Question 2 (15)

Let

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\} \quad \mathcal{D} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \quad \vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Calculate $[\vec{v}]_{\mathcal{D}}$ specifically by computing each of $\mathcal{P}_{B\leftarrow\mathcal{S}}$ ($\mathcal{S}$ is the standard basis) and $\mathcal{P}_{D\leftarrow\mathcal{B}}$ and verify your answer by showing the linear combination defined by $[\vec{v}]_{\mathcal{D}}$ does indeed get you back to $\vec{v}$

$$[\mathcal{B} \mid \mathcal{S}] = \begin{bmatrix}1&1&0&|&1&0&0\\0&1&1&|&0&1&0\\1&0&1&|&0&0&1\end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix}1&1&0&|&1&0&0\\-1&1&0&|&0&1&-1\\1&0&1&|&0&0&1\end{bmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{bmatrix}1&1&0&|&1&0&0\\0&2&0&|&1&1&-1\\1&0&1&|&0&0&1\end{bmatrix} \xrightarrow[2R_3]{2R_1} \begin{bmatrix}2&2&0&|&2&0&0\\0&2&0&|&1&1&-1\\2&0&2&|&0&0&2\end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix}2&0&0&|&1&-1&1\\0&2&0&|&1&1&-1\\2&0&2&|&0&0&2\end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix}2&0&0&|&1&-1&1\\0&2&0&|&1&1&-1\\0&0&2&|&-1&1&1\end{bmatrix}$$

$$\xrightarrow{\substack{1/2R_1\\1/2R_2\\1/2R_3}} \begin{bmatrix}1&0&0&|&1/2&-1/2&1/2\\0&1&0&|&1/2&1/2&-1/2\\0&0&1&|&-1/2&1/2&1/2\end{bmatrix} \quad \therefore\ \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{S}} \begin{bmatrix}1/2&-1/2&1/2\\1/2&1/2&-1/2\\-1/2&1/2&1/2\end{bmatrix}$$

$$[\mathcal{D} \mid \mathcal{B}] = \begin{bmatrix}1&1&1&|&1&1&0\\0&1&1&|&0&1&1\\0&0&1&|&1&0&1\end{bmatrix} \xrightarrow{\substack{R_1-R_2\\R_2-R_3}} \begin{bmatrix}1&0&0&|&1&0&-1\\0&1&0&|&-1&1&0\\0&0&1&|&1&0&1\end{bmatrix}$$

$$\therefore\ \mathcal{P}_{\mathcal{D}\leftarrow\mathcal{B}} \begin{bmatrix}1&0&-1\\-1&1&0\\1&0&1\end{bmatrix}$$

$$[\vec{v}]_{\mathcal{S}} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{aligned}[\vec{v}]_{\mathcal{B}} &= \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{S}}[\vec{v}]_{\mathcal{S}}\\ &= \begin{bmatrix}1/2&-1/2&1/2\\1/2&1/2&-1/2\\-1/2&1/2&1/2\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1/2-1+3/2\\1/2+1-3/2\\-1/2+1+3/2\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
[\vec{v}]_{\mathcal{D}} &= \mathcal{P}_{\mathcal{D}\leftarrow\mathcal{B}}[\vec{v}]_{\mathcal{B}} \\
&= \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-2 \\ -1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}
\end{aligned}
$$

Verify

$$
-1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-1+3 \\ -1+3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{v}
$$