Question 4 (20)

Assume $b \neq 0$. Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$$

by producing an orthogonal matrix Q and a diagonal matrix D s.t.
$D = Q^\top AQ$

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} a-\lambda & 0 & b \\ 0 & a-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} \\
&= (a-\lambda)\begin{vmatrix} a-\lambda & 0 \\ 0 & a-\lambda \end{vmatrix} + b\begin{vmatrix} 0 & a-\lambda \\ b & 0 \end{vmatrix} \\
&= (a-\lambda)(a-\lambda)^2 + b(-b(a-\lambda)) \\
&= (a-\lambda)^3 - b^2(a-\lambda) = (a-\lambda)((a-\lambda)^2 - b^2) \\
&= (a-\lambda)(a-\lambda+b)(a-\lambda-b) \\
&= (a-\lambda)(\lambda-(a+b))(\lambda-(a-b)) \\
\lambda &= a, (a+b), (a-b) \\
E_a &= \left(\begin{array}{ccc|c} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} b & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
\vec{x} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t \Rightarrow E_a = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \\
E_{(a+b)} &= \left(\begin{array}{ccc|c} a-(a+b) & 0 & b & 0 \\ 0 & a-(a+b) & 0 & 0 \\ b & 0 & a-(a+b) & 0 \end{array}\right) \\
&= \left(\begin{array}{ccc|c} -b & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ b & 0 & -b & 0 \end{array}\right) \xrightarrow{R_3 + R_1} \left(\begin{array}{ccc|c} -b & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
&\xrightarrow[-R_2]{-R_1} \left(\begin{array}{ccc|c} b & 0 & -b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
\end{aligned}$$

$$bx_1 - bx_3 = 0$$
$$bx_2 = 0$$
$$\Rightarrow \ \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t \Rightarrow E_{(a+b)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$E_{(a-b)} = \begin{pmatrix} a-(a-b) & 0 & b & | & 0 \\ 0 & a-(a-b) & 0 & | & 0 \\ b & 0 & a-(a-b) & | & 0 \end{pmatrix}$$
$$= \begin{pmatrix} b & 0 & b & | & 0 \\ 0 & b & 0 & | & 0 \\ b & 0 & b & | & 0 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} b & 0 & b & | & 0 \\ 0 & b & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$
$$bx_1 + bx_3 = 0$$
$$bx_2 = 0$$
$$\Rightarrow \ \vec{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t \Rightarrow E_{(a-b)} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} a & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & a-b \end{pmatrix}$$
$$\vec{u}_1 = \frac{1}{\|P_1\|} P_1 = \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\vec{u}_2 = \frac{1}{\|P_2\|} P_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$
$$\vec{u}_3 = \frac{1}{\|P_3\|} P_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & a-b \end{pmatrix}$$