Question 5 (20)

Verify that polynomials with real coefficients and degree less than or equal to 2 with the usual notions of addition and scalar multiplication are a vector space. (this means you will need to check all 10 axioms like we did in class).

Let $p(x) = a_1 + a_2x + a_3x^2 \in V$ and $q(x) = b_1 + b_2x + b_3x^2 \in V$

Axiom 1: (Closed under Addition)

$$\begin{aligned} p(x) + q(x) &= a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2 \\ &= (a_1 + b_1) + (a_2 + b_2)x + (a_3 + b_3)x^2 \\ &= c_1 + c_2x + c_3x^2 \in V \quad \text{for some } c_i \in \mathbb{R} \end{aligned}$$

Axiom 2: (Commutativity of Addition)

$$\begin{aligned} p(x) + q(x) &= a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2 \\ &= b_1 + b_2x^1 + b_3x^2 + a_1 + a_2x + a_3x^2 \\ &= q(x) + p(x) \end{aligned}$$

Axiom 3: (Associativity). Let $r(x) = c_1 + c_2x + c_3x^2 \in V$

$$\begin{aligned} (p(x) + q(x)) + w(x) &= (a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2) + c_1 + c_2x + c_3x^2 \\ &= a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2 + c_1 + c_2x + c_3x^2 \\ &= a_1 + a_2x + a_3x^2 + (b_1 + b_2x^1 + b_3x^2 + c_1 + c_2x + c_3x^2) \\ &= p(x) + (q(x) + r(x)) \end{aligned}$$

Axiom 4: (Zero Vector)

$$\begin{aligned} \vec{0} &= 0 + 0x + 0x^2 = 0 \in V \\ p(x) + \vec{0} &= a_1 + a_2x + a_3x^2 + 0 \\ &= a_1 + a_2x + a_3x^2 \\ &= p(x) \end{aligned}$$

Axiom 5: (Additive Inverse)

$$p(x) - p(x) = a_1 + a_2x + a_3x^2 - (a_1 + a_2x + a_3x^2) = 0 = \vec{0}$$

Axiom 6: (Closed under Scalar Multiplication)

$$\begin{aligned} cp(x) &= c(a_1 + a_2x + a_3x^2) \qquad \forall c \in \mathbb{R} \\ &= ca_1 + ca_2x + ca_3x^2 \\ &= d_1 + d_2x + d_3x^2 \in V \quad \text{ for some } d_i \in \mathbb{R} \end{aligned}$$

Axiom 7: (Distributivity)

$$\begin{aligned} c(p(x) + q(x)) &= c(a_1 + a_2x + a_3x^2 + b_1 + b_2x^1 + b_3x^2) \quad \forall c \in \mathbb{R} \\ &= ca_1 + ca_2x + ca_3x^2 + cb_1 + cb_2x^1 + cb_3x^2 \\ &= cp(x) + cq(x) \end{aligned}$$

Axiom 8: (Distributivity)

$$\begin{aligned} (c+d)p(x) &= (c+d)(a_1 + a_2x + a_3x^2) \quad \forall c, d \in \mathbb{R} \\ &= ca_1 + a_2x + ca_3x^2 + da_1 + da_2x + da_3x^2 \\ &= cp(x) + dp(x) \end{aligned}$$

Axiom 9: (Collection of Scalars)

$$\begin{aligned} c(dp(x)) &= c(da_1 + da_2x + da_3x^2) \quad \forall c, d \in \mathbb{R} \\ &= cda_1 + cda_2x + cda_3x^2 \\ &= cd(a_1 + a_2x + a_3x^2) \\ &= cdp(x) \end{aligned}$$

Axiom 10: (Scalar Multiplicative Identity)

$$\begin{aligned} 1p(x) &= 1(a_1 + a_2x + a_3x^2) \\ &= a_1 + a_2x + a_3x^2 \\ &= p(x) \end{aligned}$$

Therefore, V, which contains all polynomials with real coefficients and degree ≤ 2 w/ the usual notions of addition and scalar multiplication, is a vector space