Q2 - 20 marks

The aim of this question will be to produce matrices with specificed eigen values which are not just triangular! Let $p(x)$ be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and define the *companion matrix* to the polynomial as

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

a) Write down the matrix $C(p)$ of the polynomial $p(x) = x^3 - 4x^2 + 5x - 2$

$$C(p) = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

b) Find the characteristic polynomial of the matrix C(p) which you wrote in the previous step

$$C(p) - \lambda I = \begin{pmatrix} 4-\lambda & -5 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$\begin{aligned} \det(C(p) - \lambda I &= 2\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{vmatrix} \\ &= 2(1) - \lambda[(4-\lambda)(-\lambda) - (-5)] \\ &= 2 - \lambda(-4\lambda + \lambda^2 + 5) \\ &= 2 + 4\lambda^2 - \lambda^3 - 5\lambda \\ &= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0 \end{aligned}$$

c) Show that $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue 2

Let $\vec{x}$ be $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$, then

$$C(p)\vec{x} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 - 10 + 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$$

$$\lambda\vec{x} = 2 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$$

$$\therefore\ C(p)\vec{x} = \lambda\vec{x}$$

$\therefore\ \vec{x} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue $\lambda = 2$

d) Find the matrix $C(p)$ associated to the polynomial $p(x) = x^3 + ax^2 + bx + c$

$$C(p) = \begin{pmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

e) Determine the characteristic polynomial of the matrix $C(p)$ from the previous step

$$
\begin{aligned}
C(p) - \lambda I &= \begin{pmatrix} -a-\lambda & -b & -c \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} \\
\det(C(p) - \lambda I) &= -1 \begin{vmatrix} -a-\lambda & -c \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} \\
&= -1[(-a-\lambda)(0) - (-c)(1)] - \lambda[(-a-\lambda)(-\lambda) - (-b)(1)] \\
&= -1(c) - \lambda(a\lambda + \lambda^2 + b) \\
&= -c - a\lambda^2 - \lambda^3 - b\lambda \\
&= -\lambda^3 - a\lambda^2 - b\lambda - c = 0
\end{aligned}
$$

f) Show that if λ is an eigenvalue of the companion matrix $C(p)$, then $\begin{pmatrix}\lambda^2\\ \lambda\\ 1\end{pmatrix}$ is an eigenvector of $C(p)$ corresponding to λ

$$E_\lambda = \mathrm{Nul}(C(p)-\lambda I)$$

$$= \left(\begin{array}{ccc|c} -a-\lambda & -b & -c & 0\\ 1 & -\lambda & 0 & 0\\ 0 & 1 & -\lambda & 0\end{array}\right) \xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} -a-\lambda & -b & -c & 0\\ 0 & 1 & -\lambda & 0\\ 1 & -\lambda & 0 & 0\end{array}\right)$$

$$\xrightarrow{R_1\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -\lambda & 0 & 0\\ 0 & 1 & -\lambda & 0\\ -a-\lambda & -b & -c & 0\end{array}\right) \xrightarrow[R_1+\lambda R_2]{R_3+bR_2} \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0\\ 0 & 1 & -\lambda & 0\\ -a-\lambda & 0 & -c-b\lambda & 0\end{array}\right)$$

$$\xrightarrow{R_3+(a+\lambda)R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0\\ 0 & 1 & -\lambda & 0\\ 0 & 0 & -\lambda^3-a\lambda^2-b\lambda-c & 0\end{array}\right)$$

From the characteristic polynomial: $\lambda^3 = -a\lambda^2 - b\lambda - c$

$$\therefore \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0\\ 0 & 1 & -\lambda & 0\\ 0 & 0 & -(-a\lambda^2-b\lambda-c)-a\lambda^2-b\lambda-c & 0\end{array}\right)$$

$$\therefore \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0\\ 0 & 1 & -\lambda & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

$$\begin{aligned} x_1 \qquad - \lambda^2 x_3 &= 0\\ x_2 - \ \lambda x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= \lambda^2 s\\ x_2 &= \lambda s \qquad \Rightarrow \qquad \vec{x} = s\begin{bmatrix}\lambda^2\\ \lambda\\ 1\end{bmatrix}\\ x_3 &= s \end{aligned}$$

$$\therefore \mathcal{B}_{E_\lambda} = \left\{\begin{bmatrix}\lambda^2\\ \lambda\\ 1\end{bmatrix}\right\} \Rightarrow \begin{pmatrix}\lambda^2\\ \lambda\\ 1\end{pmatrix} \text{ is an eigenvector of } C(p) \text{ corresponding to } \lambda$$

g) Construct a non-triangular 3×3 matrix of eigenvalues -2, 1, 3 using companion matrices. *Briefly* justify your answer.

Let $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then,

$$[P \mid I] = \begin{pmatrix} 1 & 1 & 1 & \mid & 1 & 0 & 0 \\ 1 & 1 & 0 & \mid & 0 & 1 & 0 \\ 0 & 1 & 1 & \mid & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & \mid & 1 & 0 & -1 \\ 1 & 1 & 0 & \mid & 0 & 1 & 0 \\ 0 & 1 & 1 & \mid & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 & \mid & 1 & 0 & -1 \\ 0 & 1 & 0 & \mid & -1 & 1 & 1 \\ 0 & 1 & 1 & \mid & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 & \mid & 1 & 0 & -1 \\ 0 & 1 & 0 & \mid & -1 & 1 & 1 \\ 0 & 0 & 1 & \mid & 1 & -1 & 0 \end{pmatrix}$$

$$\therefore \; P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\text{Let } D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = PDP^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -2 & 3 \\ -3 & 1 & 3 \\ 2 & -2 & 1 \end{pmatrix}$$