Math 225 lecture 35 Dec 6^{th} 2023

Goal: cover examples & discussions regarding pre-midterm content ~~with the~~ for the final exam.

Class Q: what did you get wrong on the midterm? how will you fix that for the final?

ideas for today! pre-midterm material specifically complex #s Markov matrices, orthogonal complements & LSS via the direct method.

Complex #s things to keep in mind

- what is i
- how do we write an element of $\mathbb{C}$
- where do we expect them to appear?

1. $i = \sqrt{-1}$ its only property is that $i^2 = -1$. you can treat it like ~~an~~ a variable with that property if you would like.

2. $z = x+iy$ or $a+ib$ etc. it has a real part x & imaginary part y both of which are real numbers remember if $z = x+iy$ $\bar{z} = x-iy$ that is the complex conjugate.

3. the primary context for this class is eigen stuff especially the values. don't forget real matrices can have complex eigen values, but they always come in pairs ~~...~~ λ & $\bar{\lambda}$

for example $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\det(A - \lambda I) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$
$\lambda = \pm i$

<u>Markov matrices</u> you don't have a calculator nor a lot of time As such there will not really be computations with these. What is important is their definition & properties.

- a Markov matrix is a square matrix with entries ≥ 0 an whose columns are probability vectors (i.e. they sum to 1)

- $\lambda = 1$ is always an e.val & $|\lambda_i| < 1$ for all other evals

- it has a "steady state vector" that is ~~vector~~ a vector which is unchanged by applying the matrix (the e.vect for e.val $\lambda = 1$)

- $\lim_{k \to \infty} M^k \vec{v}_0 = \vec{s}$ where M is a markov matrix, $\vec{v}_0$ is a probability vector, & $\vec{s}$ is the steady state vector.

- see the notes from lecture 5 sept 16th for more details this is very unlikely to see a major role on the final as we only coverd it one day.

LSS via the direct method

remember theres lots of ways to get to a least squares problem but the hallmarks are a system which is not solvable that you'd like to approximate a solution to.

the direct method involves projecting the solution vetor onto the column space to do this we need an orthogonal basis of col(A) to project onto

Ex: find a LSS to $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

first orthogonalize col(A)

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad v_2 = \vec{x}_2 - proj_{\vec{v}_1}(\vec{x}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2+1}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

now ~~to~~ find $\vec{P} = proj_{\vec{v}_1}(\vec{b}) + proj_{\vec{v}_2}(\vec{b}) = \frac{\vec{v}_1 \cdot \vec{b}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{b}}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$

$$= \frac{4+6-1}{4+4+1}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \frac{2/3 - 2 - 2/3}{1/9 + 4/9 + 4/9}\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 4/3 \\ -4/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 10/3 \\ -1/3 \end{bmatrix}$$

now we find that the system $A\vec{x} = \vec{P}$ will be solvable (despite being over determined) & we do this via the usual method of solving

$$[A \vdots \vec{P}] = \left[\begin{array}{cc|c} 2 & 1 & 4/3 \\ 2 & 0 & 10/3 \\ 1 & 1 & -1/3 \end{array}\right] \xrightarrow[\text{reduction \& solving}]{\text{row}} \vec{x} = \begin{bmatrix} 5/3 \\ -2 \end{bmatrix}$$

see oct 13 & 16 letures 16 & 17 & midterm review lectures (oct 23 & 25) for more detail & examples

lastly orthogonal complements

Recall: If V is ~~an ... space~~ inner product space & $W \subset V$ is a subspace then $W^\perp$ is the subspace of all vectors orthogonal to everything in W.

i.e. $W^\perp = \{\vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = 0 \ \forall \vec{w} \in W\}$

Properties
- $W^\perp$ is a subspace
- $(W^\perp)^\perp = W$
- $W \cap W^\perp = \{\vec{0}\}$
- if $W = \text{span}\{\vec{w}_1, \ldots, \vec{w}_k\}$ then $\vec{v} \in W^\perp$ iff $\langle \vec{v}, \vec{w}_i \rangle = 0 \ \forall i$
- $(\text{row}(A))^\perp = \text{null}(A)$, $(\text{col}(A))^\perp = \text{null}(A^T)$

remember the last property is your main one for computational purposes

the other main point about $\perp$ is that $\dim(W) + \dim(W^\perp) = \dim(V)$

<u>Important!</u> $W^\perp \subset V$, as well orthogonal complements <u>must</u> live in the same space.

Types of questions you could be asked

let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, 3x+4y=0 \right\}$ find a basis for $W^\perp$

let $W = \text{span}\{\vec{w}_1, \vec{w}_2\}$ find $\text{Proj}_{W^\perp \text{ or } W} \vec{u}$ for some $\vec{u}$

let ~~V=~~ let $W = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ find the orthogonal decomposition with respect to W of $\vec{u}$ ie write $\vec{u} = \vec{w} + \vec{v}$ with $\vec{w} \in W$ & $\vec{v} \in W^\perp$

and so on (would you like to see one of these worked out on friday?)