Question 1 (25)

In this question, you will compare the QR and SVD decomposition of a matrix.

$$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

a) Find the QR decomposition for the above matrix

$$\text{col}(A) = \left\{ \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2} \end{bmatrix}, \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} \right\} = \{w_1, w_2\}$$

Using G.S.

$$\vec{v}_1 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2} \end{bmatrix}$$

$$\vec{q}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{5/2}}\vec{v}_1 = \frac{\sqrt{2}}{\sqrt{5}} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$\begin{aligned}
\vec{v}_2 = \vec{w}_2 - \text{proj}_{\vec{q}_1}\vec{w}_2 &= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} - (\vec{q}_1 \cdot \vec{w}_2)\vec{q}_1 \\
&= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} - \left(\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} - \left(\frac{\sqrt{2}}{2\sqrt{5}} - \frac{2\sqrt{2}}{\sqrt{5}} \right) \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} - \left(\frac{-3\sqrt{2}}{2\sqrt{5}} \right) \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} - \begin{bmatrix} -3\sqrt{2}/10 \\ 3\sqrt{2}/5 \end{bmatrix} = \begin{bmatrix} \dfrac{5\sqrt{2}}{10} + \dfrac{3\sqrt{2}}{10} \\ \sqrt{2} - \dfrac{3\sqrt{2}}{5} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{4\sqrt{2}}{5} \\ \dfrac{2\sqrt{2}}{5} \end{bmatrix}
\end{aligned}$$

$$\vec{q}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{32/25 + 8/25}}\vec{v}_2 = \frac{1}{\sqrt{40/25}}\vec{v}_2$$

$$= \frac{5}{2\sqrt{10}} \begin{bmatrix} \frac{4\sqrt{2}}{5} \\ \frac{2\sqrt{2}}{5} \end{bmatrix} = \begin{bmatrix} \frac{4\sqrt{2}}{2\sqrt{10}} \\ \frac{2\sqrt{2}}{2\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{2}}{\sqrt{10}} \\ \frac{\sqrt{2}}{\sqrt{10}} \end{bmatrix}$$

$$Q = [\vec{q}_1, \vec{q}_2]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2\sqrt{2}}{\sqrt{10}} \\ -\frac{2}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{10}} \end{bmatrix}$$

$$R = Q^\top A$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2\sqrt{2}}{\sqrt{10}} & \frac{\sqrt{2}}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2\sqrt{5}} + \frac{2\sqrt{2}}{\sqrt{5}} & \frac{\sqrt{2}}{2\sqrt{5}} - \frac{2\sqrt{2}}{\sqrt{5}} \\ \frac{2}{\sqrt{10}} - \frac{2}{\sqrt{10}} & \frac{2}{\sqrt{10}} + \frac{2}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5\sqrt{2}}{2\sqrt{5}} & -\frac{3\sqrt{2}}{2\sqrt{5}} \\ 0 & \frac{4}{\sqrt{10}} \end{bmatrix}$$

b) Find an SVD decomposition for the matrix A above, i.e. find orthogonal matrices U_A, V_A and a "diagonal" matrix (in the sense of SVD) Σ_A such that

$$A = U_A \Sigma_A^t V_A$$

$$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$A^\top = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2} \end{bmatrix}$$

$$\begin{aligned}
A^\top A &= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 2/4+2 & 2/4-2 \\ 2/4-2 & 2/4+2 \end{bmatrix} \\
&= \begin{bmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{bmatrix} \\
\det(A^\top A - \lambda I) &= \begin{vmatrix} 5/2-\lambda & -3/2 \\ -3/2 & 5/2-\lambda \end{vmatrix} \\
&= \left(\frac{5}{2}-\lambda\right)\left(\frac{5}{2}-\lambda\right) + \left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) \\
&= \frac{25}{4} - \frac{5}{2}\lambda - \frac{5}{2}\lambda + \lambda^2 - \frac{9}{4} \\
&= \frac{16}{4} - 5\lambda + \lambda^2 = 4 - 5\lambda + \lambda^2 \\
&= (\lambda-1)(\lambda-4) \\
\lambda_1 &= 1,\ \lambda_2 = 4 \\
\sigma_1 &= 1,\ \sigma_2 = 2 \\
\Sigma_A &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\
E_1 &= \mathrm{Nul}\left(\begin{bmatrix} 5/2-1 & -3/2 \\ -3/2 & 5/2-1 \end{bmatrix}\right) \\
&= \left[\begin{array}{cc|c} 3/2 & -3/2 & 0 \\ -3/2 & 3/2 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 3/2 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

