MATH 225, Fall 2023, Written Assignment # 1,
Due: Saturday September 23 by 11:59pm via Assign2.

1. (15 points) Consider the following matrix

$$A = \begin{pmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{pmatrix}.$$

(a) Write down the RREF of A.

$$RREF(A) = \begin{pmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) What is the dimension of the row space of A?

As the rank of A can be seen to be 2 from $RREF(A)$, this dimension is 2. In fact it has basis the two vectors formed from the two top rows of $RREF(A)$.

(c) Write down a basis of the column space of A ?

The column space has basis given by the columns of A corresponding to columns in $RREF(A)$ which have leading entries, *i.e.* by the first and third columns of A. In other words, a basis is given by

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(d) What is the nullity of A. Write down a basis of the null space of A?

The nullity of A is the equal to

$$\text{total number of variables} - \text{ rank of } A,$$

so in this case is equal to $5 - 2 = 3$. Let s, t, u be corresponding free variables. From $RREF(A)$, we see that the two equations defining the null space are

$$x_1 - 2x_2 + x_4 + \frac{1}{2}x_5 = 0 \quad (1)$$

$$x_3 + 3x_4 + \frac{7}{2}x_5 = 0. \quad (2)$$

In the second equation let us assign $x_4 = s$ and $x_5 = t$, so that

$$x_3 = -3s - 7/2t.$$

Letting $x_2 = u$, the first equation now gives that

$$x_1 = 2u - s - \frac{1}{2}t.$$

In sum we have found that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2u - s - 1/2t \\ u \\ -3s - 7/2t \\ s \\ t \end{pmatrix} = u \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{pmatrix}.$$

Hence a basis of the null space can be taken as

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{pmatrix}$$

2. (20 pts)The aim of this question will be to produce matrices with specified eigenvalues which are not just triangular! Let $p(x)$ be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

and define the *companion matrix* to the polynomial as

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

(a) Write down the matrix $C(p)$ of the polynomial $p(x) = x^3 - 4x^2 + 5x - 2$

(b) Find the characteristic polynomial of the matrix $C(p)$ which you wrote in the previous step.

(c) Show that $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue 2.

(d) Find the matrix $C(p)$ associated to the polynomial $p(x) = x^3 + ax^2 + bx + c$

(e) Determine the characteristic polynomial of the matrix $C(p)$ from the previous step

(f) Show that if λ is an eigenvalue of the companion matrix $C(p)$, then $\begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ corresponding to λ

(g) Construct a non-triangular 3×3 matrix with eigenvalues $-2, 1, 3$ using companion matrices. *Briefly* justify your answer.

(a) $C(p) = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(b) Expand along 1^{st} column: $\det \begin{pmatrix} 4-\lambda & -5 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$

$= (4-\lambda) \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} -5 & 2 \\ 1 & -\lambda \end{vmatrix}$

$= -(\lambda^3 - 4\lambda + 5\lambda - 2)$

(c) $C(p) \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

(d) $C(p) = \begin{pmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(e) $\det C(p) - \lambda I = \det \begin{pmatrix} -a-\lambda & -b & -c \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$

$= (-a-\lambda) \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} -b & -c \\ 1 & -\lambda \end{vmatrix}$

$= -(\lambda^3 + a\lambda^2 + b\lambda + c)$

(f)
$$\begin{pmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} -a\lambda^2 - b\lambda - c \\ \lambda^2 \\ \lambda \end{pmatrix}$$

But from (e) if λ is an eigenvector it must satisfy the char. polynomial of $C(p)$, i.e.

$$-(\lambda^3 + a\lambda^2 + b\lambda + c) = 0$$

$$\Rightarrow \quad -a\lambda^2 - b\lambda - c = \lambda^3$$

hence
$$C(p) \cdot \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \end{pmatrix} = \lambda \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$$

proving its an eigenvector w/ eigenvalue λ

(f) $C(p)$ for

$$p(x) = (x+2)(x-1)(x-3) = x^3 - 2x - 5x + 6$$

has char. polynomial

$$-(\lambda^3 - 2\lambda - 5\lambda + 6)$$

$$= -(\lambda+2)(\lambda-1)(\lambda-3)$$

$$\therefore \quad C(p) = \begin{pmatrix} 2 & 5 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

has required eigenvalues

3. (20 pts) Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

(a) Find matrices P and D (with D diagonal) so that $A = PDP^{-1}$

Note: The matrix P will have columns consisting of eigenvectors as we did in class. A previous version of the assignment asked you to write $A = P^{-1}DP$ for some (different) matrix P. The solutions to these two variants are given by taking an inverse (of the matrix answer).

