CMPUT 428: 3D Projective Geometry

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Contents

1 Lecture - Feb 13

lec06Proj3D.pdf

Points:

$$\mathbf{X} = (x_1, x_2, x_3, x_4); x_4 \neq 0$$

Point at inf:

$$(x_1, x_2, x_3, 0)$$

Planes:

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$$

Plane at inf:

$$\pi^\top \mathbf{X} = 0$$

proj transform $\rightarrow$ 4×4 nonsingular matrix H
Point Transformation: $X' = HX$
Plane Transformation: $\pi' = H^{-\top}\pi$

3D point:
$(X, Y, Z)^\top$ in $\mathbb{R}^3$
$X = (X_1, X_2, X_3, X_4)$ in P^3

Euclidean Rep:

$$n^\top \tilde{X} + d = 0$$

$n = (\pi_1, \pi_2, \pi_3)^\top$, $\tilde{X} = (X, Y, Z)^\top$
$\pi_4 = d$, $X_4 = 1$
d is distance

1.1 Planes from Points

Solve π for:

$$\mathbf{X}_1^\top \pi = 0, \mathbf{X}_2^\top \pi = 0, \text{ and } , \mathbf{X}_3^\top \pi = 0$$

(use SVD)
or implicitly **Coplanarity Condition** (check slides)

Canonical Form:
If we have a plane, we can get nullspace span $\mathbf{M} = \begin{bmatrix} p \\ I \end{bmatrix}$

$$p = \left[-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a}\right]^\top$$

1.2 Points from Planes

Solve $\mathbf{X}$ from

$$\pi_1^\top \mathbf{X} = 0, \pi_2^\top \mathbf{X} = 0, \text{ and } \pi_3^\top \mathbf{X} = 0$$

1.3 Lines

Join of two points (A, B)

$$W = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix}, \lambda A + \mu B$$

Intersection of two planes (P, Q)

$$W^* = \begin{bmatrix} P^\top \\ Q^\top \end{bmatrix}, \lambda P + \mu Q \text{ "pencil of planes"}$$

pencil of planes - all the planes that could generate an intersection of this line

$$W^* W^\top = W W^{*\top} = 0_{2\times 2}$$

Define basis/direction vectors as pure direction ($x_4 = 0$) rather than a finite point
W is 2×4

1.4 Points, lines, planes

Join of point X and line W is plane π

$$\mathbf{M} = \begin{bmatrix} W \\ X^\top \end{bmatrix} \quad \mathbf{M}\pi = 0$$

Intersection of line W with plane π is point X

$$\mathbf{M} = \begin{bmatrix} W^* \\ \pi^\top \end{bmatrix} \quad \mathbf{M}X = 0$$

1.5 Affine space

Difficulties with a projective space:

- Nonintuitive notion of direction
  - Parallelism not represented
  - Restore parallelism $\rightarrow$ need to know/fix some affine property
- Infinity not distinguished
- No notion of "inbetweenness"
  - Projective lines are topologically circular
- Only cross-ratios are available

– Ratios are required for many practical tasks

Solution:
Find the point at infinity

$$\pi_\infty = (0, 0, 0, 1)$$

Transform model to give π_∞ canonical coords
2D analogy: fix horizontal line $\mathbf{l}_\infty = (0, 0, 1)$

1.5.1 Proj to Affine

Get "parallel" lines in img, find projective transformation that would make lines parallel

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Affine transformation:

$$H_A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

12 DOF, π_∞ unchanged
Invariants:

Ratio of lengths on a line

Ratios of angles

Parallelism

1.6 Metric Space

Metric transformation (similarity)

$$H_S = \begin{bmatrix} sR & \mathbf{t} \\ \mathbf{O}^t & 1 \end{bmatrix}$$

7DOF, maps absolute conic to itself
Invariants:

Length ratios

Angles

Absolute conic

Highest level of geometric structure that can be retrived from images (without yard stick/ruler)

Overview 1.6.1: Main Takeaway of 428

"Stratified Geometry" vs Euclidean Geometry

Euclidean geometry is not the only repr for:

1. building models from imgs
2. building imgs from models

When 3D realism is the goal; how can we effectively build and use projective, affine, metric representations.

2 Lecture - Feb 15

lec08Proj, lec06Proj3D; **READ SLIDES!!!!!!!!!**

Projections - use similar triangles
Camera matrix

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} \approx \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

$$(U, V, W) \rightarrow \left(\frac{U}{W}, \frac{V}{W} \right) = (u, v)$$

when coord sys not aligned, P becomes arbitrary singular or nonsingular matrix

$$x = PX$$

2.1 Intrinsic Parameters

Conversion from metric to pixel coords (and reverse)

$$x_{\text{mm}} = -(x_{\text{pix}} - o_x)s_x$$

$$y_{\text{mm}} = -(y_{\text{pix}} - o_y)s_x$$

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix}_{\text{pix}} = \begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}_{\text{mm}} = \mathbf{M}_{\text{int}}\mathbf{p}$$

Expln 2.1.1

f is focal length (mm); s is scale factor (aspect ratios), o center coords

2.2 Relative Location (cam/laser)

create homog transforms that represent the alignments;
camera interal params from manufacturers
first matrix via calibration eq.
Internal params = known information; need to find ext. params (unknown)

2.3 Diff Cam Models

Orthographic → pure parallel projection
Weak Perspective → parallel projection onto imaginary plane (obj plane) and then perspective transformation. (single scaling factor)
Paraperspective →
Perspective →

2.4 Weak Perspective

Uses a scaling factor T

$$u = Tx, v = Ty, T = f/Z$$

Z is the **average** depth

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & f/Z^* \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix}$$

Z^* fixed val, usually mean/centroid distance

2.5 Full Affine Linear Camera

factor into rotation part and scale part (?)

2.6 Camera Models