Math 225 lecture 24 Nov 3^{rd} 2023

Goal: Reexamine the notions of basis, dimension, finite dimensionality and linear independance through the lense of different vectorspaces.

section 6.2

Class Q: do the concepts of basis, dimension and linear independance work outside of $\mathbb{R}^n$?

Quick notes before our main topic. I do have some small indexing errors in Wednesday's notes. When writing the linear combinations ~~we~~ arrived at via ~~coord~~ coord. vectors.

also Lemma let $\mathcal{B}$ be a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for V. & let $\vec{u}, \vec{v} \in V$ & c a scalar then we have

- $[\vec{u}+\vec{v}]_{\mathcal{B}} = [\vec{u}]_{\mathcal{B}} + [\vec{v}]_{\mathcal{B}}$
- $[c\vec{u}]_{\mathcal{B}} = c[\vec{u}]_{\mathcal{B}}$

Today we are revisiting the concepts of basis, dimension, and linear independance, but... with a twist.

Step 1 some definitions and results (rapid fire style)

Dfn a set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is said to be linearly independant if for $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = 0$ we must have $c_1 = c_2 = \dots = c_n = 0$.

equivalently if one of the $\vec{v}_i$'s can be written as a linear combination of the others then ~~it~~ the set is linearly dependant.

Dfn this allows us to define a basis.
a subset $\mathcal{B} \subset V$ is a basis if
① $\mathcal{B}$ spans V
② $\mathcal{B}$ is linearly independant.

- writing a vector as a linear combination of a basis is also unique.

- bases are not unique but the number of elements in them are.

↳ we call this number the dimension of the space. moreover if a space has a basis and it is finite we call the space finite-dimensional.

- the dimension of $\{\vec{0}\}$ is 0

and finally if $W \subset V$ is a subspace & V is finite dimensional then
① W has a basis
② W is finite dimensional with $\dim(W) \leq \dim(V)$
③ $\dim(W) = \dim(V)$ iff $W = V$.

we could prove these facts or belabor the definitions that sounds boring for you and thus not productive for me.

as an alternative we will outline these ideas and look at examples of them but almost exclusively outside $\mathbb{R}^n$

Ex: let $P_2(\mathbb{R})$ be the set of ~~real valued~~ polynomials with real coefficients of degree ≤ 2

let $B = \{1+x, x+x^2, 1+x^2\}$ is B linearly independant in $P_2(\mathbb{R})$?

$$c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = 0$$
$$= (c_1+c_3) + (c_1+c_2)x + (c_2+c_3)x^2 = 0$$

$\Rightarrow$ $c_1 + c_3 = 0$, $c_1 + c_2 = 0$, $c_2 + c_3 = 0$

we could check but the only solution to that is $c_1 = c_2 = c_3 = 0$ $\Rightarrow$ B forms a linearly independant set in $P_2(\mathbb{R})$.

similarly we could ask if $S = \{1, x, x^2\}$

is linearly independant.

Proof 1 $P(x) = c_1 + c_2 x + c_3 x^2 = 0$ for all values of x but if a polynomial is 0 at more points than its degree its 0 everywhere making $c_1 = c_2 = c_3 = 0$ $\Rightarrow$ S is linearly independant.

(Proof 2 can be done with derivatives see pg 445)

is S a basis for $P_2(\mathbb{R})$? Yes! why?

consider a generic degree two polynomial. it can always be written as $ax^2 + bx + c$ but that is a linear combination of that basis!

making $\dim(P_2(\mathbb{R})) = 3$

was B also a basis?

consider $W \in \mathcal{P}_2(\mathbb{R})$ to be the polynomials of degree 1 or less. this is a subspace of $\mathcal{P}_2(\mathbb{R})$

what is $\dim(W)$? 2

find a basis for W what about $\{1, x\}$? ✓
$\{x, x+1\}$? ✓ $\{x+1, x+2\}$? ✓
$\{8, 19x+24\}$ ✓ and so on.

what about the other things we can do with a basis?

let $\mathcal{B} = \{1, x+1\}$ what is $[2x+4]_{\mathcal{B}}$?

$$c_1(1) + c_2(x+1) = (c_1+c_2) + c_2 x = 2x+4$$

$$\Rightarrow c_2 = 2 \quad c_1 = 2 \quad \therefore [2x+4]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

all of these concepts apply across vectorspaces in general

Ex: let $M_{2\times2}(\mathbb{R})$ be 2×2 matrices with real entries

what is the dimension of this space? 4 what is a basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

~~case~~ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ & $C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

linearly independant here?

NO, ~~aa~~ $A+B=C$

you can do this with functions too ie
$\{\sin^2 x, \cos^2 x, \cos 2x\}$ these are linearly

dependant too as $\cos 2x = \cos^2 x - \sin^2 x$

Examples are endless! but what is the point?

I want you thinking in terms of vectorspaces, not just $\mathbb{R}^n$. everything we do here is valid in that more general context.