Practice problems week 3, Solutions

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Solution 1 For the first part we want a vector that when added to something else changes nothing as such we want a vector (a, b) such that $(u_1, u_2) + (a, b) = (u_1, u_2)$ using the odd notion of addition defined in the problem. As such lets solve

$$(u_1 + a + 1, u_2 + b - 1) = (u_1, u_2)$$

from here we can see that if we take $a = -1$, $b = 1$ we satisfy the equation. meaning the "0-vector" in this space is $(a, b) = (-1, 1)$.

Similarly for part two we want a vector (a, b) which can be added to $(2, 5)$ to get our 0-vector from part one $(-1, 1)$. Thus we have to solve

$$(2 + a + 1, 5 + b - 1) = (-1, 1)$$

this leads us to the conclusion that we need $(a, b) = (-4, -3)$ to make the above true. This allows us to conclude that $-(2, 5) = (-4, -3)$ for this particular vector space.

For part three lets go through each of the conditions in sequence

Conditions 1-3: (closure under $+$, commutativity, and associativity of $+$) all follow in exactly the same manor as the examples done in class and the properties of $\mathbb{R}$

Condition 4: we proved in part one

Condition 5: if we repeat the process done for part two for a general vector (u_1, u_2) we would get $(-u_1 - 2, -u_2 + 2)$

Condition 6: is clear from $\mathbb{R}$ being the base field

Condition 7: this is done via direct computation, let $c \in \mathbb{R}$, and $\vec{v}, \vec{u} \in \mathbb{R}^2$ with these operations then we have examine

$$c(\vec{v} + \vec{u}) = c\big((u_1, u_2) + (v_1, v_2)\big) = c\big((u_1 + v_1 + 1, u_2 + v_2 - 1)\big)$$

$$= (cu_1 + cv_1 + c + c - 1, cu_2 + cv_2 - c - c + 1) =$$

$$(cu_1 + c - 1 + cv_1 + c - 1 + 1, cu_2 - c + 1 + cv_2 + c + 1 - 1) =$$

$$(cu_1 + c - 1, cu_2 - c + 1) + (cv_1 + c - 1, cv_2 - c + 1) = c\vec{v} + c\vec{u}$$

which is as we required.

Condition 8: this is the same process as condition 7

Condition 9: this again done via direct computation. let $c, d \in \mathbb{R}$ and $\vec{u} \in \mathbb{R}^2$ with our operations.

$$c(d\vec{u}) = c(du_1+d-1, du_2-d+1) = (cdu_1+cd-c+c-1, cdu_2-cd+c-c+1) =$$

$$\big((cd)u_1 + (cd) - 1, (cd)u_2 - (cd) + 1\big) = (cd)\vec{u}$$

and lastly, condition 10: this is again a quick computation

$$1\vec{u} = (1u_1 + 1 - 1, 1u_2 - 1 + 1) = (u_1, u_2) = \vec{u}$$

Solution 2 This fails the following conditions, conditions: 3 (associativity), 4 (existence of a zero vector), and 5 (existence of additive inverse)

to see this consider the following,

Condition 3 take $\vec{v} = (1,1), \vec{u} = (3,3), \vec{w} = (5,5)$ and compute associativity, i.e

$$(\vec{v} + \vec{u}) + \vec{w} = (26, 8)$$

where as

$$\vec{v} + (\vec{u} + \vec{w}) = (36, 8)$$

which are clearly not equal.

For condition 4 let (a, b) be the hypothetical zero vector. Consider $(1,1) + (a,b) = (2 + 2a, 1 + b)$ for the condition needed for (a, b) to be the additive identity to hold this forces $(a, b) = (-\frac{1}{2}, 0)$ which gives us $(1,1) + (a,b) = (1,1)$. However to be the additive identity it must be unique and we can see from

$$(2,2) + (-\frac{1}{2}, 0) = (3,2)$$

That this vector cannot be the zero vector as it must have the desired property for any vector in the space. Hence it simultaneously must be and cannot be $(-\frac{1}{2}, 0)$ meaning it cannot exist.

For condition 5 we cannot properly define additive inverses without an additive identity

Solution 3 Recall that to be an orthogonal matrix by our definition the columns must be both orthogonal and normal. We also showed this as being equivalent to $A^{-1} = A^T$.

for A we will use the first condition and note that that the columns each have norm $\sqrt{1^2 + 2^2 + 2^2} = \sqrt{5} \neq 1$ which automatically precludes A from being orthogonal.

For B lets show the property via verification that the columns form an orthonormal set. (denote the columns of B as $\vec{b}_1, \vec{b}_2$) we have to check each of $\vec{b}_1 \cdot \vec{b}_1 = 1, \vec{b}_2 \cdot \vec{b}_2 = 1, \vec{b}_1 \cdot \vec{b}_2 = 0$ hold, this is done below.

$$\vec{b}_1 \cdot \vec{b}_1 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$$

$$\vec{b}_2 \cdot \vec{b}_2 = \left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$\vec{b}_1 \cdot \vec{b}_2 = \frac{\sqrt{3}}{2} \cdot \frac{-1}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{-\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0$$

Each of which are as desired to account for orthonormality. This allows us to conclude that B is in fact orthogonal.

Lastly for C we will multiply C by C^T and verify that this product is in fact I_4.

$$C^T C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which means that $C^T = C^{-1}$. As previously stated this is equivalent to C being orthogonal.