Let us consider the following, reaction-diffusion, partial differential equation in 2 dimensions,

$$\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi + \rho - \kappa\varphi,$$

where φ indicates the concentration of some chemical, ρ denotes a source term (which depends on space), and κ is a positive constant, determining the degradation rate for that chemical, while D is the diffusion coefficient.

1. Write down a suitable finite difference algorithm to integrate the equation above subject to a suitable initial condition, and periodic boundary conditions in space. **[5]**

2. Based on this algorithm, write a Python code to solve this equation equation on a 50×50 grid, subject to the initial condition that $\varphi = 0.5 + \eta$, with η a spatially dependent random noise uniformly distributed between -0.1 and 0.1. The source term ρ is given by

$$\rho = \exp\left(-\frac{\mathbf{r}^2}{\sigma^2}\right),$$

   with σ a positive constant determining the size of the source term, and $\mathbf{r}$ the position vector linking any point of the grid to the grid centre. It should be possible to set the values of κ and σ at run time. Your code should display the density field φ in real time, as it is running. Here and in what follows you can set $D = 1$; you can further set the spatial discretisation $\Delta x = 1$ (the time step Δt has then to be small enough for the algorithm to converge). **[20]**

3. Use your code to study the behaviour of a system with $\sigma = 10$, and $\kappa = 0.01$. Show/sketch a few snapshots of the system over time, discussing the behaviour you see. Plot the value of the average value of φ over the simulation domain over time, estimating when the system reaches a steady state (i.e., when it stops changing over time). **[8]**

4. For the same system, plot the value of φ as a function of $r = |\mathbf{r}|$ at steady state. You should determine whether this function is best fitted by a power law or an exponential, giving values of the parameters which provide the best fit. **[7]**

5. Now consider the case whether the field is advected by a spatially dependent velocity $\mathbf{v}$,

$$\frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla\varphi = D\nabla^2\varphi + \rho - \kappa\varphi,$$

   where $\mathbf{v} = (v_x, 0)$, and v_x only depends on the y coordinate as follows,

$$v_x(y) = -v_0 \sin(2\pi y/N),$$

   where $y = 0, \ldots, N-1$ (and $N = 50$). Modify your code so as to solve this equation by using a suitable finite difference algorithm. **[5]**

6. Find the 2D solution for $v_0 = 0.01$, $v_0 = 0.1$ and $v_0 = 0.5$ in steady state. Discuss the physical meaning of the shape of the contour plots for φ you have found. **[5]**