$$\xrightarrow{\frac{2}{3}R_1} \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_2 &= s \end{aligned} \Rightarrow \begin{aligned} x_1 &= s \\ x_2 &= s \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s$$

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow V_1 = \operatorname{span} \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

$$\begin{aligned} E_4 &= \operatorname{Nul}\left(\begin{bmatrix} 5/2 - 4 & -3/2 \\ -3/2 & 5/2 - 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} -3/2 & -3/2 & | & 0 \\ -3/2 & -3/2 & | & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} -3/2 & -3/2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \\ &\xrightarrow{-\frac{2}{3}R_1} \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= s \end{aligned} \Rightarrow \begin{aligned} x_1 &= -s \\ x_2 &= \ \ s \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} s$$

$$\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow V_4 = \operatorname{span} \left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

$$V_A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{aligned} A\vec{v}_1 &= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 + 1/2 \\ -1 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{u}_1 \\ A\vec{v}_2 &= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} -1/2 + 1/2 \\ 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$U_A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\therefore\ U_A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ \Sigma_A = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix},\ V_A = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

c) For the matrix R found in part a), find an SVD decomposition, i.e. matrices U_R, Σ_R, V_R such that

$$R = U_R \Sigma_R^t V_R$$

$$R = \begin{bmatrix}\frac{5\sqrt{2}}{2\sqrt{5}} & -\frac{3\sqrt{2}}{2\sqrt{5}}\\ 0 & \frac{4}{\sqrt{10}}\end{bmatrix}$$

$$R^\top = \begin{bmatrix}\frac{5\sqrt{2}}{2\sqrt{5}} & 0\\ -\frac{3\sqrt{2}}{2\sqrt{5}} & \frac{4}{\sqrt{10}}\end{bmatrix}$$

$$R^\top R = \begin{bmatrix}\frac{5\sqrt{2}}{2\sqrt{5}} & 0\\ -\frac{3\sqrt{2}}{2\sqrt{5}} & \frac{4}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{5\sqrt{2}}{2\sqrt{5}} & -\frac{3\sqrt{2}}{2\sqrt{5}}\\ 0 & \frac{4}{\sqrt{10}}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{25(2)}{4(5)} + 0 & -\frac{15(2)}{4(5)} + 0\\ -\frac{15(2)}{4(5)} + 0 & \frac{9(2)}{4(5)} + \frac{16}{10}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{5}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \frac{5}{2}\end{bmatrix}$$

