Quantum Walks

Theory and Application

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Sumário

What is graph theory?

Graphs

Graphs

Mathematical definition

Given a graph G with a set of vertices $V(G)$ and a set of edges $E(G)$:

- Laplacian of a graph:

$$(L)_{ij} = \begin{cases} -1 & \text{if } (i,j) \in E(G), \\ d_i & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}; \tag{1}$$

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- Adjacency matrix of a graph:

$$(A)_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E(G), \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

Continuous-time quantum walks

Continuous-time quantum walks (CTQW)

- Evolution in time of a classical random walk

$$\frac{d}{dt} p_a = \gamma \sum_k L_{jk} p_b, \tag{3}$$

where p_k is the probability of finding the walker in the vertex k.

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- Is it ok if we use $H = A$?.

Nature and graphs

Theorem[1]

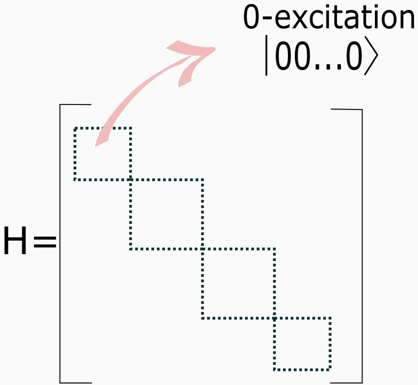
The XY-Hamiltonian can be defined as

$$H_{xy} = \sum_{ij \in E(G)} X_i X_j + Y_i Y_j, \tag{5}$$

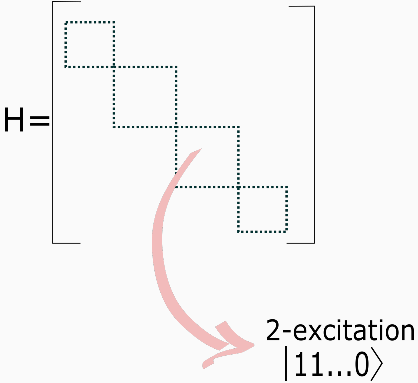
where X, Y are Pauli matrices. H_{xy} is a block matrix where each block is a k-excitation space. Additionally, the 1-excitation space is equal to the adjacency matrix, A, of the graph G.

[1] Godsil and Coutinho, Graph Spectra and Continuous Quantum Walks

Nature and graphs

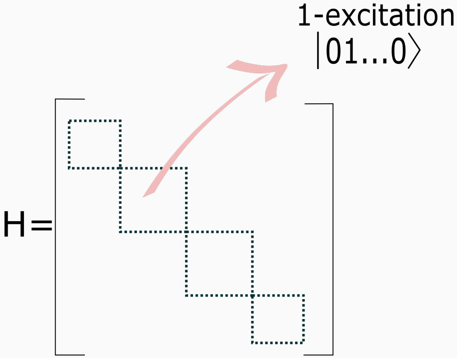


$$H\left|00\ldots0\right\rangle = \left|00\ldots0\right\rangle \tag{6}$$



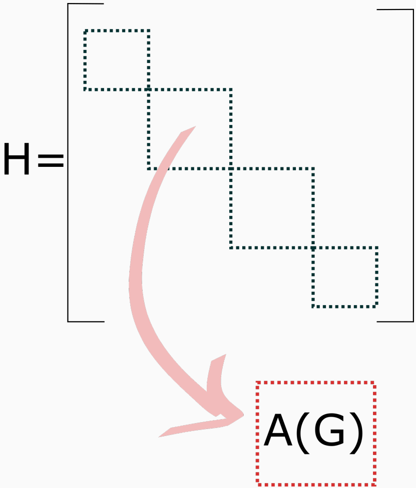
$$H\,|x_2\rangle = \sum_{x_2'} \alpha_{x_2'}\,|x_2'\rangle \tag{7}$$

Nature and graphs



$$H\,|x_1\rangle = \sum_{x_1'} \alpha_{x_1'}\,|x_1'\rangle \tag{8}$$

Nature and graphs



Change of basis

- From a Hilbert space of dimension 2^n;

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- To one of its subspaces of dimension n;

