Math 225 lecture 12 Oct 2^{nd} 2023

Goal: students should be able compute a QR decomposition & understand some of the theoretical benifits there to.

Section 5.3

Class Q: list 2 or more potential benifits of the QR factorization.

Recall: last time we did G-S now we will have to put these in a matrix (ideally in a clever way)

Thm: let A be an $m \times n$ matrix with linearly independant columns then A can be factored as $A = QR$ with Q an $m \times n$ matrix with orthogonal columns and R invertible and upper triangular

(spidey sences should be tingling! how to orthogonalize? G-S!)

This can be proven in general with no specific examples. time permitting we will explore this (later, catch up day, etc.)

Recall upper triangular matrices are those which have non-zero elements on or above the diagonal only & orthogonal matrices have $QQ^T = I$

$$\begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \ddots & * \end{pmatrix}$$

potentially non-zero above

main diagonal non-zero

all 0's below.

Ex: lets start with actually figuring out how to find Q & R (in theory we know it can be done)

let $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ we know that Col(A) is spanned by the columns of A. since these are linearly indep. (check this) we can apply G-S

after one applies G-S & normalizes (we did lots of this last class) we get

$$\left\{ \underset{\vec{q}_1}{\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}}, \underset{\vec{q}_2}{\begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}}, \underset{\vec{q}_3}{\begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}} \right\}$$

let $Q = [\vec{q}_1\ \vec{q}_2\ \vec{q}_3]$

we know Q's columns are orthonormal by construction which means we can use that fact along with our proposed factorization of $A = QR$ to get

$$Q^T A = Q^T QR = IR = R \quad (\text{note } Q^T = Q^{-1})$$

in this case "one sided"

& we can see via matrix mult.

$$\begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

Notes & observations:

- the diagonal elements of R are each non-zero why? ~~...~~ since R was invertible this must be true.

- if A ~~...~~ is $n \times n$ can we find A^{-1} in a nice way? Yes! $A^{-1} = (QR)^{-1} = R^{-1}Q^{-1}$ ← why we needed $n \times n$
$= R^{-1}Q^{T}$

Q^T is easy to find & if R^{-1} is easy then computing A^{-1} is better this way

Ex: in $\mathbb{R}^3$
since we know $R = \begin{bmatrix} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & a/d_1 & b/d_1 \\ 0 & 1 & c/d_2 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow R^{-1} = \begin{bmatrix} 1 & a/d_1 & b/d_1 \\ 0 & 1 & c/d_2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix}$$

meaning finding R^{-1} is the same as solving

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

solving directly we get $a + x = 0 \Rightarrow a = -x$
$c + z = 0 \Rightarrow c = -z$
$b + cx + y = 0 \Rightarrow b = -y + xz$

Ex: find $\begin{bmatrix} 3 & 6 & 10 \\ 0 & 5 & 15 \\ 0 & 0 & 7 \end{bmatrix}^{-1} = \left[\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\right]^{-1}$

$= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/7 \end{bmatrix}$ apply the above.

$= \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/7 \end{bmatrix}$

$= \begin{bmatrix} 1/3 & -2/5 & 4/7 \\ 0 & 1/5 & -3/7 \\ 0 & 0 & 1/7 \end{bmatrix}$ is this easy? not super easy but easier than the general case.

you can work out the general upper triangular matrix with 1's on the diagonal inverse we wont do that here but there is an exact formula.

Whats the point?

- this turns out to be useful for numeric approximations with eigenvalues (just for interest)
- this with least squares approximation (starting next week)
- the inversion thing is actually beneficial