Math 225 lecture 3 sept 11 2023

Goal: students are expected to be able to compute the determinant of a matrix and list basic properties of the determinant. Additionally students should be aware of complex numbers and be able to do basic computations with them.

Class Q: $\frac{1}{i}$ to what power gives you 1?

section 4.2

Determinants:

where/when? only square matrices

Purpose? most importantly it tells us when a matrix is invertible

why? lets us define eigenvalues & eigenvectors (next class)

"fancy" defn

the determinant of a matrix A, denoted $\det(A)$ or $|A|$ is a real number we associate to the matrix such that

i) $\det(\mathbb{1}_n) = 1$

ii) $\det(A) = -\det(A')$ if A' is obtained by interchanging 2 rows or 2 columns of A

iii) $\det(A') = c\det(A)$ if A' is A but with all the entries of one column of A by $c \in \mathbb{R}$

iv) $\det(A) = \det(A')$ if A' is obtained from A by adding a row to another

v) if A has a row of 0's then $\det(A) = 0$

Ex: let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$

aside: this can be proven using i) – iv) feel free to try it!

<u>Note</u>: additional properties of the determinant

$$\det(AB) = \det(BA) \quad (\text{recall } AB \neq BA)$$
$$\det(A) = \det(A^T) \quad (\text{this is transpose swap rows/cols})$$
$$\det(AB) = \det(A)\det(B)$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$

that said the "moral"/practical def'n of a determinant is the expansion by minors

<u>Ex</u>: $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$\uparrow$

expand along col 1 $\Rightarrow \det(A) =$

$$1\begin{vmatrix} 7 & 0 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 7 & 0 \end{vmatrix}$$

$$= 7 - 3(2-5) = 7+9 = 16$$

answer is independant of choice of expansion row/column for example expand by the third row & get

$$\det(A) = 0\begin{vmatrix} 2 & 1 \\ 7 & 0 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}$$

$$= -5(-3) + 1(7-6) = 15 + 1 = 16$$

<u>Exersize</u>: try another

with this fact what do we do naturally moving forward? pick a good row/col to expand in i.e. one with lots of 0's

Important: the key piece of information that is provided to us via the determinant is that

$\det(A) \neq 0 \Leftrightarrow A$ is invertible

if and only if , iff

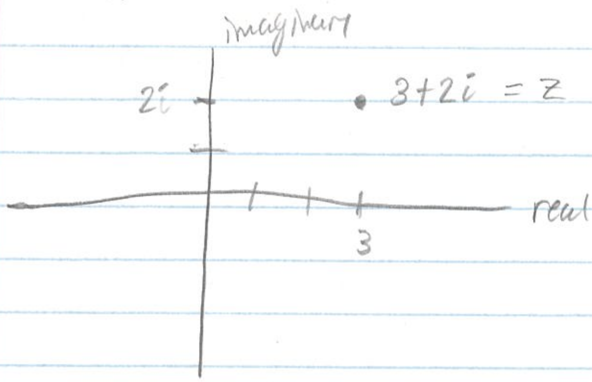
i.e. if $\det(A) \neq 0$ then A is invertible <u>and</u>
if A is invertible then $\det(A) \neq 0$

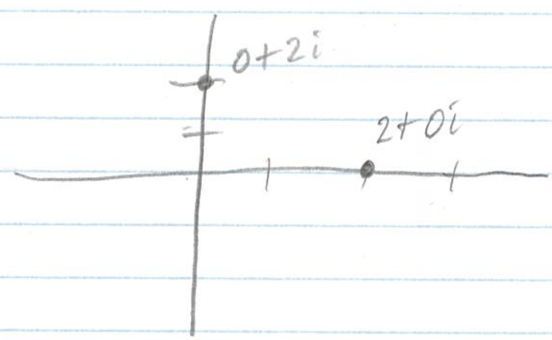
Appendix C

complex "imaginary" numbers. (brief into, will use occassionally)

denoted $\mathbb{C}$ are based around the concept of $\sqrt{-1} = i$
i.e. $i^2 = -1$

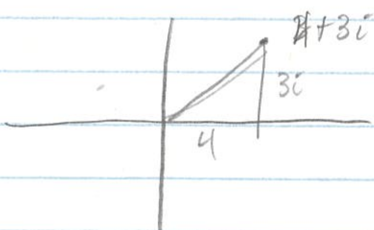
they are written as $z = \underset{\text{"Real part"}}{a} + \underset{\text{"Imaginary Part"}}{bi}$ and visualized as $\mathbb{R}^2$





$|z| = \sqrt{a^2+b^2}$ this is the magnitude of z it is the length of z when visualized as a vector in $\mathbb{R}^2$ and is a real # obtained via the Pythagorian theorem.

Ex:



$|z| = \sqrt{4^2+3^2} = 5$

$\bar{z}$ is complex conjugate of z & is $a-bi$

Hilroy

all of our basic operations can be done in $\mathbb{C}$ addition, subtraction, multiplication, and division.

Ex: $+$, $(a+bi)+(c+di) = (a+c)+(b+d)i$
$-$, $(a+bi)-(c+di) = (a-c)+(b-d)i$

real with real, imaginary with imaginary

$\cdot$, $(a+bi)\cdot(c+di) = (ac-bd)+i(ad+bc)$

$\div$, $\dfrac{(a+bi)}{(c+di)} = \dfrac{ac+bd}{c^2+d^2} + i\dfrac{(bc-ad)}{c^2+d^2}$

less straight forward, how do we get there?

$$(a+bi)\cdot(c+di) = ac + adi + bci + bdi^2$$
(foil)
$$= (ac-bd)+i(ad+bc)$$

for division use congugates

$$\frac{(a+bi)}{(c+di)}\cdot\frac{(c-di)}{(c-di)} = \frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-d^2i^2}$$
$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}$$

there is more that can be done with complex #s. lots of resources, great topic for office hours & review sessions.

Note: why do we need this?

- arrize naturally when working with real #'s
- appear as eigenvalues
- perfectly valid place to work
- in many ways nicer than $\mathbb{R}$

these will be used in the course but sparingly and no more than basic operations should be needed.

any additional info about them will be introduced at need

if you're interested key terms to look up &/or ask about are

- polar form
- euler's formula
- switching coordinates
- complex valued functions
- complex numbers as a field
- algebraic completion of $\mathbb{R}$
- roots of unity

these are intended to be optional for those interested and will likely be ignored in the course with minimal exception.