Math 225 lecture 28 Nov 20^{th} 2023

Goal: Students should understand the basic definitions, properties and examples of inner products & inner product spaces

Class Q: What is an example of an inner product space which is not $\mathbb{R}^n$ with the dot product.

Recall: we talked about vector spaces being the umbrella term for spaces which ~~we~~ had addition, scalar mult & so on.

one thing that was not mentioned was the dot product.

Brainstorming 4 properties of the dot product

↳ An (positive definite) inner product on a vector space V is an operation which assigns to every pair of vectors $\vec{v}, \vec{u} \in V$ a real number $\langle \vec{v}, \vec{u} \rangle$ s.t.

1. $\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle$ (technically with a conjugate if over $\mathbb{C}$ but we won't be)
2. $\langle \vec{v}, \vec{u} + \vec{w} \rangle = \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{w} \rangle$
3. $\langle c\vec{v}, \vec{u} \rangle = c \langle \vec{v}, \vec{u} \rangle \quad \forall c \in \mathbb{R}$
4. $\langle \vec{v}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$

a vector space with an inner product is called an inner product space.

Ex: $\mathbb{R}^n$ with the usual dot product.
$\mathcal{C}[a,b]$ continuous functions on the interval $[a,b]$ with
$\langle f,g \rangle = \int_a^b f(x)\,g(x)\,dx$

$\mathbb{R}^n$ with a weighted dot product
$P_2(\mathbb{R})$ with $\langle f,g \rangle = f_0 g_0 + f_1 g_1 + f_2 g_2$
and many others. we will focus on verification and examples on wednesday.

Properties: let $\vec{u}, \vec{v}, \vec{w} \in V$ an inner prod. space and $c \in \mathbb{R}$ then

1. $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
2. $\langle \vec{u}, c\vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
3. $\langle \vec{u}, \vec{0} \rangle = \langle \vec{0}, \vec{v} \rangle = 0$ ← the # not the vector.

Note that these all follow directly from the definition and are not included there for that reason.

What concepts from class did we define with a dot product, and thus can now be generalized. think small first.

1. length / norm of $\vec{v}$ is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

2. distance between $\vec{u}$ and $\vec{v}$ is $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

3. $\vec{u}$ and $\vec{v}$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

from a practical standpoint why should we care?

results: Thm (the pythagoras' Thm) let $\vec{u}$ & $\vec{v}$ be vectors in an inner product space V. then $\vec{u}$ and $\vec{v}$ are orthogonal iff

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Proof: $\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle = \langle \vec{u}, \vec{u}+\vec{v} \rangle + \langle \vec{v}, \vec{u}+\vec{v} \rangle$

$= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2$

which is $\|\vec{u}\|^2 + \|\vec{v}\|^2$ iff $\langle \vec{u}, \vec{v} \rangle = 0$, i.e. $\vec{u}$ and $\vec{v}$ are orth. $\blacksquare$

more generally this leads us to another result,

Thm (the triangle inequality) let $\vec{u}, \vec{v} \in V$ an inner prod space then

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

for proof see page 540.

last result for today

Thm: (the Cauchy-Schwarz inequality) let $\vec{u}, \vec{v} \in V$ an inner prod space then

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|$$

with equality iff $\vec{u}$ and $\vec{v}$ are scalar mults of eachother.

the proof of this is based around projections. (included in friday's talk)

Next class we will verify that some of the examples from before are indeed inner prod. spaces and do some examples of work in these spaces.