Practice problems week 4, Solutions

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Solution 1 There are several methods for doing this. One can manually verify that the dot products of each vector with itself is one and that the dot product of any one vector with any other is 0. This method however is painful. If we recall a faster way to check this would be to put it in a matrix. Since we are in $\mathbb{R}^4$ and we are dealing with 4 vectors they form an orthonromal basis if and only if the matrix A consisting of columns given by these vectors is orthogonal. This is much easier to check as we can tell if a matrix is orthogonal if $A^T A = I$ which as we can see

$$\begin{bmatrix} 1/\sqrt{2} & 0 & -1/2 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 & 1/2 \\ 1/\sqrt{2} & 0 & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & -1/2 & 11/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

does turn out to be the case

Solution 2 The easiest method here will be to find a vector in W and something orthogonal to that vector (i.e. something in $W^\perp$). The reason this will work so nicely in this case is because we are working in $\mathbb{R}^2$ and we can see that $dim(W) = 1$ so $dim(W^\perp)$ in this case will also have to be 1. By inspection we can take $\vec{w} = (x, y) = (4, -3)$ as it satisfies the condition of being in W. Also by inspection we can see that $\vec{v} = (3, 4)$ will be orthogonal to $\vec{w}$. Given those previous dimension counts we must have

$$W^\perp = \left\{ t \begin{bmatrix} 3 \\ 4 \end{bmatrix} \middle| t \in \mathbb{R} \right\}, \; \mathcal{B}^\perp = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}.$$

If you wanted to extend this argument to a more general method you could do the following in $\mathbb{R}^n$. Find a basis for W this is done by solving the system that defines the condition for membership (just as we did by inspection above). Then count the dimensions and take $n - dim(W)$ this gives you the dimension of $W^\perp$. At that point you need that many linearly independent vectors which are orthogonal to your basis for W. Which can be done either by inspection, or subtracting off the projections onto W of other linearly independent vectors (and possibly orthogonalizing after that via something like Gram-Schmidt)

Solution 3 This is a direct application of the Gram-Schmidt process

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} - \left(\frac{6+1+8}{4+1+1+4}\right)\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -3/2 \\ 3/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_2}\right)\vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2-1+1+2}{4+1+1+4}\right)\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{0+1/2-3/2+1}{0+1/4+9/4+1}\right)\begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/5 \\ -4/10 \\ 4/10 \\ 4/5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 7/5 \\ 3/5 \\ 1/5 \end{bmatrix}$$

At this point we renormalize so that our vectors have length 1

$$\vec{u}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1 = \frac{1}{\sqrt{2^2+(-1)^2+1^2+2^2}}\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{10} \\ -1/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{||\vec{v}_2||}\vec{v}_2 = \frac{1}{\sqrt{0^2+(1/2)^2+(-3/2)^2+1^2}}\begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2}/2\sqrt{7} \\ -3\sqrt{2}/2\sqrt{7} \\ \sqrt{2}/\sqrt{7} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{||\vec{v}_3||}\vec{v}_3 = \frac{1}{\sqrt{(1/5)^2+(7/5)^2+(3/5)^2+(1/5)^2}}\begin{bmatrix} 1/5 \\ 7/5 \\ 3/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} \sqrt{5}/10\sqrt{2} \\ 7\sqrt{5}/10\sqrt{2} \\ 3\sqrt{5}/10\sqrt{2} \\ \sqrt{5}/10\sqrt{2} \end{bmatrix}$$

It is also acceptable to leave the $\vec{u}_i$'s in a factored form.

Solution 4 As the hint suggested we want to do a double inclusion argument.

Proof. Step 1 take $\vec{w} \in W \cap W^{\perp}$. By assumption $\vec{w} \in W$. We also know by assumption that $\vec{w} \in W^{\perp}$ which means it is orthogonal to anything in W, for example $\vec{w}$. Together this means $\vec{w}$ is orthogonal to itself. i.e. $\vec{w} \cdot \vec{w} = \vec{0}$. However the only vector with this property is $\vec{0}$, based on our definition of the dot product. Meaning $\vec{w} \in \{\vec{0}\}$ and thus $W \cap W^{\perp} \subset \{\vec{0}\}$.

For part two take an arbitrary element in $\{\vec{0}\}$, (in this case there is only one choice $\vec{0}$). Firstly $\vec{0} \in W$ for any choice of W as W is a subspace and that condition is baked into the definition. However since $\vec{u} \cdot \vec{0} = \vec{0}$ for all possible vectors $\vec{u} \in \mathbb{R}^n$ in particular it works for all the vectors in W. Thus making $\vec{0} \in W^{\perp}$ as desired. Meaning $\vec{0} \in W \cap W^{\perp}$ and hence $\{\vec{0}\} \subset W \cap W^{\perp}$. Moreover if two sets can be seen to be subsets of one another in either direction this proves that they are equal as sets giving us the desired statement that $W \cap W^{\perp} = \{\vec{0}\}$.

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