Expanding along the first column, we find that

$$\det(A - \lambda I_4) = (2-\lambda)\left((2-\lambda)(-2-\lambda)^2\right) = (2-\lambda)^2(2+\lambda)^2.$$

From this we see that the eigenvalues are $\lambda = 2$ and $\lambda = -2$ both with algebraic multiplicity 2. To compute the geometric multiplicity (and basis of eigenspaces) we need to compute

$$\text{Nullspace}(A - 2I_4) \text{ and } \text{Nullspace}(A + 2I_4).$$

We begin by computing the following.

$$\text{Nullspace}(A - 2I_4) = \text{Nullspace}\begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \tag{3}$$

$$= \text{Nullspace}\begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\} \tag{4}$$

$$\text{Nullspace}(A + 2I_4) = \text{Nullspace}\begin{pmatrix} 4 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\} \tag{5}$$

Forming the matrix with columns given by these (linearly independent) eigenvectors, we find the matrix

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

will then satisfy

$$P^{-1} \cdot AP = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

(b) For each positive integer n, write down a formula for A^n

We may use the formula $A = PDP^{-1}$ with D the diagonal matrix above. It follows that $A^n = PD^nP^{-1}$, and we can easily compute D^n as the diagonal matrix $\text{diag}(2^n, 2^n, (-2)^n, (-2)^n)$. Note that if n is even, this is just $2^n I_4$, so that in this case $PD^nP^{-1} = 2^n P \cdot I \cdot P^{-1} = 2^n I$, i.e. we have

$$A^n = 2^n I_4 \text{ when } n \text{ is even .}$$

Suppose then that n is odd. In this case, has

$$D^n = \begin{pmatrix} 2^n & 0 & 0 & 0 \\ 0 & 2^n & 0 & 0 \\ 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{pmatrix} = 2^{n-1}D.$$

Hence when n is odd, we find

$$A^n = PD^nP^{-1} = 2^{n-1}PDP^{-1} = 2^{n-1}A = \begin{pmatrix} 2^n & 0 & 0 & 2^{n+1} \\ 0 & 2^n & 0 & 0 \\ 0 & 0 & -2^n & 0 \\ 0 & 0 & 0 & -2^n \end{pmatrix}$$

4. (20 pts) A study of pine nuts in the American southwest from 1940 to 1947 hypothesized that nut production followed a Markov chain. The data suggested that if one year's crop was good, then the probabilities that the following year's crop would be good, fair, or poor were 0.08, 0.07, 0.85 respectively; if one year's crop was fair, then the probabilities that the following year's crop would be good, fair, or poor were 0.09, 0.11, and 0.80, respectively; if one year's crop was poor, then the probabilities that the following year's crop would be good, fair, or poor were 0.11, 0.05, 0.84 respectively.

   (a) Write down the transition matrix for this Markov process

   Abbreviating the states as G, fair F, and poor P, the matrix with respect to this ordering G, F, P becomes

   $$A = \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & .11 & .05 \\ 0.85 & 0.8 & 0.84 \end{pmatrix}$$

   (b) If the pine cut crop was good in 1940, find the probabilities of a good crop in the years 1941 through 1945

   This amounts to computing $A \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ which is just the first column of A, i.e. the probability of being good is 0.08, of being fair is .07, and of being poor is .85.

   (c) In the long run, what proportion of the crops will be good, fair, and poor?

   As discussed in the lectures, the steady state vector is the unique eigenvector of A with eigenvalue 1 that has all positive entries. In other words, we need to first compute

   $$\text{Nullspace}(A - I)$$

   and then (possibly) rescale the single basis vector so that the the sum of entries is equal to 1 (all the entries will automatically be positive by general theory!)

We can find using the usual method that

$$(A - I) = \text{Span}\begin{pmatrix} 1 \\ 537/1024 \\ 8125/1024 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ .52 \\ 7.93 \end{pmatrix}.$$

So we need to rescale by diving this vector by $1 + .52 + 7.93 \simeq 9.46$ to obtain the vector of

$$\begin{pmatrix} .11 \\ .06 \\ 0.83 \end{pmatrix}$$