Q1 - 15 marks

Consider the following matrix

$$A = \begin{pmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{pmatrix}$$

a) Write down the RREF of A

$$A = \begin{pmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & -2 & 1 & 4 & 4 \\ -1 & 2 & 1 & 2 & 3 \\ 2 & -4 & 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 2 & -4 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_3-2R_1} \begin{pmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & -2 & -6 & -7 \end{pmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{pmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 1 & 3 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{pmatrix} 1 & -2 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 3 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b) What is the dimension of the row space of A

$$\dim(A) = 2$$

c) Write down a basis of the column space of A

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

d) What is the nullity of A. Write down a basis of the nullspace A?

nullity$(A) = 3$

From a):

$$\begin{pmatrix} 1 & -2 & 0 & 1 & \frac{1}{2} & | & 0 \\ 0 & 0 & 1 & 3 & \frac{7}{2} & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - 2x_2 \qquad\quad + \; x_4 + \tfrac{1}{2}x_5 &= 0 \\ x_3 + 3x_4 + \tfrac{7}{2}x_5 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= 2s - t - \tfrac{1}{2}u \\ x_2 &= \quad\; s \\ x_3 &= \; -3t + \tfrac{7}{2}u \\ x_4 &= \quad\; t \\ x_5 &= \quad\; u \end{aligned}$$

$$\vec{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{7}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \; \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{7}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Q2 - 20 marks

The aim of this question will be to produce matrices with specificed eigen values which are not just triangular! Let $p(x)$ be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and define the *companion matrix* to the polynomial as

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

a) Write down the matrix $C(p)$ of the polynomial $p(x) = x^3 - 4x^2 + 5x - 2$

$$C(p) = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

b) Find the characteristic polynomial of the matrix C(p) which you wrote in the previous step

$$C(p) - \lambda I = \begin{pmatrix} 4-\lambda & -5 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$\begin{aligned} \det(C(p) - \lambda I &= 2\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{vmatrix} \\ &= 2(1) - \lambda[(4-\lambda)(-\lambda) - (-5)] \\ &= 2 - \lambda(-4\lambda + \lambda^2 + 5) \\ &= 2 + 4\lambda^2 - \lambda^3 - 5\lambda \\ &= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0 \end{aligned}$$

c) Show that $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue 2

Let $\vec{x}$ be $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$, then

$$C(p)\vec{x} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 - 10 + 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$$

$$\lambda\vec{x} = 2 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$$

$$\therefore\ C(p)\vec{x} = \lambda\vec{x}$$

$\therefore\ \vec{x} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ with eigenvalue $\lambda = 2$

d) Find the matrix $C(p)$ associated to the polynomial $p(x) = x^3 + ax^2 + bx + c$

$$C(p) = \begin{pmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

e) Determine the characteristic polynomial of the matrix $C(p)$ from the previous step

$$
\begin{aligned}
C(p) - \lambda I &= \begin{pmatrix} -a-\lambda & -b & -c \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} \\
\det(C(p) - \lambda I) &= -1 \begin{vmatrix} -a-\lambda & -c \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} \\
&= -1[(-a-\lambda)(0) - (-c)(1)] - \lambda[(-a-\lambda)(-\lambda) - (-b)(1)] \\
&= -1(c) - \lambda(a\lambda + \lambda^2 + b) \\
&= -c - a\lambda^2 - \lambda^3 - b\lambda \\
&= -\lambda^3 - a\lambda^2 - b\lambda - c = 0
\end{aligned}
$$

f) Show that if λ is an eigenvalue of the companion matrix $C(p)$, then $\begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$ is an eigenvector of $C(p)$ corresponding to λ

$$
\begin{aligned}
E_\lambda &= \text{Nul}(C(p) - \lambda I) \\
&= \left(\begin{array}{ccc|c} -a-\lambda & -b & -c & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -a-\lambda & -b & -c & 0 \\ 0 & 1 & -\lambda & 0 \\ 1 & -\lambda & 0 & 0 \end{array}\right) \\
&\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ -a-\lambda & -b & -c & 0 \end{array}\right) \xrightarrow[R_1+\lambda R_2]{R_3+bR_2} \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ -a-\lambda & 0 & -c-b\lambda & 0 \end{array}\right) \\
&\xrightarrow{R_3+(a+\lambda)R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & -\lambda^3 - a\lambda^2 - b\lambda - c & 0 \end{array}\right)
\end{aligned}
$$

From the characteristic polynomial: $\lambda^3 = -a\lambda^2 - b\lambda - c$

$$
\therefore \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & -(-a\lambda^2 - b\lambda - c) - a\lambda^2 - b\lambda - c & 0 \end{array}\right)
$$

$$
\therefore \left(\begin{array}{ccc|c} 1 & 0 & -\lambda^2 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

$$
\begin{aligned}
x_1 \qquad - \lambda^2 x_3 &= 0 \\
x_2 - \lambda x_3 &= 0
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \lambda^2 s \\
x_2 &= \lambda s \qquad \Rightarrow \qquad \vec{x} = s \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix} \\
x_3 &= s
\end{aligned}
$$

$$
\therefore \mathcal{B}_{E_\lambda} = \left\{ \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix} \right\} \Rightarrow \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix} \text{ is an eigenvector of } C(p) \text{ corresponding to } \lambda
$$

g) Construct a non-triangular 3×3 matrix of eigenvalues -2, 1, 3 using companion matrices. *Briefly* justify your answer.

