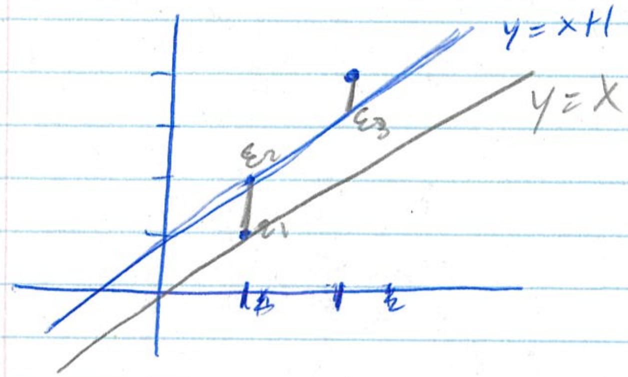
Math 225 lecture 15 Oct 11th 2023

Goal: students should see why least squares solutions are the "best fit" and be able to compute them using the direct method.

Section 7.3

Class Q: how do we decide mathematically which line/parabola, etc. is the "best fit"

suppose you have a set of points (1,1), (1,2), (2,4) can a line connect them all? NO



you cannot have a line through all 3. How close can we get? i.e. what is the "best fit"

guess $y=x+1$

x-value	desired y-value	y=x+1 value	difference/error ~~actual~~ desired − guess
1	1	2	$-1 = \varepsilon_1$
1	2	2	$0 = \varepsilon_2$
2	4	3	$1 = \varepsilon_3$

$$|\varepsilon| = \sqrt{\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2} = \sqrt{2}$$

(note the sign on ε is irrelevant which is why we do it this way

x-val	desired y-val	y=x val	error
1	1	1	0
1	2	1	1
2	4	2	2

$$|\varepsilon| = \sqrt{0^2+1^2+2^2} = \sqrt{5}$$

this is more error $\therefore$ worse guess.

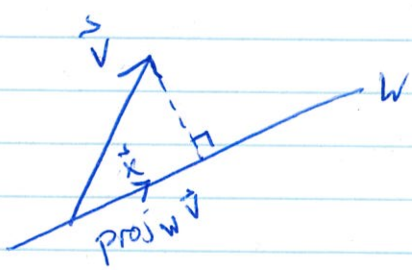
How do we minimize $|\varepsilon|$? Put it in a matrix! How?

x-val	desired y-val	$y = ax+b$	error
1	1	$a+b$	$a+b-1$
1	2	$a+b$	$a+b-2$
2	4	$2a+b$	$2a+b-4$

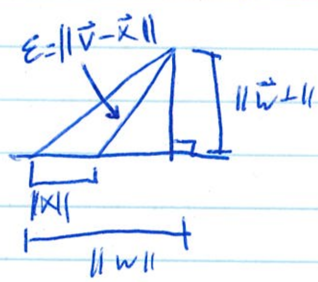
what is $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} a+b \\ a+b \\ 2a+b \end{bmatrix}}_{\vec{b}}$ we want this as close to $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ as possible

if possible do $A\vec{x} = b$! However we can't do that so the follow up is how close can we get?

aside find a point on W which is closest to the vector $\vec{v}$



recall we can write $\vec{v} = \vec{w} + \vec{w}^{\perp}$



$\|\vec{w} - \vec{x}\|^2 + \|\vec{w}^{\perp}\|^2 = \|\vec{v} - \vec{x}\|^2 \Rightarrow \|\vec{w}^{\perp}\| \leq \|\vec{v} - \vec{x}\|^2$

Hence the closest point to $\vec{v}$ in W is $\text{proj}_W(\vec{v})$ (note this works in higher dimensions too)

$\therefore$ where can we get to with A?

$$A\vec{x} = b\begin{bmatrix}1\\1\\1\end{bmatrix} + a\begin{bmatrix}1\\1\\2\end{bmatrix} = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}\right\} = \text{col}(A)$$

hence $\text{Proj}_{\text{col}(A)}(\vec{b})$ is the best we can do!

recall to compute $\text{proj}_{\text{col}(A)}$ we need an orthogonal basis for col(A) i.e. apply G-S to the columns of A

$$W = \text{span}\left\{\underset{\vec{x}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\vec{x}_2}{\begin{bmatrix}1\\1\\2\end{bmatrix}}\right\}$$

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$\therefore \vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix} - \left(\frac{1+1+2}{1+1+1}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\-1/3\\2/3\end{bmatrix}$$

$$\text{Proj}_W(\vec{b}) = \frac{\vec{b}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{b}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

Can use $\begin{bmatrix}-1\\-1\\2\end{bmatrix}$ as we only need orth. not normal.

$$= \left(\frac{7}{3}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} + \left(\frac{5}{6}\right)\begin{bmatrix}-1\\-1\\2\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\4\end{bmatrix} = P$$

this will be in Col(A) by construction as such

$A\vec{x} = \vec{P}$ will have a solution and said solution

$$\begin{bmatrix}1 & 1\\1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\4\end{bmatrix}$$

$\Rightarrow a = 5/2 \quad b = -1 \qquad y = \frac{5}{2}x - 1$ is the line of best fit

In general

A is an $m \times n$ matrix $\vec{b}$ is an m-vector $\vec{x}$ is an n-vector

assuming $A\vec{x} = \vec{b}$ is not consistent.

① produce a consistent system by replacing $\vec{b}$ with $\vec{p} = \text{proj}_{\text{col}(A)}(\vec{b})$

② solve $A\vec{x} = \vec{p}$

don't forget in order to compute the needed projection for step ① we need an orthogonal basis for $\text{col}(A)$.

Next time we will look at a different method of computing this as well as answer an important question.

- is there a unique solution and if so when?