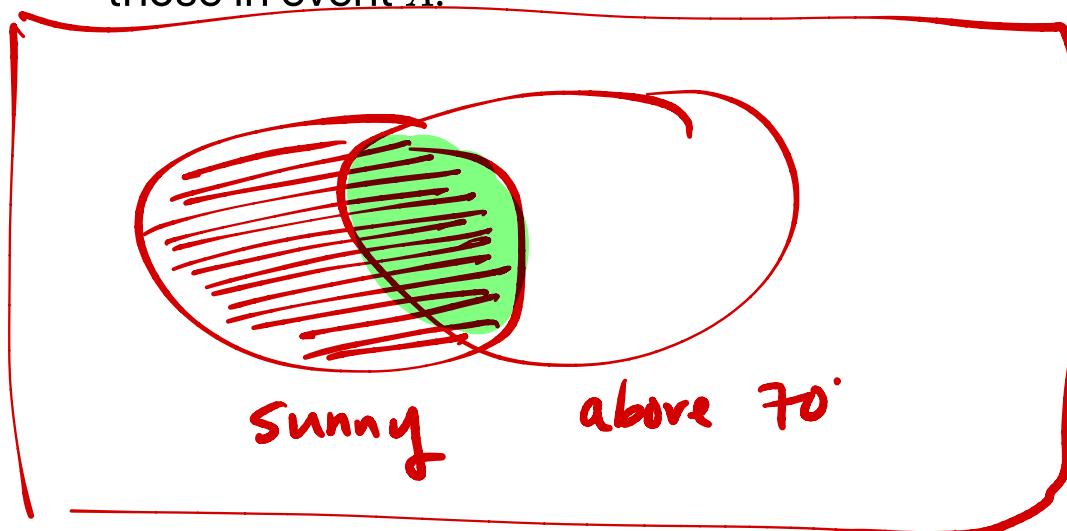
no multiplication
needed!

Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of B ,
 $P(B) = \frac{\# \text{ of outcomes satisfying } B}{\text{total } \# \text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A .



$P(\text{above } 70^\circ \text{ given sunny})$

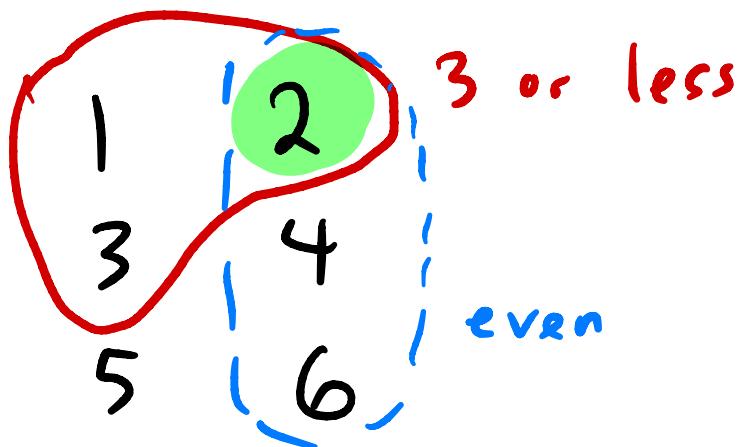
Concept Check ✓ – Answer at cc.dsc10.com

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

$$P(\text{even given 3 or less}) = \frac{1}{3}$$



Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

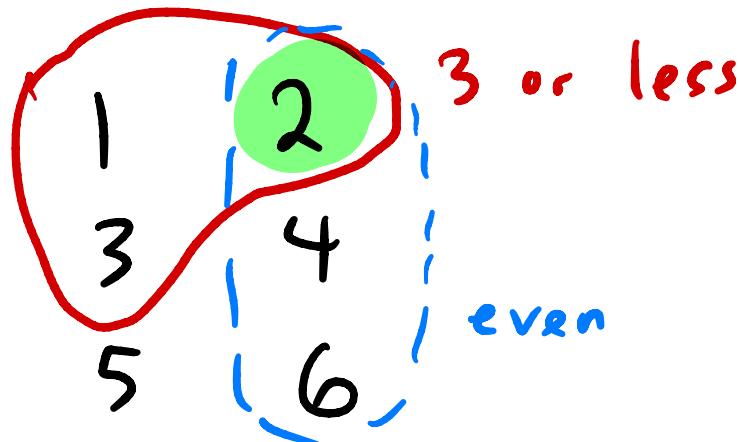
$$P(A \text{ and } B) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \# \text{ of outcomes}}$$

- Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

outcomes 3 or less
AND even

total outcomes

$$P(3 \text{ or less AND even}) =$$



=

$$\frac{1}{6}$$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

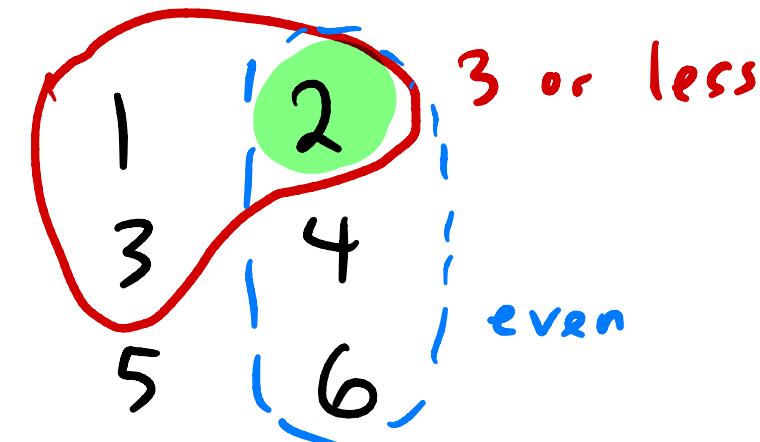
$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

always holds true!

- Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$$P(3 \text{ or less AND even})$$

$$= P(3 \text{ or less}) \cdot P(\text{even} \text{ GIVEN } 3 \text{ or less})$$



$$\begin{aligned} &= \frac{1}{2} \cdot \frac{1}{3} \quad \leftarrow \text{from a few slides ago} \\ &= \boxed{\frac{1}{6}} \end{aligned}$$

What if A isn't affected by B ? 🤔

- The multiplication rule states that, for any two events A and B ,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
 - Suppose we flip a fair coin three times.
 - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

if A and B have nothing to do with each other,

$$P(B \text{ given } A) = P(B)$$

AND → multiply

Independent events

$P(2^{\text{nd}} H)$ = probability that
second flip is
heads

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$
, special case :
not always true!

- Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

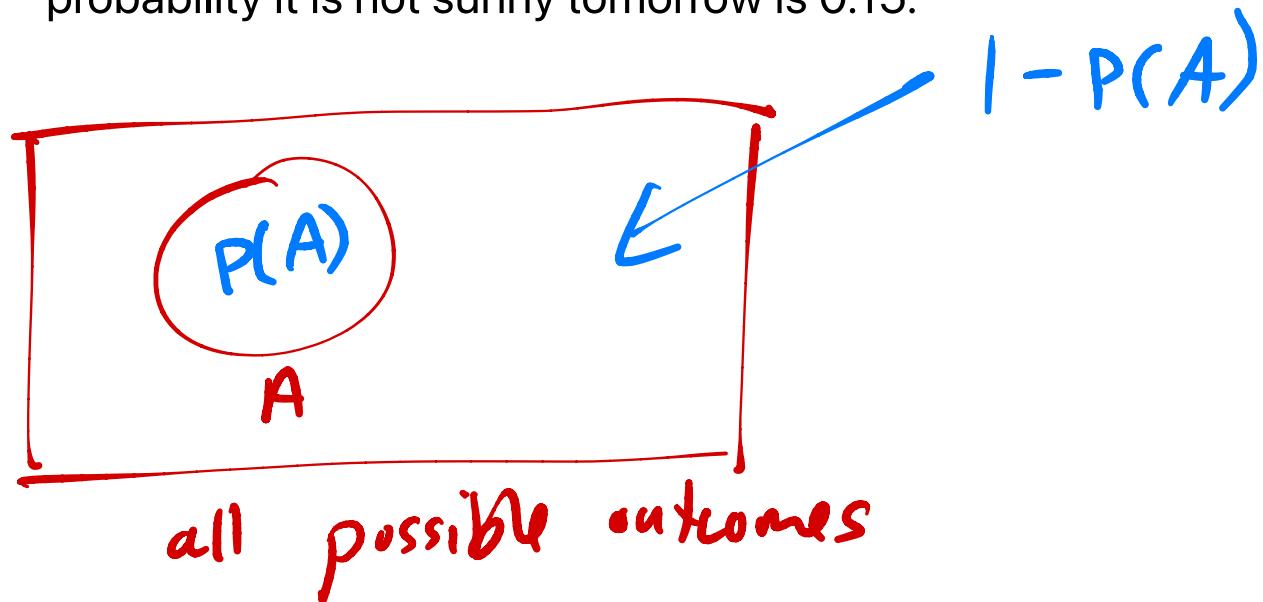
$$\begin{aligned} P(\text{all 5 heads}) &= P(1^{\text{st}} H \text{ AND } 2^{\text{nd}} H \text{ AND } \dots \text{ AND } 5^{\text{th}} H) \\ &= P(1^{\text{st}} H) \cdot P(2^{\text{nd}} H) \cdot \dots \cdot P(5^{\text{th}} H) \\ &= 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 \\ &\quad \boxed{0.7^5} \end{aligned}$$

NOT:
 $0.7 \cdot 5$

"Complement rule"

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.



