Math 225 lecture 10 Sept 27^{th} 2023

Goal: students should understand the result of the G-S process and be able to execute the computations involved for reasonably sized examples.

Class Q: what do you use as inputs to the Gram-Shmidt Process and what do you get back?

section 6.3

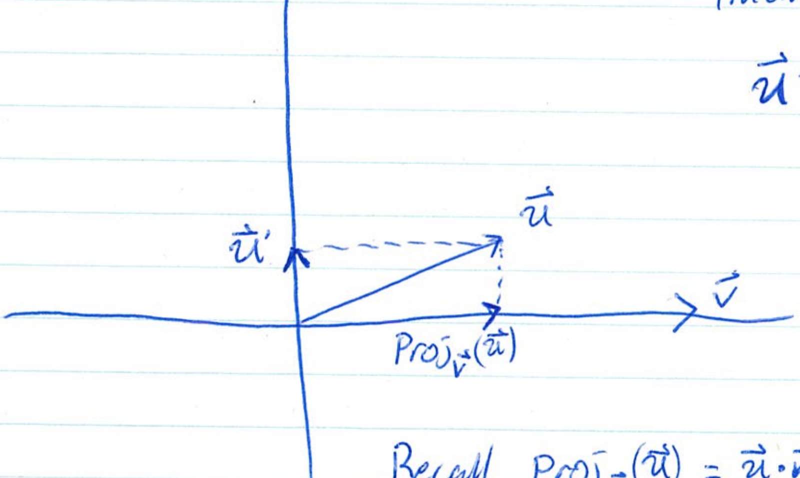
Q: given a set of linearly independant vectors can we find a set which are orthogonal (or orthonormal) which are in some sense equivalent?

Part 1 correct sense of equivalent
- the idea is that they should cover the same span ie for a set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ the new set we construct $\{\vec{v}_1' \dots, v_k'\}$ should have the same number of vectors and the same span.

Part 2 how do we orthogonalize? (making them normal is easy via rescaling at the end)

lets expriment in $\mathbb{R}^2$ visually

suppose we start with $\{\vec{v}, \vec{u}\}$ linearly indep.



$\vec{u}' = \mathrm{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \mathrm{Proj}_{\vec{v}}(\vec{u})$

does this work?

are they still linearly indep?

Recall $\mathrm{Proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$

are they orthogonal?

same span?

did it work?

- if $\vec{w} = c_1 \vec{u} + c_2 \vec{v}$ is in $\mathbb{R}^2$ and we recall that $\vec{u} = \vec{u}' - \underbrace{\text{proj}_v(\vec{u})}_{\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}}$ then $\vec{w} = c_1 \vec{u}' + \left(c_2 - c_1 \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$ which is still a linear combination.

with this we can see that the spans of $\{\vec{u}', \vec{v}\}$ & $\{\vec{u}, \vec{v}\}$ are the same (there is a formal proof to be made here but we will ignore it for now)

- by construction $\vec{u}' \cdot \vec{v} = 0$ we can see this as $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ removes the parts of $\vec{u}$ in the direction of $\vec{v}$ leaving us with only $\vec{u}'$ which is perpendicular to $\vec{v}$

New questions?

1 - could this be done in a larger vector space?
2 - could it be done with more vectors?
3 - how do we move from orthogonal to orthonormal?

3 this one is the "easy" one the norm or length of a vector $\vec{v}$ is denoted $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$ as such we can divide by that to give each of our vectors unit length i.e. replace $\vec{v}$ with $\frac{1}{\|\vec{v}\|}\vec{v}$ & similarly for each other vector.

1 other than our ability to visualize there was nothing unique or otherwise special about $\mathbb{R}^2$. this process would work equally well in $\mathbb{R}^n$, $\mathbb{C}^n$, or a great variety of other spaces

② yes! and doing so is captured via the "gram-shmidt" process

<u>Thm</u>: the Gram-Shmidt process

let $\{\vec{x}_1, \dots, \vec{x}_k\}$ be a basis for the subspace $W \subset \mathbb{R}^n$ then we have

$$\vec{v}_1 = \vec{x}_1 \qquad W_1 = \text{span}\{\vec{x}_1\}$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 \qquad W_2 = \text{span}\{\vec{x}_1, \vec{x}_2\}$$

$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 \qquad W_3 = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$$

$$\vdots$$

$$\vec{v}_k = \vec{x}_k - \left(\frac{\vec{v}_1 \cdot x_k}{\vec{v}_1 \cdot \vec{v}_1}\right)v_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_k}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 - \dots - \left(\frac{\vec{v}_{k-1} \cdot \vec{x}_k}{\vec{v}_{k-1} \cdot \vec{v}_{k-1}}\right)\vec{v}_{k-1}$$

$$W_k = \text{span}\{x_1, \dots, x_k\} = W$$

for each i $\{\vec{v}_1, \dots, \vec{v}_i\}$ is an orthogonal basis for W_i making $\{\vec{v}_1, \dots, \vec{v}_k\}$ an orthogonal basis for W

at this point we can transition to an orthonormal basis by rescaling all of the $\vec{v}_i$'s via i.e.
let $\vec{u}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$