<u>MATH 225 – Fall 2023– Section B1</u>

Review questions (not to be turned in)
Week 1-2 to be discussed by TA on Sep. 7 & 14, 2023

Please note that the following questions do not need to be turned in. They are for your own practice, so please attempt them. They represent a review of what you need from a first course in linear algebra. Solutions will be posted on eClass, and also the material will be reviewed by the TA on Sep. 7 and Sep. 14 during his office hours.

1. Which statement is true?
   - ◯ Every vector space has at most one basis
   - ◯ If $\mathcal{B} = \{w_1, w_2, w_3\}$ is a basis of $\mathbb{R}^3$, then every vector $v \in \mathbb{R}^2$ can be written uniquely as a linear combination of w_1, w_2 and w_3
   - ◯ A basis for every subspace of $\mathbb{R}^n$ must have n-vectors
   - ◯ If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and w_1, w_2 form a basis of $\mathbb{R}^2$, then $\{T(w_1), T(w_2)\}$ form a basis of $\mathbb{R}^2$ as well.

2. Let A be an 3×3 matrix whose whose characteristic polynomial is $p(\lambda) = -(\lambda - 1)(\lambda - 2)^2$. Which of the following statements can now be deduced?
   - ◯ A can be diagonalized
   - ◯ A is invertible
   - ◯ The nullity of A is at least 1
   - ◯ The diagonal entries of A are either 1 or 2

3. Let A is a 2×3 matrix.
   - ◯ The dimension of $\text{row}(A)$ must be less than the dimension of $\text{col}(A)$
   - ◯ The dimension of $\text{row}(A)$ could be greater than the dimension of $\text{col}(A)$
   - ◯ A must have a null space
   - ◯ None of the above statements are true.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Which of the following is true?
   - ◯ T could be an invertible linear transformation
   - ◯ There is a vector $v \in \mathbb{R}^3$ other than the zero vector such that $T(v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
   - ◯ Every vector in $\mathbb{R}^2$ must be in the image of T
   - ◯ We don't have enough information to decide if any of the above choices is always true.

5. Suppose A is a square matrix such that $A^2 = A$. Then which of the following is true
   - ◯ A is symmetric
   - ◯ A has eigenvalues $0, -1$ and 1
   - ◯ A is invertible
   - ◯ The linear system $Av = v$ has an infinite number of solutions

6. Suppose A is a 3×3 matrix that has the following vectors as eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 2 \\ 4a \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}$$

Find a value of a for which the matrix A is **diagonalizable**. Try to justify your answer.

7. Suppose A is a 2×3 matrix which has null space consisting of all vectors of the form

$$\begin{pmatrix} -3t \\ -2t \\ t \end{pmatrix}$$

What does the RREF of A look like?

8. Let $\vec{n} := (1,2,3) \in \mathbb{R}^3$ and consider the set of all vectors perpendicular to $\vec{n}$, i.e.

$$W = \{\vec{v} = (x,y,z) \in \mathbb{R}^3 \mid \vec{n} \cdot \vec{v} = 0\}.$$

Is W a subspace of $\mathbb{R}^3$? If so, verify the definition of a subspace is satisfied. If not, produce a counterexample.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which is the composition $T = U \circ V$ where U is reflection about the yz-plane, and V is reflection about the xy-plane. Write down the standard matrix $[T]$ of this transformation.

   (Hint: draw a picture first)

10. Assume that the following determinant,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Compute the determinant

$$\begin{vmatrix} 0 & d & e & f \\ 1 & d-3 & e-4 & f-5 \\ 0 & 2a & 2b & 2c \\ 0 & g+d & h+e & i+f \end{vmatrix}$$

11. Let A be the matrix

$$\begin{pmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 2 & 3 \\ 4 & -8 & 3 & 2 \end{pmatrix}$$

   (a) Compute the RREF of A and use this to find a basis for $\mathrm{row}(A)$
   (b) Does the vector $(5,-10,3,7)$ lie in $\mathrm{row}(A)$? If so, write it explicitly as a linear combination of the basis you found in the previous part.
   (c) Write down a basis for $\mathrm{col}(A)$
   (d) Write down a basis for $\mathrm{null}(A)$.

12. This question is concerned with the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Compute the characteristic polynomial of A and write down all eigenvalues of A with their algebraic multiplicities. For each eigenvalue, find the geometric multiplicity and write down a **basis** of the eigenspace.

(b) Compute A^8 using the previous parts. (If you try to compute this directly by multiplying out the matrix by itself 8 times, you will get no credit for this part of the problem. Also you may want to use the fact that $2^8 = 256$.)