MATH 225, Fall 2023, Written Assignment # 2,
Due: Saturday October 14 by 11:59pm via Assign2.

1. (20 **pts**) Verify that the vectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$
form an orthogonal basis of $\mathbb{R}^3$. Then find the orthogonal decomposition of the vector $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to this basis, *i.e.* compute $[\mathbf{w}]_{\mathcal{B}}$.

2. (a) (5 **pts**) Find the orthogonal complement $W^\perp$ of the subspace
$$W = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 1/2t, y = -1/2t, z = 2t, t \in \mathbb{R} \}.$$
   (b) Consider the subspace $W = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 + x_3 + x_4 = 0\}$ of $\mathbb{R}^4$
      i. (3 **pts**) Find a basis for $W^\perp$. (Hint: this question requires almost no work! Think about normal vectors to planes)
      ii. (7 **pts**) Find a basis for W and use it to construct an orthogonal basis for W using the Gram-Schmidt process.
      iii. Let $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Find $\text{proj}_W(\mathbf{v})$ in two ways:
         A. (2 **pts**) By first computing $\text{proj}_{W^\perp} v$
         B. (3 **pts**) Using the orthogonal basis of W found above.

3. (a) (10 **pts**) Apply Gram-Schmidt to the vectors
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
   to find an orthogonal basis for $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
   (b) (10 **pts**) Find a QR factorization to the matrix $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$ whose columns are given by the vectors from part (a).

4. (20 **pts**) Assume $b \neq 0$. Orthogonally diagonalize the matrix
$$A = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$$
by producing an orthogonal matrix Q and a diagonal matrix D such that $D = Q^t A Q$.

5. (20 **pts**) Verify that polynomials with real coefficients and degree less than or equal to 2 with the usual notions of addition and scalar multiplication are a vector space. (this means you will need to check all 10 axioms like we did in class).