Final

Rules: You cannot work in groups for the Final, each student must turn in their own work.

1. **Cooking a rectangular two-dimensional potato**

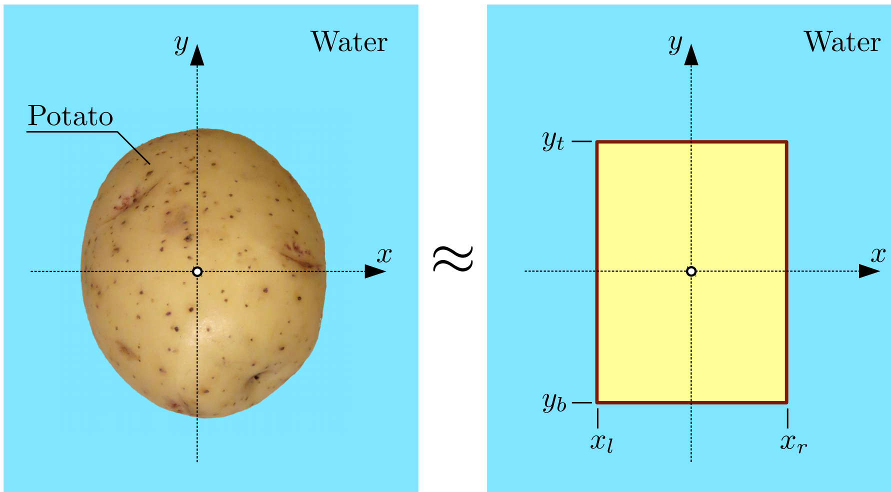

Figure 1: Rectangular two-dimensional approximation of a potato

The goal of this exercise is to determine the optimal time for boiling a potato. To simplify this task let us consider the problem in two spatial dimensions and approximate the potato by a rectangle (yep!) of size 4 cm by 5 cm (see Figure 1). At time $t_{\text{start}} = 0$ a pan filled with water and containing the potato is placed on a stove top. Initially the potato and the water are at room temperature $T_{\text{room}} = 20\,°\text{C}$. The temperature of the water rises from 20 °C to 100 °C in 60 seconds and stays at 100 °C afterwards, that is, $T_{\text{water}} = \min\left(20 + 80\frac{t}{60}, 100\right)$. Due to the thermal conduction heat propagates from water inside the potato. At the temperature of $T_{\text{cooking}} = 65\,°\text{C}$ the cellular structure of the potato begins to change and the starch starts to gelatinize. We assume that it takes 300 seconds (5 minutes) for the potato's material to get fully cooked after it has reached temperature T_{cooking}.

The temperature $T = T(t, x, y)$ inside the potato satisfies the heat equation (= diffusion equation)

$$\frac{\partial T}{\partial t} = \lambda \Delta T, \quad (x, y) \in \Omega \tag{1}$$

with the boundary conditions

$$T(t, x, y) = T_{\text{water}}(t), \quad (x, y) \in \partial\Omega,$$

and the initial conditions

$$T(0, x, y) = T_{\text{room}}, \quad (x, y) \in \Omega,$$

where λ is the thermal diffusivity of the potato's material, Ω and $\partial\Omega$ denote the domain in space occupied by the potato and the boundary of this domain.

(a) Consider a generalized problem of heat transfer in a rectangular domain $\Omega = [x_l; x_r] \times [y_b; y_t]$

$$\begin{cases} \dfrac{\partial T}{\partial t} = \lambda \Delta T + f, & (x,y) \in \Omega, \\ T(t,x,y) = T_{\text{bc}}(t,x,y), & (x,y) \in \partial\Omega, \\ T(t_{\text{start}},x,y) = T_{\text{start}}(x,y), & (x,y) \in \Omega, \end{cases} \tag{2}$$

where λ is the thermal diffusivity of the material, $f = f(t,x,y)$, $T_{\text{bc}}(t,x,y)$ and $T_{\text{start}}(x,y)$ are given functions describing the source term, boundary conditions and initial conditions, respectively.

Use the **implicit** scheme to discretize (2) and explain the structure of the linear system satisfied by $\vec{T}^{n+1} = \begin{pmatrix} T^{n+1}_{1,1} & T^{n+1}_{1,2} & \dots & T^{n+1}_{N_x,N_y} \end{pmatrix}$.

