MATH 225 HW2

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Question 1 (20)

Verify that the vectors $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

form an orthogonal basis of $\mathbb{R}^3$. Then find the orthogonal decomposition of the vector $\vec{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to the basis, *i.e.* compute $[\vec{w}]_{\mathcal{B}}$

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 - 1 + 0 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1 - 1 + 0 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1 + 1 - 2 = 0$$

$$\therefore \quad \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are orthogonal } \Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are linearly independent}$$

Since any 3 lin. independent vectors of $\mathbb{R}^3$ form a basis in $\mathbb{R}^3$, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms an orthogonal basis of $\mathbb{R}^3$

$$\begin{aligned}
\vec{v} &= \mathrm{proj}_{\mathcal{B}}(\vec{w}) = \mathrm{proj}_{\vec{v}_1}(\vec{w}) + \mathrm{proj}_{\vec{v}_2}(\vec{w}) + \mathrm{proj}_{\vec{v}_3}(\vec{w}) \\
&= \frac{\vec{v}_1 \cdot \vec{w}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{w}}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{v}_3 \cdot \vec{w}}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3 = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\end{aligned}$$

$$\vec{v}^{\perp} = \mathrm{perp}_{\mathcal{B}}(\vec{w}) = \vec{w} - \mathrm{proj}_{\mathcal{B}}(\vec{w}) = \vec{0}$$

$$\vec{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Question 2 (20)

a) Find the orthogonal complement $W^\perp$ of the subspace

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x = \tfrac{1}{2}t, y = -\tfrac{1}{2}t, z = 2t, t \in \mathbb{R} \right\}$$

$$W = \text{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{pmatrix} \right\}$$

$$W^\perp = \{\vec{v} \in \mathbb{R}^3 : \vec{v} \cdot \vec{w} = 0 \;\; \forall \vec{w} \in W\}$$

$$\because \; \vec{w} \in \text{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{pmatrix} \right\}$$

$$\vec{v} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{pmatrix} = 0$$

$$\therefore \; W^\perp = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \frac{1}{2}x - \frac{1}{2}y + 2z = 0 \right\}$$

b) Consider the subspace $W = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 + x_3 + x_4 = 0\}$ of $\mathbb{R}^4$

i. Find a basis for $W^\perp$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1 + x_2 + x_3 + x_4 = 0$$

Since normal vector $\vec{n}$ is orthogonal to the hyperplane

$$\mathcal{B}^\perp = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

ii. Find a basis for W and use it to construct an orthogonal basis for W using the Gram-Schmidt process.

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \mid & 0 \end{pmatrix}$$

$$x_1 = -x_2 - x_3 - x_4$$

$$\begin{aligned} x_1 &= -s - t - u \\ x_2 &= \quad s \\ x_3 &= \quad t \\ x_4 &= \quad u \end{aligned}$$

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$\therefore \mathcal{B}_W = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Using Graham-Schmidt,

$$\begin{aligned}
\vec{v}_1 &= \vec{w}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\vec{v}_2 &= \vec{w}_2 - \text{proj}_{\vec{v}_1}(\vec{w}_2) = \vec{w}_2 - \frac{v_1 \cdot w_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\
&= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \\
\vec{v}_3 &= \vec{w}_3 - \text{proj}_{\vec{v}_1}\vec{w}_3 - \text{proj}_{\vec{v}_2}\vec{w}_3 = \vec{w}_3 - \frac{\vec{v}_1 \cdot \vec{w}_3}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_3}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\
&= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\frac{1}{2}}{\frac{3}{2}}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 + \frac{1}{2} + \frac{1}{6} \\ 0 - \frac{1}{2} + \frac{1}{6} \\ 0 - 0 - \frac{1}{3} \\ 1 - 0 - 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}
\end{aligned}$$

Thus, the orthogonal basis of W is

$$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \right\}$$

iii. Let $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Find $\text{proj}_W(\vec{v})$ in two ways:

