Math 225 lecture 8 Sept 22nd 2023

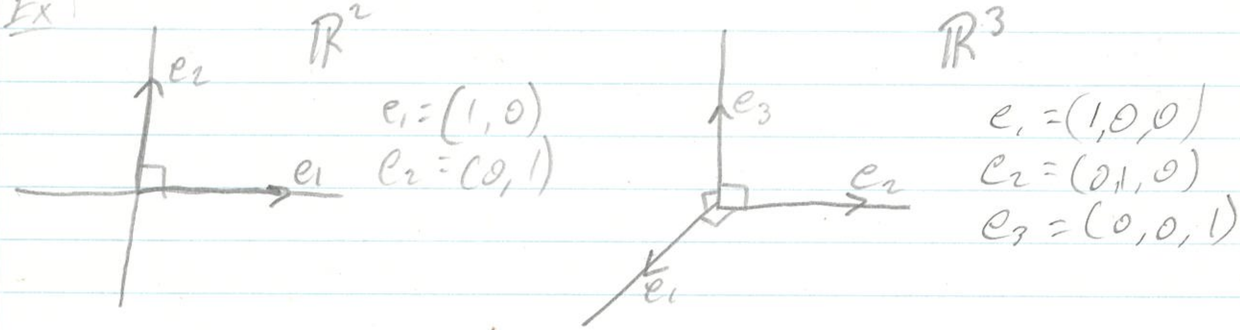
Goal: students should understand the basic concepts behind orthogonality. Furthermore students should be able to verify when a set of vectors are orthogonal.

Class Q: How can you check when vectors are $\perp$ to one another?

section 5.1

Perpendicular & orthogonal are the same

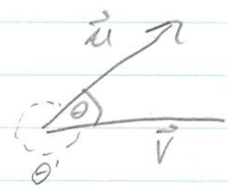
Ex 1



Recall: if $\vec{u}, \vec{v} \in \mathbb{R}^2$ (or $\mathbb{R}^n$ more generally) then

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$ but geometrically this is interpreted as

$\vec{u} \cdot \vec{v} = |u|\,|v| \cos\theta$



(note it doesn't matter which θ as $\cos(2\pi - \theta) = \cos\theta$)

You may recognize this as $\vec{u}$ & $\vec{v}$ are are orthogonal iff $\vec{u} \cdot \vec{v} = 0$

these are equivalent because if $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta = 0$ then $\cos(\theta) = 0$ (as $\|\vec{u}\|\|\vec{v}\| > 0$ if $\vec{u} \neq 0$ and $\vec{v} \neq 0$)

$\Rightarrow \theta = \pi/2$

it turns out this works in general not just $\mathbb{R}^2$

For a proof see office hours

↳ We say a set of vectors in $\mathbb{R}^n$ $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is orthogonal if $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$ i.e. when the vectors are pairwise orthogonal or any two are orthogonal

Example $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\} \subset \mathbb{R}^3$ $\{\vec{e}_1, \vec{e}_2\} \subset \mathbb{R}^2$

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \subset \mathbb{R}^3$ (& so on)

↳ Moreover such a set of vectors is said to be orthonormal if in addition to being orthogonal they are "normal" (have length 1 $|\vec{v}_i| = 1$)

Example $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \subset \mathbb{R}^3$ $\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\} \subset \mathbb{R}^2$

verify $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \cos\theta(-\sin\theta) + \sin\theta\cos\theta = 0$

$$\left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \sqrt{1} = 1$$

$$\left\| \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = \sqrt{(\sin\theta)^2 + (\cos\theta)^2} = \sqrt{1} = 1$$

<u>Lemma</u>: any set of n different orthogonal vectors in $\mathbb{R}^n$ forms a basis

Proof: for $n=3$ (exact same process for arbitrary n)

let $\vec{V}_1, \vec{V}_2, \vec{V}_3$ be orthogonal vectors & suppose

$$c_1\vec{V}_1 + c_2\vec{V}_2 + c_3\vec{V}_3 = \vec{0}$$ lets dot with V_1

$$c_1\vec{V}_1\cdot\vec{V}_1 + c_2\underbrace{\vec{V}_2\cdot\vec{V}_1}_{=\vec{0}} + c_3\underbrace{\vec{V}_3\cdot\vec{V}_1}_{=\vec{0}} = \underbrace{\vec{0}\cdot\vec{V}_1}_{=\vec{0}}$$

$$\Rightarrow c_1|V_1|^2 = 0 \quad \Rightarrow c_1 = 0$$

if you repeat this process with $\vec{V}_2$ & $\vec{V}_3$ you get that $c_2 = c_3 = 0$. this gives us 3 linearly independent vectors in $\mathbb{R}^3$ which means (as $\dim(\mathbb{R}^3)=3$) they form a basis!

why do we care? orthogonal things are nice!

Thm: let $\{\vec{V}_1, \dots, \vec{V}_n\}$ be an orthogonal basis for $\mathbb{R}^n$ and $\vec{u}\in\mathbb{R}^n$ then the unique scalars $c_1, \dots, c_n$ s.t.

$$\vec{u} = c_1\vec{V}_1 + \dots + c_n\vec{V}_n \text{ are given by } c_i = \frac{\vec{u}\cdot\vec{V}_i}{\vec{V}_i\cdot\vec{V}_i} \quad \forall i$$

Proof: by assumption $\vec{u} = c_1\vec{V}_1 + \dots + c_n\vec{V}_n$ since the $\vec{V}_i$'s form a basis. to get the formula we will do the same trick & look at

$$\vec{u}\cdot\vec{V}_i = (c_1\vec{V}_1 + \dots + c_n\vec{V}_n)\cdot V_i$$

$$= c_1\vec{V}_1\cdot\vec{V}_i + \dots + c_n\vec{V}_n\cdot\vec{V}_i$$ by orthogonality

$$= c_i(V_i\cdot V_i) \quad \text{since } V_i\cdot V_j = 0 \text{ unless } i=j$$

since $V_i \neq 0$ $V_i\cdot V_i \neq 0$ & we can divide by it to get $c_i = \dfrac{\vec{u}\cdot\vec{V}_i}{\vec{V}_i\cdot\vec{V}_i}$

this idea leads to projections (mondays class)

Naturally one might then consider what we can do with orthonormal vectors? put them in a matrix! when we do we get some remarkable properties.

↳ an $n \times n$ matrix A whose columns form an orthonormal set is called an <u>orthogonal matrix</u>

<u>Properties:</u>

- A is orthogonal iff $A^{-1} = A^T$
- $\|A\vec{x}\| = \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{R}^n$
- $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$
- A^{-1} is orthogonal
- $\det(A) = \pm 1$
- if λ is an e val of A then $|\lambda| = 1$
- if A & B are orthogonal then so too is AB

<u>Proof:</u> (of #1) in this context showing that $A^{-1} = A^T$ is equivalent to $A^TA = I_n$ (as A is invertible & inverses are unique)

Consider A^TA if we want it to be the identity that is showing that $(A^TA)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$

let $\vec{a}_i$ denote the columns of A and hence the rows of A^T by the definition of matrix multiplication

$$A^TA = \begin{bmatrix} - \vec{a}_1 - \\ \vdots \\ - \vec{a}_n - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} = \left[a_i \cdot a_j \right]_{ij}$$

however the $\vec{a}_i$'s form an orthonormal set by assumption.
thus $\vec{a}_i \cdot \vec{a}_j = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$ which is as we showed also equal to $[A^TA]_{ij}$ thus completing the proof

$\blacksquare$