Math 225 lecture 21 oct 25^{th} 2023

Goal: help students. Work through examples for the midterm

Class Q: which concepts are you still least sure on? what can we do to help you with that?

Ex: LSS construct a least squares solution for the system $A\vec{x}=\vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 2 \end{bmatrix}$$

we will do this via the normal system, i.e.
$\vec{x} = (A^TA)^{-1}A^T\vec{b}$

$$(A^TA)^{-1} = \left(\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\right)^{-1}$$

$$= \frac{1}{27}\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2/9 & 1/9 \\ 1/9 & 2/9 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \end{bmatrix}$$

$$\vec{x} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 2/9 & 1/9 \\ 1/9 & 2/9 \end{bmatrix}\begin{bmatrix} 12 \\ -2 \end{bmatrix} = \begin{bmatrix} 22/9 \\ 8/9 \end{bmatrix}$$

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orthogonal complements

let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ find a basis for $W^\perp$

Recall, a basis is a minimal spanning set. how many elements will a basis of $W^\perp$ have? 2 since $W + W^\perp = \mathbb{R}^4$

easiest method for constructing $W^\perp$ is to let $W = \text{col}(B)$ with $B = [\vec{w}_1, \vec{w}_2]$ then we know $W^\perp = \text{null}(B^T)$

$$\therefore W^\perp = \text{null}\left(\begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \right) = \text{null}\left(\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \right)$$

$\Rightarrow$ ~~x_2~~ $x_3 = s$ $x_4 = t$ parameters as neither column has a leading 1 & the system is

$$\begin{aligned} x_1 + 0 + s - t &= 0 \\ 0 + x_2 - 2s + t &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -s + t \\ x_2 &= 2s - t \end{aligned}$$

meaning $\text{null}(B^T) = \left\{ \begin{bmatrix} -s+t \\ 2s-t \\ s \\ t \end{bmatrix} \middle| \ s,t \in \mathbb{R} \right\}$

or $\left\{ s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \middle| \ s,t \in \mathbb{R} \right\}$ or $\text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

making these two vectors a basis of $W^\perp$. we can verify this as those vectors are linearly indep. and orthogonal to W.

Orthogonally diagonalize $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix}$

recall we need A to be symmetric to do this (which it is)

first find the e. vals of A

$$\det(A - \lambda I) = \left| \begin{bmatrix} (2-\lambda) & 3 & 0 \\ 3 & (2-\lambda) & 4 \\ 0 & 4 & (2-\lambda) \end{bmatrix} \right| = (2-\lambda) \begin{vmatrix} (2-\lambda) & 4 \\ 4 & (2-\lambda) \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 4 & (2-\lambda) \end{vmatrix}$$

$$= (2-\lambda)\left((2-\lambda)^2 - 16\right) - 3\left(3(2-\lambda)\right)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda - 12) - 18 + 9\lambda$$

$$= -\lambda^3 + 6\lambda^2 + 13\lambda - 42 = (\lambda - 2)\ \lambda^2 - 4\lambda - 21$$

$$= (\lambda-2)(\lambda-7)(\lambda+3)$$

this can be done via polynomial long division

$\Rightarrow \lambda_1 = 2 \quad \lambda_2 = 7 \quad \lambda_3 = -3$

find the eigen vectors

$$E_2 = \text{null}\left(\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}\right)$$

$$E_7 = \text{null}\left(\begin{bmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}\right\}$$

$$E_{-3} = \text{null}\left(\begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}\right\}$$

note these are all vectors from destinct e-spaces as such they are all orthogonal already! (why?)

meaning we only need to normalize

$$q_1 = \frac{1}{\sqrt{16+0+9}} \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 0 \\ 3/5 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{9+25+16}} \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5\sqrt{2} \\ 1/\sqrt{2} \\ 4/5\sqrt{2} \end{bmatrix}$$

$$q_3 = \frac{1}{\sqrt{9+25+16}} \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5\sqrt{2} \\ -1/\sqrt{2} \\ 4/5\sqrt{2} \end{bmatrix}$$

meaning $Q = \begin{bmatrix} -4/5 & 3/5\sqrt{2} & 3/5\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 3/5 & 4/5\sqrt{2} & 4/5\sqrt{2} \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

note if eigen values w/ geo mult ≥ 2 you may need to apply G-S to that eigen space.

<u>SVD</u> find the singular value decomposition of ~~A~~

$A = [3, 4]$

first $A^T A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} [3\ 4] = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$

find the eigen values ie. ~~det~~ det $\left| \begin{bmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{bmatrix} \right| = (9-\lambda)(16-\lambda) - 144$

$= \lambda^2 - 25\lambda$

$\lambda(\lambda - 25)$

$\therefore \lambda_1 = 25 \Rightarrow \sigma_1 = 5 \Rightarrow \Sigma = \begin{bmatrix} 5 & 0 \end{bmatrix}$

we then find eigenvectors

$$E_{25} = null\left(\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix}\right) = span\left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$$

$$E_0 = null\left(\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\right) = span\left\{\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right\}$$

renormalize to get

$$\vec{v}_1 = \frac{1}{\sqrt{9+16}}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$\vec{V}_2 = \frac{1}{\sqrt{16+9}}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{5}\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = 1/5 \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

only need one u

$$\therefore A = \begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$$