Question 2 (20)

a) Find the orthogonal complement $W^\perp$ of the subspace

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle|\ x = \tfrac{1}{2}t, y = -\tfrac{1}{2}t, z = 2t, t \in \mathbb{R} \right\}$$

$$W = \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{pmatrix} \right\}$$

$$W^\perp = \{\vec{v} \in \mathbb{R}^3 : \vec{v} \cdot \vec{w} = 0 \ \ \forall \vec{w} \in W\}$$

$$\because \ \vec{w} \in \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{pmatrix} \right\}$$

$$\vec{v} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{pmatrix} = 0$$

$$\therefore \ W^\perp = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \frac{1}{2}x - \frac{1}{2}y + 2z = 0 \right\}$$

b) Consider the subspace $W = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 + x_3 + x_4 = 0\}$ of $\mathbb{R}^4$

i. Find a basis for $W^\perp$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1 + x_2 + x_3 + x_4 = 0$$

Since normal vector $\vec{n}$ is orthogonal to the hyperplane

$$\mathcal{B}^\perp = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

ii. Find a basis for W and use it to construct an orthogonal basis for W using the Gram-Schmidt process.

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & | & 0 \end{pmatrix}$$

$$x_1 = -x_2 - x_3 - x_4$$

$$\begin{aligned} x_1 &= -s - t - u \\ x_2 &= \quad s \\ x_3 &= \quad t \\ x_4 &= \quad u \end{aligned}$$

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$\therefore \mathcal{B}_W = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Using Graham-Schmidt,

$$\vec{v}_1 = \vec{w}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = \vec{w}_2 - \text{proj}_{\vec{v}_1}(\vec{w}_2) = \vec{w}_2 - \frac{v_1 \cdot w_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_3 = \vec{w}_3 - \text{proj}_{\vec{v}_1}\vec{w}_3 - \text{proj}_{\vec{v}_2}\vec{w}_3 = \vec{w}_3 - \frac{\vec{v}_1 \cdot \vec{w}_3}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_3}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\frac{1}{2}}{\frac{3}{2}}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 + \frac{1}{2} + \frac{1}{6} \\ 0 - \frac{1}{2} + \frac{1}{6} \\ 0 - 0 - \frac{1}{3} \\ 1 - 0 - 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

Thus, the orthogonal basis of W is

$$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \right\}$$

iii. Let $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Find $\text{proj}_W(\vec{v})$ in two ways:

A. By first computing $\text{proj}_{W^\perp}(\vec{v})$

$$\because\ \mathcal{B}^\perp = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\vec{x}_1\},$$

$$\text{proj}_{W^\perp}(\vec{v}) = \frac{\vec{x}_1 \cdot \vec{v}}{\vec{x}_1 \cdot \vec{x}_1}\vec{x}_1 = \frac{1+1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Since $\text{perp}_{W^\perp}(\vec{v}) = \vec{v} - \text{proj}_{W^\perp}(\vec{v})$ finds the part of the vector perpendicular to $W^\perp$ and $(W^\perp)^\perp = W$

$$\text{proj}_W(\vec{v}) = \vec{v} - \text{proj}_{W^\perp}(\vec{v}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

B. Using the orthogonal basis of W found above.

$$\text{Recall: } \mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \right\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

$$\begin{aligned}
\text{proj}_W(\vec{v}) &= \frac{\vec{v}_1 \cdot \vec{v}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{v}}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{v}_3 \cdot \vec{v}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\
&= \frac{-1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\frac{1}{2}}{\frac{3}{2}}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \frac{-\frac{2}{3}}{\frac{4}{3}}\begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \\
&= -\frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ -\frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ 0 + \frac{1}{3} + \frac{1}{6} \\ 0 + 0 - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ -\frac{1}{2} - \frac{1}{6} + \frac{1}{6} \\ 0 + \frac{2}{6} + \frac{1}{6} \\ 0 + 0 - \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\end{aligned}$$