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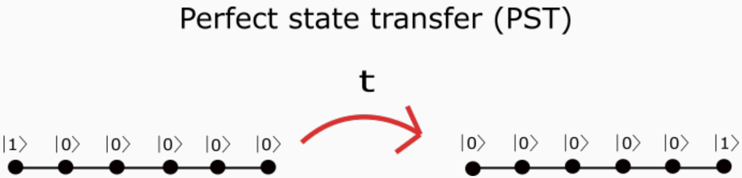
- Define the characteristic vector of vertex a as $|a\rangle$ and the quantum walk as

$$e^{-itA}|a\rangle = \sum_{k \in V(G)} \alpha_k |k\rangle . \tag{9}$$

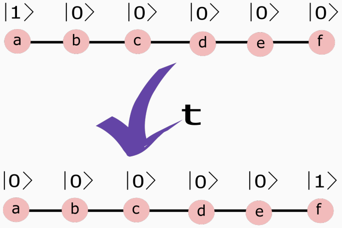
State transfer[2]



[2] Phys. Rev. Lett. 92, 187902 (2004)

PST

$$U(t)|a\rangle=\beta|f\rangle$$



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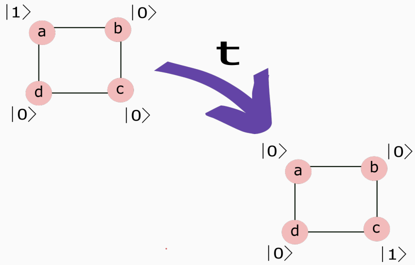

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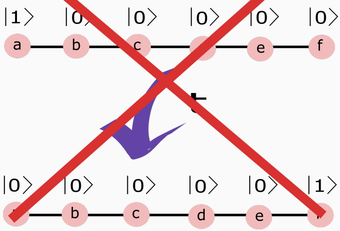

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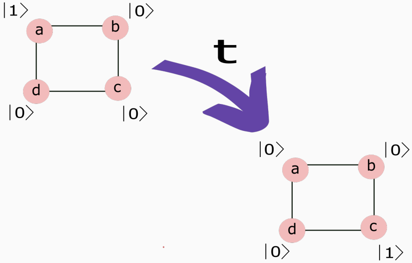
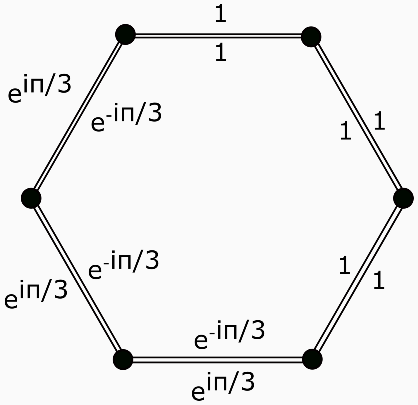


What is the difference?

Zero transfer in even cycles



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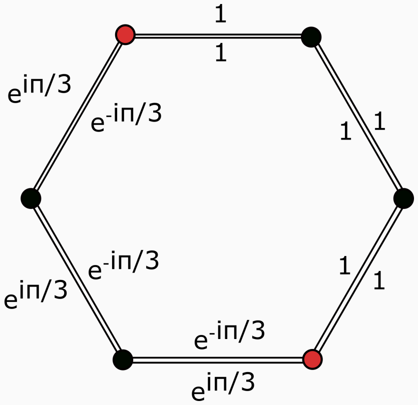

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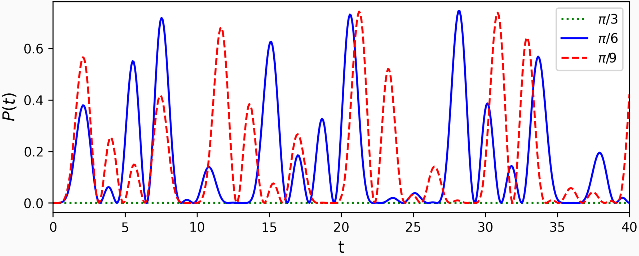

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- Graph theory gives good explanation on why it happens;
- Not applicable to QC :(

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- Time is discrete and evolves in steps;

[3] Portugal, Quantum Walks and search algorithms

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- Tensor product of two Hilberts spaces, $\mathcal{H}_c \otimes \mathcal{H}_V$;

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- Shift operator (S):

$$S = |0\rangle\langle 0| \otimes \sum_{x \in V(G)} |x-1\rangle\langle x| + |1\rangle\langle 1| \otimes \sum_{x \in V(G)} |x+1\rangle\langle x| \tag{11}$$

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- Coin operator (C) for us is $RX(\theta)$;
- Usually associated to search algorithms due to faster hitting time[3].

