MATH 225 HW4

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Question 1 (20)

Let $V = \mathcal{P}_2(\mathbb{R})$ that is the polynomials of degree less than or equal to two with real coefficients. Let the map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be given by $p(x) \mapsto p'(x)$ i.e. the first derivative.

a) Verify T is a linear map

Let $\mathbf{0}(x) = 0x^2 + 0x + 0 \in \mathcal{P}_2(\mathbb{R})$

$$\begin{aligned} T(\mathbf{0}(x)) &= (0x^2 + 0x + 0)' = 0 = 0x^2 + 0x + 0 \in \mathcal{P}_2(\mathbb{R}) \\ &\therefore \quad T(\mathbf{0}(x)) = \mathbf{0}(x) \end{aligned}$$

Let $f(x) = f_0x^2 + f_1x + f_2 \in \mathcal{P}_2(\mathbb{R})$ and $c \in \mathbb{R}$

$$\begin{aligned} T(cf) &= (cf_0x^2 + cf_1x + cf_2)' = (2cf_0x + cf_1 + 0) \\ &= c(2f_0x + f_1) = c(f_0x^2 + f_1x + f_2)' = cT(f) \\ &\therefore T(cf) = cT(f) \end{aligned}$$

Let $f = f_0x^2 + f_1x + f_2 \in \mathcal{P}_2(\mathbb{R})$ and $g = g_0x^2 + g_1x + g_2 \in \mathcal{P}_2(\mathbb{R})$

$$\begin{aligned} T(f+g) &= (f_0x^2 + f_1x + f_2 + g_0x^2 + g_1x + g_2)' \\ &= 2f_0x + f_1 + 2g_0x + g_1 \\ T(f) + T(g) &= (f_0x^2 + f_1x + f_2)' + (g_0x^2 + g_1x + g_2)' \\ &= 2f_0x + f_1 + 2g_0x + g_1 \\ &\therefore T(f+g) = T(f) + T(g) \end{aligned}$$

Thus, T is a linear map/transformation

b) Find the kernel of T

Since $\forall c \in \mathcal{P}_0(\mathbb{R})$ (polynomials with degree 0) $\in \mathcal{P}_2(\mathbb{R})$, $c' = 0$

$$\begin{aligned} \ker(T) &= \{ax^2 + bx + c : a = b = 0\} \\ &= \{c : c \in \mathbb{R}\} \end{aligned}$$

c) Find the range of T

Let $f = ax^2 + bx + c \in \mathcal{P}_2(\mathbb{R})$. Since $f' = 2ax + b \in \mathcal{P}_2(\mathbb{R})\ \forall a, b \in \mathbb{R}$:

$$\begin{aligned}\text{range}(T) &= \{T(f) \in \mathcal{P}_2(\mathbb{R}) : f \in \mathcal{P}_2(\mathbb{R})\} \\ &= \{2ax + b : a, b \in \mathbb{R}\}\end{aligned}$$

d) is T onto? is it one-to-one?

$$\begin{aligned}\text{range}(T) = \{2ax + b : a, b \in \mathbb{R}\} &= p'(x) \quad \forall p(x) \in \mathcal{P}_2(\mathbb{R}) \\ \therefore T \text{ is onto}\end{aligned}$$

Since $\forall c \in \mathbb{R}\ \forall f = 0x^2 + 0x + c \in P_2(\mathbb{R}),\ f' = 0$

ie. For $c_1 = 1, c_2 = 2$ s.t. $f = 0x^2 + 0x + 1,\ g = 0x^2 + 0x + 2$

$$f' = g' = 0 \Rightarrow T(f) = T(g) = 0 \quad \therefore T \text{ is not onto}$$

Question 2 (15)

