Math 225 lecture 26 NOV 8^{th} 2023

Section 6.3

Goal: students should understand the underlying principals involved with a change of basis and see how it connects back to previously discussed concepts like diagonalization and SVD.

Class Q: how are the concepts of diagonalization, SVD and change of basis ~~relateted~~ related?

Change of basis: for each ~~[illegible]~~ transformation the same thing is happening (marker toss demo) (thumb, middle pointer finger basis) but your perspective changes. this is whats happening mathematically.

Recall 125 approach is $[T] = \begin{bmatrix} | & & | \\ [T(\vec{e}_1)] & \cdots & [T(\vec{e}_n)] \\ | & & | \end{bmatrix} = [T]_{\mathcal{S} \leftarrow \mathcal{S}}$

our approach $T: V \to W$ with ~~[illegible]~~ bases $\mathcal{B} = \{\vec{v}_1, \dots, v_n\}$ $\mathcal{D} = \{\vec{w}_1, \dots, \vec{w}_m\}$ ~~[illegible]~~ for V & W resp.
then

$$[T]_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathcal{D}} & \cdots & [T(v_n)]_{\mathcal{D}} \\ | & & | \end{bmatrix}$$

Today we consider $T = Id$. i.e. $T: V \to V$
$v \mapsto v$

$$\underset{\mathcal{B}}{V} \xrightarrow{Id} \underset{\mathcal{D}}{V} \qquad P_{\mathcal{D} \leftarrow \mathcal{B}} = [Id]_{\mathcal{D} \leftarrow \mathcal{B}}$$

is what we call the "change of basis matrix from $\mathcal{B}$ to $\mathcal{D}$"

Q's: why do we care?
- some bases are nicer to work with than others.
- sometimes information is given to us in non-standard coordinates and we have to use it anyway.

etc.

formally: if $\mathcal{B}=\{\vec{v}_1,\dots,\vec{v}_n\}$ & $\mathcal{D}=\{\vec{w}_1,\dots,\vec{w}_n\}$ ~~then~~ are bases of V let ~~then~~ $P_{\mathcal{D}\leftarrow\mathcal{B}}$ is the change of basis from $\mathcal{B}$ to $\mathcal{D}$ then

① $P_{\mathcal{D}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{D}} \quad \forall \vec{x}\in V$

② $P_{\mathcal{D}\leftarrow\mathcal{B}}$ is the unique matrix with that property

③ $P_{\mathcal{D}\leftarrow\mathcal{B}}$ is invertible with $(P_{\mathcal{D}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{D}}$

Ex: let $V=\mathbb{R}^2$ $\mathcal{B}=\{\vec{e}_1,\vec{e}_2\}$ $\mathcal{D}=\left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right\}$

find $P_{\mathcal{D}\leftarrow\mathcal{B}}$ to do this we need $[\vec{e}_1]_{\mathcal{D}}$ & $[\vec{e}_2]_{\mathcal{D}}$

$$[\vec{e}_1]_{\mathcal{D}} \qquad \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$\parallel$$

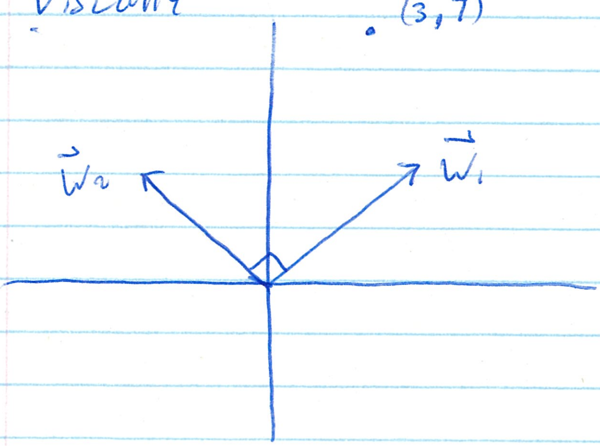
$$\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} + 1/\sqrt{2}\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \Rightarrow [\vec{e}_2]_{\mathcal{D}} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

recall $P_{\mathcal{D}\leftarrow\mathcal{B}} = [Id]_{\mathcal{D}\leftarrow\mathcal{B}} = \begin{bmatrix} | & | \\ [Id(\vec{e}_1)]_{\mathcal{D}} & [Id(\vec{e}_2)]_{\mathcal{D}} \\ | & | \end{bmatrix}$

