Formalizing Modal Embeddings of Call-by-Name and Call-by-Value

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Research Question

How can the unification of call-by-name and call-by-value evaluation strategies using modal logic be formalised in Agda?

Introduction

Call-by-name

Let f be defined as

$f(x) = x * x$

Call-by-name

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Call-by-name evaluation of f (3 + 3)

Call-by-name

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$f(x) = x * x$

Call-by-name evaluation of f (3 + 3)

$$f(3+3) \rightarrow (3+3) * (3+3)$$

Call-by-name

Let f be defined as

$f(x) = x * x$

Call-by-name evaluation of f (3 + 3)

$$\begin{aligned} f(3+3) &\rightarrow (3+3) * (3+3) \\ &\rightarrow 6 * (3+3) \end{aligned}$$

Call-by-name

Let f be defined as

$f(x) = x * x$

Call-by-name evaluation of f (3 + 3)

$$\begin{aligned} f(3+3) &\rightarrow (3+3) * (3+3) \\ &\rightarrow 6 * (3+3) \\ &\rightarrow 6 * 6 \end{aligned}$$

Call-by-name

Let f be defined as

$f(x) = x * x$

Call-by-name evaluation of f (3 + 3)

$$\begin{aligned} f(3+3) &\rightarrow (3+3) * (3+3) \\ &\rightarrow 6 * (3+3) \\ &\rightarrow 6 * 6 \\ &\rightarrow 36 \end{aligned}$$

Call-by-value

Let f be defined as

$f(x) = x * x$

Call-by-value

Let f be defined as

$f(x) = x * x$

Call-by-value evaluation of f (3 + 3)

Call-by-value

Let f be defined as

$f(x) = x * x$

Call-by-value evaluation of f (3 + 3)

$$f(3 + 3) \rightarrow f(6)$$

Call-by-value

Let f be defined as

$f(x) = x * x$

Call-by-value evaluation of f (3 + 3)

$$\begin{aligned} f(3+3) &\rightarrow f(6) \\ &\rightarrow 6 * 6 \end{aligned}$$

Call-by-value

Let f be defined as

$f(x) = x * x$

Call-by-value evaluation of f (3 + 3)

$$\begin{aligned} f(3+3) &\rightarrow f(6) \\ &\rightarrow 6 * 6 \\ &\rightarrow 36 \end{aligned}$$

Unifying cbn and cbv

- Why unify cbn and cbv?
- Some approaches to unification:
  - Modal logic
  - Linear logic
  - Thunks

Background

Grammar Lambda Calculus (λ-calculus)

$$A \quad ::= \quad X \mid \; A \to A' \qquad\qquad M, N, P, Q \quad ::= \quad x \mid \; \lambda x.M \mid \; MN$$

Example terms

Grammar Lambda Calculus (λ-calculus)

$$A \quad ::= \quad X \mid \; A \to A' \qquad\qquad M, N, P, Q \quad ::= \quad x \mid \; \lambda x.M \mid \; MN$$

Example terms

x

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$$A \quad ::= \quad X \mid \; A \to A' \qquad\qquad M, N, P, Q \quad ::= \quad x \mid \; \lambda x.M \mid \; MN$$

Example terms

x

$\lambda x.x$

Grammar Lambda Calculus (λ-calculus)

$$A \quad ::= \quad X \mid \; A \to A' \qquad\qquad M, N, P, Q \quad ::= \quad x \mid \; \lambda x.M \mid \; MN$$

Example terms

x

$\lambda x.x$

$\lambda y.yz$

Grammar Lambda Calculus (λ-calculus)

$$A \quad ::= \quad X \mid \; A \to A' \qquad\qquad M, N, P, Q \quad ::= \quad x \mid \; \lambda x.M \mid \; MN$$

Example terms

x

$\lambda x.x$

$\lambda y.yz$

$(\lambda y.y)z$

Grammar Lambda Calculus (λ-calculus)

$$A \quad ::= \quad X \mid \; A \to A' \qquad\qquad M, N, P, Q \quad ::= \quad x \mid \; \lambda x.M \mid \; MN$$

Example terms

x

$\lambda x.x$

$\lambda y.yz$

$(\lambda y.y)z$

$\lambda x.\lambda t.xt(\lambda s.w)wy$

Call-by-name and call-by-value λ-calculus

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Closure Rules

Here we define closure rules

Evaluation Relations

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Call-by-box λ-calculus

Grammer

$$A \quad ::= \quad X \mid \; B \to A \mid \; B \qquad\qquad B \quad ::= \quad \Box A$$

$$M, N, P, Q \quad ::= \quad \varepsilon(x) \mid \; \lambda x.M \mid \; MN \mid \; \mathrm{box}(N)$$

Evaluation Relations of λ_b

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Embeddings into λ_b

Girard's Translation

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Gödel's Translation

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Challenges of Formalisation

Overview Challenges

- Variables
- Ill typed terms
- Formal definition of raise

De Bruijn Indices

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Restriction to well-typed terms

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Formal definition of raise

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Propositions

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Conclusion

Conclusion

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