MATH 225, Fall 2023, Written Assignment # 1,
Due: Saturday September 23 by 11:59pm via Assign2.

1. (15 points) Consider the following matrix

$$A = \begin{pmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{pmatrix}.$$

   (a) Write down the RREF of A.

   (b) What is the dimension of the row space of A?

   (c) Write down a basis of the column space of A ?

   (d) What is the nullity of A. Write down a basis of the null space of A?

2. (20 pts)The aim of this question will be to produce matrices with specified eigenvalues which are not just triangular! Let $p(x)$ be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

and define the *companion matrix* to the polynomial as

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

   (a) Write down the matrix $C(p)$ of the polynomial $p(x) = x^3 - 4x^2 + 5x - 2$

   (b) Find the characteristic polynomial of the matrix $C(p)$ which you wrote in the previous step.

   (c) Show that $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue 2.

   (d) Find the matrix $C(p)$ associated to the polynomial $p(x) = x^3 + ax^2 + bx + c$

   (e) Determine the characteristic polynomial of the matrix $C(p)$ from the previous step

   (f) Show that if λ is an eigenvalue of the companion matrix $C(p)$, then $\begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ corresponding to λ

   (g) Construct a non-triangular 3×3 matrix with eigenvalues $-2, 1, 3$ using companion matrices. *Briefly* justify your answer.

3. (20 pts) Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

   (a) Find matrices P and D (with D diagonal) so that $A = PDP^{-1}$

   Note: The matrix P will have columns consisting of eigenvectors as we did in class. A previous version of the assignment asked you to write $A = P^{-1}DP$ for some (different) matrix P. The solutions to these two variants are given by taking an inverse (of the matrix answer).

(b) For each positive integer n, write down a formula for A^n

4. (20 pts) A study of pine nuts in the American southwest from 1940 to 1947 hypothesized that nut production followed a Markov chain. The data suggested that if one year's crop was good, then the probabilities that the following year's crop would be good, fair, or poor were 0.08, 0.07, 0.85 respectively; if one year's crop was fair, then the probabilities that the following year's crop would be good, fair, or poor were 0.09, 0.11, and 0.80, respectively; if one year's crop was poor, then the probabilities that the following year's crop would be good, fair, or poor were 0.11, 0.05, 0.84 respectively.

   (a) Write down the transition matrix for this Markov process

   (b) If the pine cut crop was good in 1940, find the probabilities of a good crop in the years 1941 through 1945

   (c) In the long run, what proportion of the crops will be good, fair, and poor?