(b) Implement the **implicit** scheme for solving the heat equation (2). In particular, your implementation should make a good use of the Matlab `sparse` structure. Test your code using the following example:

$$\begin{aligned} \text{Domain:} \quad & \Omega = [-1; 1] \times [-0.5; 1.7], \\ \text{Thermal diffusivity:} \quad & \lambda = 0.75, \\ \text{Exact solution:} \quad & T_{\text{exact}} = \sin(x)\cos(y)\exp(-t), \end{aligned}$$

and where initial conditions $T_{\text{start}}(x,y)$, boundary conditions $T_{\text{bc}}(t,x,y)$ and source term $f(t,x,y)$ should be calculated from the given exact solution T_{exact}. Solve the heat equation (2) from $t_{\text{start}} = 0$ to $t_{\text{final}} = 1$ for grid resolutions $(N_x, N_y) = (25, 30)$, $(50, 60)$ and $(100, 120)$ and time-step $\Delta t = 0.5\Delta x$. Calculate errors of numerical solutions as the maximum (among all grid points) absolute deviation from the exact solution at the moment of time $t = t_{\text{final}}$ and determine the order of accuracy of the numerical method.

(c) Use your code to simulate the process of boiling a potato, that is, set parameters to

$$\begin{aligned} \text{Domain:} \quad & \Omega = [-2; 2] \times [-2.5; 2.5] \text{ cm}, \\ \text{Thermal diffusivity:} \quad & \lambda = 1.5 \times 10^{-3} \text{ cm}^2/\text{s}, \\ \text{Initial conditions:} \quad & T_{\text{start}}(x,y) = 20\,^\circ\text{C}, \\ \text{Boundary conditions:} \quad & T_{\text{bc}}(t,x,y) = \min\left(20 + 80\frac{t}{60}, 100\right)\,^\circ\text{C}, \\ \text{Source term:} \quad & f(t,x,y) = 0, \end{aligned}$$

Solve the heat equation from $t_{\text{start}} = 0$ s to $t_{\text{final}} = 1500$ s using $N_x = 80$ grid points in the x-direction and $N_y = 100$ grid points in the y-direction and time-step $\Delta t = 5$ s. Plot the temperature at the center of the potato ($(x,y) = (0,0)$) vs time and determine when this temperature reaches $T_{\text{cooking}} = 65\,^\circ$C. Add 300 seconds to the found time to obtain the total time needed to cook a potato. Take snapshots of the temperature distribution inside the potato at moments of time $t = 0$, 200, 400, 600 s.

What to turn in: your code and a professional report should be uploaded to Gauchospace.

2. **Refugio Oil Spill**

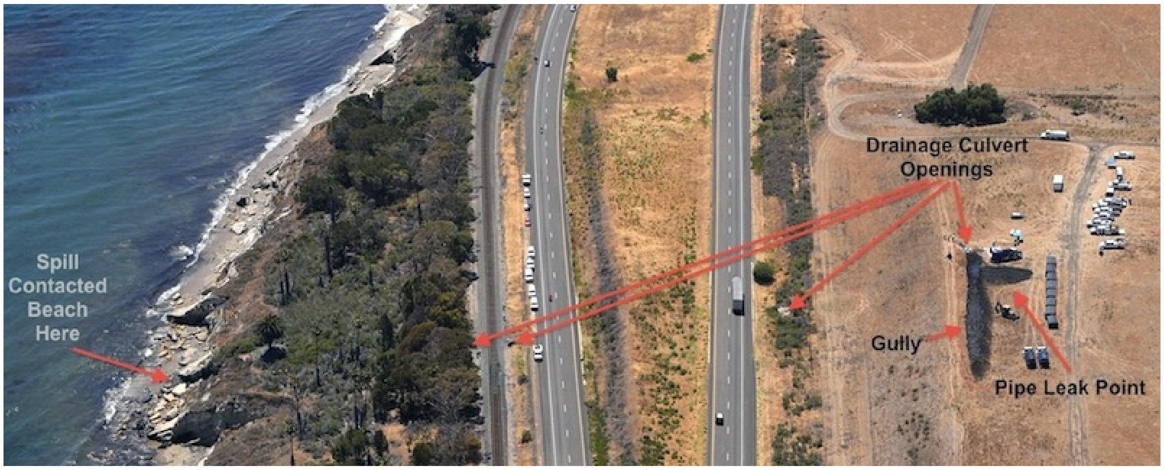

Figure 2: Aerial photo of Refugio Oil Spill (source: John L. Wiley http://flickr.com/jw4pix)

It is May 20, 2015. Yesterday, May 19, 2015, a broken onshore pipeline near Santa Barbara spewed oil down a storm drain and into the ocean for several hours before it was shut off. Currents and natural diffusion of contaminants are two effects that account for how the oil spreads. In order to avoid unsafe bathing, the city of Santa Barbara has tasked you to estimate what beaches should be closed to the public. According to health officials, a beach where the concentration of oil is (strictly) greater than $c_{\text{limit}} = .006$ is deemed unsafe.

