Math 225 lecture 17th Oct 16th 2023

Goal: show final examples of LSS via each of the 3 methods. And to introduce the topic of SVD (Singular value decomposition)

section 7.3/7.4

Class Q: What is your prefered method of doing LSS? Also how would you describe to someone what that process does?

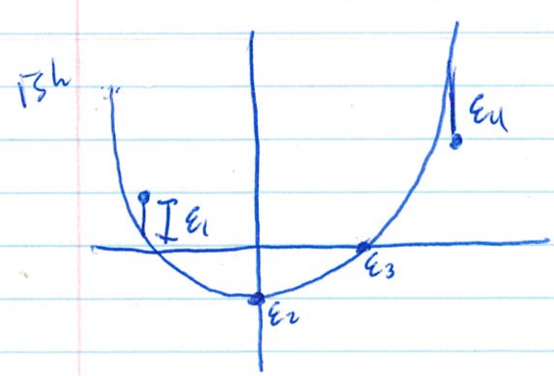
last week: we showed how LSS work via three different methods.

Warning! something will look wrong toward the end this is intentional and will be explained!

- directly (via solving the projected system)
- normalsystem (via $\vec{x} = (A^TA)^{-1} \cdot A^T\vec{b}$)
- QR method (via computing the QR factorization & solving $R\vec{x} = Q^T\vec{b}$)

Finish our thoughts on this with 1 more example slightly larger via each of our 3 methods.

Ex: find the parabola of best fit for the points $(-1,1), (0,-1), (1,0), (2,2)$



each parabola is $y = ax^2+bx+c$
$\therefore A\vec{x} = \vec{b}$ is set up as

$$\begin{array}{ccc} x^2 & x & 1 \end{array}$$
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\varepsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2}$$

note: we reorder the vectors to make the computation easier this is fine for 1 & 2 nore our issue with problem 3

2

Method 1. (direct)

for this we need to apply G.-S to the columns of A. since we need this for both this method and QR we will do this

let $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}$

$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \underbrace{\left(\frac{-1+0+1+2}{1+1+1+1}\right)}_{=\left(\frac{1}{2}\right)}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$ $= \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix} - \underbrace{\left(\frac{1+0+1+4}{1+1+1+1}\right)}_{=\left(\frac{3}{2}\right)}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\left(\frac{-\frac{3}{2}+0+\frac{1}{2}+\frac{12}{2}}{\frac{9}{4}+\frac{1}{4}+\frac{1}{4}+\frac{9}{4}}\right)}_{=\left(\frac{5}{5}\right)=1}\begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{bmatrix} - \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

normalize for $\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} \frac{-3\sqrt{5}}{10} \\ \frac{-\sqrt{5}}{10} \\ \sqrt{5}/10 \\ 3\sqrt{5}/10 \end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

Hilroy

for the direct method we need $\vec{p} = \text{proj}_{\text{col}(A)}(\vec{b})$

that is $\vec{p} = \text{proj}_{\vec{u}_1}(\vec{b}) + \text{proj}_{\vec{u}_2}(\vec{b}) + \text{proj}_{\vec{u}_3}(\vec{b})$

$$= \underbrace{\left(\frac{\frac{1}{2} - \frac{1}{2} + 0 + 1}{1}\right)}_{(=1)} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \underbrace{\left(\frac{\frac{-3\sqrt{5}}{10} + \frac{\sqrt{5}}{10} + 0 + \frac{6\sqrt{5}}{10}}{1}\right)}_{\left(=\frac{2\sqrt{5}}{5}\right)} \begin{bmatrix} \frac{-3\sqrt{5}}{10} \\ \frac{-\sqrt{5}}{10} \\ \frac{\sqrt{5}}{10} \\ \frac{3\sqrt{5}}{10} \end{bmatrix} + \underbrace{\left(\frac{\frac{1}{2} + \frac{1}{2} + 0 + 1}{1}\right)}_{(=2)} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

sorry, computations are hard for me too ☹

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{10} \\ -\frac{7}{10} \\ -\frac{3}{10} \\ \frac{21}{10} \end{bmatrix}$$

now we solve $A\vec{x} = \vec{p}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{9}{10} \\ -\frac{7}{10} \\ -\frac{3}{10} \\ \frac{21}{10} \end{bmatrix}$$

this yields a solution of $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{5} \\ -\frac{7}{10} \end{bmatrix}$

<u>method 2</u> (normal system)

Here we solve $\vec{x} = (A^T A)^{-1} \cdot A^T \vec{b}$

$$(A^T A)^{-1} = \left(\begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & -1/4 & -1/4 \\ -1/4 & 9/20 & 3/20 \\ -1/4 & 3/20 & 11/20 \end{bmatrix}$$

(recall we solve $[A \mid I]$ to $[I \mid A^{-1}]$)

$$A^T\vec{b} = \begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}$$

$$\vec{x} = (A^TA)^{-1} \cdot A^T\vec{b}$$

$$= \begin{bmatrix} 1/4 & -1/4 & -1/4 \\ -1/4 & 9/20 & 3/20 \\ -1/4 & 3/20 & 11/20 \end{bmatrix} \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3/5 \\ -7/10 \end{bmatrix}$$ as expected!

<u>method 3</u> (QR method)

here we solve $R\vec{x} = Q^T\vec{b}$

note, we already did the work to find Q

its $Q = \begin{bmatrix} 1/2 & -3\sqrt{5}/10 & 1/2 \\ -1/2 & -\sqrt{5}/10 & 1/2 \\ -1/2 & \sqrt{5}/10 & 1/2 \\ 1/2 & 3\sqrt{5}/10 & 1/2 \end{bmatrix}$

This is NOT a standard QR do not do it this way

$$\therefore R = Q^TA = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -3\sqrt{5}/10 & -\sqrt{5}/10 & \sqrt{5}/10 & 3\sqrt{5}/10 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ \sqrt{5} & \sqrt{5} & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

now solving $R\vec{x} = Q^T\vec{b}$ yields the correct solution of $\begin{bmatrix} 1 \\ -3/5 \\ -7/10 \end{bmatrix}$ However, R is lower triangular when it should be upper, what happened?

when we did G-S we took the vectors in reverse order. while this yields an orthogonal basis for Col(A) it ~~messes up~~ wrecks the structure of our QR factorization.

Had we done G-S in the correct order we would get our standard QR & $R\vec{x} = Q^T\vec{b}$ would look like

$$\begin{bmatrix} 3\sqrt{2} & \frac{4\sqrt{2}}{3} & \sqrt{2} \\ 0 & \frac{\sqrt{22}}{3} & \frac{-\sqrt{22}}{11} \\ 0 & 0 & \frac{2\sqrt{55}}{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{6} & 0 & \frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{-13\sqrt{22}}{66} & 0 & \frac{5\sqrt{22}}{66} & \frac{\sqrt{22}}{33} \\ \frac{3\sqrt{55}}{110} & \frac{\sqrt{55}}{10} & \frac{9\sqrt{55}}{110} & \frac{-3\sqrt{55}}{110} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

and would still yield $\begin{bmatrix} 1 \\ -3/5 \\ -7/10 \end{bmatrix}$ as a solution vector.

<u>Punchline!</u> if doing the QR factorization (or method of LSS using QR) <u>don't</u> change the order of the columns of A when doing G-S. <u>if</u> all you need is an orth basis for col(A) use the columns in any order.