Math 225 lecture 9 sept 25th 2023

Goal: Students should be able to explain what an orthogonal complement is and how to find one, in addition to basic properties. lastly students should be able to compute basic orthogonal projections.

Section 5.2

Class Q: if $\vec{w} \in W$ and $\vec{v}_1, \vec{v}_2 \in W^{\perp}$ then what is $\vec{w} \cdot (\vec{v}_1 + \vec{v}_2)$?

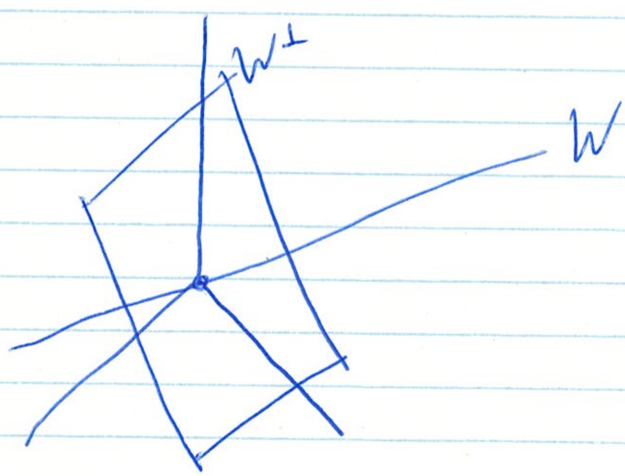
last time we talked about what it means to be orthogonal we will extend that thinking today with orthogonal complements and projections.

↳ let $W \subset \mathbb{R}^n (V)$ be a subspace then the orthogonal complement of W denoted $W^{\perp}$ "W perp" is

$$W^{\perp} := \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \ \forall \vec{w} \in W\}$$

in other words $W^{\perp}$ is the set of all vectors which are orthogonal to everything in W.

Example let $V = \mathbb{R}^3$ and W be a line through the origin. what is $W^{\perp}$?



Visually this is a line and plane.

what do we notice
- dim count?
- subspace?
- intersection?

Properties: of orthogonal complement.

- $W^\perp$ is a subspace of $\mathbb{R}^n$ (V) ①
- $(W^\perp)^\perp = W$
- $W \cap W^\perp = \{\vec{0}\}$
- if $W = \text{span}\{\vec{w}_1, \ldots, w_k\}$ then $\vec{v} \in W^\perp$ iff $\vec{v} \cdot \vec{w}_i = 0 \ \forall i$
- ☆ $(\text{row}(A))^\perp = \text{null}(A)$ & $(\text{col}(A))^\perp = \text{null}(A^T)$ ☆ (main computation one)

Try some of these!

more proofs Th 5.2

Proof of ① let $\vec{u}, \vec{v} \in W^\perp$ this means $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0$ $\forall \vec{w} \in W$

consider $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = 0 + 0 = 0$
$\Rightarrow \vec{u} + \vec{v} \in W^\perp$

for $c \in \mathbb{R}$ take $(c\vec{u}) \cdot \vec{w} = c(\vec{u} \cdot \vec{w}) = c\,0 = 0$
$\Rightarrow c\vec{u} \in W^\perp$

by definition $\vec{0} \cdot \vec{w} = \vec{0} \Rightarrow \vec{0} \in W^\perp$

together this proves $W^\perp$ is a subspace ■

Ex:
let $W = \text{span}\left\{\begin{bmatrix}2\\4\\12\end{bmatrix}, \begin{bmatrix}3\\8\\5\end{bmatrix}\right\}$ find a basis for $W^\perp$

$W = \text{col}\left(\begin{bmatrix}2 & 3\\4 & 8\\12 & 5\end{bmatrix}\right) \Rightarrow W^\perp = \text{null}\left(\begin{bmatrix}2 & 4 & 12\\3 & 8 & 5\end{bmatrix}\right)$

$\begin{bmatrix}2 & 4 & 12\\3 & 8 & 5\end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix}1 & 2 & 6\\3 & 8 & 5\end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix}1 & 2 & 6\\0 & 2 & -13\end{bmatrix}$