Let

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\} \quad \mathcal{D} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \quad \vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Calculate $[\vec{v}]_{\mathcal{D}}$ specifically by computing each of $\mathcal{P}_{B\leftarrow\mathcal{S}}$ ($\mathcal{S}$ is the standard basis) and $\mathcal{P}_{D\leftarrow\mathcal{B}}$ and verify your answer by showing the linear combination defined by $[\vec{v}]_{\mathcal{D}}$ does indeed get you back to $\vec{v}$

$$[\mathcal{B} \mid \mathcal{S}] = \begin{bmatrix} 1 & 1 & 0 & \mid & 1 & 0 & 0 \\ 0 & 1 & 1 & \mid & 0 & 1 & 0 \\ 1 & 0 & 1 & \mid & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 1 & 0 & \mid & 1 & 0 & 0 \\ -1 & 1 & 0 & \mid & 0 & 1 & -1 \\ 1 & 0 & 1 & \mid & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 1 & 0 & \mid & 1 & 0 & 0 \\ 0 & 2 & 0 & \mid & 1 & 1 & -1 \\ 1 & 0 & 1 & \mid & 0 & 0 & 1 \end{bmatrix} \xrightarrow[2R_3]{2R_1} \begin{bmatrix} 2 & 2 & 0 & \mid & 2 & 0 & 0 \\ 0 & 2 & 0 & \mid & 1 & 1 & -1 \\ 2 & 0 & 2 & \mid & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 2 & 0 & 0 & \mid & 1 & -1 & 1 \\ 0 & 2 & 0 & \mid & 1 & 1 & -1 \\ 2 & 0 & 2 & \mid & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 2 & 0 & 0 & \mid & 1 & -1 & 1 \\ 0 & 2 & 0 & \mid & 1 & 1 & -1 \\ 0 & 0 & 2 & \mid & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{1/2R_2\\1/2R_3}]{1/2R_1} \begin{bmatrix} 1 & 0 & 0 & \mid & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & \mid & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & \mid & -1/2 & 1/2 & 1/2 \end{bmatrix} \quad \therefore\ \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{S}} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$[\mathcal{D} \mid \mathcal{B}] = \begin{bmatrix} 1 & 1 & 1 & \mid & 1 & 1 & 0 \\ 0 & 1 & 1 & \mid & 0 & 1 & 1 \\ 0 & 0 & 1 & \mid & 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & \mid & 1 & 0 & -1 \\ 0 & 1 & 0 & \mid & -1 & 1 & 0 \\ 0 & 0 & 1 & \mid & 1 & 0 & 1 \end{bmatrix}$$

$$\therefore\ \mathcal{P}_{\mathcal{D}\leftarrow\mathcal{B}} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$[\vec{v}]_{\mathcal{S}} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{aligned} [\vec{v}]_{\mathcal{B}} &= \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{S}}[\vec{v}]_{\mathcal{S}} \\ &= \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix} 1/2 - 1 + 3/2 \\ 1/2 + 1 - 3/2 \\ -1/2 + 1 + 3/2 \end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
[\vec{v}]_{\mathcal{D}} &= \mathcal{P}_{\mathcal{D}\leftarrow\mathcal{B}}[\vec{v}]_{\mathcal{B}} \\
&= \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-2 \\ -1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}
\end{aligned}
$$

Verify

$$
-1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-1+3 \\ -1+3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{v}
$$

Question 3 (15)

Let $\mathcal{P}_2(\mathbb{R})$ be as it was in problem 1 and define the following,

$$\langle f(x), g(x) \rangle = f_0 g_0 + f_1 g_1 + f_2 g_2$$

where $f(x) = f_0 + f_1 x + f_2 x^2$ (similar for g)

a) Verify that $\mathcal{P}_2(\mathbb{R})$ with this product is an inner product space

$$\begin{aligned}
\langle f, g \rangle &= f_0 g_0 + f_1 g_1 + f_2 g_2 \\
\langle g, f \rangle &= g_0 f_0 + g_1 f_1 + g_2 f_2 = f_0 g_0 + f_1 g_1 + f_2 g_2 \\
\therefore \langle f, g \rangle &= \langle g, f \rangle
\end{aligned}$$

