Question 3 (15)

Let $\mathcal{P}_2(\mathbb{R})$ be as it was in problem 1 and define the following,

$$\langle f(x), g(x)\rangle = f_0g_0 + f_1g_1 + f_2g_2$$

where $f(x) = f_0 + f_1x + f_2x^2$ (similar for g)

a) Verify that $\mathcal{P}_2(\mathbb{R})$ with this product is an inner product space

$$\begin{aligned}
\langle f, g\rangle &= f_0g_0 + f_1g_1 + f_2g_2 \\
\langle g, f\rangle &= g_0f_0 + g_1f_1 + g_2f_2 = f_0g_0 + f_1g_1 + f_2g_2 \\
\therefore \langle f, g\rangle &= \langle g, f\rangle
\end{aligned}$$

Let $h(x) = h_0 + h_1x + h_2x^2$

$$\begin{aligned}
g + h &= g_0 + g_1x + g_2x^2 + h_0 + h_1x + h_2x^2 \\
&= (g_0 + h_0) + (g_1 + h_1)x + (g_2 + h_2)x^2 \\
\langle f, g + h\rangle &= f_0(g_0 + h_0) + f_1(g_1 + h_1) + f_2(g_2 + h_2) \\
\langle f, g\rangle &= f_0g_0 + f_1g_1 + f_2g_2 \\
\langle f, h\rangle &= f_0h_0 + f_1h_1 + f_2h_2 \\
\langle f, g\rangle + \langle f, h\rangle &= f_0(g_0 + h_0) + f_1(g_1 + h_1) + f_2(g_2 + h_2) \\
\therefore \langle f, g + h\rangle &= \langle f, g\rangle + \langle f, h\rangle
\end{aligned}$$

Suppose $c \in \mathbb{R}$ s.t. $cf(x) = cf_0 + cf_1x + cf_2x^2$

$$\begin{aligned}
\langle cf, g\rangle &= cf_0g_0 + cf_1g_1 + cf_2g_2 = c(f_0g_0 + f_1g_1 + f_2g_2) \\
&= c\langle f, g\rangle
\end{aligned}$$

Since $f_if_i \geq 0 \quad \forall f_i \in \mathbb{R}$ and $f_if_i = 0$ iff $f_i = 0$,

$$\langle f, f\rangle \geq 0 \text{ and } \langle f, f\rangle = 0 \iff f = 0 + 0x + 0x^2 = \mathbf{0}(x)$$

Given that $\mathcal{P}_2(\mathbb{R})$ is a vector space and since the properties hold, $\mathcal{P}_2(\mathbb{R})$ with this product forms an inner product space

b) In this space what is the length/norm of the "vector" $f(x) = 3+2x+4x^2$

$$\begin{aligned}
\|f\| &= \sqrt{\langle f, f\rangle} = \sqrt{3\cdot 3 + 2\cdot 2 + 4\cdot 4} \\
&= \sqrt{9 + 4 + 16} = \sqrt{29}
\end{aligned}$$

c) In this space what is the distance between $f(x) = 4 - 5x + 2x^2$ and $g(x) = -1 + 8x + 7x^2$

$$\begin{aligned} d = \|f - g\| &= \|4 - 5x + 2x^2 - (-1 + 8x + 7x^2)\| \\ &= \|4 - 5x + 2x^2 + 1 - 8x - 7x^2\| \\ &= \|5 - 13x - 5x^2\| \\ &= \sqrt{5 \cdot 5 + (-13)(-13) + (-5)(-5)} \\ &= \sqrt{25 + 169 + 25} = \sqrt{219} \end{aligned}$$

d) In this space find TWO different "vectors" which are orthogonal to $f(x) = 1 + x + x^2$

Since g orthogonal to f iff $\langle f, g\rangle = 0$,

$$\langle f, g\rangle = 1g_0 + 1g_1 + 1g_2 = 0$$

Let $g_0 = 0, g_1 = 1, g_2 = -1$ s.t. $g = 0 + x - x^2$

$$\begin{aligned} \therefore\ \langle f, g\rangle &= 1 \cdot 0 + 1 \cdot 1 + 1 \cdot (-1) \\ &= 1 - 1 = 0 \Rightarrow 0 + x - x^2 \text{ is orthogonal to } f(x) \end{aligned}$$

Let $g_0 = 1, g_1 = -1, g_2 = 0$ s.t. $g = 1 - x + 0x^2$

$$\begin{aligned} \therefore\ \langle f, g\rangle &= 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 0 \\ &= 1 - 1 = 0 \Rightarrow 1 - x + 0x^2 \text{ is orthogonal to } f(x) \end{aligned}$$