Concept Check ✓ – Answer at cc.dsc10.com

Every time I call my grandma 🧼, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A) $\frac{1}{3}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- D) ~~1~~
- E) None of the above.

not! $\frac{1}{3} \cdot 3$

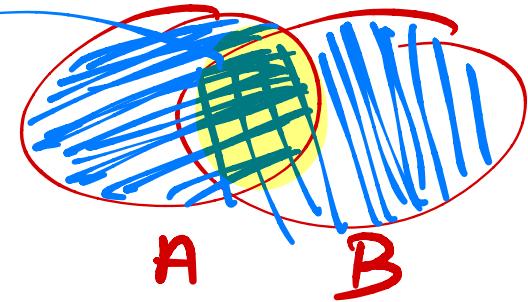
$$P(\text{at least once}) = 1 - P(\text{never})$$

$$= 1 - P(\text{no } 1^{\text{st}} \text{ AND no } 2^{\text{nd}} \text{ AND no } 3^{\text{rd}})$$

$$= 1 - P(\text{no } 1^{\text{st}}) \cdot P(\text{no } 2^{\text{nd}}) \cdot P(\text{no } 3^{\text{rd}})$$

$$= 1 - \left(1 - \frac{1}{3}\right)^3 = 1 - \left(\frac{2}{3}\right)^3 = 1 - \frac{8}{27} = \boxed{\frac{19}{27}}$$

double counted!



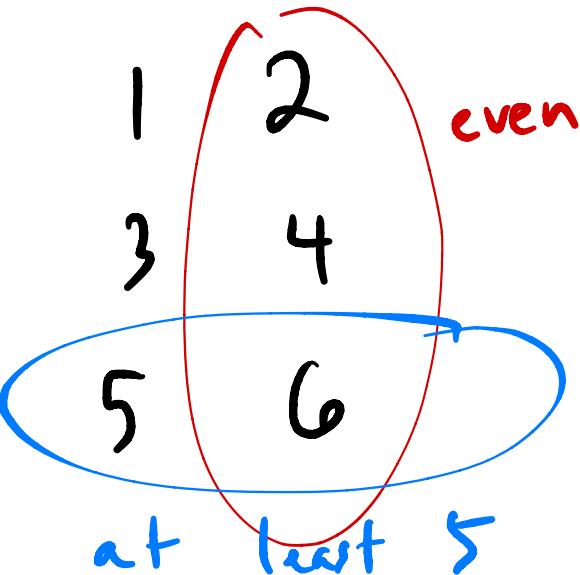
Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

$$P(\text{even or at least 5}) = \frac{4}{6} = \frac{2}{3}$$



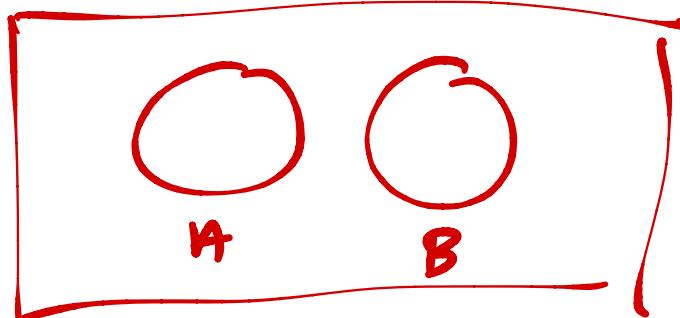
$$P(\text{even}) = \frac{3}{6}$$

$$P(\text{at least 5}) = \frac{2}{6}$$

can't add directly :
the outcome 6 would be double counted!

OR → add

The addition rule



- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
 - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

special case !

- **Example 5:** Suppose I have two biased coins, coin A and coin B . Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$P(A=\text{heads} \text{ AND } B=\text{tails}) \text{ OR } P(A=\text{tails} \text{ AND } B=\text{heads})$

$$= P(A \text{ heads}) \cdot P(B \text{ tails}) + P(A \text{ tails}) \cdot P(B \text{ heads})$$

$$= 0.6 \cdot 0.7 + 0.4 \cdot 0.3 = 0.42 + 0.12 = 0.54$$

$$\uparrow \\ 1 - P(B \text{ heads})$$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned} P(A \text{ or } B) &= \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A) + (\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= P(A) + P(B) \end{aligned}$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
 - The **multiplication rule**, which states that for any two events,
$$P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A).$$
 - The **addition rule**, which states that for any two **mutually exclusive** events,
$$P(A \text{ or } B) = P(A) + P(B).$$
- **Next time:** Simulations.