Math 225 lecture 19 oct 20^{th} 2023

section 7.4

<u>Goal</u>: Show & explore how to fix $\mathcal{U}$ when there aren't enough vectors. In addition, students should be able to compute an SVD.

<u>Class Q</u>: when will $\mathcal{U}$ not have enough vectors & how do we fix that?

<u>Recall</u>: last day we said any matrix A has an SVD given by

$A = \mathcal{U}\Sigma V^T$ defined as follows

V is $n \times n$ given as an orthonormalized eigen vectors of A^TA

Σ is $m \times n$ and "diagonal" with the singular values of A (square roots of e.vals of A^TA)

$\mathcal{U}$ is $m \times m$ and given by the normalizations of the vectors $\frac{1}{\sigma_i} A\vec{v}_i$

This had a potential problem, what was it?

there may not be enough $\mathcal{U}$ vectors! how do we deal with this?

we will need someting orthogonal to the vectors already in $\mathcal{U}$ in order to complete this set to a basis.

let $W = \text{span}\{u_1, \dots, u_r\}$

<u>Recall</u>: $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^m)$ i.e. you're in W or $W^\perp$

So if we're looking for something orthogonal to everything in W we look at $W^\perp$

As such the process of "completing to a basis" ammounts to starting with $u_1, \dots, u_r$, finding $m-r$ linearly independant vectors in $W^\perp$, then orthonormalizing the whole set!

<u>Ex</u>: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ last day we showed that $A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & has singular values $\sigma_1 = \sqrt{3}$ $\sigma_2 = 1$

$\lambda_1 = 3$ $\lambda_2 = 1$

$$E_3 = \text{null}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$E_1 = \text{null}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

$\therefore$ V is the normalization of these i.e. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(this is a 2×2 orthogonal matrix as required)

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

U start with $A v_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

and $A\vec{v}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore \vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 2/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\quad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \left(\frac{1}{1}\right) \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

we need a u_3! let $B = [\vec{u}_1 \ \vec{u}_2]$

recall $W + W^\perp = \mathbb{R}^3$ if $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$ then $W = \text{col}(B)$ & $W^\perp = \text{null}(B^T)$

$= \text{null}\left(\begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/2 & 1/2 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$

we now need to apply G-S to orthonormalize the u_i's with a basis for $W^\perp$.

However... the u_i's as we showed will already be orthogonal & in fact normal. in addition since we pulled $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ from $W^\perp$ it will also be orthogonal already as such the only thing we actually need to do to find U is normalize our additional vector. this yields

$$U = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

in total we have found our SVD, namely

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

There are many applications to SVD
- data compression
- LSS
- approximating functions & more!
- see section 7.4 / 7.5 & google it interested.

while there is value here we wont be focusing on these applications. time permitting this is something that might be revisited during the applications section at the end of the course.

General idea behind SVD make A "diagonal"

much like diagonalization we want A to act in its prefered way, i.e. an eigen basis (basis consisting of e. vects) this process ammounts to finding vectors which when transformed remain orthogonal.

look up "What is singular value decomposition?" by stochastic less than 8 minutes please do it!