section 7.1

①

Math 225 lecture 29 NOV 22^{nd} 2023

Goal: students should be comfortable with computations involving inner products including but not limited to the dot product.

Class Q: since inner products let us define distance, norm, and orthogonality in more generality what related concepts should we now have notions of? (foreshadowing ☺)

Recall: an inner product space is a vector space with an inner product $\langle , \rangle$ s.t. $\forall \vec{u}, \vec{v}, \vec{w} \in V$ and $c \in \mathbb{R}$

① $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

② $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

③ $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$

④ $\langle \vec{u}, \vec{u} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$

Today we will verify that a few spaces are in fact inner prod. spaces and do a few examples of computations of norm, distance, and orthogonality.

Example 1 (or rather example 0) $V = \{\vec{0}\}$ with $\langle \vec{0}, \vec{0} \rangle = 0$ being the inner product.

① $0 = \langle \vec{0}, \vec{0} \rangle = \langle \vec{0}, \vec{0} \rangle = 0$

the other order

② $\langle \vec{0}, \vec{0}+\vec{0} \rangle = \langle \vec{0}, \vec{0} \rangle = 0 = 0 + 0 = \langle \vec{0}, \vec{0} \rangle + \langle \vec{0}, \vec{0} \rangle$

③ $\langle c\vec{0}, \vec{0} \rangle = \langle \vec{0}, \vec{0} \rangle = 0 = c\,0 = c\langle \vec{0}, \vec{0} \rangle$

④ $\langle \vec{0}, \vec{0} \rangle \geq 0$ and $\langle \vec{0}, \vec{0} \rangle = 0$ as $\vec{0} = \vec{0}$

Ex 1 for real let $V = \mathbb{R}^n$ and $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$

① $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n = v_1 u_1 + \dots + v_n u_n = \vec{v} \cdot \vec{u} = \langle \vec{v}, \vec{u} \rangle$

② $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \vec{u} \cdot (\vec{v} + \vec{w}) = u_1(v_1 + w_1) + \dots + u_n(v_n + w_n)$

$= u_1 v_1 + \dots + u_n v_n + u_1 w_1 + \dots + u_n w_n$

$= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

③ $\langle c\vec{u}, \vec{v} \rangle = cu_1 v_1 + \dots + cu_n v_n = c(u_1 v_1 + \dots + u_n v_n) = c(\vec{u} \cdot \vec{v}) = c\langle \vec{u}, \vec{v} \rangle$

④ $\langle \vec{u}, \vec{u} \rangle = u_1^2 + \dots + u_n^2 \geq 0$ and is 0 iff $u_i = 0$ i.e. $\vec{u} = \vec{0}$

Ex 2 let $V = \mathbb{R}$ and $\langle x, y \rangle = xy$

① $\langle x, y \rangle = xy = yx = \langle y, x \rangle$

② $\langle x, y + z \rangle = x(y + z) = xy + xz = \langle x, y \rangle + \langle x, z \rangle$

③ $\langle cx, y \rangle = cxy = c(xy) = c\langle x, y \rangle$

④ $\langle x, x \rangle = x^2 \geq 0$ and is zero iff $x = 0$

Ex 3 Non example $V = P_1(\mathbb{R})$ with $\langle f(x), g(x) \rangle = f(0)g(1) + f(1)g(0)$

counter example?

let $f(x) = -x + 1$ consider $\langle f(x), f(x) \rangle = f(0)f(1) + f(1)f(0)$

$= (1)(0) + 0(1) = 0$

this contradicts property 3! could we fix the innerproduct to make it work?

Ex 4: $V = P_1(\mathbb{R})$ $\langle f(x), g(x)\rangle = f(0)g(0) + f(1)g(1)$

1. $\langle f(x), g(x)\rangle = f(0)g(0) + f(1)g(1) = g(0)f(0) + g(1)f(1) = \langle g(x), f(x)\rangle$

2. $\langle f(x), g(x)+h(x)\rangle = f(0)\big(g(0)+h(0)\big) + f(1)\big(g(1)+h(1)\big) = f(0)g(0) + f(1)g(1) + f(0)h(0) + f(1)h(1)$
$= \langle f(x), g(x)\rangle + \langle f(x), h(x)\rangle$

3. $\langle c f(x), g(x)\rangle = cf(0)g(0) + cf(1)g(1) = c\big(f(0)g(0) + f(1)g(1)\big) = c\langle f(x), g(x)\rangle$

4. $\langle f(x), f(x)\rangle = f(0)^2 + f(1)^2 \geq 0$ and is zero iff $f(0) = f(1) = 0$

however to make a line equal zero at two points that entire line must be zero hence iff $f(x) = \vec{0}$
(this is why we need two different evaluation points how many would we need for a cubic?)

in this space from Ex 4 lets do some computations you try!

1. $\| 2x+1 \|$

2. the distance between $3x+4$ and x

3. $2+x$ is orthogonal to which of $5x+3$, $4x$, $1-7x$

1. $\|2x+1\| = \sqrt{\langle 2x+1, 2x+1\rangle} = \sqrt{1\cdot 1 + 3\cdot 3} = \sqrt{10}$

2. $d(3x+4, x) = \|3x+4-x\| = \|2x+4\| = \sqrt{4^2+6^2} = \sqrt{52}$
or $2\sqrt{13}$

3. its $-5x+3$ as $\langle 2+x, -5x+3\rangle = 2(3) + 3(-2) = 6-6 = 0$

remember orthogonality is based on a choice of inner product

other examples include ~~we~~

$\mathbb{R}^n$ with a weighted dot product
$P_2(\mathbb{R})$ with $\int_0^1 f(x)\,g(x)\,dx$

among many many others!

on friday experience class! think about the class question from today and the options having an inner product affords you!