Question 1 (20)

Let $V = \mathcal{P}_2(\mathbb{R})$ that is the polynomials of degree less than or equal to two with real coefficients. Let the map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be given by $p(x) \mapsto p'(x)$ i.e. the first derivative.

a) Verify T is a linear map

Let $\mathbf{0}(x) = 0x^2 + 0x + 0 \in \mathcal{P}_2(\mathbb{R})$

$$\begin{aligned} T(\mathbf{0}(x)) &= (0x^2 + 0x + 0)' = 0 = 0x^2 + 0x + 0 \in \mathcal{P}_2(\mathbb{R}) \\ &\therefore \quad T(\mathbf{0}(x)) = \mathbf{0}(x) \end{aligned}$$

Let $f(x) = f_0x^2 + f_1x + f_2 \in \mathcal{P}_2(\mathbb{R})$ and $c \in \mathbb{R}$

$$\begin{aligned} T(cf) &= (cf_0x^2 + cf_1x + cf_2)' = (2cf_0x + cf_1 + 0) \\ &= c(2f_0x + f_1) = c(f_0x^2 + f_1x + f_2)' = cT(f) \\ &\therefore T(cf) = cT(f) \end{aligned}$$

Let $f = f_0x^2 + f_1x + f_2 \in \mathcal{P}_2(\mathbb{R})$ and $g = g_0x^2 + g_1x + g_2 \in \mathcal{P}_2(\mathbb{R})$

$$\begin{aligned} T(f+g) &= (f_0x^2 + f_1x + f_2 + g_0x^2 + g_1x + g_2)' \\ &= 2f_0x + f_1 + 2g_0x + g_1 \\ T(f) + T(g) &= (f_0x^2 + f_1x + f_2)' + (g_0x^2 + g_1x + g_2)' \\ &= 2f_0x + f_1 + 2g_0x + g_1 \\ &\therefore T(f+g) = T(f) + T(g) \end{aligned}$$

Thus, T is a linear map/transformation

b) Find the kernel of T

Since $\forall c \in \mathcal{P}_0(\mathbb{R})$ (polynomials with degree 0) $\in \mathcal{P}_2(\mathbb{R})$, $c' = 0$

$$\begin{aligned} \ker(T) &= \{ax^2 + bx + c : a = b = 0\} \\ &= \{c : c \in \mathbb{R}\} \end{aligned}$$

c) Find the range of T

Let $f = ax^2 + bx + c \in \mathcal{P}_2(\mathbb{R})$. Since $f' = 2ax + b \in \mathcal{P}_2(\mathbb{R})\ \forall a, b \in \mathbb{R}$:

$$\begin{aligned} \text{range}(T) &= \{T(f) \in \mathcal{P}_2(\mathbb{R}) : f \in \mathcal{P}_2(\mathbb{R})\} \\ &= \{2ax + b : a, b \in \mathbb{R}\} \end{aligned}$$

d) is T onto? is it one-to-one?

$$\begin{aligned} \text{range}(T) = \{2ax + b : a, b \in \mathbb{R}\} = p'(x) \quad \forall p(x) \in \mathcal{P}_2(\mathbb{R}) \\ \therefore T \text{ is onto} \end{aligned}$$

Since $\forall c \in \mathbb{R}\ \forall f = 0x^2 + 0x + c \in P_2(\mathbb{R}),\ f' = 0$

ie. For $c_1 = 1, c_2 = 2$ s.t. $f = 0x^2 + 0x + 1,\ g = 0x^2 + 0x + 2$

$$f' = g' = 0 \Rightarrow T(f) = T(g) = 0 \quad \therefore T \text{ is not onto}$$