$$\begin{aligned}
\det(R^\top R - \lambda I) &= \begin{vmatrix} \frac{5}{2} - \lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} - \lambda \end{vmatrix} \\
&= \left(\frac{5}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) - \left(\frac{3}{2}\right)\left(\frac{3}{2}\right) \\
&= \frac{25}{4} - \frac{5}{2}\lambda - \frac{5}{2}\lambda + \lambda^2 - \frac{9}{4} \\
&= \frac{16}{4} - 5\lambda + \lambda^2 \\
&= 4 - 5\lambda + \lambda^2 = (\lambda - 1)(\lambda - 4) \\
&\lambda_1 = 1,\ \lambda_2 = 4 \\
&\sigma_1 = 1,\ \sigma_2 = 2 \\
\Sigma_R &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\
E_1 &= \operatorname{Nul}\left(\begin{bmatrix} 5/2 - 1 & -3/2 \\ -3/2 & 5/2 - 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 3/2 & -3/2 & | & 0 \\ -3/2 & 3/2 & | & 0 \end{bmatrix} \\
&\xrightarrow{R_2 + R_1} \begin{bmatrix} 3/2 & -3/2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \\
&\xrightarrow{\frac{2}{3}R_1} \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \\
\begin{matrix} x_1 - x_2 = 0 \\ x_2 = s \end{matrix} &\Rightarrow \begin{matrix} x_1 = s \\ x_2 = s \end{matrix} \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s \\
\vec{v}_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow V_1 = \operatorname{span}\left\{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\right\} \\
E_4 &= \operatorname{Nul}\left(\begin{bmatrix} 5/2 - 4 & -3/2 \\ -3/2 & 5/2 - 4 \end{bmatrix}\right) \\
&= \begin{bmatrix} -3/2 & -3/2 & | & 0 \\ -3/2 & -3/2 & | & 0 \end{bmatrix}
\end{aligned}$$

$$\xrightarrow{R_2-R_1}\begin{bmatrix}-3/2 & -3/2 & | & 0\\ 0 & 0 & | & 0\end{bmatrix}$$

$$\xrightarrow{-\frac{2}{3}R_1}\begin{bmatrix}1 & 1 & | & 0\\ 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{aligned} x_1+x_2&=0\\ x_2&=s\end{aligned} \Rightarrow \begin{aligned} x_1&=-s\\ x_2&=\ \ s\end{aligned} \Rightarrow \vec{x}=\begin{bmatrix}-1\\ 1\end{bmatrix}s$$

$$\vec{v}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}-\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\end{bmatrix}\Rightarrow V_4=\text{span}\left\{\begin{bmatrix}-\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\end{bmatrix}\right\}$$

$$V_R=\begin{bmatrix}\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}}\end{bmatrix}$$

$$R\vec{v}_1=\begin{bmatrix}\dfrac{5\sqrt{2}}{2\sqrt{5}} & -\dfrac{3\sqrt{2}}{2\sqrt{5}}\\ 0 & \dfrac{4}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{5}{2\sqrt{5}}-\dfrac{3}{2\sqrt{5}}\\ \dfrac{4}{2\sqrt{5}}\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\sqrt{5}}\\ \dfrac{2}{\sqrt{5}}\end{bmatrix}=\vec{u}_1$$

$$R\vec{v}_2=\begin{bmatrix}\dfrac{5\sqrt{2}}{2\sqrt{5}} & -\dfrac{3\sqrt{2}}{2\sqrt{5}}\\ 0 & \dfrac{4}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}-\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}-\dfrac{5}{2\sqrt{5}}-\dfrac{3}{2\sqrt{5}}\\ \dfrac{4}{\sqrt{20}}\end{bmatrix}=\begin{bmatrix}-4/\sqrt{5}\\ 2/\sqrt{5}\end{bmatrix}$$

$$\vec{u}_2=\frac{1}{\sigma_2}\begin{bmatrix}-4/\sqrt{5}\\ 2/\sqrt{5}\end{bmatrix}=\begin{bmatrix}\dfrac{-2}{\sqrt{5}}\\ \dfrac{1}{\sqrt{5}}\end{bmatrix}$$

$$\therefore\ U_R = \begin{bmatrix} \dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \\ \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \end{bmatrix},\ \Sigma_R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ V_R = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

d) In general, suppose A is an $n \times n$ matrix w/ linearly independent columns, and so has a QR decomposition with Q an orthogonal matrix. If we compare the SVD decomposition for R and A, what changes? In other words: what is the relation between the singular values of R and those of A, what is the relation between U_R and U_A? what is the relation between V_R and V_A

The singular values of R and A are identical. Similarly, the right singular vectors of R and A are also identical. Additionally, the left singular vector of A (U_A) is equivalent to QU_R.