Math 225 lecture ~~13~~ 14 Oct 6th 2023

Goal: students should be comfortable with computing orthogonal diagonalizations of symmetric matrices. In addition to recalling the primary theorem.

Section 5.4

Class Q: do eigen vectors from the same eigen space have to be orthogonal for free?

Recall: • eigen vectors from different eigenspaces of symmetric matrices are orthogonal.

• Real symmetric matrices have real eigenvalues.

• a ~~[illegible]~~ matrix is symmetric iff it is orthogonally diagonalizable

we proved the first two. we will do half of the proof for part 3 the other half can be found on pages 403 & 404 its more involved than I would like to show.

Proof of reverse direction

suppose A is orthogonally diagonalizable.

$\therefore \exists$ Q orthogonal & D diagonal s.t.

$Q^T A Q = D$ & since $Q^T = Q^{-1}$ we have $Q^T Q = I = Q Q^T$

meaning $QDQ^T = Q(Q^T A Q)Q^T = IAI = A$

however

$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$

$\therefore$ A is symmetric. $\blacksquare$

lets do some orthogonal diagonalizations.

Ex1: let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

find e. vals

$$\det(A - \lambda I) = \left| \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} \right| = (4-\lambda)^2 - 1$$
$$= 16 - 8\lambda^2 + \lambda^2 - 1$$
$$= \lambda^2 - 8\lambda + 16$$
$$= (\lambda - 5)(\lambda - 3)$$

$\therefore$ evals are $\lambda_1 = 5$ $\lambda_2 = 3$

$E_5 = \text{null}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$E_3 = \text{null}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

do we have orthogonal vectors? yes ✓
are they normal? NO!

normalize them let $\vec{v}_1 = \frac{1}{\sqrt{1^2+1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

let $\vec{v}_2 = \frac{1}{\sqrt{1^2+(-1)^2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

now we have $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

<u>Ex 2</u>: let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

<u>step 1</u> find e. vals you can go through this to find a char poly of $-\lambda^3 + 6\lambda^2 - 9\lambda + 4$
$= -(\lambda - 4)(\lambda - 1)^2$

$\therefore \lambda_1 = 4, \ \lambda_2 = 1$

<u>step 2</u>: find e. spaces

$E_4 = \text{null}\left(\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

$E_1 = \text{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = \text{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$\Rightarrow x_1 + x_2 + x_3 = 0$ 3 unknowns 1 restriction $\therefore$ 2 free vars, let $x_2 = s \quad x_3 = t$

$\therefore x_1 + s + t = 0$
$x_1 = -s - t$

$\therefore \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

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note that each of $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ are orthogonal

to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as expected. however they themselves are not orthogonal how do we fix this?

<u>Step 3</u> apply G-S to $E_1 = \text{span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ $\qquad \vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(x_2) = \begin{bmatrix}-1\\0\\1\end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$\therefore E_1 = \text{span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}\right\}$ $\qquad = \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$

which is orthogonal & still orthogonal to E_4

once we normalize each of these we obtain 3 vectors mutually orthogonal & normed which means they form an orthogonal matrix giving us

$$Q = \begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix} \quad \text{with } D = \begin{bmatrix}4 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Ex: $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ (assuming $b \neq 0$)

$$\det(A - \lambda I) = \left| \begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix} \right| = (a-\lambda)^2 - b^2$$
$$= a^2 - 2a + \lambda^2 - b^2$$
$$= \lambda^2 - 2a + a^2 - b^2$$
$$= (\lambda - (a+b))(\lambda - (a-b))$$

$\therefore \lambda_1 = a+b \qquad \lambda_2 = a-b$

$$E_{a+b} = \text{null}\left(\begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$E_{a-b} = \text{null}\left(\begin{bmatrix} b & b \\ b & b \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

normalize $E_{a+b} = \text{span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

$E_{a-b} = \text{span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$

$\therefore Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$