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Math 225 lecture 2 Sept 8^{th} 2023

Goal: students should recognize the differences between column space, row space, and null space. Additionally they should be able to define and verify a subspace

Class Q: name three of the subspaces we will deal with in this class.

"fancy"

section 3.5

dfn ↳ a subspace is a subset $W \subseteq V$ (think $\mathbb{R}^n$ for now) such that

i) $\vec{0} \in W$ ii) if $v, w \in W$ then $v+w \in W$

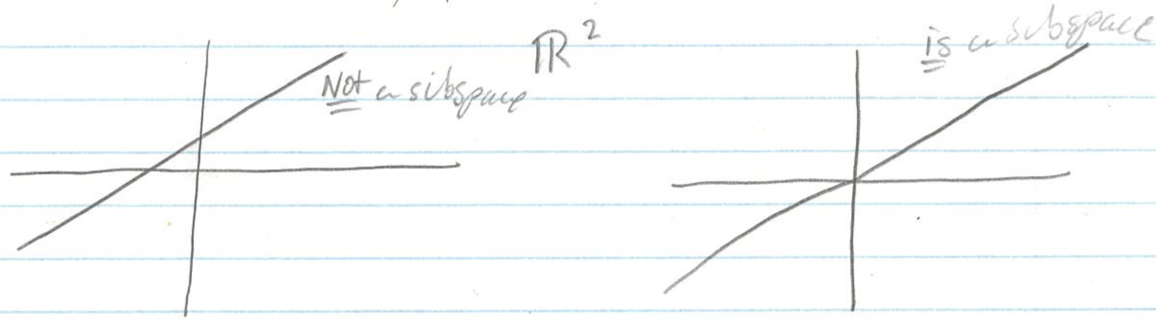
iii) for any $c \in \mathbb{R}$ and $w \in W$ $cw \in W$

i.e. closed under addition & scaler mult.

- note importance to the course

Breaking down the points:

① easily verified, pictorially



② closure under vector addition
i.e. can't escape the subspace via normal ways of combining vectors.
(examples to follow)

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(3) Closure under scalar multiplication, i.e. rescaling or resizing a vetor keeps you in the space

real/moral[**] dfn ↳ a subspace is a subset you can't escape from with normal means.

Ex: let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} \subset \mathbb{R}^3$

(aside about set notation if needed)

is it a subspace?

(1) $0+0+0=0 \Rightarrow \vec{0} \in W$

(2) let $w_1 = (x_1, y_1, z_1)$ $w_2 = (x_2, y_2, z_2) \in W$ (recall meaning)

$w_1 + w_2 = (x_1+x_2, y_1+y_2, z_1+z_2) \in W$?

Yes, $(x_1+x_2) + (y_1+y_2) + (z_1+z_2)$

$= x_1 + y_1 + z_1 + x_2 + y_2 + z_2 = 0 + 0 = 0$

(3) let $c \in \mathbb{R}$ consider $cw_1 = (cx_1, cy_1, cz_1)$

Check membership $cx_1 + cy_1 + cz_1 = c(x_1 + y_1 + z_1) = c0 = 0$

all the axioms hold $\Rightarrow$ W is a subspace

NON Ex: let $V = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y\}$

remember it only takes one counterexample to show a space isn't a subspace

ASK Which one does it break?

if 2 use $w_1 = w_2 = (1, 1)$ if 3 use $w_1 = (1, 1)$
$c = 2$

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reiterate the importance of subspaces and allude to abstract V-spaces.

Basic properties & related concepts

let $W \subset \mathbb{R}^n$ be a subspace

$\{v_1, \ldots, v_n\}$ span W if $\forall w \in W$ $w = c_1 v_1 + \cdots + c_n v_n$
(mention lin comb.)

$\mathcal{B} = \{v_1, \ldots, v_n\}$ is a Basis of W if

i) $\mathcal{B}$ spans W ii) no smaller subset of $\mathcal{B}$ would

or iib) if $c_1 v_1 + \cdots + c_n v_n = 0 \Rightarrow c_1 = \cdots = c_n = 0$

Facts: ① every subspace has a basis (∞ - many)

② basies are not unique but the number of elements in them is
(that # is called the dimension of the subspace)

we will talk lots about subspaces especially ones associated to linear transformations some important naturally occuring subspaces are as follows. for A an $m \times n$ matrix $= \begin{bmatrix} - \vec{r}_1 - \\ \vdots \\ - \vec{r}_m - \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

$null(A)$ = the null space of $A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}$

$Col(A)$ = the column space of A = $Span\{\vec{v}_1, \ldots, \vec{v}_n\} \subset \mathbb{R}^m$

$Row(A)$ = the row space of A = $Span\{\vec{r}_1, \ldots, \vec{r}_m\} \in \mathbb{R}^n$

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<u>Note</u>: ① dim of null space = "nullity" = (# of free vars = # of vars − rank)

② dim col(A) = dim row(A)

③ the non zero rows in REF of A form a basis for row(A)

④ the columns of A <u>corresponding to</u> the columns of REF(A) with leading 1's form a basis for col(A)

<u>Ex</u>: let $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \xrightarrow[\substack{R_3+3R_1 \\ R_4-4R_1}]{R_2-2R_1} \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 5 & 10 & 1 & 19 \\ 0 & -3 & -6 & -3 & -21 \end{bmatrix}$

$\xrightarrow{R_4 - R_2} \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 5 & 10 & 1 & 19 \\ 0 & 0 & 0 & -2 & -8 \end{bmatrix} \xrightarrow{-2 R_4} \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 5 & 10 & 1 & 19 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_3-R_4]{R_2+R_4} \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & 0 & -9 \\ 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{5R_2+3R_3} \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[\substack{R_4 \to R_3 \\ \frac{1}{5}R_3}]{\substack{R_2 \to R_4 \\ R_3 \to R_2}} \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

as per ③ $r_1 = [1\ 1\ 3\ 1\ 6]$, $r_2 = [0\ 1\ 2\ 0\ 3]$, $r_3 = [0\ 0\ 0\ 1\ 4]$

are a basis for Row(A)

via ④ $v_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ & $v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ are a basis for col(A)

via ① $\dim \operatorname{null}(A) = \#$ of free vars $= \#$ of vars $-$ rank
$= 5 - 3 = 2$

i.e. find the basis for the solutions of the system

$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 6 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{array}{rr} x_1 + x_2 + 3x_3 + x_4 + 6x_5 = 0 & ① \\ x_2 + 2x_3 + 3x_5 = 0 & ② \\ x_4 + 4x_5 = 0 & ③ \end{array}$$

with $x_3 = s \quad x_5 = t$
(free vars via no leading 1)

③ $\Rightarrow x_4 = -4t$

② $\Rightarrow x_2 + 2s + 3t = 0 \Rightarrow x_2 = -2s - 3t$

① $\Rightarrow x_1 - 2s - 3t + 3s - 4t + 6t = 0 \Rightarrow x_1 = -s + t$

$\therefore$ all solutions look like

$$\begin{bmatrix} -s+t \\ -2s-3t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

meaning $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$ form a basis for null A