<u>MATH 225 – Fall 2023– Section B1</u>

Review questions (not to be turned in)
Week 1-2 to be discussed by TA on Sep. 7 & 14, 2023

Please note that the following questions do not need to be turned in. They are for your own practice, so please attempt them. They represent a review of what you need from a first course in linear algebra. Solutions will be posted on eClass, and also the material will be reviewed by the TA on Sep. 7 and Sep. 14 during his office hours.

1. Which statement is true?
   - ○ Every vector space has at most one basis
   - ○ If $\mathcal{B} = \{w_1, w_2, w_3\}$ is a basis of $\mathbb{R}^3$, then every vector $v \in \mathbb{R}^2$ can be written uniquely as a linear combination of w_1, w_2 and w_3
   - ● A basis for every subspace of $\mathbb{R}^n$ must have n-vectors
   - ○ If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and w_1, w_2 form a basis of $\mathbb{R}^2$, then $\{T(w_1), T(w_2)\}$ form a basis of $\mathbb{R}^2$ as well.

2. Let A be an 3×3 matrix whose whose characteristic polynomial is $p(\lambda) = -(\lambda - 1)(\lambda - 2)^2$. Which of the following statements can now be deduced?
   - ○ A can be diagonalized
   - ● A is invertible
   - ○ The nullity of A is at least 1
   - ○ The diagonal entries of A are either 1 or 2

3. Let A is a 2×3 matrix.
   - ○ The dimension of row(A) must be less than the dimension of col(A)
   - ○ The dimension of row(A) could be greater than the dimension of col(A)
   - ● A must have a null space
   - ○ None of the above statements are true.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Which of the following is true?
   - ○ T could be an invertible linear transformation
   - ● There is a vector $v \in \mathbb{R}^3$ other than the zero vector such that $T(v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
   - ○ Every vector in $\mathbb{R}^2$ must be in the image of T
   - ○ We don't have enough information to decide if any of the above choices is always true.

5. Suppose A is a square matrix such that $A^2 = A$. Then which of the following is true
   - ○ A is symmetric
   - ● A has eigenvalues $0, -1$ and 1
   - ○ A is invertible
   - ○ The linear system $Av = v$ has an infinite number of solutions

6. Suppose A is a 3×3 matrix that has the following vectors as eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 2 \\ 4a \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}$$

Find a value of a for which the matrix A is **diagonalizable**. Try to justify your answer.

$A = \begin{pmatrix} 1 & 2 & a \\ 0 & 4a & 2 \\ a & 0 & 1 \end{pmatrix}$ has $\det(A) = -4a^3 + 8a$

If $\det(A) \neq 0$ these 3 eigenvectors are linearly independent & hence A will be diagonalizable. So

for $a \neq 0, 2$.

7. Suppose A is a 2×3 matrix which has null space consisting of all vectors of the form

$$\begin{pmatrix} -3t \\ -2t \\ t \end{pmatrix} \quad \begin{matrix} x \\ y \\ z \end{matrix}$$

What does the RREF of A look like?

nullity 1 $\Rightarrow$ rank $A = 2$

$$A = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}$$ could work

need $A \cdot \begin{pmatrix} -3t \\ -2t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{matrix} -3t + b \cdot t = 0 \\ -2t + a \cdot t = 0 \end{matrix}$

so $b = -3$, $a = -2$ works

$$\boxed{\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}}$$ could work

8. Let $\vec{n} := (1, 2, 3) \in \mathbb{R}^3$ and consider the set of all vectors perpendicular to $\vec{n}$, i.e.

$$W = \{\vec{v} = (x, y, z) \in \mathbb{R}^3 \mid \vec{n} \cdot \vec{v} = 0\}.$$

~~Is W a subspace of $\mathbb{R}^3$?~~ If so, verify the definition of a subspace is satisfied. If not, produce a counterexample.

Yes

- $v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$ as $\vec{n} \cdot \vec{v} = \vec{n} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$ ✓

- if $v, w \in W$ then $v + w \in W$ since

$$\begin{matrix} \vec{n} \cdot v = 0 \\ \vec{n} \cdot w = 0 \end{matrix} \Rightarrow \vec{n} \cdot (v + w) = \vec{n} \cdot (v) + \vec{n} \cdot w = 0 + 0 = 0$$

- if $v \in W$, then $cv \in W$ for all $c \in \mathbb{R}$ since

$$\vec{n} \cdot v = 0 \Rightarrow \vec{n} \cdot (cv) = c(\vec{n} \cdot v) = c \cdot 0 = 0$$

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which is the composition $T = U \circ V$ where U is reflection about the yz-plane, and V is reflection about the xy-plane. Write down the standard matrix $[T]$ of this transformation.

