Math 225 lecture 4 Sept 13th 2023

Goal: students should be able to describe characteristics of an eigen vector eigenvalue pair & recognize the differences between algebraic and geometric multiplicities of such eigenvalues.

Class Q: describe the properties that make a number an eigen value. list 2 uses for such a number

sections 4.1, 4.3, 4.4

if $A \sim B$ "A is similar to B" how are they related?

↳ if A & B are $n \times n$ matrices then $\underline{A \sim B}$ if $\exists$ P $n \times n$ and invertible such that $P^{-1}AP = B$ (secretly basis change)

Moreover A is called "diagonalizable" if it is similar to some diagonal matrix D

- these are useful concepts (trust me for now) we will return to but how do we see when this can happen?

Eigen vectors, values, & spaces

↳ let A be an $n \times n$ matrix then $\vec{v} \in \mathbb{R}^n$ is an eigen vector for A if $\vec{v} \neq \vec{0}$ and $A\vec{v} = \lambda v$ for some number λ, we call this number $\vec{v}$'s corresponding eigen value

Important concepts, will be used lots

natural questions: why do we care, and how do we find them

How: use determinants

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A - \lambda I_n)\vec{v} = 0$$

this means $(A - \lambda I_n)$ has a non-zero vector $\vec{v}$ in its null space meaning it <u>cannot</u> be invertible $\therefore \det(A - \lambda I_n) = 0$

[if $A\vec{v} = 0$ & A^{-1} exists $A^{-1}A\vec{v} = 0 \Rightarrow \vec{v} = 0$ a contradiction]

<u>Ex</u>: let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\cdot A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$

☆ we call this the characteristic polynomial of A denoted $p(\lambda)$ $\longrightarrow$

$$\det(A - \lambda I_2) = (1-\lambda)(3-\lambda) - 8$$
$$= 3 - 4\lambda + \lambda^2 - 8$$
$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

i.e. $\lambda \in \{5, -1\}$ meaning $\lambda = 5$ and $\lambda = 1$ are possible sol'n's

lets do $\lambda = 5$ what are its eigenvectors? <u>Ask</u> the null space of $A - \lambda I_n$ recall from last friday it means solving $[A - \lambda I_2 \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow -4X_1 + 2X_2 = 0$$

with $X_2 = t$ a parameter (no leading entry)

$-4X_1 + 2t = 0$

$-4X_1 = -2t \Rightarrow X_1 = \frac{1}{2}t$ making the solutions $\begin{bmatrix} t \\ \frac{1}{2}t \end{bmatrix} = t\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$

meaning $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 5$

<u>Note</u> any multipel of $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ works too

<u>In general</u>: a polynomial $p(\lambda) = \det(A - \lambda I)$ has destinct roots $\lambda_1, \ldots, \lambda_k$

(i.e. $p(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}$) with the n_i's being the "algebraic multiplicity of λ_i"

Note $n_1 + \dots + n_k = n$ as $P(\lambda)$ will have degree n as a polynomial

↳ for each λ_i we can also consider its <u>eigen space</u> denoted
$E_{\lambda_i} := \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda_i \vec{v}\}$

in our last example $E_5 = \{\begin{bmatrix} t \\ \frac{1}{2}t \end{bmatrix} \mid t \in \mathbb{R}\}$

↳ we then define $d_i = \dim(E_{\lambda_i})$ as the <u>geometric multiplicity of λ_i</u>

<u>quick facts</u>: • $1 \leq d_i \leq n_i$ for all i (geo mult $\leq$ alg mult)

• vectors from different eigenspaces are linearly independent (ie. their bases are disjoint)

a natural question would then be when are the geometric and algebraic multiplicities of A the same? this leads us back to why we care about eigenvalues

<u>Thm</u>: the diagonalization theorem. let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_k$ then the following are equivalent

① A is diagonalizable
② the union of all the bases of the eigen spaces of A contains n vectors (i.e. $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_k$ has n vectors & is a basis of $\mathbb{R}^n$)
③ the algebraic and geometric multiplicities of each eigenvalue match

not a full proof, but lets see ③ ⇒ ① in action as this shows how we will use this in a practical way

full proof in ____

since we have that $n_i = d_i$ for all i and we know $n_1 + n_2 + \dots + n_k = n$ we get that $d_1 + d_2 + \dots + d_k = n$

$\Rightarrow \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ forms a set of n linearly independent vectors in $\mathbb{R}^n$ and thus a basis for it

let $$P = \begin{bmatrix} | & & | & | & & | \\ v_{\lambda_1,1} & \dots & v_{\lambda_1,d_1} & v_{\lambda_2,1} & \dots & v_{\lambda_k,d_k} \\ | & & | & | & & | \end{bmatrix}$$

i.e. the $n \times n$ matrix formed from lining up all the basis elements of each Eigen-space, i.e. all the eigen vectors.

but notice what happens when we apply A to this matrix

$$AP = \begin{bmatrix} | & & | \\ Av_{\lambda_1,1} & \dots & v_{\lambda_k,d_k} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 v_{\lambda_1,1} & \dots & \lambda_k v_{\lambda_k,d_k} \\ | & & | \end{bmatrix}$$

$$= P \underbrace{\begin{bmatrix} \underbrace{\begin{matrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_1\end{matrix}}_{d_1}\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}d_1 & & & \\ & \underbrace{\begin{matrix}\lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_2\end{matrix}}_{d_2}\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}d_2 & & \\ & & \ddots & \\ & & & d_k\left\{\overbrace{\begin{matrix}\lambda_k & & 0 \\ & \ddots & \\ 0 & & \lambda_k\end{matrix}}^{d_k}\right. \end{bmatrix}}_{D}$$

in other words $AP = PD$ which since P must be invertible (why, ask?) is equivalent to saying that $P^{-1}AP = D$ which means $A \sim D$ which is the definition of being diagonalizable

Note: as linear transformations on an abstract vector space A & $P^{-1}AP$ are morally the same

<u>quick facts:</u> • the eigenvalues of a triangular (or diagonal) matrix are its diagonal entries

- a matrix is invertible iff 0 is not an eigenvalue

- if A is diagonalizable powers of A are easy to compute

$$P^{-1}AP = D \text{ or } A = PDP^{-1} \Rightarrow A^m = \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{n \text{ times}}$$

since D is diagonal $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ $= PD^mP^{-1}$

$$D^m = \begin{bmatrix} d_1^m & & 0 \\ & \ddots & \\ 0 & & d_n^m \end{bmatrix}$$

- similarly $A^k\vec{v}$ can be computed easily if we write $\vec{v}$ as a linear combination of eigenvectors (this will be explained and used next time) try it! ☺

<u>optional example!</u>