Question 3 (20)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by a reflection about the y-axis. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}, \ \mathcal{D} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\}$$

be bases for $\mathbb{R}^2$. Find $[T]_{\mathcal{D}\leftarrow\mathcal{B}}$ and confirm that it works as desired (like we did in class on November 1st) by applying it to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (**Important**: don't forget that the vector you are testing it on is written with respect to the standard basis)

$$\begin{aligned} &(a, b) \to (-a, b) \\ &T(\vec{v}_1) = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &T(\vec{v}_2) = T \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \end{aligned}$$

Solve for $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = c_{11} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_{21} \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}^{-1} : \begin{bmatrix} -1 & 4 & | & 1 & 0 \\ 1 & -2 & | & 0 & 1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 0 & 2 & | & 1 & 1 \\ 1 & -2 & | & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{bmatrix} 0 & 2 & | & 1 & 1 \\ 1 & 0 & | & 1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0 & 1 & | & 1/2 & 1/2 \\ 1 & 0 & | & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & | & 1 & 2 \\ 0 & 1 & | & 1/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} = [T\vec{v}_1]_{\mathcal{D}}$$

Solve for $\begin{bmatrix}-2\\4\end{bmatrix} = c_{12}\begin{bmatrix}-1\\1\end{bmatrix} + c_{22}\begin{bmatrix}4\\-2\end{bmatrix}$

$$\begin{bmatrix}-1 & 4\\1 & -2\end{bmatrix}\begin{bmatrix}c_{12}\\c_{22}\end{bmatrix} = \begin{bmatrix}-2\\4\end{bmatrix}$$

$$\therefore \begin{bmatrix}c_{12}\\c_{22}\end{bmatrix} = \begin{bmatrix}-1 & 4\\1 & -2\end{bmatrix}^{-1}\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}1 & 2\\1/2 & 1/2\end{bmatrix}\begin{bmatrix}-2\\4\end{bmatrix}$$

$$= \begin{bmatrix}6\\1\end{bmatrix}$$

$$\therefore [T]_{\mathcal{D}\leftarrow\mathcal{B}} = \begin{bmatrix}-1 & 6\\ -\frac{1}{2} & 1\end{bmatrix}$$

Reflection of $\begin{bmatrix}1\\1\end{bmatrix}$ about the y-axis should be $\begin{bmatrix}-1\\1\end{bmatrix}$

$$\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}1/2\\1/4\end{bmatrix}$$

$$[T]_{\mathcal{D}\leftarrow\mathcal{B}}\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}-1 & 6\\-1/2 & 1\end{bmatrix}\begin{bmatrix}1/2\\1/4\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}_{\mathcal{D}}$$

$$\Rightarrow 1\begin{bmatrix}-1\\1\end{bmatrix} + 0\begin{bmatrix}4\\-2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix} \text{ as predicted}$$