Problem Set 3: Dynamics
PhD IO, Fall 2022
Due Wed. Dec 21, 2022

This problem set is designed to help you to understand the NFP and CCP algorithm for estimation of single agent dynamic discrete choice model.

1 Model

As in Rust (1987), firms use machines to produce output. Older machines are more likely to break down and in each period the operator has the option of replacing the machine. Let a_t be the age of the machine at time t and the let the current flow payoff, given state a_t, action i_t and shocks $\epsilon_{0t}, \epsilon_{1t}$, be

$$\Pi(a_t, i_t, \varepsilon_{0t}, \epsilon_{1t}) = \begin{cases} R + \varepsilon_{1t} & \text{if } i_t\text{=1} \\ \mu a_t + \varepsilon_{0t} & \text{if } i_t\text{=0.} \end{cases}$$

where $i_t = 1$ if the firm decides to replace the machine at t, R is the cost of the new machine, and $\epsilon_{0t}, \epsilon_{1t}$ are time-specific shocks to the pay-offs from not replacing and replacing respectively.

Assume $\epsilon_{0t}, \epsilon_{1t}$ are iid T1EV errors. Assume a simple law of motion for the age state

$$a_{t+1} = \begin{cases} 1 & \text{if } i_t\text{=1} \\ min\{5, a_t + 1\} & \text{if } i_t\text{=0.} \end{cases}$$

That is, if not replaced, the machine ages by one year up to a maximum of 5 years. After 5 years, the machine's age is fixed at 5 until replacement. If replaced in the current year, the machine's age next year is 1. Note that $i_t = 1$ is a renewal action because $a_{t+1}(i_t = 1, a_t) = a_{t+1}(i_t = 1) = 1$. Also note that the law of motion is deterministic.

2 Exercises

The goal is to estimate $\theta \equiv \{\mu, R\}$

1. Write down the sequence problem for the firm.

2. Write down Bellman's equation for the value function of the firm. - $V(.;\theta) = f(V(.;\theta))$ where $V(.;\theta)$ is a 5×1 vector . Express the value

function in terms of choice-specific conditional value functions $\bar{V}_0(.,\theta)$ and $\bar{V}_1(.,\theta)$.

3. **Contraction Mapping**: Solve this dynamic programming problem through value function iteration given parameter values $(\mu, R) = (-1, -3)$. Assume $\beta = 0.9$. The mapping should iterate on the two choice-specific conditional value functions. Recall that the logit-error assumption implies an analytic solution to the expectation of the max in these equations (the "logsum" formula), and that Euler's constant is approximately 0.5775.

   (a) Suppose $a_t = 2$. For what value of $\epsilon_{0t} - \epsilon_{1t}$ is the firm indifferent between replacing its machine and not? What is the probability (to an econometrician who doesn't observe the ϵ draws) that this firm will replace its machine? What is the value of a firm at state $\{a_t = 4, \epsilon_{0t} = 1, \epsilon_{1t} = 1.5\}$?

4. **Simulate Data**: Generate a dataset of observable states a_t and choices i_t for $T = 20,000$ periods from the model in (3). Note that cross-sectional and longitudinal data points are perfect substitutes here.

5. Use the data from (4) to estimate θ using Rust's NFP approach.

   (a) Start with a guess of θ.

   (b) Solve for the choice-specific conditional value functions $\bar{V}_0(.,\theta)$ and $\bar{V}_1(.,\theta)$ using contraction mapping.

   (c) Plug the $\bar{V}_0(.,\theta)$ and $\bar{V}_1(.,\theta)$ in the logit discrete choice formula to construct likelihood $P(a_t;\theta)$

   (d) Search over θ to maximise the likelihood objective function.

6. **THIS PART IS OPTIONAL** Estimate θ using Hotz and Miller's CCP approach. Below is guidance on how to estimate θ using analytical formulas (covered in the recitation) and forward simulation (covered in the lecture). You can use either.

   (a) Estimate the replacement probabilities at each state $\hat{P}(.)$ using the average replacement rate in the sample. (These estimates are non-parametric estimates of the replacement probabilities.)

   (b) **Analytical Formula** Express $\bar{V}_1(.,\theta) - \bar{V}_0(.,\theta)$ as a function of $P(.;\theta)$, by following the steps below.

i. Write out the expression for $\bar{V}_1(a_t, \theta) - \bar{V}_0(a_t, \theta)$ in terms of differences in current flow utilities and continuation values.

ii. Write down the Arcidiacono-Miller inversion formula for this model. That is, write $V(a_t; \theta) - \bar{V}_1(a_t; \theta)$ as a function of $P(a_t; \theta)$.

iii. Replace the continuation values $V(a_{t+1}(a_t, i_t); \theta)$ in (i) with expressions (in terms of $\bar{V}_1(a_{t+1}(a_t, i_t); \theta)$ and $P(a_{t+1}(a_t, i_t); \theta))$ from the inversion formula.

iv. Express $\bar{V}_1(a_{t+1}(a_t, 1); \theta)$ and $\bar{V}_1(a_{t+1}(a_t, 0); \theta)$ as sums of period $t+1$ flow pay-off and continuation values. Differencing will make continuation value terms drop out because

$$a_{t+2}(a_{t+1}((a_t, 1)), 1) = a_{t+2}(a_{t+1}((a_t, 0)), 1) = 1$$

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Plug-in the estimated replacement probabilities $\hat{P}(.)$ in the analytic formula for $\bar{V}_1(., \theta) - \bar{V}_0(., \theta)$.

(c) **Forward Simulation** Compute $\bar{V}_0\,(.\theta)$ and $\bar{V}_1\,(.\theta)$ given estimated replacement probabilities.

i. Write down the conditional state transition matrices F_0 and F_1 (of 5×5 dimension), which represent the transition probabilities of the state conditional on the $\{0, 1\}$ replacement choice. (Note that the law of motion is deterministic in this case, hence, F_0 and F_1 will be populated by 1s and 0s.)

ii. Using the estimated replacement probabilities and the F_0 and F_1 matrices, calculate the unconditional state transition matrix (of 5×5 dimension). This unconditional matrix accounts for both the probability of replacement and the transition probabilities.

iii. Write a procedure that (forward) simulates $\bar{V}_0\,(.\theta)$ and $\bar{V}_1\,(.\theta)$ using estimated replacement probabilities and unconditional transition probabilities.

(d) Once you have $\bar{V}_1(., \theta) - \bar{V}_0(., \theta)$, proceed as in (5) - c,d. [1]

[1] Note that the likelihood you construct is only a "pseudo"-likelihood in that it is a valid likelihood function only at the true parameter values. See Arcidiacono-Miller for details on the consistency of this estimator.