Math 225 lecture 25 NOV 6th 2023

Section 6.5

Goal: autopsy the midterm and briefly discuss the concepts of kernel and range.

Class Q: What could you have done differently to prep for the midterm to remember for the final?

Midterm Autopsy, the good the bad & the ugly

- the good (what was done well)
  - overall problems 7 & 8 were relatively well done
    - remember I give hints for a reason
  - 7 in particular most remembered you can find an orth. comp. using a nullspace computation.
  - 8 lots of really clean solutions

- the bad (what was done poorly)
  - problem 5 was tough and perhaps too long that said some parts could have been avoided. no need for G-S if clever about basis
    - 1 was a clear root as $A - I$ is $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$
    -
  - the multiple choice NO COMPUTATION was expected for these
    - 1, 3, 5 was meant to be oddman out.
    - 2 reviewing $\Leftrightarrow$ of SVD
    - 4 (single problem that caused the most issues) symmetry is meant to be apparent
      linear independence of the columns gives a QR
      orth diag $\Leftrightarrow$ symmetric $\Rightarrow$ real evals,
      all matrices have an SVD
      $A - 2I$ has two identical rows making 2 an eval.

moving forward, my multipal choice should involve little to no computation at all. focused on concepts.
skip things & come back if its taking too long

the ugly (red flag responces)

- please don't leave MC blank!
- orthogonal complements live in the same spaces
- I won't ask you to do ridiculus things no inversion of 5x5 (or probably even 3x3) matrizes in general

general comments / Questions

- lower than I expected, was I aiming too high or was it too hard?
- what could I have done to better prepare you for this exam
- what could you have done to better prepare you for this exam.
- regrading will be done on a rolling basis please email me before the end of the week.
- questions comments and concerns ~~from~~ for me?

Most of today is terminology definitions and a few small results.

For all of today let $T: V \to W$ be a linear transformation between vectorspaces. and $[T]$ be the transformation matrix of T.

definitions: the kernel of T is the set of vectors in V which T sends to $\vec{0}$ in W i.e.

$$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$$

the range of T is the set of vectors in W that are the images of vectors in V under T. that is

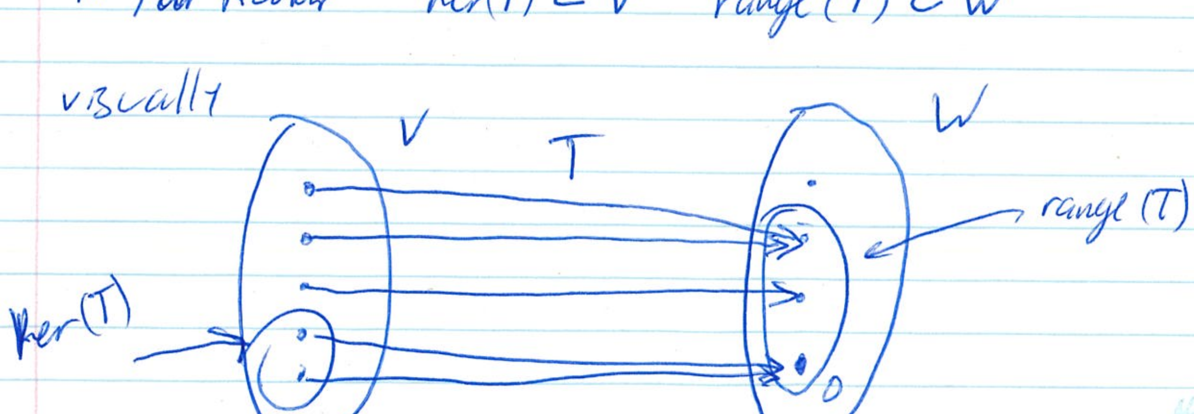
$$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\} \text{ or } \{\vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V\}$$

in terms of $[T]$ what are the above sets?

$\ker(T) = \text{null}([T])$
$\text{range}(T) = \text{col}([T])$

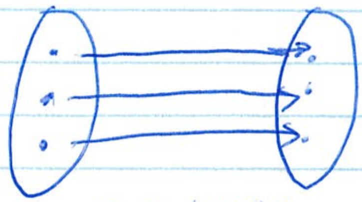
meaning both the kernel and range are subspaces in particular $\ker(T) \subset V$ $\text{range}(T) \subset W$

visually

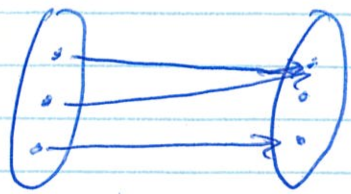


more definitions

we say T is one-to-one if $\vec{u} \neq \vec{v} \Rightarrow T(\vec{u}) \neq T(\vec{v})$
ie. destinct points get maped to destinct points



one-to-one



not one-to-one

Thm: T is one-to-one iff $\ker(T) = \{\vec{0}\}$

Proof

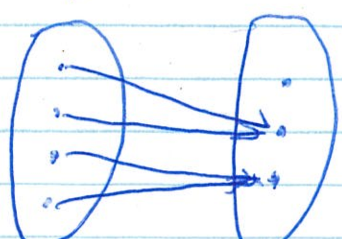
$(\Rightarrow)$ assume T is one-to-one. take $\vec{v} \in \ker(T)$ $T(\vec{v}) = \vec{0}$

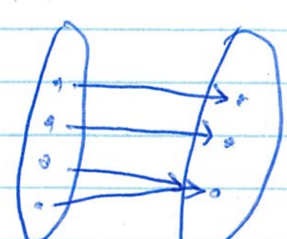
but $T(\vec{0}) = \vec{0}$ by definition $\Rightarrow T(\vec{v}) = T(\vec{0})$ meaning $\vec{v} = \vec{0}$ as destinct points are maped to destinct points

$(\Leftarrow)$ take $\vec{u}, \vec{v}$ st. $T(\vec{u}) = T(\vec{v})$ then $T(u-v) = T(\vec{u}) - T(\vec{v}) = \vec{0}$
placing $\vec{u} - \vec{v} \in \ker(T)$ but here we assumed $\ker(T) = \vec{0}$
$\Rightarrow \vec{u} - \vec{v} = \vec{0} \Rightarrow \vec{u} = \vec{v}$ making T one-to-one

$\boxtimes$

similarly T is called onto if $\text{range}(T) = W$ ie.
everything in W can be reached via T.





T is invertible iff it is one-to-one and onto such T's are called isomorphisms and we say it makes W & V isomorphic vectorspaces $V \cong W$

which transformations cannot be one-to-one? onto? isomorphic?

Thm: ~~el~~ let V & W be finite dim'l V. spaces (over the same base field) then $V \cong W$ iff $\dim(V) = \dim(W)$