[3] Portugal, Quantum Walks and search algorithms

Dirac Cellular Automata (DCA)[4]

- Cellular automata: a tool for computation with discrete time and space;

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- Cellular automata: a tool for computation with discrete time and space;
- DCA time evolution for spin-1/2 particle based on Dirac's equation:

$$U_{DCA} = \alpha\big(|0\rangle\langle 0| \otimes T_- + |1\rangle\langle 1| \otimes T_+ \big) - i\beta\big(X \otimes I\big), \tag{12}$$

where $T_\pm = \sum_{x\in V(G)} |x \pm 1\rangle\langle x|$;

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- If we define a DWQ with operators given by

$$U_{DW} = S_+ C(\theta_2) S_- C(\theta_1), \tag{13}$$

where $S_+ = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1|\, T_+$ and $S_- = |0\rangle\langle 0| \otimes T_- + |1\rangle\langle 1| \otimes I$;

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- If we set $\theta_1 = 0$, then

$$U_{DQW} = \cos\theta_2\big(|0\rangle\langle 0| \otimes T_- + |1\rangle\langle 1| \otimes T_+\big) - i\sin\theta_2\big(X \otimes I\big). \qquad (14)$$

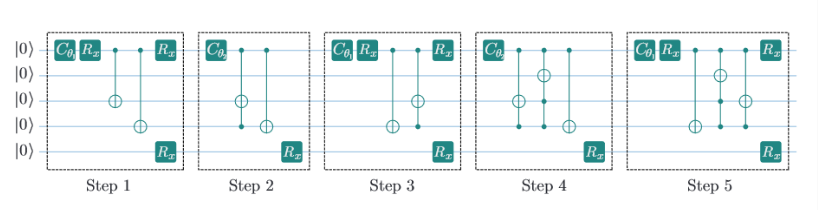
[4] arXiv:2002.02537

DCA for a path graph

$\vert x=0\rangle \equiv \vert 0000\rangle$	
$\vert x=1\rangle \equiv \vert 0001\rangle$	$\vert x=-1\rangle \equiv \vert 0111\rangle$
$\vert x=2\rangle \equiv \vert 0010\rangle$	$\vert x=-2\rangle \equiv \vert 0110\rangle$
$\vert x=3\rangle \equiv \vert 0011\rangle$	$\vert x=-3\rangle \equiv \vert 0101\rangle$
$\vert x=4\rangle \equiv \vert 1100\rangle$	$\vert x=-4\rangle \equiv \vert 0100\rangle$
$\vert x=5\rangle \equiv \vert 1101\rangle$	$\vert x=-5\rangle \equiv \vert 1011\rangle$
$\vert x=6\rangle \equiv \vert 1110\rangle$	$\vert x=-6\rangle \equiv \vert 1010\rangle$
$\vert x=7\rangle \equiv \vert 1111\rangle$	$\vert x=-7\rangle \equiv \vert 1001\rangle$

DCA for a path graph

Proposed circuit by the authors:



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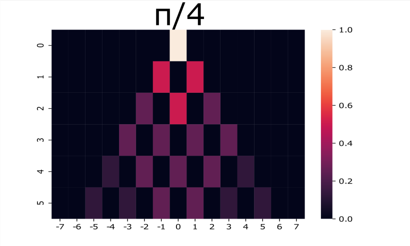

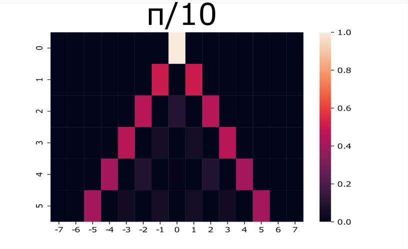

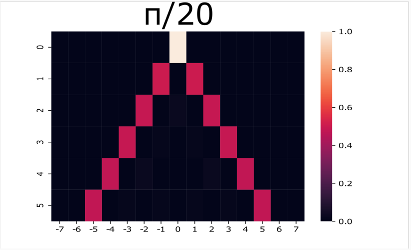

End

Thank you!