Math 225 lecture ~~12~~ 13 Oct 4^{th} 2023

Goal: lay the groundwork & reminders for orthogonal diagonalization of symmetric matrices. show students this process in brief.

Class Q: If A is orthogonally diagonalizable then where do its eigen values live.

Section 5.4

lets do some recalling practice problems & a few proofs regarding symmetric matrices.

Recall: how do we diagonalize a matrix?

Ex: let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ lets diagonalize it!

$$\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \right| = (1-\lambda)(-2-\lambda) - 4$$
$$= -2 - \lambda + 2\lambda + \lambda^2 - 4$$
$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

$\therefore \lambda_1 = -3 \quad \lambda_2 = 2$

lets look for eigenvectors.

$E_{-3} = null\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right)$ by inspection $\begin{bmatrix} 1 \\ -2 \end{bmatrix} t$ or $span\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$

$E_2 = null\left(\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\right)$ by inspection $\begin{bmatrix} 2 \\ 1 \end{bmatrix} t$ or $span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

this means A is diagonalizable with

$P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ & $P^{-1}AP = D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

what do we notice? $E_{-3} \perp E_2$
coincidence? (foreshadowing ☺)

i.e. $\vec{P}_1 \perp \vec{P}_2$ if we normalize we get

$\vec{u}_1 = \frac{1}{\|\vec{P}_1\|}\vec{P}_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$ $\vec{u}_2 = \frac{1}{\|\vec{P}_2\|} P_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

now if we take $Q = [\vec{u}_1, \vec{u}_2]$ we have
still that $Q^{-1}AQ = D$ but also Q is
an orthogonal matrix (i.e. $Q^{-1} = Q^T$)

↳ we say matricies A that have such a
Q which is orthogonal s.t. $Q^{-1}AQ = D$ are called
orthogonally diagonalizable

Symmetric matricies:

these are matricies for which $A^T = A$
∴ symmetrical about the main diagonal.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or I_n $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} a & b & c & d \\ b & a & b & c \\ c & b & a & b \\ d & c & b & a \end{bmatrix}$

Properties of symmetric matricies

Lemma: let A be a symmetric matrix w/ e. vals λ_1, λ_2, $\lambda_1 \neq \lambda_2$ & $\vec{v}_1 \in E_{\lambda_1}$ $\vec{v}_2\ E_{\lambda_2}$ then $\vec{v}_1 \cdot \vec{v}_2 = 0$ $(\lambda_1, \lambda_2 \neq 0)$
(i.e. eigen values of different eigenspaces here are orth)

Proof: Recall $A v_i = \lambda_i v_i$ for $i = 1, 2$
$\therefore$ $v_1 = \frac{1}{\lambda_1} A v_1$ & $v_2 = \frac{1}{\lambda_2} A v_2$

consider $\vec{v}_1 \cdot \vec{v}_2 = (\frac{1}{\lambda_1} A\vec{v}_1)^T \vec{v}_2$ (as matrices)

$$= \frac{1}{\lambda_1} (A\vec{v}_1)^T \cdot \vec{v}_2$$

$\begin{pmatrix} (AB)^T = B^T A^T \\ (AB)^{-1} = B^{-1} A^{-1} \end{pmatrix}$

$$= \frac{1}{\lambda_1} \vec{v}_1^{\,T} A^T \vec{v}_2$$

$$= \frac{1}{\lambda_1} \vec{v}_1^{\,T} A \vec{v}_2$$ (by symmetry)

$$= \frac{1}{\lambda_1} \vec{v}_1^{\,T} \lambda_2 \vec{v}_2$$

$$= \frac{\lambda_2}{\lambda_1} \vec{v}_1^{\,T} \vec{v}_2 = \frac{\lambda_2}{\lambda_1} (\vec{v}_1 \cdot \vec{v}_2)$$

note $\frac{\lambda_2}{\lambda_1} \neq 0, 1$ $\therefore$

$$(\vec{v}_1 \cdot \vec{v}_2) - \frac{\lambda_2}{\lambda_1}(\vec{v}_1 \cdot \vec{v}_2) = 0 \Rightarrow \underbrace{(1 - \tfrac{\lambda_2}{\lambda_1})}_{\neq 0} (\vec{v}_1 \cdot \vec{v}_2) = 0$$

$\therefore$ $\vec{v}_1 \cdot \vec{v}_2 = 0$ $\quad \vec{v}_1 \perp \vec{v}_2$

$\blacksquare$

Lemma: if A is real & symmetric then the eigen values of A are real

(aside: z is real iff $z = \bar{z}$ i.e. $x + iy = x - iy$)

Proof: suppose λ is an e. val for A and $\vec{v}$ its e. vec. i.e. $A\vec{v} = \lambda \vec{v}$

we can take complex conjugates & get

~~Suppose~~ $A\bar{\vec{v}} = \bar{A}\bar{\vec{v}} = \underbrace{\overline{A\vec{v}} = \overline{\lambda\vec{v}}}_{\text{start here}} = \bar{\lambda}\bar{\vec{v}}$

we can also take the transpose to see

$\bar{\vec{v}}^T A = \bar{\vec{v}}^T A^T = (A\bar{\vec{v}})^T = (\bar{\lambda}\bar{\vec{v}})^T = \bar{\lambda}\bar{\vec{v}}^T$

lets look at

$\lambda(\bar{\vec{v}}^T\vec{v}) = \bar{\vec{v}}^T(\lambda\vec{v}) = \bar{\vec{v}}^T(A\vec{v}) = (\bar{\vec{v}}^T A)\vec{v} = \bar{\lambda}(\bar{\vec{v}}^T\vec{v})$

$\Rightarrow (\lambda - \bar{\lambda})(\bar{\vec{v}}^T v) = 0$

since $\vec{v} = \begin{bmatrix} a_1 + ib_1 \\ \vdots \\ a_n + ib_n \end{bmatrix}$ $\bar{\vec{v}} = \begin{bmatrix} a_1 - ib_1 \\ \vdots \\ a_n - ib_n \end{bmatrix}$

$\Rightarrow \bar{\vec{v}}^T\vec{v} = (a_1^2 + b_1^2) + \ldots + (a_n^2 + b_n^2) \neq 0$ since $\vec{v} \neq \vec{0}$ on account of being an e.vec.

hence we must have $(\lambda - \bar{\lambda}) = 0 \Rightarrow \lambda = \bar{\lambda}$
i.e. λ is real!

$\blacksquare$

You will <u>not</u> be expected to do proofs like this

<u>Thm:</u> let A be an $n \times n$ real matrix. then A is symmetric iff it is orthogonally diagonalizable.