Q4 - 20 marks

A study of pine nuts in the American southwest from 1940 to 1947 hypothesized that nut production followed a Markov chain. The data suggested that if one year's crop was good, then the probabilities that the following year's crop would be good, fair, or poor were 0.08, 0.07, 0.85 respectively; if one year's crop was fair, then the probabilities that the following year's crop would be good, fair, or poor were 0.09, 0.11, and 0.80, respectively; if one year's crop was poor, then the probabilities that the following year's crop would be good, fair, or poor were 0.11, 0.05, 0.84 respectively.

a) Write down the transition matrix for this Markov process

$$A = \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}$$

b) If the pine cut crop was good in 1940, find the probabilities of a good crop in the years 1941 through 1945

$$1941:\ A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.07 \\ 0.85 \end{pmatrix} \Rightarrow 0.08$$

$$1942:\ A^2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = AA\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\begin{pmatrix} 0.08 \\ 0.07 \\ 0.85 \end{pmatrix}$$

$$= \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}\begin{pmatrix} 0.08 \\ 0.07 \\ 0.85 \end{pmatrix} = \begin{pmatrix} 0.1062 \\ 0.0558 \\ 0.838 \end{pmatrix}$$

$$\Rightarrow 0.1062$$

$$1943:\ A^3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = AA^2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\begin{pmatrix} 0.1062 \\ 0.0558 \\ 0.838 \end{pmatrix}$$

$$= \begin{pmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{pmatrix}\begin{pmatrix} 0.1062 \\ 0.0558 \\ 0.838 \end{pmatrix} = \begin{pmatrix} 0.105698 \\ 0.055472 \\ 0.83883 \end{pmatrix}$$

$$\Rightarrow 0.105698$$

$$
\begin{aligned}
1944:\ A^4 \begin{pmatrix}1\\0\\0\end{pmatrix} &= AA^3 \begin{pmatrix}1\\0\\0\end{pmatrix} = A \begin{pmatrix}0.105698\\0.055472\\0.83883\end{pmatrix} \\
&= \begin{pmatrix}0.08 & 0.09 & 0.11\\0.07 & 0.11 & 0.05\\0.85 & 0.80 & 0.84\end{pmatrix} \begin{pmatrix}0.105698\\0.055472\\0.83883\end{pmatrix} = \begin{pmatrix}0.10571962\\0.05544228\\0.8388381\end{pmatrix} \\
&\Rightarrow 0.10571962 \\
1945:\ A^5 \begin{pmatrix}1\\0\\0\end{pmatrix} &= AA^4 \begin{pmatrix}1\\0\\0\end{pmatrix} = A \begin{pmatrix}0.10571962\\0.05544228\\0.8388381\end{pmatrix} \\
&= \begin{pmatrix}0.08 & 0.09 & 0.11\\0.07 & 0.11 & 0.05\\0.85 & 0.80 & 0.84\end{pmatrix} \begin{pmatrix}0.10571962\\0.05544228\\0.8388381\end{pmatrix} \\
&= \begin{pmatrix}0.1057195658\\0.0554409292\\0.838839505\end{pmatrix} \Rightarrow 0.10571962
\end{aligned}
$$

c) In the long run, what proportion of the crops will be good, fair, and poor?

Since the steady state vector is the eigenvector corresponding to $\lambda = 1$,

$$
\begin{aligned}
E_1 = \mathrm{Nul}(A - 1I) &= \left(\begin{array}{ccc|c}0.08-1 & 0.09 & 0.11 & 0\\0.07 & 0.11-1 & 0.05 & 0\\0.85 & 0.80 & 0.84-1 & 0\end{array}\right) \\
&= \left(\begin{array}{ccc|c}-0.92 & 0.09 & 0.11 & 0\\0.07 & -0.89 & 0.05 & 0\\0.85 & 0.80 & -0.16 & 0\end{array}\right) \\
&\xrightarrow{\frac{1}{-0.92}R_1} \left(\begin{array}{ccc|c}1 & -\frac{9}{92} & -\frac{11}{92} & 0\\0.07 & -0.89 & 0.05 & 0\\0.85 & 0.80 & -0.16 & 0\end{array}\right) \\
&\xrightarrow[R_3-0.85R_1]{R_2-0.07R_1} \left(\begin{array}{ccc|c}1 & -\frac{9}{92} & -\frac{11}{92} & 0\\0 & -\frac{325}{368} & \frac{537}{9200} & 0\\0 & \frac{325}{368} & -\frac{537}{9200} & 0\end{array}\right)
\end{aligned}
$$

$$\xrightarrow{R_3+R_1} \begin{pmatrix} 1 & -\frac{9}{92} & -\frac{11}{92} & | & 0 \\ 0 & -\frac{325}{368} & \frac{537}{9200} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{368}{325}R_2} \begin{pmatrix} 1 & -\frac{9}{92} & -\frac{11}{92} & | & 0 \\ 0 & 1 & -\frac{537}{8125} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{R_1+\frac{9}{92}R_2} \begin{pmatrix} 1 & 0 & -\frac{1024}{8125} & | & 0 \\ 0 & 1 & -\frac{537}{8125} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 \qquad - \tfrac{1024}{8125}x_3 &= 0 \\ x_2 - \tfrac{537}{8125}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= \tfrac{1024}{8125}s \\ x_2 &= \tfrac{537}{8125}s \\ x_3 &= \quad s \end{aligned} \qquad \Rightarrow \qquad \vec{x} = s \begin{bmatrix} \frac{1024}{8125} \\ \frac{537}{8125} \\ 1 \end{bmatrix}$$

$$a\frac{1024}{8125} + a\frac{537}{8125} + a = 1 \Rightarrow a = \frac{8125}{9686}$$

$$\Rightarrow \vec{v} = a \begin{bmatrix} \frac{1024}{8125} \\ \frac{537}{8125} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{512}{4838} \\ \frac{537}{9686} \\ \frac{8125}{9686} \end{bmatrix}$$

$\therefore$ In the long run, $\frac{512}{4838}$ crops will be good, $\frac{537}{9686}$ crops will be fair, and $\frac{8125}{9686}$ crops will be poor