Math 225 lecture 36 Dec 8^{th} 2023

Goal: finish final exam examples, adress any questions or concerns, enjoy the last class!

Class Q: What was your ~~favorit~~ favorite part of Math 225 & what was the most interesting thing you learned?

Today: final examples & farewell ☹

as per last class lets do a computation with an orthogonal complement.

Ex: let $W \subset \mathbb{R}^3$ be given by $W = \text{span}\left\{\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

let $\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ find $\text{Proj}_{W^\perp}(\vec{u})$

2 ways option 1 find $W^\perp$ & project onto it

$W = \text{col}\left(\begin{bmatrix}2 & 0\\0 & 1\\0 & 1\end{bmatrix}\right) \Rightarrow W^\perp = \text{null}\left(\begin{bmatrix}2 & 0 & 0\\0 & 1 & 1\end{bmatrix}\right)$

this can be done systematically by solving $2x_1 = 0$ & $x_2 + x_3 = 0$ or by eye to get

$W^\perp = \text{span}\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

now $proj_{W^\perp}(\vec{u}) = proj_{\vec{w}_1}(\vec{u}) = \dfrac{-1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\-1/2\\1/2\end{bmatrix}$

option 2 project onto W & subtract off. recall
$\vec{u} = proj_W(\vec{u}) + proj_{W^\perp}(\vec{u})$ ∴ we can find the projection onto $W^\perp$ with just W

Note $\vec{w}_1, \vec{w}_2$ are orthogonal, thus

$$proj_W(\vec{u}) = proj_{\vec{w}_1}(\vec{u}) + proj_{\vec{w}_2}(\vec{u}) = \frac{2}{4}\begin{bmatrix}2\\0\\0\end{bmatrix} + \frac{5}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\5/2\\5/2\end{bmatrix}$$

$$\therefore\ proj_{W^\perp}(\vec{u}) = \vec{u} - proj_W(\vec{u}) = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1\\5/2\\5/2\end{bmatrix} = \begin{bmatrix}0\\-1/2\\1/2\end{bmatrix}$$ as expected.

<u>Linear transformations</u> (specifically kernel/range/one-to-one/onto)

First let $T: V \to W$ be a map. we call it a linear transformation when it obeys the following

1. $T(\vec{0}) = \vec{0}$ ($\vec{0}$ in V first then $\vec{0}$ in W)
2. $T(c\vec{v}) = cT(\vec{v})$
3. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

Recall the kernel of T, $\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$

where does $\ker(T)$ live? what does it remind us of?

the range of T $\operatorname{range}(T) = \{\vec{w} \in W \mid T(\vec{v}) = \vec{w} \text{ for some } \vec{v} \in V\}$

where does range(T) live? what does it remind us of?

related are one-to-one, we say T is one-to-one if $\vec{v}, \vec{u} \in V$ have $\vec{v} \neq \vec{u} \Rightarrow T(\vec{v}) \neq T(\vec{u})$ distinct points remain distinct. (only possible if $\dim(V) \leq \dim(W)$)

if not, a counter example is a $\vec{u}, \vec{v} \in V$ s.t. $\vec{u} \neq \vec{v}$ but $T(\vec{u}) = T(\vec{v})$

similarly onto, we say T is onto if range(T) = W i.e. you can reach everything in the target space via T (only possible if $\dim(V) \geq \dim(W)$)

if not, a counterexample is a $\vec{w} \in W$ s.t. $T(\vec{v}) \neq \vec{w}$ for all $\vec{v} \in V$. (or $\vec{w} \in W$ s.t. $\vec{w} \notin \operatorname{range}(T)$)

Ex: for each of the following find the kernel, & range. is each one-to-one, onto? prove or counter example.

① $T: \mathbb{R} \to \mathbb{R}^3$ its linear (You can check)

$a \mapsto \begin{bmatrix} a \\ a \\ a \end{bmatrix}$

$\ker(T) = \{0\}$ $\text{range}(T) = \left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix} \in \mathbb{R}^3 \mid a \in \mathbb{R} \right\}$

or $= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x = y = z \right\}$ or $\text{span}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

is T one-to-one?

yes! take $a, b \in \mathbb{R}$, with $a \neq b$ then

$T(a) = \begin{bmatrix} a \\ a \\ a \end{bmatrix} \neq \begin{bmatrix} b \\ b \\ b \end{bmatrix} = T(b)$

onto? NO! take $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \text{range}(T)$

② $T: \mathbb{R}^2 \to \mathbb{R}$ $\ker(T) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x + y = 0 \right\} = \left\{ \begin{bmatrix} x \\ -x \end{bmatrix} \mid x \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x + y$

again linear you can check

$\text{range}(T) = \mathbb{R}$

one-to-one? NO, consider $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ these are not equal and yet their image under T is (i.e. $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0$)

onto? yes! $\text{range}(T) = \mathbb{R}$ to see this consider $a \in \mathbb{R}$

$T\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) = a$ $\therefore$ any element of $\mathbb{R}$ can be seen as the image of something under T.

- Thats it! Thank you all so much for a ~~wonderful~~ wonderful term you have been the best first class I could have asked for. You all will hold a special place in my heart and I wish you all the best! _ Brandon Gill