$\therefore P_{\mathcal{D}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{D}}$ $\qquad = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

Visually



$$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}3\\7\end{bmatrix} = \begin{bmatrix}10/\sqrt{2}\\4/\sqrt{2}\end{bmatrix}$$

$$\Rightarrow \frac{10}{\sqrt{2}}[\vec{w}_1] + 4/\sqrt{2}[\vec{w}_2] = \begin{bmatrix}3\\7\end{bmatrix}$$

<u>Recall</u>: I mentioned linear maps have a favorite basis what was that? the eigen basis! (diagonalization)

You have been tricked! you've done change of basis before we've just called it different things.

<u>Diagonalization</u>: in this context whats going on?

$A = P^{-1}DP$

$$\begin{tikzcd} V \arrow[r, "{[A]_{S\leftarrow S}}"] \arrow[d, "{[P]_{B\leftarrow S}}"'] & V \arrow[d, "{[P]_{B\leftarrow S}}"] \\ V \arrow[r, "{[D]_{B\leftarrow B}}"'] & V \end{tikzcd}$$

where $S = \{\vec{e}_1, \dots, \vec{e}_n\}$

$B = \{\vec{v}_1, \dots, \vec{v}_n\}$

where $\vec{v}_i$ is an eigen vector for A

<u>Ex</u>: let $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} a \\ b \end{bmatrix} \longmapsto \begin{bmatrix} a+2b \\ 2a+b \end{bmatrix}$$

this makes $[T]_{S\leftarrow S} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

our e.vals will be -1 and 3

$E_{-1} = \text{null}\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$E_3 = \text{null}\left(\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

you can do the full process or note that those choices of λ give $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ both of which have non-empty null spaces let $\mathcal{D} = \left\{ \overset{\vec{w}_1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}, \overset{\vec{w}_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right\}$ what is $[T]_{\mathcal{D} \leftarrow \mathcal{D}}$?

$$\begin{bmatrix} | & | \\ [T(\vec{w}_1)]_{\mathcal{D}} & [T(\vec{w}_2)]_{\mathcal{D}} \\ | & | \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{D}} & \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{D}}$ is $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ as $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ similarly we have

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 0 \overset{\vec{w}_1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} + 3 \overset{\vec{w}_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

notice this is diagonal! meaning when the transformation T is viewed through the basis $\mathcal{D}$ it simply scales vectors in those directions.

whats really going on is

$A = P^{-1} D P$ or ~~$[T]_{\mathcal{S} \leftarrow \mathcal{S}} = P$~~ as maps

$$[T]_{\mathcal{S} \leftarrow \mathcal{S}} = \underset{\substack{\parallel \\ [Id]_{\mathcal{S} \leftarrow \mathcal{D}}}}{[P]_{\mathcal{S} \leftarrow \mathcal{D}}} [T]_{\mathcal{D} \leftarrow \mathcal{D}} \underset{\substack{\parallel \\ [Id]_{\mathcal{D} \leftarrow \mathcal{S}}}}{[P]_{\mathcal{D} \leftarrow \mathcal{S}}}$$

it turns out when T is diagonalizeable it means you can find a basis $\mathcal{D}$ for which the action of T is a diagonal matrix.

$\mathcal{D}$ is the basis of eigenvalues for T!

we had a similar notion for all matrices, SVD!

$A = U \Sigma V^T$

let V & W be v. spaces

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \mathcal{B} & & \mathcal{D} \end{array} \quad \text{or} \quad \begin{array}{ccc} V & \to & W \\ \mathcal{S}_V & \to & \mathcal{S}_W \end{array}$$

$$[T]_{\mathcal{D} \leftarrow \mathcal{B}} \qquad [T]_{\mathcal{S}_W \leftarrow \mathcal{S}_V}$$

$$[T]_{\mathcal{S}_W \leftarrow \mathcal{S}_V} = [P]_{\mathcal{S}_W \leftarrow \mathcal{D}} [T]_{\mathcal{D} \leftarrow \mathcal{B}} [P]_{\mathcal{B} \leftarrow \mathcal{S}_V}$$

i.e. making the closest we can get to a diagonal action.

Next day we see how to compute change of basis in general an lots of examples.