Question 2 (25)

For this problem we will be looking at the following

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and solving the least squares problem for the system $A\vec{x} = \vec{b}$ in each of the three ways described in class

a) Solve $A\vec{x} = \vec{b}$ by solving the normal system. This method is the most direct and should be what you compare your other answers with.

$$A^\top A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$$

$$A^\top b = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$A^\top A\vec{x} = A^\top b$$

$$\begin{bmatrix} 6 & -3 & | & 1 & 0 \\ -3 & 2 & | & 0 & 1 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 0 & 1 & | & 1 & 2 \\ -3 & 2 & | & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2-2R_1} \begin{bmatrix} 0 & 1 & | & 1 & 2 \\ -3 & 0 & | & -2 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 0 & 1 & | & 1 & 2 \\ 1 & 0 & | & 2/3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} 1 & 0 & | & 2/3 & 1 \\ 0 & 1 & | & 1 & 2 \end{bmatrix}$$

$$(A^\top A)^{-1} = \begin{bmatrix} 2/3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\vec{x} = (A^\top A)^{-1}(A^\top b) = \begin{bmatrix} 2/3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 2 \end{bmatrix}$$

$$\therefore \; y = \frac{4}{3}x + 2$$

b) Solve $A\vec{x} = \vec{b}$ via the direct method by projecting onto column space. (Remember, what do you need to project onto a subspace)

Using G.S.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \mathrm{proj}_{\vec{v}_1}(\vec{x}_1) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} \right\}$$

$$\mathrm{proj}_{\mathrm{Col}(A)}(\vec{b}) = \mathrm{proj}_{\vec{v}_1}(\vec{b}) + \mathrm{proj}_{\vec{v}_2}(\vec{b}) = \frac{\vec{v}_1 \cdot \vec{b}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{b}}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \frac{2}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{1/2} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$A\vec{x} = \mathrm{proj}_{\mathrm{Col}(A)}(\vec{b})$$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 4/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$
\begin{aligned}
x_1 &= 4/3 \\
2x_1 - x_2 &= 2/3 \\
-x_1 + x_2 &= 2/3 \\
\Rightarrow x_2 &= \ \ 2 \\
\therefore \vec{x} &= \begin{bmatrix} \frac{4}{3} \\ 2 \end{bmatrix} \\
\therefore \ \ y &= \frac{4}{3}x + 2
\end{aligned}
$$

c) Lastly solve $A\vec{x} = \vec{b}$ using the QR factorization A. (Note, you should already have done the hardest part of finding the QR decomposition above)

$$
A = QR
$$

From b)

$$
\begin{aligned}
\vec{v}_1 &= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} \\
\vec{q}_1 &= \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \\
\vec{q}_2 &= \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{1/2}}\vec{v}_2 = \sqrt{2} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \\
Q &= \begin{bmatrix} 1/\sqrt{6} & \sqrt{2}/2 \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & \sqrt{2}/2 \end{bmatrix} \\
R = Q^\top A &= \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{6} + 4/\sqrt{6} + 1/\sqrt{6} & -2/\sqrt{6} - 1/\sqrt{6} \\ \sqrt{2}/2 - \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} 6/\sqrt{6} & -3/\sqrt{6} \\ 0 & \sqrt{2}/2 \end{bmatrix} \\
QR\vec{x} &= \vec{b} \Rightarrow R\vec{x} = Q^\top \vec{b} \\
Q^\top \vec{b} &= \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ \sqrt{2} \end{bmatrix} \\
&\begin{bmatrix} 6/\sqrt{6} & -3/\sqrt{6} \\ 0 & \sqrt{2}/2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2/\sqrt{6} \\ \sqrt{2} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{2}/2x_2 &= \sqrt{2} \\
6/\sqrt{6}x_1 - 3/\sqrt{6}x_2 &= 2/\sqrt{6}
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= 2 \\
6/\sqrt{6}x_1 &= 2/\sqrt{6} + 6/\sqrt{6} \Rightarrow x_1 = 8/6 = 4/3 \\
\therefore\ \vec{x} &= \begin{bmatrix} \frac{4}{3} \\ 2 \end{bmatrix} \\
\therefore\ y &= \frac{4}{3}x + 2
\end{aligned}
$$