Question 1 (20)

Verify that the vectors $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$$\vec{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

form an orthogonal basis of $\mathbb{R}^3$. Then find the orthogonal decomposition of the vector $\vec{w} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ with respect to the basis, *i.e.* compute $[\vec{w}]_{\mathcal{B}}$

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix} = 1 - 1 + 0 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} = 1 - 1 + 0 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} = 1 + 1 - 2 = 0$$

$$\therefore \quad \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are orthogonal } \Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are linearly independent}$$

Since any 3 lin. independent vectors of $\mathbb{R}^3$ form a basis in $\mathbb{R}^3$, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms an orthogonal basis of $\mathbb{R}^3$

$$\begin{aligned}
\vec{v} &= \mathrm{proj}_{\mathcal{B}}(\vec{w}) = \mathrm{proj}_{\vec{v}_1}(\vec{w}) + \mathrm{proj}_{\vec{v}_2}(\vec{w}) + \mathrm{proj}_{\vec{v}_3}(\vec{w}) \\
&= \frac{\vec{v}_1 \cdot \vec{w}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{w}}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{v}_3 \cdot \vec{w}}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3 = 2\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\-2\end{pmatrix} \\
&= \begin{pmatrix}2\\2\\2\end{pmatrix} - \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{pmatrix} - \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}
\end{aligned}$$

$$\vec{v}^{\perp} = \mathrm{perp}_{\mathcal{B}}(\vec{w}) = \vec{w} - \mathrm{proj}_{\mathcal{B}}(\vec{w}) = \vec{0}$$

$$\vec{w} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}0\\0\\0\end{pmatrix}$$