Question 3 (20)

a) Apply Graham-Schmidt to the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

to find an orthogonal basis for $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{w}_1 \cdot \vec{v}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix}$$

$$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}(\vec{v}_2) - \text{proj}_{\vec{w}_2}(\vec{v}_3) = \vec{v}_3 - \frac{\vec{w}_1 \cdot \vec{v}_3}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{w}_2 \cdot \vec{v}_3}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-\frac{1}{4}}{\frac{3}{4}}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 0 - \frac{3}{4} + \frac{1}{12} \\ 1 - \frac{3}{4} + \frac{1}{12} \\ 1 - \frac{3}{4} + \frac{1}{12} \\ 1 - \frac{3}{4} - \frac{1}{4} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$$

Thus, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \right\}$

b) Find a QR factorization to the matrix $A = (\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3)$ whose columns are given by the vectors from part (a)

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\|\vec{w}_1\| = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

$$\|\vec{w}_2\| = \left\| \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} \right\| = \sqrt{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{9}{16}} = \sqrt{\frac{12}{16}} = \frac{\sqrt{3}}{2}$$

$$\|\vec{w}_3\| = \left\| \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Normalize vectors from G-S

$$\frac{1}{\|\vec{w}_1\|}\vec{w}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\frac{1}{\|\vec{w}_2\|}\vec{w}_2 = \frac{1}{\frac{\sqrt{3}}{2}}\vec{w}_2 = \frac{2}{\sqrt{3}}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{2}{4\sqrt{3}} \\ \frac{2}{4\sqrt{3}} \\ \frac{2}{4\sqrt{3}} \\ -\frac{6}{4\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \\ -\frac{3}{2\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\frac{1}{\|\vec{w}_3\|}\vec{w}_3 = \frac{1}{\frac{\sqrt{6}}{3}}\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} = \frac{3}{\sqrt{6}}\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} = \frac{\sqrt{6}}{2}\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ 0 \end{pmatrix}$$

$$\text{So, let } Q = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6} \\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix} = \begin{pmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{pmatrix}$$

$$A = QR \Rightarrow Q^\top A = Q^\top QR \Rightarrow Q^\top A = R$$

$$Q^\top A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{2} \\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$
= \begin{pmatrix} \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0 & 0+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \\ \frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}-\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}+0 & 0+\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{6}-\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{6}}{3}+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+0 & -\frac{\sqrt{6}}{3}+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+0 & 0+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{pmatrix} = R
$$