Q2 - 20 marks

The aim of this question will be to produce matrices with specificed eigen values which are not just triangular! Let $p(x)$ be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and define the *companion matrix* to the polynomial as

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

a) Write down the matrix $C(p)$ of the polynomial $p(x) = x^3 - 4x^2 + 5x - 2$

$$C(p) = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

b) Find the characteristic polynomial of the matrix C(p) which you wrote in the previous step

$$C(p) - \lambda I = \begin{pmatrix} 4-\lambda & -5 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$\begin{aligned} \det(C(p) - \lambda I &= 2 \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{vmatrix} \\ &= 2(1) - \lambda[(4-\lambda)(-\lambda) - (-5)] \\ &= 2 - \lambda(-4\lambda + \lambda^2 + 5) \\ &= 2 + 4\lambda^2 - \lambda^3 - 5\lambda \\ &= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0 \end{aligned}$$

c) Show that $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue 2

Let $\vec{x}$ be $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$, then

$$C(p)\vec{x} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 - 10 + 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$$

$$\lambda\vec{x} = 2 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$$

$$\therefore \; C(p)\vec{x} = \lambda\vec{x}$$

$\therefore \; \vec{x} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue $\lambda = 2$

d) Find the matrix $C(p)$ associated to the polynomial $p(x) = x^3 + ax^2 + bx + c$

$$C(p) = \begin{pmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

e) Determine the characteristic polynomial of the matrix $C(p)$ from the previous step

$$C(p) - \lambda I = \begin{pmatrix} -a-\lambda & -b & -c \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$\begin{aligned}
\det(C(p) - \lambda I) &= -1 \begin{vmatrix} -a-\lambda & -c \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} \\
&= -1[(-a-\lambda)(0) - (-c)(1)] - \lambda[(-a-\lambda)(-\lambda) - (-b)(1)] \\
&= -1(c) - \lambda(a\lambda + \lambda^2 + b) \\
&= -c - a\lambda^2 - \lambda^3 - b\lambda \\
&= -\lambda^3 - a\lambda^2 - b\lambda - c = 0
\end{aligned}$$

f) Show that if λ is an eigenvalue of the companion matrix $C(p)$, then $\begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ corresponding to λ

$$
\begin{aligned}
E_\lambda &= \text{Nul}(C(p) - \lambda I) \\
&= \left(\begin{array}{ccc|c} -a-\lambda & -b & -c & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -a-\lambda & -b & -c & 0 \\ 0 & 1 & -\lambda & 0 \\ 1 & -\lambda & 0 & 0 \end{array}\right) \\
&\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ -a-\lambda & -b & -c & 0 \end{array}\right) \\
&\xrightarrow[R_1+\lambda R_2]{R_3+bR_2} \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ -a-\lambda & 0 & -c-b\lambda & 0 \end{array}\right) \\
&\xrightarrow{R_3+(a+\lambda)R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & -\lambda^3 - a\lambda^2 - b\lambda - c & 0 \end{array}\right)
\end{aligned}
$$

From the characteristic polynomial: $\lambda^3 = -a\lambda^2 - b\lambda - c$

$$
\therefore \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & -(-a\lambda^2 - b\lambda - c) - a\lambda^2 - b\lambda - c & 0 \end{array}\right)
$$

$$
\therefore \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 \quad - \lambda^2 x_3 &= 0 \\ x_2 - \ \lambda x_3 &= 0 \end{aligned}
$$

$$
\begin{aligned} x_1 &= \lambda^2 s \\ x_2 &= \lambda s \\ x_3 &= \ s \end{aligned} \quad \Rightarrow \quad \vec{x} = s \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}
$$

$$
\therefore \mathcal{B}_{E_\lambda} = \left\{ \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix} \right\}
$$

$\Rightarrow \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ corresponding to λ

g) Construct a non-triangular 3×3 matrix of eigenvalues -2, 1, 3 using companion matrices. *Briefly* justify your answer.

From f)

$$E_\lambda = \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$$

Thus,

$$\lambda = -2, 1, 3$$

$$\mathcal{B}_{E_{-2}} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \qquad \mathcal{B}_{E_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathcal{B}_{E_3} = \begin{pmatrix} 9 \\ 3 \\ 1 \end{pmatrix}$$

$$\therefore\ P = \begin{pmatrix} 4 & 1 & 9 \\ -2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$[\ P \mid I\ ] = \begin{pmatrix} 4 & 1 & 9 & | & 1 & 0 & 0 \\ -2 & 1 & 3 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2-R_3]{R_1-R_3} \begin{pmatrix} 3 & 0 & 8 & | & 1 & 0 & -1 \\ -3 & 0 & 2 & | & 0 & 1 & -1 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{pmatrix} 3 & 0 & 8 & | & 1 & 0 & -1 \\ 0 & 0 & 10 & | & 1 & 1 & -2 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{10}R_2} \begin{pmatrix} 3 & 0 & 8 & | & 1 & 0 & -1 \\ 0 & 0 & 1 & | & \frac{1}{10} & \frac{1}{10} & \frac{-2}{10} \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1-8R_2} \begin{pmatrix} 3 & 0 & 0 & | & \frac{2}{10} & \frac{-8}{10} & \frac{6}{10} \\ 0 & 0 & 1 & | & \frac{1}{10} & \frac{1}{10} & \frac{-2}{10} \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{3}R_1]{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{30} & \frac{-8}{30} & \frac{6}{30} \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{30} & \frac{3}{30} & \frac{-6}{30} \end{array}\right)$$

$$\xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{30} & \frac{-8}{30} & \frac{6}{30} \\ 0 & 1 & 1 & \frac{-2}{30} & \frac{8}{30} & \frac{24}{30} \\ 0 & 0 & 1 & \frac{3}{30} & \frac{3}{30} & \frac{-6}{30} \end{array}\right)$$

$$\xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{30} & \frac{-8}{30} & \frac{6}{30} \\ 0 & 1 & 0 & \frac{-5}{30} & \frac{5}{30} & 1 \\ 0 & 0 & 1 & \frac{3}{30} & \frac{3}{30} & \frac{-6}{30} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{15} & \frac{-4}{15} & \frac{1}{5} \\ 0 & 1 & 0 & \frac{-1}{6} & \frac{1}{6} & 1 \\ 0 & 0 & 1 & \frac{1}{10} & \frac{1}{10} & \frac{-1}{5} \end{array}\right)$$

$$P^{-1} \begin{pmatrix} \frac{1}{15} & -\frac{4}{15} & \frac{1}{5} \\ -\frac{1}{6} & \frac{1}{6} & 1 \\ \frac{1}{10} & \frac{1}{10} & -\frac{1}{5} \end{pmatrix}$$

$$A = PDP^{-1}$$

$$= \begin{pmatrix} 4 & 1 & 9 \\ -2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{15} & -\frac{4}{15} & \frac{1}{5} \\ -\frac{1}{6} & \frac{1}{6} & 1 \\ \frac{1}{10} & \frac{1}{10} & -\frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 1 & 27 \\ 4 & 1 & 9 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{15} & -\frac{4}{15} & \frac{1}{5} \\ -\frac{1}{6} & \frac{1}{6} & 1 \\ \frac{1}{10} & \frac{1}{10} & -\frac{1}{5} \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 5 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$