A. By first computing $\text{proj}_{W^\perp}(\vec{v})$

$$\because \ \mathcal{B}^\perp = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\vec{x}_1\},$$

$$\text{proj}_{W^\perp}(\vec{v}) = \frac{\vec{x}_1 \cdot \vec{v}}{\vec{x}_1 \cdot \vec{x}_1}\vec{x}_1 = \frac{1+1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Since $\text{perp}_{W^\perp}(\vec{v}) = \vec{v} - \text{proj}_{W^\perp}(\vec{v})$ finds the part of the vector perpendicular to $W^\perp$ and $(W^\perp)^\perp = W$

$$\text{proj}_W(\vec{v}) = \vec{v} - \text{proj}_{W^\perp}(\vec{v}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

B. Using the orthogonal basis of W found above.

$$\text{Recall: } \mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \right\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

$$\begin{aligned}
\text{proj}_W(\vec{v}) &= \frac{\vec{v}_1 \cdot \vec{v}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{v}}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{v}_3 \cdot \vec{v}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\
&= \frac{-1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\frac{1}{2}}{\frac{3}{2}} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \frac{-\frac{2}{3}}{\frac{4}{3}} \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ -\frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ 0 + \frac{1}{3} + \frac{1}{6} \\ 0 + 0 - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ -\frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ 0 + \frac{2}{6} + \frac{1}{6} \\ 0 + 0 - \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\end{aligned}$$

Question 3 (20)

a) Apply Graham-Schmidt to the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

to find an orthogonal basis for $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\begin{aligned}
\vec{w}_1 &= \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
\vec{w}_2 &= \vec{v}_2 - \text{proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{w}_1 \cdot \vec{v}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 \\
&= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} \\
\vec{w}_3 &= \vec{v}_3 - \text{proj}_{\vec{w}_1}(\vec{v}_2) - \text{proj}_{\vec{w}_2}(\vec{v}_3) = \vec{v}_3 - \frac{\vec{w}_1 \cdot \vec{v}_3}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{w}_2 \cdot \vec{v}_3}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 \\
&= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-\frac{1}{4}}{\frac{3}{4}}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} \\
&= \begin{pmatrix} 0 - \frac{3}{4} + \frac{1}{12} \\ 1 - \frac{3}{4} + \frac{1}{12} \\ 1 - \frac{3}{4} + \frac{1}{12} \\ 1 - \frac{3}{4} - \frac{1}{4} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}
\end{aligned}$$

Thus, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \right\}$

b) Find a QR factorization to the matrix $A = (\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3)$ whose columns are given by the vectors from part (a)

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\|\vec{w}_1\| = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

$$\|\vec{w}_2\| = \left\| \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} \right\| = \sqrt{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{9}{16}} = \sqrt{\frac{12}{16}} = \frac{\sqrt{3}}{2}$$

$$\|\vec{w}_3\| = \left\| \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Normalize vectors from G-S

$$\frac{1}{\|\vec{w}_1\|}\vec{w}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\frac{1}{\|\vec{w}_2\|}\vec{w}_2 = \frac{1}{\frac{\sqrt{3}}{2}}\vec{w}_2 = \frac{2}{\sqrt{3}}\begin{pmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ -\frac{3}{4}\end{pmatrix} = \begin{pmatrix}\frac{2}{4\sqrt{3}}\\ \frac{2}{4\sqrt{3}}\\ \frac{2}{4\sqrt{3}}\\ -\frac{6}{4\sqrt{3}}\end{pmatrix} = \begin{pmatrix}\frac{1}{2\sqrt{3}}\\ \frac{1}{2\sqrt{3}}\\ \frac{1}{2\sqrt{3}}\\ -\frac{3}{2\sqrt{3}}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\sqrt{3}}{6}\\ \frac{\sqrt{3}}{6}\\ \frac{\sqrt{3}}{6}\\ -\frac{\sqrt{3}}{2}\end{pmatrix}$$

