Math 225 lecture 23 Nov 1st 2023

section 6.4/6.6

Goal: adress the issues described at the end of last class. Show examples of this fix, i.e. $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ additionally introduce properties of composition and inverses.

Class Q: is it possible to uniquely identify a linear transformation by its matrix alone? if so, how? if not, why?

last time: We decided that linear transformations ~~work~~ on subspaces and when we care about different bases were the weak points of the math 125 approach.

we also explained that if $\mathcal{B}$ was a basis then $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ means $\vec{w} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$

lets put that into action with a simple example to start,

let $W = \left(\text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right)^{\perp}$ & $\vec{w} = \begin{bmatrix}1\\-1\\0\end{bmatrix} \in W$

find a basis for W = null$\left(\begin{bmatrix}1 & 1 & 1\end{bmatrix}\right)$ take $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

what is $[\vec{w}]_{\mathcal{B}}$?

$[\vec{w}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ with $\begin{bmatrix}1\\-1\\0\end{bmatrix} = c_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\-1\end{bmatrix} \Rightarrow \begin{matrix} c_1 = 1 \\ c_2 = 1\end{matrix}$

this makes $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ "the coordinate vector"

What happens if we chose $\mathcal{B}' = \left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix} \right\}$?

in this case we get $[\vec{w}]_{\mathcal{B}'} = \begin{bmatrix} -1/2 \\ 1/5 \end{bmatrix}$

this describes for us a new & improved more general version of taking a matrix of transformation. (at least in terms of notation)

① Pick bases $\mathcal{B}$ of V $\vec{v}_1, \dots, v_k$ &
$\mathcal{C}$ of W $\vec{w}_1, \dots, \vec{w}_\ell$

② note that $[\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad [\vec{v}_k]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

③ find $[T\vec{v}_i]_{\mathcal{C}} = \begin{bmatrix} c_{i1} \\ \vdots \\ c_{i\ell} \end{bmatrix}$ i.e. $T\vec{v}_i = c_{i1}\vec{w}_1 + \dots + c_{i\ell}\vec{w}_\ell$

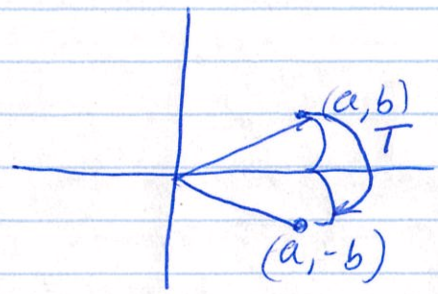
$$A = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ \vdots & \ddots & & \vdots \\ c_{\ell 1} & \cdots & & c_{\ell k} \end{bmatrix} = \begin{bmatrix} | & & | \\ [T\vec{v}_1]_{\mathcal{C}} & \cdots & [T\vec{v}_k]_{\mathcal{C}} \\ | & & | \end{bmatrix} = [T]_{\mathcal{C} \leftarrow \mathcal{B}}$$

<u>Ex:</u> let $V = \mathbb{R}^2$ $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2\}$

& $W = \mathbb{R}^2$ $\mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \{\vec{w}_1, \vec{w}_2\}$

and let T be reflecting about the x- axis
find $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$

whats happening?



find $T(\vec{v}_1) = T\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

& $T(\vec{v}_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

now convert these into $[T(\vec{v}_1)]_{\mathcal{C}}$ & $[T(\vec{v}_2)]_{\mathcal{C}}$

i.e. solve $\begin{bmatrix} 1 \\ -3 \end{bmatrix} = c_{11}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_{21}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

or equivalently $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

we can invert this $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$\therefore \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = [T\vec{v}_1]_{\mathcal{C}}$

similarly

$[T(\vec{v}_2)]_{\mathcal{C}} = \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

making T w.r.t. $\mathcal{B}$ & $\mathcal{C}$

$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -2 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{bmatrix}$

is this right? lets check an example.

what is the reflection of $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$? it should be $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

To see this we need $[T]_{\mathcal{C}\leftarrow\mathcal{B}} \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{\mathcal{B}}$

recall $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

making $\begin{bmatrix} 3 \\ 5 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ by inspection.

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -2 & -1/2 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

thats not $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$, what happened? these are in

$\mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ coordinates

$\Rightarrow -4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ should be our true answer! which is as predicted.

common mistakes to avoid

$\begin{bmatrix} -2 & -1/2 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} =$ reflection **NO** wrong coordinates for input

$\begin{bmatrix} -2 & -1/2 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix} =$ reflection **NO** wrong coords. for output be careful!

the final notes of importance relating these are composition and inverses i.e.

let $T: V \to W$ and $S: W \to \mathcal{U}$

With basises given by $\mathcal{B}$ for V, $\mathcal{C}$ for W & $\mathcal{D}$ for $\mathcal{U}$

then $S \circ T: V \to \mathcal{U}$ and can be formed as

$$[T]_{\mathcal{C} \leftarrow \mathcal{B}} \sim \text{"A"} \quad \& \quad [S]_{\mathcal{D} \leftarrow \mathcal{C}} \sim \text{"B"}$$

$$\text{"BA"} \sim [S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}}$$

note these are the same.

lastly $\left([T]_{\mathcal{C} \leftarrow \mathcal{B}}\right)^{-1} = [T^{-1}]_{\mathcal{B} \leftarrow \mathcal{C}}$