Let $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then,

$$[P \mid I] = \begin{pmatrix} 1 & 1 & 1 & \mid & 1 & 0 & 0 \\ 1 & 1 & 0 & \mid & 0 & 1 & 0 \\ 0 & 1 & 1 & \mid & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & \mid & 1 & 0 & -1 \\ 1 & 1 & 0 & \mid & 0 & 1 & 0 \\ 0 & 1 & 1 & \mid & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 & \mid & 1 & 0 & -1 \\ 0 & 1 & 0 & \mid & -1 & 1 & 1 \\ 0 & 1 & 1 & \mid & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 & \mid & 1 & 0 & -1 \\ 0 & 1 & 0 & \mid & -1 & 1 & 1 \\ 0 & 0 & 1 & \mid & 1 & -1 & 0 \end{pmatrix}$$

$$\therefore\ P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Let $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$A = PDP^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -2 & 3 \\ -3 & 1 & 3 \\ 2 & -2 & 1 \end{pmatrix}$$

Q3 - 20 marks

Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

a) Find matrices P and D (with D diagonal) so that $A = PDP^{-1}$

$$C_A(\lambda) = (\lambda - 2)^2(\lambda + 2)^2 \qquad \lambda = 2, -2$$

For $\lambda = 2$,

$$\begin{aligned} A - 2I &= \begin{pmatrix} 2-2 & 0 & 0 & 4 \\ 0 & 2-2 & 0 & 0 \\ 0 & 0 & -2-2 & 0 \\ 0 & 0 & 0 & -2-2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \end{aligned}$$

$$E_2 = \mathrm{Nul}(A - 2I) = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{\substack{\frac{1}{4}R_1 \\ R_1 \leftrightarrow R_2 \\ -\frac{1}{4}R_3 \\ -\frac{1}{4}R_4}} \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\substack{R_3 \leftrightarrow R_1 \\ R_4 - R_1}} \left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x_1 &= \text{free} & x_1 &= s \\ x_2 &= \text{free} & x_2 &= t \\ x_3 &= 0 & x_3 &= 0 \\ x_4 &= 0 & x_4 &= 0 \end{aligned} \qquad \Rightarrow \qquad \vec{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \mathcal{B}_{E_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = -2$,

$$A + 2I = \begin{pmatrix} 4 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-2} = Nul(A + 2I) = \begin{pmatrix} 4 & 0 & 0 & 4 & | & 0 \\ 0 & 4 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow[\frac{1}{4}R_2]{\frac{1}{4}R_1} \begin{pmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_4 &= 0 \\ x_2 &= 0 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \end{aligned} \qquad \begin{aligned} x_1 &= -t \\ x_2 &= 0 \\ x_3 &= s \\ x_4 &= t \end{aligned} \qquad \Rightarrow \quad \vec{x} = s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \mathcal{B}_{E_{-2}} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thus, $P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

b) For each positive integer n, write down a formula for A^n

$$
\begin{aligned}
&\forall n \in \mathbb{Z}^+,\\
&A^n = (PDP^{-1})^n\\
&\quad = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\dots(\cancel{P}DP^{-1}) = PD^nP^{-1}\\
&\quad = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 & 0 & 0 \\ 0 & 2^n & 0 & 0 \\ 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Q4 - 20 marks

A study of pine nuts in the American southwest from 1940 to 1947 hypothesized that nut production followed a Markov chain. The data suggested that if one year's crop was good, then the probabilities that the following year's crop would be good, fair, or poor were 0.08, 0.07, 0.85 respectively; if one year's crop was fair, then the probabilities that the following year's crop would be good, fair, or poor were 0.09, 0.11, and 0.80, respectively; if one year's crop was poor, then the probabilities that the following year's crop would be good, fair, or poor were 0.11, 0.05, 0.84 respectively.

a) Write down the transition matrix for this Markov process

$$A = \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}$$

b) If the pine cut crop was good in 1940, find the probabilities of a good crop in the years 1941 through 1945

$$\text{1941: } A\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0.08\\0.07\\0.85\end{pmatrix} \Rightarrow 0.08$$

$$\text{1942: } A^2\begin{pmatrix}1\\0\\0\end{pmatrix} = AA\begin{pmatrix}1\\0\\0\end{pmatrix} = A\begin{pmatrix}0.08\\0.07\\0.85\end{pmatrix}$$

$$= \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}\begin{pmatrix}0.08\\0.07\\0.85\end{pmatrix} = \begin{pmatrix}0.1062\\0.0558\\0.838\end{pmatrix}$$

$$\Rightarrow 0.1062$$

$$\text{1943: } A^3\begin{pmatrix}1\\0\\0\end{pmatrix} = AA^2\begin{pmatrix}1\\0\\0\end{pmatrix} = A\begin{pmatrix}0.1062\\0.0558\\0.838\end{pmatrix}$$