$\Rightarrow x_1 + 2x_2 + 6x_3 = 0$
$2x_2 - 13x_3 = 0$

let $x_3 = t$

$\Rightarrow 2x_2 - 13t = 0 \quad \therefore x_2 = \frac{13}{2}t$

$x_1 + 2\left(\frac{13}{2}t\right) + 6(t) = 0 \Rightarrow x_1 = -19t$

$\therefore$ ~~basis~~ $null(A^T) = \left\{ t\begin{bmatrix} -19 \\ 13/2 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

making $\begin{bmatrix} -19 \\ 13/2 \\ 1 \end{bmatrix}$ a basis for $W^\perp$

we get this property by considering the fact that $null(A^T)$ would be the vectors for which the dot product with the columns of A is zero.

this leads to

<u>Thm</u>: if $W \subset \mathbb{R}^n$ is a subspace then

$$\dim(W) + \dim(W^\perp) = n$$

i.e. you are always in W or $W^\perp$

<u>cor</u>: $\text{rank}(A) + \text{nullity}(A) = n$

this is why its called orthogonal "complement"

since rank(A) is the dimension of row(A) & nullity is the dimension of null(A) & $row(A)^\perp = null(A)$

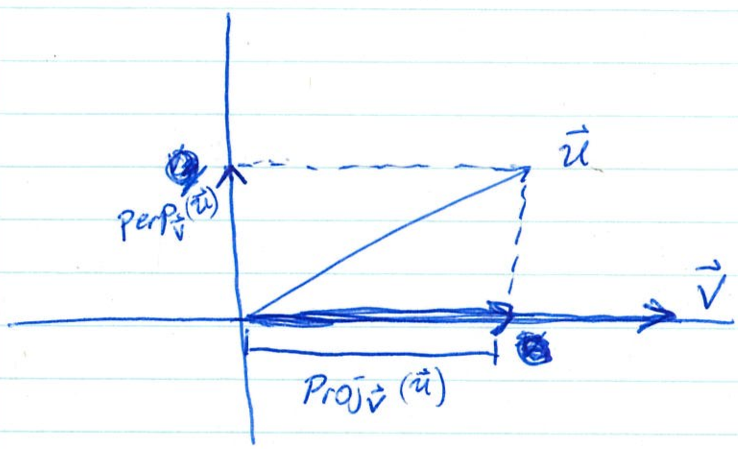
consider our examples from earlier to see this

① a line in $\mathbb{R}^3$ has dim 1 & a plane dim 2

② we took the span of 2 vetors in $\mathbb{R}^3$, this will have dimension 2 & its orthogonal complement has dim 1

$2+1=3$

Projections: in $\mathbb{R}^2$



the concept of (orthogonal) projection is describing how much of $\vec{u}$ is "in the direction of $\vec{v}$"

in $\mathbb{R}^2$ we see this visually, but we can compute it too.

Recall from friday we had a nice form for a vector when we had an orthogonal basis, it looked like

$$\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \dots + \frac{\vec{u}\cdot\vec{v}_n}{\vec{v}_n\cdot\vec{v}_n}\vec{v}_n$$

for $\{v_1, \dots, v_n\}$ an orthogonal basis.

what is this doing?

measuring how much of $\vec{u}$ is in each direction $\vec{v}_i$, which is just the projection!

$\therefore$ we see that the projection of $\vec{u}$ onto $\vec{v}$ denoted

$$proj_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \quad , \quad perp_{\vec{v}}(\vec{u}) = \vec{u} - proj_{\vec{v}}(\vec{u})$$

↑ component orthogonal to $\vec{v}$

more generally if $W \subset \mathbb{R}^n$ is a subspace with an orthogonal basis given by $\{\vec{w}_1, \dots, \vec{w}_k\}$ then

$$Proj_W(\vec{u}) = \frac{\vec{u}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \dots + \frac{\vec{u}\cdot\vec{w}_k}{\vec{w}_k\cdot\vec{w}_k}\vec{w}_k$$

$$Perp_W(\vec{u}) = \vec{u} - Proj_W(\vec{u})$$

Ex: let $W \subset \mathbb{R}^3$ be given as $W = \text{span}\left\{ \overset{\vec{w}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{w}_2}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}} \right\}$

this basis for W is orthogonal

let $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ find $\text{proj}_W(\vec{u})$ & $\text{perp}_W(\vec{u})$

$$\text{Proj}_W(\vec{u}) = \frac{\vec{u}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \frac{\vec{u}\cdot\vec{w}_2}{\vec{w}_2\cdot\vec{w}_2}w_2 = \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{-2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\text{Perp}_W(\vec{u}) = \vec{u} - \text{proj}_W(\vec{u}) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$$

(note such a decomposition of $\vec{u} = \vec{w} + \vec{w}^\perp$ is always possible)