$$\frac{1}{\|\vec{w}_3\|}\vec{w}_3 = \frac{1}{\frac{\sqrt{6}}{3}}\begin{pmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{1}{3}\\ 0\end{pmatrix} = \frac{3}{\sqrt{6}}\begin{pmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{1}{3}\\ 0\end{pmatrix} = \frac{\sqrt{6}}{2}\begin{pmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{1}{3}\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{\sqrt{6}}{3}\\ \frac{\sqrt{6}}{6}\\ \frac{\sqrt{6}}{6}\\ 0\end{pmatrix}$$

$$\text{So, let } Q = \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{3}\\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6}\\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6}\\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0\end{pmatrix} = \begin{pmatrix}\vec{q}_1 & \vec{q}_2 & \vec{q}_3\end{pmatrix}$$

$$A = QR \Rightarrow Q^\top A = Q^\top QR \Rightarrow Q^\top A = R$$

$$Q^\top A = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{2}\\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}$$

$$
= \begin{pmatrix} \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0 & 0+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \\ \frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}-\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}+0 & 0+\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}-\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{6}}{3}+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+0 & -\frac{\sqrt{6}}{3}+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+0 & 0+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{pmatrix} = R
$$

Question 4 (20)

Assume $b \neq 0$. Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$$

by producing an orthogonal matrix Q and a diagonal matrix D s.t. $D = Q^\top A Q$

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} a-\lambda & 0 & b \\ 0 & a-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} \\
&= (a-\lambda) \begin{vmatrix} a-\lambda & 0 \\ 0 & a-\lambda \end{vmatrix} + b \begin{vmatrix} 0 & a-\lambda \\ b & 0 \end{vmatrix} \\
&= (a-\lambda)(a-\lambda)^2 + b(-b(a-\lambda)) \\
&= (a-\lambda)^3 - b^2(a-\lambda) = (a-\lambda)((a-\lambda)^2 - b^2) \\
&= (a-\lambda)(a-\lambda+b)(a-\lambda-b) \\
&= (a-\lambda)(\lambda-(a+b))(\lambda-(a-b)) \\
\lambda &= a, (a+b), (a-b) \\
E_a &= \left(\begin{array}{ccc|c} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} b & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
\vec{x} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t \Rightarrow E_a = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \\
E_{(a+b)} &= \left(\begin{array}{ccc|c} a-(a+b) & 0 & b & 0 \\ 0 & a-(a+b) & 0 & 0 \\ b & 0 & a-(a+b) & 0 \end{array}\right) \\
&= \left(\begin{array}{ccc|c} -b & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ b & 0 & -b & 0 \end{array}\right) \xrightarrow{R_3 + R_1} \left(\begin{array}{ccc|c} -b & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
&\xrightarrow[-R_2]{-R_1} \left(\begin{array}{ccc|c} b & 0 & -b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
\end{aligned}$$

$$bx_1 - bx_3 = 0$$
$$bx_2 = 0$$
$$\Rightarrow \ \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t \Rightarrow E_{(a+b)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$E_{(a-b)} = \left(\begin{array}{ccc|c} a-(a-b) & 0 & b & 0 \\ 0 & a-(a-b) & 0 & 0 \\ b & 0 & a-(a-b) & 0 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} b & 0 & b & 0 \\ 0 & b & 0 & 0 \\ b & 0 & b & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} b & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$bx_1 + bx_3 = 0$$
$$bx_2 = 0$$
$$\Rightarrow \ \vec{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t \Rightarrow E_{(a-b)} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} a & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & a-b \end{pmatrix}$$
$$\vec{u}_1 = \frac{1}{\|P_1\|} P_1 = \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\vec{u}_2 = \frac{1}{\|P_2\|} P_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$
$$\vec{u}_3 = \frac{1}{\|P_3\|} P_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & a-b \end{pmatrix}$$

Question 5 (20)

Verify that polynomials with real coefficients and degree less than or equal to 2 with the usual notions of addition and scalar multiplication are a vector space. (this means you will need to check all 10 axioms like we did in class).

