Math 225 lecture 11 Sept 29th 2023

Goal: ground students understanding of the Gram-Schmidt process in examples. ~~[scribbled out]~~ Additionally students should be confident with these types of computations by the end of class.

Section 5.3

Class Q: for each of the following sets, can you use the Gram-Schmidt process? if so, do it, if not explain why $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-5\\9\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}\sqrt{2}\\-\sqrt{2}\end{bmatrix}\right\}$

Recall: last day we explained the G-S process lets remind ourselves what it says

let $\{\vec{x}_1, \dots, \vec{x}_k\}$ be a basis for $W \subset \mathbb{R}^n$ a subspace then

$\vec{v}_1 = \vec{x}_1$ $\qquad W_1 = \text{span}\{\vec{x}_1\}$

$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$ $\qquad W_2 = \text{span}\{\vec{x}_1, \vec{x}_2\}$

$\vdots$

$\vec{v}_k = \vec{x}_k - \left(\frac{\vec{v}_1 \cdot \vec{x}_k}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \dots - \left(\frac{\vec{v}_{k-1} \cdot \vec{x}_k}{\vec{v}_{k-1} \cdot \vec{v}_{k-1}}\right)\vec{v}_{k-1}$ $\qquad W_k = W = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$

for each i $\{\vec{v}_1, \dots, \vec{v}_i\}$ is an orthogonal basis for W_i making $\{\vec{v}_1, \dots, \vec{v}_k\}$ an orthogonal basis for W

if we let $\vec{u}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$ then both of the above statements hold accept the bases will be orthonormal as opposed to orthogonal for today lets do a few concrete examples.

① let $W = \text{span}\{\vec{x}_1, \vec{x}_2\}$ $\quad \vec{x}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

use G-S

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \left(\frac{9+(-3)}{9+9}\right)\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \left(\frac{1}{3}\right)\begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\therefore \cancel{\text{span}}\{\vec{v}_1, \vec{v}_2\} = \cancel{\text{span}}\left\{\begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$ is an orthogonal basis for W

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{(3)^2+(-3)^2}}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{18}}\begin{bmatrix} 3 \\ -3 \end{bmatrix} \text{ or } \frac{1}{3\sqrt{2}}\begin{bmatrix} 3 \\ -3 \end{bmatrix} \text{ or } \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{(2)^2+(2)^2}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\therefore \{\vec{u}_1, \vec{u}_2\} = \left\{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$ is an orthonormal basis for W

② let $W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ $\quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \left(\frac{1-1-1+5}{1+1+1+1}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1+2-0+1}{1+1+1+1}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{0-4+0+4}{0+4+4+16}\right)\begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \vec{0} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$ forms an orthogonal basis for W

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{1^2+1^2+(-1)^2+1^2}}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{0+(-2)^2+(2)^2+4^2}}\begin{bmatrix}0\\-2\\2\\4\end{bmatrix} = \frac{1}{\sqrt{24}}\begin{bmatrix}0\\-2\\2\\4\end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix}0\\-2\\2\\4\end{bmatrix} = \begin{bmatrix}0\\-1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \frac{1}{\sqrt{0+1^2+1^2+0}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}$$

making $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{\begin{bmatrix}1/2\\1/2\\-1/2\\1/2\end{bmatrix}, \begin{bmatrix}0\\-1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}\right\}$ an orthonormal basis for W

(3) lastly lets find an orthogonal basis for $\mathbb{R}^3$ which contains $\begin{bmatrix}3\\1\\5\end{bmatrix}$

to start take any linearly independant set containing this vetor, lets say $\left\{\begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}7\\0\\0\end{bmatrix}, \begin{bmatrix}0\\7\\0\end{bmatrix}\right\}$ this is a basis for $\mathbb{R}^3$ (you can check for yourself)

and set $\vec{x}_1 = \begin{bmatrix}3\\1\\5\end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix}7\\0\\0\end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix}0\\7\\0\end{bmatrix}$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix}3\\1\\5\end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix}7\\0\\0\end{bmatrix} - \left(\frac{21}{9+1+25}\right)\begin{bmatrix}3\\1\\5\end{bmatrix} = \begin{bmatrix}7\\0\\0\end{bmatrix} - \begin{bmatrix}9/5\\3/5\\3\end{bmatrix}$$

$$= \begin{bmatrix}26/5\\-3/5\\-3\end{bmatrix} \xrightarrow{\text{rescale}} \begin{bmatrix}26\\-3\\-15\end{bmatrix}$$

note we can rescale the $\vec{v}_i$'s along the way as this does not change orthogonality & we usually rescale at the end, just be careful

$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 = \begin{bmatrix}0\\7\\0\end{bmatrix} - \left(\frac{7}{35}\right)\begin{bmatrix}3\\1\\5\end{bmatrix} - \left(\frac{-21}{676+9+225}\right)\begin{bmatrix}26\\-3\\-15\end{bmatrix}$$

$$= \begin{bmatrix}0\\7\\0\end{bmatrix} - \left(\frac{1}{5}\right)\begin{bmatrix}3\\1\\5\end{bmatrix} + \frac{3}{130}\begin{bmatrix}26\\-3\\-15\end{bmatrix} = \begin{bmatrix}0\\7\\0\end{bmatrix} - \begin{bmatrix}3/5\\1/5\\1\end{bmatrix} + \begin{bmatrix}78/130\\-9/130\\-45/130\end{bmatrix} =$$

$$= \begin{bmatrix}0\\ \frac{875}{130}\\ \frac{-175}{130}\end{bmatrix} = \begin{bmatrix}0\\ 175/26\\ -35/26\end{bmatrix}$$

$$\therefore \{v_1, v_2, v_3\} = \left\{\begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}26\\-3\\-15\end{bmatrix}, \begin{bmatrix}0\\175/26\\-35/26\end{bmatrix}\right\}$$ is an orthogonal basis for $\mathbb{R}^3$ containing $\begin{bmatrix}3\\1\\5\end{bmatrix}$

Note! if I ask you to do something similar the obvious road should have slightly nicer numbers.

Natural question: once we have a set of $\vec{v}_i$'s what do we do with them? Put them in a matrix!

until next time ~