Math 225 lecture 27 Nov 10^{th} 2023

Section 6.3

Goal: by the end of class students should understand how and be able to compute change of basis matrices using both the manual method and the Gauss-Jordan method.

Class Q: Is the notion of change of basis restricted to $\mathbb{R}^n$? if so, why? if not, given an example.

last day: we covered what a change of basis matrix was and did some computations manually.

in particular we said a change of basis matrix $P_{D \leftarrow B}$ is the transformation matrix of the identity i.e.

$$P_{D \leftarrow B} = [Id]_{D \leftarrow B}$$

and why might we do this? different bases have their advantages, and sometimes we are given info in one basis and asked to ~~find~~ give an answer in another.

to do this manually we solved systems ~~of~~ of the form

$[\vec{v}_i]_B = c_1\vec{d}_1 + c_2\vec{d}_2 + \dots + c_n\vec{d}_n$ for the c_i's which involves

an augmented matrix $\left[\begin{array}{ccc|c} c_1 d_{11} & \cdots & c_n d_{1n} & v_{i1} \\ \vdots & & \vdots & \vdots \\ c_n d_{n1} & \cdots & c_n d_{nn} & v_{in} \end{array}\right]$

and we would do this for each $\vec{v}_i$ this is tedious but what do we notice? how can we do this a better way?

a larger augmented matrix! (contractual obligation!) we can solve them all at once!

↳ the Gauss-Jordan method. let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ & $\mathcal{D} = \{\vec{w}_1, \dots, \vec{w}_n\}$ be bases for V then we can set up

$$[\mathcal{D} \mid \mathcal{B}] \xrightarrow[\text{reduce}]{\text{row}} [Id \mid P_{\mathcal{D}\leftarrow\mathcal{B}}]$$

and solve for the change of basis matrix. (note! the order on $\mathcal{B}$ & $\mathcal{D}$ both for P and in the augmented matrix)

Ex: let $V = \mathbb{R}^2$ $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $\mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

find $P_{\mathcal{D}\leftarrow\mathcal{B}}$

$$\left[\begin{array}{cc|cc} 0 & 2 & 1 & 1 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] = [Id \mid P_{\mathcal{D}\leftarrow\mathcal{B}}]$$

How can we check this? let $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(this is done by inspection, it could be done systematically or done by computing $P_{\mathcal{B}\leftarrow\mathcal{S}}$)

if we then compute $P_{\mathcal{D}\leftarrow\mathcal{B}}[\vec{v}]_{\mathcal{B}}$ we should get $[\vec{v}]_{\mathcal{D}}$

$$P_{\mathcal{D}\leftarrow\mathcal{B}}[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we can tell this is correct since

$$\begin{bmatrix}2\\3\end{bmatrix} = 0\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\vec{w}_1} + 1\underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{\vec{w}_2}$$

it is important to remember that we can do this more generally in a vectorspace context i.e. $P_n(\mathbb{R})$, $Mat_{2,2}(\mathbb{R})$ and <u>many</u> others.

<u>Ex</u>: let $P_2(\mathbb{R}) = V$ (polynomials of deg ≤ 2 with real coeffs)

$\mathcal{B} = \{x, 1+x^2, x+x^2\}$ $\mathcal{D} = \{1, 1+x, x^2\}$

(here $\mathcal{S} = \{1, x, x^2\}$) this looks different but is in fact the same process

$$[\mathcal{D} \mid \mathcal{B}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad Id \qquad P_{\mathcal{D} \leftarrow \mathcal{B}}$$

lets test this on $4 - 2x - x^2$. to do this we need $[4 - 2x - x^2]_{\mathcal{B}}$ lets do this systematically. that is we need

$$c_1 x + c_2(1+x^2) + c_3(x+x^2) = 4 - 2x - x^2$$

or

$c_2 + (c_1+c_3)x + (c_2+c_3)x^2 = 4 - 2x - x^2$ with translates to the system

$$\begin{aligned} c_2 \quad &= 4 \\ c_1 \quad + c_3 &= -2 \\ c_2 + c_3 &= -1 \end{aligned}$$

which can be solved as

$c_1 = 3$ $c_2 = 4$ $c_3 = -5$

$\therefore [4-2x-x^2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$ making (hopefully) $[4-2x-x^2]_{\mathcal{D}} =$

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}$$

in other words we know how to write $4-2x-x^2$ in terms of $\mathcal{D}$ lets check

$$6(1)+(-2)(1+x)+(-1)(x^2) = 6-2-2x-x^2 = 4-2x-x^2 \text{ as desired.}$$

final notes

- when constructing $[\mathcal{D} \mid \mathcal{B}]$ it does not matter what basis you write the columns in as long as they are all the same basis
- all of the subscript notation is there for us not the math its to keep track of where things are and where they go to.

<u>Exersise</u> do the same as our previous 2 examples with

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

test this on $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

you can use $\mathcal{S}$ as the standard basis

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Have a great reading week ☺