Let $p(x) = a_1 + a_2x + a_3x^2 \in V$ and $q(x) = b_1 + b_2x + b_3x^2 \in V$

Axiom 1: (Closed under Addition)

$$\begin{aligned} p(x) + q(x) &= a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2 \\ &= (a_1 + b_1) + (a_2 + b_2)x + (a_3 + b_3)x^2 \\ &= c_1 + c_2x + c_3x^2 \in V \quad \text{ for some } c_i \in \mathbb{R} \end{aligned}$$

Axiom 2: (Commutativity of Addition)

$$\begin{aligned} p(x) + q(x) &= a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2 \\ &= b_1 + b_2x^1 + b_3x^2 + a_1 + a_2x + a_3x^2 \\ &= q(x) + p(x) \end{aligned}$$

Axiom 3: (Associativity). Let $r(x) = c_1 + c_2x + c_3x^2 \in V$

$$\begin{aligned} (p(x) + q(x)) + w(x) &= (a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2) + c_1 + c_2x + c_3x^2 \\ &= a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2 + c_1 + c_2x + c_3x^2 \\ &= a_1 + a_2x + a_3x^2 + (b_1 + b_2x^1 + b_3x^2 + c_1 + c_2x + c_3x^2) \\ &= p(x) + (q(x) + r(x)) \end{aligned}$$

Axiom 4: (Zero Vector)

$$\begin{aligned} \vec{0} &= 0 + 0x + 0x^2 = 0 \in V \\ p(x) + \vec{0} &= a_1 + a_2x + a_3x^2 + 0 \\ &= a_1 + a_2x + a_3x^2 \\ &= p(x) \end{aligned}$$

Axiom 5: (Additive Inverse)

$$p(x) - p(x) = a_1 + a_2x + a_3x^2 - (a_1 + a_2x + a_3x^2) = 0 = \vec{0}$$

Axiom 6: (Closed under Scalar Multiplication)

$$\begin{aligned} cp(x) &= c(a_1 + a_2x + a_3x^2) \qquad \forall c \in \mathbb{R} \\ &= ca_1 + ca_2x + ca_3x^2 \\ &= d_1 + d_2x + d_3x^2 \in V \quad \text{ for some } d_i \in \mathbb{R} \end{aligned}$$

Axiom 7: (Distributivity)

$$\begin{aligned} c(p(x) + q(x)) &= c(a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2) \quad \forall c \in \mathbb{R} \\ &= ca_1 + ca_2x + ca_3x^2 + cb_1 + cb_2x^1 + cb_3x^2 \\ &= cp(x) + cq(x) \end{aligned}$$

Axiom 8: (Distributivity)

$$\begin{aligned} (c + d)p(x) &= (c + d)(a_1 + a_2x + a_3x^2) \quad \forall c, d \in \mathbb{R} \\ &= ca_1 + a_2x + ca_3x^2 + da_1 + da_2x + da_3x^2 \\ &= cp(x) + dp(x) \end{aligned}$$

Axiom 9: (Collection of Scalars)

$$\begin{aligned} c(dp(x)) &= c(da_1 + da_2x + da_3x^2) \quad \forall c, d \in \mathbb{R} \\ &= cda_1 + cda_2x + cda_3x^2 \\ &= cd(a_1 + a_2x + a_3x^2) \\ &= cdp(x) \end{aligned}$$

Axiom 10: (Scalar Multiplicative Identity)

$$\begin{aligned} 1p(x) &= 1(a_1 + a_2x + a_3x^2) \\ &= a_1 + a_2x + a_3x^2 \\ &= p(x) \end{aligned}$$

Therefore, V, which contains all polynomials with real coefficients and degree ≤ 2 w/ the usual notions of addition and scalar multiplication, is a vector space