Math 225 lecture 22 Oct 30^{th} 2023

<u>Goal</u>: students should understand the basic notion of linear transformations & the limitations to the 125 approach

section 6.4

<u>Class Q</u>: Where can the MATH 125 approach of finding a transformation matrix fail?

What is a linear transformation?

let V & W be vectorspaces (say $V=\mathbb{R}^n$ & $W=\mathbb{R}^m$) then a map $T: V \to W$ is called a linear transformation if it obeys the following

① $T(\underbrace{\vec{0}}_{\vec{0}\in\mathbb{R}^n}) = \underbrace{\vec{0}}_{\vec{0}\in\mathbb{R}^m}$

② $T(c\vec{v}) = cT(\vec{v}) \quad \forall c\in\mathbb{R}$ & $\vec{v}\in V$

③ $T(\vec{v}_1+\vec{v}_2) = T(\vec{v}_1)+T(\vec{v}_2) \qquad \vec{v}_i \in V$

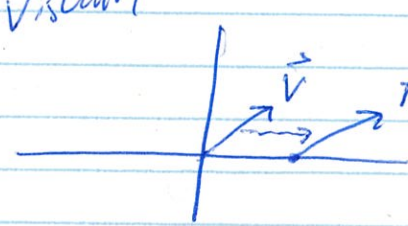
(note we can combine ② & ③ into $T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1)+c_2T(\vec{v}_2)$

why are these the conditions? they preserve the vector space structure.

<u>Ex</u>: lets see which of the following are linear transformations

① $T: \mathbb{R}^2 \to \mathbb{R}^2$
$\vec{v} \mapsto \vec{v} + \begin{bmatrix}1\\0\end{bmatrix}$

visually



this is <u>not</u> a linear transformation

$$T(\vec{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(2) $T: \mathbb{R}^2 \to \mathbb{R}^2$
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$$

this fails condition 2

$$T(c\vec{v}) \neq cT(\vec{v})$$

$$\begin{bmatrix} c^2x^2 \\ c^2y^2 \end{bmatrix} = T\begin{bmatrix} cx \\ cy \end{bmatrix} \qquad c\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \begin{bmatrix} cx^2 \\ cy^2 \end{bmatrix}$$

(3) let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{v} \mapsto A\vec{v}$

$$T(\vec{0}) = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$T(c\vec{v}) = A(c\vec{v}) = cA\vec{v}$$

$$T(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$$

we've used these properties many times & overall we understand what's going on here well.

Moreover if A is any $m \times n$ matrix the linear transformation $T_A: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{v} \mapsto A\vec{v}$ satisfies all of the conditions!

In fact we can go further, associated to <u>any</u> linear transformation $T: V \to W$ we can associate to it a matrix (sometimes called the "transformation matrix")

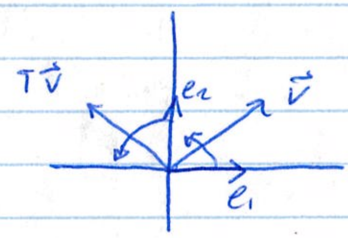
How do we do that? in math 125 we did this

let $V = \mathbb{R}^n$ $W = \mathbb{R}^m$ & $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\cdots$ $\vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

$T(\vec{e}_1) = \vec{v}_1 \in \mathbb{R}^m \quad \cdots \quad T(\vec{e}_n) = \vec{v}_n \in \mathbb{R}^m$

we then call $A = [\vec{v}_1 \cdots \vec{v}_n]$ the transformation matrix of T

<u>Ex</u>: rotate by $\pi/2$



$T(0) = 0$
$T(c\vec{v}) = cT\vec{v}$
$T(v+w) = T(\vec{v}) + T(\vec{w})$

what is the matrix $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

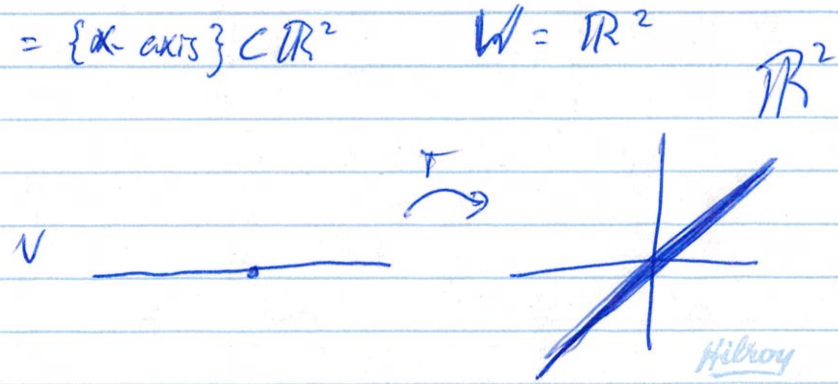
$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\therefore A_T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

There are problems / limitations to this approach
first suppose $V \subset \mathbb{R}^n$; $W \subset \mathbb{R}^m$ & we want

$T: V \to W$

<u>Ex</u>: $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \{x\text{-axis}\} \subset \mathbb{R}^2$ $\quad W = \mathbb{R}^2$

$T: \{x\text{-axis}\} \to \mathbb{R}^2$
$x \mapsto \begin{bmatrix} x \\ x \end{bmatrix}$



this is still a linear transformation & should thus still have a matrix however $T(\vec{e}_2)$ is not defined. How do we fix this.

<u>Answer</u>: let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for V (note $\dim(V) = k$)
let $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_\ell\}$ be a basis of W (note $\dim(W) = \ell$)

recall $T(\vec{v}_i) \in W$ meaning we can write the following

$$T(\vec{v}_1) = c_{11}\vec{w}_1 + \dots + c_{1\ell}\vec{w}_\ell$$

$$T(\vec{v}_2) = c_{21}\vec{w}_1 + \dots + c_{2\ell}\vec{w}_\ell$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T(\vec{v}_k) = c_{k1}\vec{w}_1 + \dots + c_{k\ell}\vec{w}_\ell$$

what are we then contractually obligated to do? Put it in a matrix

let $A = \begin{bmatrix} c_{11} & \cdots & c_{1\ell} \\ c_{21} & \cdots & c_{2\ell} \\ \vdots & \ddots & \vdots \\ c_{k1} & \cdots & c_{k\ell} \end{bmatrix}$ this is a $k \times \ell$ matrix & we say that A is the matrix of linear transformation (or simply transformation matrix) for $T: V \to W$ with respect to (w.r.t.) the bases $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ of V & $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_\ell\}$ of W.

<u>Notation!</u> this A is usually written as $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$

A second issue is what if we consider the same linear transformation but change $\mathcal{B}$ & $\mathcal{C}$?

the transformation is the same but with a different matrix ~~we~~

we will examine both of these. one last important notion before the end is "coordinates with respect to a basis"

let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for V. This means every vector $\vec{v} \in V$ can be written as

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ uniquely based solely on $\vec{v}$ & $\mathcal{B}$

this gives us the notation $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$

i.e. the coordinates of v w.r.t. $\mathcal{B}$ or how do you write $\vec{v}$ as a linear comb. of the elements of $\mathcal{B}$.