$$= \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}\begin{pmatrix}0.1062\\0.0558\\0.838\end{pmatrix} = \begin{pmatrix}0.105698\\0.055472\\0.83883\end{pmatrix}$$

$$\Rightarrow 0.105698$$

$$1944: \ A^4\begin{pmatrix}1\\0\\0\end{pmatrix} = AA^3\begin{pmatrix}1\\0\\0\end{pmatrix} = A\begin{pmatrix}0.105698\\0.055472\\0.83883\end{pmatrix}$$

$$= \begin{pmatrix}0.08 & 0.09 & 0.11\\0.07 & 0.11 & 0.05\\0.85 & 0.80 & 0.84\end{pmatrix}\begin{pmatrix}0.105698\\0.055472\\0.83883\end{pmatrix} = \begin{pmatrix}0.10571962\\0.05544228\\0.8388381\end{pmatrix}$$

$$\Rightarrow 0.10571962$$

$$1945: \ A^5\begin{pmatrix}1\\0\\0\end{pmatrix} = AA^4\begin{pmatrix}1\\0\\0\end{pmatrix} = A\begin{pmatrix}0.10571962\\0.05544228\\0.8388381\end{pmatrix}$$

$$= \begin{pmatrix}0.08 & 0.09 & 0.11\\0.07 & 0.11 & 0.05\\0.85 & 0.80 & 0.84\end{pmatrix}\begin{pmatrix}0.10571962\\0.05544228\\0.8388381\end{pmatrix}$$

$$= \begin{pmatrix}0.1057195658\\0.0554409292\\0.838839505\end{pmatrix} \Rightarrow 0.10571962$$

c) In the long run, what proportion of the crops will be good, fair, and poor?

Since the steady state vector is the eigenvector corresponding to $\lambda = 1$,

$$E_1 = \mathrm{Nul}(A - 1I) = \left(\begin{array}{ccc|c}0.08-1 & 0.09 & 0.11 & 0\\0.07 & 0.11-1 & 0.05 & 0\\0.85 & 0.80 & 0.84-1 & 0\end{array}\right)$$

$$= \left(\begin{array}{ccc|c}-0.92 & 0.09 & 0.11 & 0\\0.07 & -0.89 & 0.05 & 0\\0.85 & 0.80 & -0.16 & 0\end{array}\right)$$

$$\xrightarrow{\frac{1}{-0.92}R_1} \left(\begin{array}{ccc|c}1 & -\frac{9}{92} & -\frac{11}{92} & 0\\0.07 & -0.89 & 0.05 & 0\\0.85 & 0.80 & -0.16 & 0\end{array}\right)$$

$$\xrightarrow[R_3-0.85R_1]{R_2-0.07R_1} \left(\begin{array}{ccc|c}1 & -\frac{9}{92} & -\frac{11}{92} & 0\\0 & -\frac{325}{368} & \frac{537}{9200} & 0\\0 & \frac{325}{368} & -\frac{537}{9200} & 0\end{array}\right)$$

$$\xrightarrow{R_3+R_1}\begin{pmatrix}1 & -\frac{9}{92} & -\frac{11}{92} & | & 0\\ 0 & -\frac{325}{368} & \frac{537}{9200} & | & 0\\ 0 & 0 & 0 & | & 0\end{pmatrix}$$

$$\xrightarrow{-\frac{368}{325}R_2}\begin{pmatrix}1 & -\frac{9}{92} & -\frac{11}{92} & | & 0\\ 0 & 1 & -\frac{537}{8125} & | & 0\\ 0 & 0 & 0 & | & 0\end{pmatrix}$$

$$\xrightarrow{R_1+\frac{9}{92}R_2}\begin{pmatrix}1 & 0 & -\frac{1024}{8125} & | & 0\\ 0 & 1 & -\frac{537}{8125} & | & 0\\ 0 & 0 & 0 & | & 0\end{pmatrix}$$

$$\begin{aligned} x_1 \qquad - \tfrac{1024}{8125}x_3 &= 0 \\ x_2 - \tfrac{537}{8125}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= \tfrac{1024}{8125}s \\ x_2 &= \tfrac{537}{8125}s \\ x_3 &= \quad s \end{aligned} \qquad \Rightarrow \qquad \vec{x} = s\begin{bmatrix}\frac{1024}{8125}\\ \frac{537}{8125}\\ 1\end{bmatrix}$$

$$a\frac{1024}{8125} + a\frac{537}{8125} + a = 1 \Rightarrow a = \frac{8125}{9686}$$

$$\Rightarrow \vec{v} = a\begin{bmatrix}\frac{1024}{8125}\\ \frac{537}{8125}\\ 1\end{bmatrix} = \begin{bmatrix}\frac{512}{4838}\\ \frac{537}{9686}\\ \frac{8125}{9686}\end{bmatrix}$$

$\therefore$ In the long run, $\frac{512}{4838}$ crops will be good, $\frac{537}{9686}$ crops will be fair, and $\frac{8125}{9686}$ crops will be poor