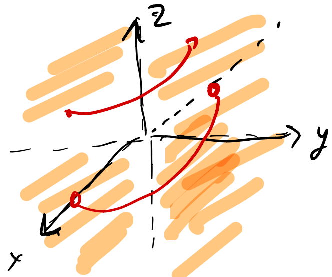
(Hint: draw a picture first)

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ unit in x-direction

$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ " y "

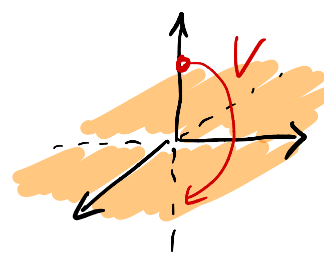
$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ " z – "

U :



$$U(e_1) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad U(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore [U] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Similarly $$[V] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\therefore [U \circ V] = [U] \cdot [V] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

10. Assume that the following determinant,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Compute the determinant

$$\begin{vmatrix} 0 & d & e & f \\ 1 & d-3 & e-4 & f-5 \\ 0 & 2a & 2b & 2c \\ 0 & g+d & h+e & i+f \end{vmatrix}$$

$$\begin{vmatrix} + & - & \cdots \\ - & \cdots & \\ + & \cdots & \\ - & \cdots & \end{vmatrix}$$

expand along 1st column

$$-1 \cdot \begin{vmatrix} d & e & f \\ 2a & 2b & 2c \\ g+d & h+e & i+f \end{vmatrix} = -1 \cdot 2 \cdot \begin{vmatrix} d & e & f \\ a & b & c \\ g+d & h+e & i+f \end{vmatrix}$$

$$= -2 \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$ as 3rd row is sum of 1st & 3rd. $\mathscr{g}$

$$= +2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2 \cdot 7 = 14$$

11. Let A be the matrix

$$\begin{pmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 2 & 3 \\ 4 & -8 & 3 & 2 \end{pmatrix}$$

(a) Compute the RREF of A and use this to find a basis for row(A)

(b) Does the vector $(5, -10, 3, 7)$ lie in row(A)? If so, write it explicitly as a linear combination of the basis you found in the previous part.

(c) Write down a basis for col(A)

(d) Write down a basis for null(A).

a) $\text{RREF}(A) = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

b) row(A) spanned by $(1, -2, 0, 0)$ & $(0, 0, 1, 0)$ & $(0,0,0,1)$

v " $\quad w$ " $\quad \xi$ "

$5v + 3w + 7\xi$

c) $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix}$

d) nullity $= 1$. $\quad s =$ free var

$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$w = 2x$

$y = 0$

$z = 0$

$\therefore$ null space

$x = s$

$w = 2s$

$y = z = 0.$

$$\begin{pmatrix} 2s \\ s \\ 0 \\ 0 \end{pmatrix} = s\boxed{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

Basis of null-space.

12. This question is concerned with the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Compute the characteristic polynomial of A and write down all eigenvalues of A with their algebraic multiplicities. For each eigenvalue, find the geometric multiplicity and write down a **basis** of the eigenspace.

(b) Compute A^8 using the previous parts. (If you try to compute this directly by multiplying out the matrix by itself 8 times, you will get no credit for this part of the problem. Also you may want to use the fact that $2^8 = 256$.)

(a) char pol $= \det(A - \lambda I) = \lambda^3 - 3\lambda - 2$

$= (\lambda-2)(\lambda+1)^2$

eigenvalues	algebraic mult	geometric
$\lambda = 2$	1	1
$\lambda = -1$	2	2

$\text{null}(A + I) = \text{null}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

rank 1 $\therefore$ nullity 2

s, t free var.

$3x + y + z = 0$ $\quad y = s$, $z = t$

$\Rightarrow x = \dfrac{-s-t}{3} \rightsquigarrow \begin{pmatrix} \frac{-s-t}{3} \\ s \\ t \end{pmatrix}$

$= s\begin{pmatrix} -1/3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$

OR: (rescaling) $\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right\}$ basis of null space

$\text{null}(A - 2I) = \text{null}\begin{pmatrix} 0 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

$\text{RREF}\begin{pmatrix} 0 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \therefore$

rk 2, nullity 1

basis of $E_{\lambda=2} = \text{null}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) So A is diagonalizable & $P = \begin{pmatrix} -1 & -1 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$

e.v. -1 e.v. 2

Satisfies $$P^{-1} A P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\therefore \quad A = P \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^{-1}$$

$$A^8 = P \begin{pmatrix} -1^8 & 0 & 0 \\ 0 & -1^8 & 0 \\ 0 & 0 & 2^8 \end{pmatrix} \cdot P^{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 256 \end{pmatrix} P^{-1}$$

To compute this we need to determine P^{-1}.

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 1 & 1/3 & 1/3 \end{array}\right)$$

$$\therefore \quad P^{-1} = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \\ 1 & 1/3 & 1/3 \end{pmatrix}$$

$$\& \quad P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 256 \end{pmatrix} P^{-1} = \begin{pmatrix} 256 & 85 & 85 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$