Math 225 lecture 6 sept 18^{th} 2023

Goal: have students engage and actively discuss and contribute to the conversation to form the definition of a vector space.

Class Q: what is a vector space?

section 6.1

Examples: $\{0\}$, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}$, $\mathbb{C}^n$, $Mat_{m,n}(\mathbb{R})$, $\mathbb{Z}/p\mathbb{Z}$, cts. functions from $[0,1] \to \mathbb{R}$, $\mathbb{R}(X)$, $\mathbb{C}(X)$, Polynomials of degree ≤ 2

& many many more!

"Fancy" definition ↳ a (real) Vector space is a set V with two operations, addition, $+$, and scalar multiplication, $c\vec{v}$, which satisfy the following. let $\vec{u}, \vec{v}, \vec{w} \in V$ and $c, d \in \mathbb{R}$ (for now)

1. $\vec{u} + \vec{v} \in V$ — closure under $+$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ — commutativity of $+$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ — associativity of $+$
4. $\exists\ \vec{0} \in V$ called the zerovector s.t. $\vec{u} + \vec{0} = \vec{u}$ — (additive identity)
5. $\forall \vec{u} \in V\ \exists -\vec{u} \in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}$ — (additive inverses)
6. $c\vec{u} \in V$ — closure under scalar mult.
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ — left distributivity
8. $(\vec{u} + \vec{v})c = c\vec{u} + c\vec{v}$ — right distributivity
9. $c(d\vec{u}) = (cd)\vec{u}$ — collection of scalars
10. $1\vec{u} = \vec{u}$ — scalar mult identity

lets verify the one we are most familiar with

Ex: $\mathbb{R}^n$

① $\vec{u}+\vec{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} \in \mathbb{R}^n$

② $\vec{u}+\vec{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ \vdots \\ v_n+u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \vec{v}+\vec{u}$

③ $(\vec{u}+\vec{v})+\vec{w} = \left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1+v_1+w_1 \\ \vdots \\ u_1+v_1+w_1 \end{bmatrix}$

$= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1+w_1 \\ \vdots \\ v_n+w_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}\right) = \vec{u}+(\vec{v}+\vec{w})$

④ $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$ and $\vec{u}+\vec{0} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} u_1+0 \\ \vdots \\ u_n+0 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \vec{u}$

⑤ let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$ then $-u = \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix} \in \mathbb{R}^n$ w. $u+(-u) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix}$

$= \begin{bmatrix} u_1-u_1 \\ \vdots \\ u_n-u_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$

⑥ if $c \in \mathbb{R}$ then $c\vec{u} = c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} \in \mathbb{R}^n$

(7) $c(\vec{u}+\vec{v}) = c\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = c\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} cu_1+cv_1 \\ \vdots \\ cu_n+cv_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$

$= c\vec{u}+c\vec{v}$

(8) $(\vec{u}+\vec{v})c = \left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right)c = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}c = \begin{bmatrix} cu_1+cv_1 \\ \vdots \\ cu_n+cv_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$

$= c\vec{u}+c\vec{v}$

(9) $c(d\vec{u}) = c\left(d\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}\right) = c\begin{bmatrix} du_1 \\ \vdots \\ du_n \end{bmatrix} = \begin{bmatrix} cdu_1 \\ \vdots \\ cdu_n \end{bmatrix} = (cd)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = (cd)\vec{u}$

(10) $1\vec{u} = 1\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 1u_1 \\ \vdots \\ 1u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \vec{u}$