Let $h(x) = h_0 + h_1 x + h_2 x^2$

$$\begin{aligned}
g + h &= g_0 + g_1 x + g_2 x^2 + h_0 + h_1 x + h_2 x^2 \\
&= (g_0 + h_0) + (g_1 + h_1)x + (g_2 + h_2)x^2 \\
\langle f, g + h \rangle &= f_0(g_0 + h_0) + f_1(g_1 + h_1) + f_2(g_2 + h_2) \\
\langle f, g \rangle &= f_0 g_0 + f_1 g_1 + f_2 g_2 \\
\langle f, h \rangle &= f_0 h_0 + f_1 h_1 + f_2 h_2 \\
\langle f, g \rangle + \langle f, h \rangle &= f_0(g_0 + h_0) + f_1(g_1 + h_1) + f_2(g_2 + h_2) \\
\therefore \langle f, g + h \rangle &= \langle f, g \rangle + \langle f, h \rangle
\end{aligned}$$

Suppose $c \in \mathbb{R}$ s.t. $cf(x) = cf_0 + cf_1 x + cf_2 x^2$

$$\begin{aligned}
\langle cf, g \rangle &= cf_0 g_0 + cf_1 g_1 + cf_2 g_2 = c(f_0 g_0 + f_1 g_1 + f_2 g_2) \\
&= c\langle f, g \rangle
\end{aligned}$$

Since $f_i f_i \geq 0 \quad \forall f_i \in \mathbb{R}$ and $f_i f_i = 0$ iff $f_i = 0$,

$$\langle f, f \rangle \geq 0 \text{ and } \langle f, f \rangle = 0 \iff f = 0 + 0x + 0x^2 = \mathbf{0}(x)$$

Given that $\mathcal{P}_2(\mathbb{R})$ is a vector space and since the properties hold, $\mathcal{P}_2(\mathbb{R})$ with this product forms an inner product space

b) In this space what is the length/norm of the "vector" $f(x) = 3 + 2x + 4x^2$

$$\begin{aligned}
\|f\| &= \sqrt{\langle f, f \rangle} = \sqrt{3 \cdot 3 + 2 \cdot 2 + 4 \cdot 4} \\
&= \sqrt{9 + 4 + 16} = \sqrt{29}
\end{aligned}$$

c) In this space what is the distance between $f(x) = 4 - 5x + 2x^2$ and $g(x) = -1 + 8x + 7x^2$

$$\begin{aligned} d = \|f - g\| &= \|4 - 5x + 2x^2 - (-1 + 8x + 7x^2)\| \\ &= \|4 - 5x + 2x^2 + 1 - 8x - 7x^2\| \\ &= \|5 - 13x - 5x^2\| \\ &= \sqrt{5 \cdot 5 + (-13)(-13) + (-5)(-5)} \\ &= \sqrt{25 + 169 + 25} = \sqrt{219} \end{aligned}$$

d) In this space find TWO different "vectors" which are orthogonal to $f(x) = 1 + x + x^2$

Since g orthogonal to f iff $\langle f, g \rangle = 0$,

$$\langle f, g \rangle = 1g_0 + 1g_1 + 1g_2 = 0$$

Let $g_0 = 0, g_1 = 1, g_2 = -1$ s.t. $g = 0 + x - x^2$

$$\begin{aligned} \therefore\ \langle f, g \rangle &= 1 \cdot 0 + 1 \cdot 1 + 1 \cdot (-1) \\ &= 1 - 1 = 0 \Rightarrow 0 + x - x^2 \text{ is orthogonal to } f(x) \end{aligned}$$

Let $g_0 = 1, g_1 = -1, g_2 = 0$ s.t. $g = 1 - x + 0x^2$

$$\begin{aligned} \therefore\ \langle f, g \rangle &= 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 0 \\ &= 1 - 1 = 0 \Rightarrow 1 - x + 0x^2 \text{ is orthogonal to } f(x) \end{aligned}$$