Math 225 lecture 33 Dec 1st 2023

Goal: Demonstrate an interesting application so the students see other areas where linear algebra is applicable

Class Q: draw a graph which has spectral radius exactly 2

House keeping: - please do the spot survey
- Notes error from last class, correct on eclass
- taking things on faith today / handwaving
(heard of rep theory? ask about it)

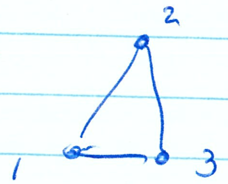
Big Thm today! need things, first,

Recall: on monday we talked about the spectral radius / Frobenius-Perron eigenvalue. that is the largest eigenvalue of the adjacency matrix for a graph G. questions,

where does this come from? will it always exist? is it unique?

Thm (Frobenius - Perron) a real square matrix with positive entries (like our A_G's) has a unique e. val of largest magnitude & that eval. is real!

there are several different proofs of this & there are other assertions one can make with this information. we will not cover these here in detail although there are interesting inequalities & such ~~are~~ that exist because of it.

Ex:  G $\quad A_G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

What are the eigenvalues of A_G? lets do this by inspection, (example of how you might save work on an exam)

$A_G - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$ if $\lambda = -1$ then $A + \lambda = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

which has an obvious null space of dimension 2

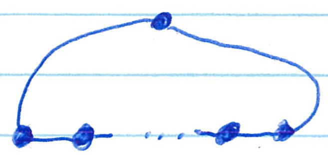
then $\lambda = 2$ gives $A - 2\lambda = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ which has all rows sum to zero placing $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the null space.

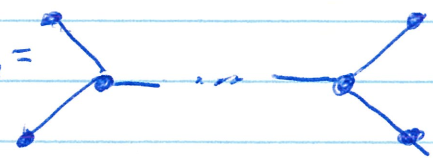
$\therefore$ $\lambda = -1, -1, 2$ are the e.vals of A_G making 2 the spectral radius

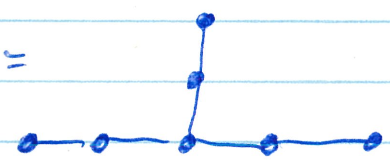
Note, 2 is the largest e.val, has multiplicity 1 and is real!

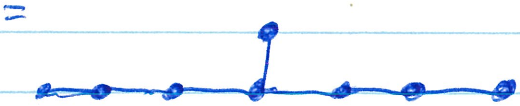
2nd ingredient the simply-laced extended Dynkin diagrams

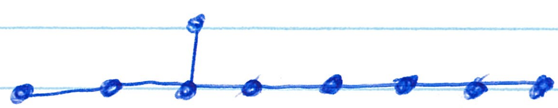
for those unfamiliar the "Dynkin diagrams" are a set of multi graphs that appear naturally when studying semisimple Lie algebras, over algebraically closed ~~fields~~ fields, finite reflection groups, root systems & more the "simply-laced" are those that ~~...~~ are our kinds of graphs. these form the famous A-D-E classification (will talk about time permiting) they are

A_n = 

D_n = 

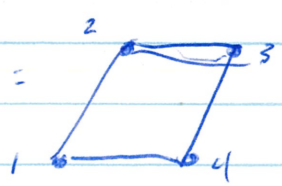
E_6 = 

E_7 = 

E_8 = 

the A & D infinite families & 3 exceptionals (the weirdos)

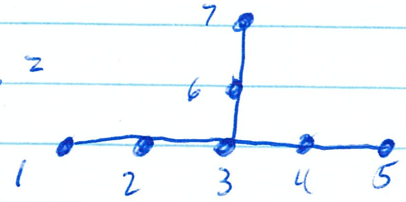
I want to connect these to our other idea, namely spectral radius. lets look at A_3 & E_6

A_3 = 

$$A_{A_3} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

with some effort we can find $\lambda = -2, 0, 0, 2$ to be the evals. Making the spectral radius of $A_3 = 2$

$E_6 =$



$$A_{E_6} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with some effort we get $\lambda = -2, -1, -1, 0, 1, 1, 2$ for e.vals and what do we note about these? 2 is again the spectral radius! (forshadowing ☺)

<u>Big Theorem!</u> the simply-laced Dynkin diagrams compleatly classify all connected graphs with spectral radius 2

<u>Proof sketch</u> first off with some work you can actually show that each of those graphs have spectral radius 2 (the families can be shown inductively).

second ~~us~~ using the F-P Thm any graph with one of our classification as a proper subgraph will have spectral radius > 2 (we see this as ours appearing as a subgraph of the new one) similarly proper subgraphs of ours will have S-R < 2

Third by the existance of the A_ns only trees could possibly be graphs not containing ours w. S-R $= 2$. However one can stare long enough to see that the list of trees ~~which~~ which are not in our classification and neither contained within nor contain (both via subgraphs) is empty making our list exhaustive!

You may ask where these came from. for the purposes of this class how we obtained our A's D's & E's will be called professorial sorcery. (It's actually rep theory but thats a whole additional class)

Lastly we can also note this so called "A-D-E" style classification is far reaching & also classify quivers of finite type, generalized quadrangles, & elementary ~~cha~~ catastrophes & many more. for us this is mathematical curiosity nothing more.