Math 225 lecture 31 Nov 27^{th} 2023

Goal: students should grasp the fundamental concepts of graph theory, specifically definitions of, path, graph, subgraph connectedness, tree, and adjacency matrix.

Class Q: What can always be done to an adjacency matrix?

Disclaimer: all of this week is considered an application and outside the standard scope of the course. As such NEXT week I will be extremely specific about what will and will not be on the exam and to what extent.

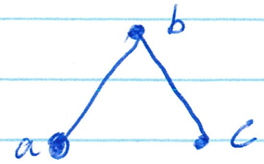
Today Graph theory basics, definitions and related objects.
Wednesday properties & examples small results.
Friday big picture and main theorem

lots of formal definitions today followed by real/moral definitions for clarification after

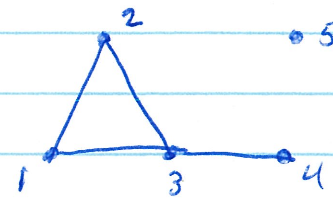
↳ a Graph is a pair (V, E) where V is a non-empty set and E is a set of ordered pairs of elements from V.
we call the elements of V the vertices (points, nodes, etc.) of the graph. and the pairs in E edges (arcs, arrows, directed arrows etc.)

we ~~denote~~ denote $V(G)$ and $E(G)$ as the vertex and edge sets of the graph G.

Ex: $V = \{a, b, c\}$
$E = \{(a,b), (b,c)\}$

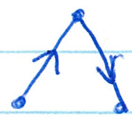


$V = \{1, 2, 3, 4, 5\}$
$E = \{(1,2), (2,3), (3,1), (3,4)\}$



★ Notable exclusions / info for this class.

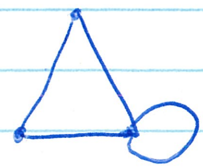
NO directed edges



vs

(this is a digraph)

NO loops
if $(a,b) \in E$
then $a \neq b$



← none of this

NO multi edges

← none of this

(this is a multi graph)

★

a path / walk is a sequence $v_1 e_1 v_2 e_2 \dots v_n e_n v_{n+1}$
where the $v_i \in V$ and $e_i \in E$ s.t. $e_i = (v_i, v_{i+1})$

moraly this is walking along a graph moving from one vertex to another

we call a path simple if each vertex we visit is unique.

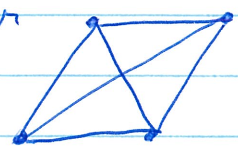
a path is <u>closed</u> if $v_1 = v_{n+1}$ i.e. you finish where you start

and a path which has all unique edges and vertices which is closed is called a <u>cycle</u>

<u>additional terminology</u> we say a graph G is <u>connected</u> if for any two vertices $a, b \in V(G)$ $\exists$ a path from a to b in G

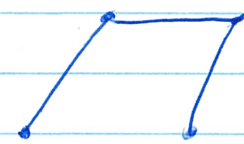
<u>Ex!</u>

G



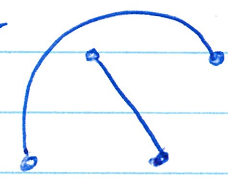
connected

H



connected

J

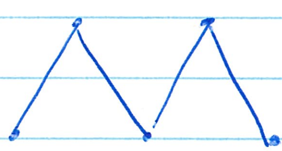


not connected

a <u>tree</u> is a connected graph with no cycles

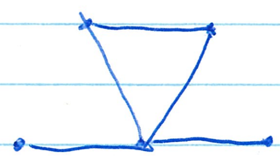
<u>Ex</u>

G



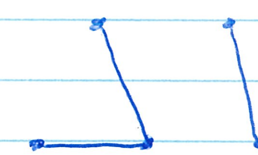
Tree

H



no tree

J



no tree

lastly we have the notion of subgraph. this is a graph $H \subset G$ s.t. $V(H) \subseteq V(G)$ $E(H) \subseteq E(G)$ s.t. H still forms a graph. i.e. if v_i appears in $E(H)$ then $v_i \in V(H)$. Moreover we call the subgraph proper if $H \neq G$

Now that we have a new interesting object what are we contractually obligated to do? PUT IT IN A MATRIX!

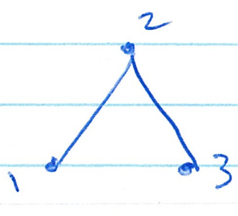
how do we do that? the adjacentcy matrix

↳ let G be a graph with its vertex set labled $1-n$ then the <u>adjacentcy matrix</u> of G denoted A_G or $A(G)$ is the $n \times n$ matrix given by

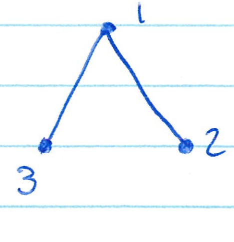
$$[a_{ij}] = \begin{cases} 1 & \text{if } (i,j) \in E(G) \\ 0 & \text{else} \end{cases}$$

(Note: this definition is not very general & needs more work for multigraphs, digraphs, etc.)

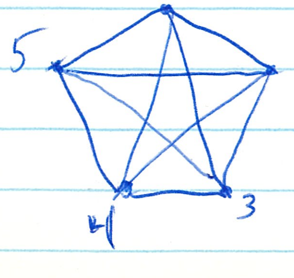
<u>Ex!</u>



$$\longrightarrow \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$



$$\longrightarrow \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$



$$\longrightarrow \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

<u>Properties:</u> in our context the following will always hold

1. $A(G)$ is real
2. $A(G)$ is symmetric
3. $a_{ii} = 0 \ \forall \ A(G)$ (no loops)

<u>Final definition for today:</u> the <u>spectral radius / Frobenius-Perron eigen value</u> is the largest ~~eig~~ (by absolute value) eigen value of an adjacency matrix A_G

<u>Wednesday</u> we will go over results involving A_G and talk about things you can learn about a graph from A_G

<u>Friday</u> big theorem regarding the spectral radius of a graph. result from my graph theory class project.

<u>Bonus</u> (things to google for interest)

there are other matrices you can associate to a graph

- adjacency but in more generality
- incidence matrix (~~counts the vertices~~ $|V| \times |E|$ used in ~~network~~ networks)
- Laplacian (essentially counts degrees of vertices (valency) and also has relations from its eigen stuff to the graph)