Q3 - 20 marks

Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

a) Find matrices P and D (with D diagonal) so that $A = PDP^{-1}$

$$C_A(\lambda) = (\lambda - 2)^2(\lambda + 2)^2 \qquad \lambda = 2, -2$$

For $\lambda = 2$,

$$\begin{aligned} A - 2I &= \begin{pmatrix} 2-2 & 0 & 0 & 4 \\ 0 & 2-2 & 0 & 0 \\ 0 & 0 & -2-2 & 0 \\ 0 & 0 & 0 & -2-2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \end{aligned}$$

$$E_2 = \mathrm{Nul}(A - 2I) = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{4}R_4]{\substack{\frac{1}{4}R_1 \\ R_1 \leftrightarrow R_2 \\ -\frac{1}{4}R_3}} \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\substack{R_3 \leftrightarrow R_1 \\ R_4 - R_1}} \left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x_1 &= \text{free} & x_1 &= s \\ x_2 &= \text{free} & x_2 &= t \\ x_3 &= \ 0 & x_3 &= 0 \\ x_4 &= \ 0 & x_4 &= 0 \end{aligned} \qquad \Rightarrow \qquad \vec{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \mathcal{B}_{E_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = -2$,

$$A + 2I = \begin{pmatrix} 4 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-2} = Nul(A+2I) = \begin{pmatrix} 4 & 0 & 0 & 4 & | & 0 \\ 0 & 4 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow[]{\substack{\frac{1}{4}R_1 \\ \frac{1}{4}R_2}} \begin{pmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_4 &= 0 \\ x_2 &= 0 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \end{aligned} \qquad \begin{aligned} x_1 &= -t \\ x_2 &= 0 \\ x_3 &= s \\ x_4 &= t \end{aligned} \qquad \Rightarrow \qquad \vec{x} = s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \mathcal{B}_{E_{-2}} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thus, $P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

b) For each positive integer n, write down a formula for A^n

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\begin{aligned}
&\forall n \in \mathbb{Z}^+,\\
&A^n = (PDP^{-1})^n\\
&\quad = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\dots(\cancel{P}DP^{-1}) = PD^nP^{-1}\\
&\quad = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 & 0 & 0 \\ 0 & 2^n & 0 & 0 \\ 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&\quad = \begin{pmatrix} 2^n & 0 & 0 & -((-2)^n) \\ 0 & 2^n & 0 & 0 \\ 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&\quad = \begin{pmatrix} 2^n & 0 & 0 & 2^n - (-2)^n \\ 0 & 2^n & 0 & 0 \\ 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{pmatrix}
\end{aligned}
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