Q3 - 20 marks

Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

a) Find matrices P and D (with D diagonal) so that $A = PDP^{-1}$

$$C_A(\lambda) = (\lambda - 2)^2 (\lambda + 2)^2$$
 $\lambda = 2, -2$

For $\lambda = 2$,

$$A - 2I = \begin{pmatrix} 2 - 2 & 0 & 0 & 4 \\ 0 & 2 - 2 & 0 & 0 \\ 0 & 0 & -2 - 2 & 0 \\ 0 & 0 & 0 & -2 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

$$E_{2} = \text{Nul}(A - 2I) = \begin{pmatrix} 0 & 0 & 0 & 4 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & -4 & 0 & | & 0 \\ 0 & 0 & -4 & 0 & | & 0 \\ 0 & 0 & 0 & -4 & | & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{\frac{1}{4}R_{1} \\ R_{1} \leftrightarrow R_{2} \\ -\frac{1}{4}R_{3} \\ -\frac{1}{4}R_{4} \\ \longrightarrow \end{pmatrix}}} \begin{pmatrix} 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}}$$

$$x_{1} = \text{free} \qquad x_{1} = s$$

$$x_{2} = \text{free} \qquad x_{2} = t$$

$$x_{3} = 0 \qquad x_{3} = 0 \qquad \Rightarrow \qquad \vec{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

 $x_4 = 0$

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$$\therefore \mathcal{B}_{E_2} = \left\{ egin{bmatrix} 1 \ 0 \ 0 \ 0 \end{bmatrix}, egin{bmatrix} 0 \ 1 \ 0 \ 0 \end{bmatrix}
ight\}$$

For $\lambda = -2$,

Thus,
$$P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
 and $D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

b) For each positive integer n, write down a formula for A^n

$$\begin{split} \forall n \in \mathbb{Z}^+, \\ A^n &= (PDP^{-1})^n \\ &= (PDP^{-1})(\cancel{P}D\cancel{P}^{-1}) \dots (\cancel{P}DP^{-1}) = PD^nP^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{split}$$