As an engineer, you model this process by an advection-diffusion equation in two spatial dimensions. In addition, you assume that the stretch of the coast you are interested in is straight (see Figure 3). You thus consider a domain $\Omega = [x_l; x_r] \times [y_b; y_t]$, a given velocity vector, $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, representing the ocean's currents, a source term, $f = f(t, x, y)$, representing the amount of oil spilling into the ocean and a concentration, $c = c(t, x, y)$, representing the concentration of oil in the ocean. The concentration thus satisfies the advection-diffusion equation

$$\frac{\partial c}{\partial t} + \vec{v} \cdot \nabla c = D \Delta c + f, \quad \text{for all } (x, y) \in \Omega. \tag{3}$$

where D is the rate of diffusion of oil in water, with the initial conditions

$$c(t_{\text{start}}, x, y) = 0.$$

Let us assume that the left, right and top boundaries of domain Ω are far enough, so that the oil concentration stays zero at those boundaries during the course of simulation, that is,

$$c(t, x, y) = 0, \quad \text{if } x = x_l,\ x = x_r \text{ or } y = y_t.$$

However, at the bottom boundary of domain Ω the oil concentration has to satisfy the no-flux condition

$$D \frac{\partial c}{\partial y} - v_y c = 0, \quad \text{if } y = y_b.$$

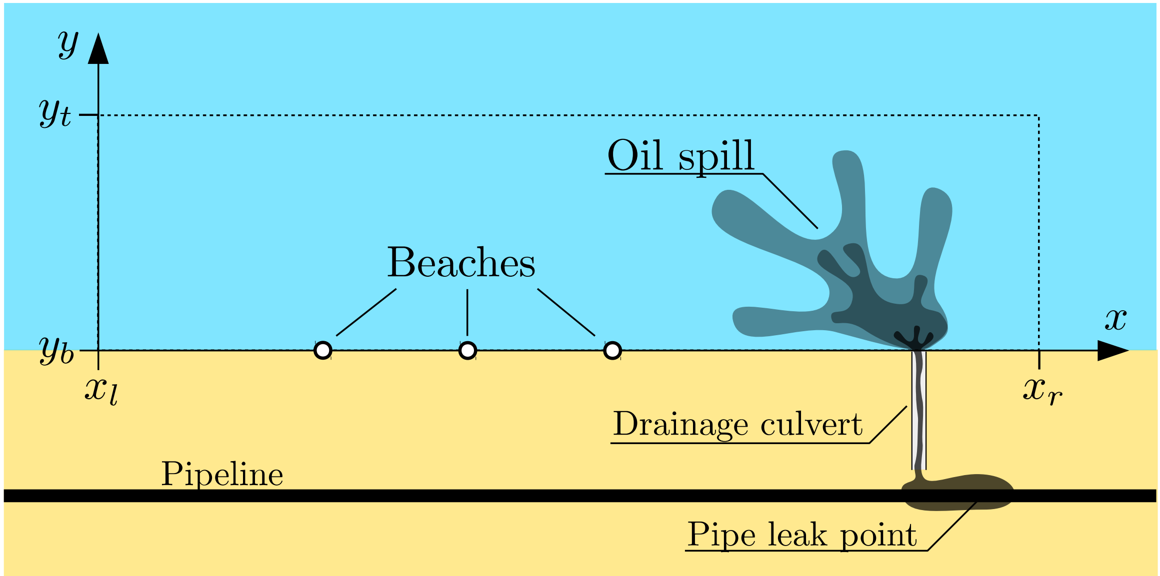

Figure 3: Schematic representation of the Refugio Oil Spill

(a) Consider a generalized advection-diffusion problem in a rectangular domain $\Omega = [x_l; x_r] \times [y_b; y_t]$

$$\begin{cases} \text{PDE:} & \dfrac{\partial c}{\partial t} + \vec{v} \cdot \nabla c = D\Delta c + f, & (x, y) \in \Omega, \\ \text{BC:} & c(t, x, y) = c_{\text{bc}}(t, x, y), & \text{if } x = x_l,\ x = x_r \text{ or } y = y_t, \\ & D\dfrac{\partial c}{\partial y} - v_y c = g(t, x, y), & \text{if } y = y_b, \\ \text{IC:} & c(t_{\text{start}}, x, y) = c_{\text{start}}(x, y), & (x, y) \in \Omega, \end{cases} \tag{4}$$

where D is the diffusivity, $f = f(t, x, y)$, $c_{\text{bc}} = c_{\text{bc}}(t, x, y)$, $g = g(t, x, y)$ and $c_{\text{start}} = c_{\text{start}}(x, y)$ are given functions describing the source term, boundary conditions and initial conditions.

Use the **explicit upwind** approximation to discretize the advection term $\vec{v} \cdot \nabla c$ and the **implicit** approximation to discretize the diffusion term $D\Delta c$. Explain the structure of the linear system satisfied by $\vec{c}^{n+1} = \begin{pmatrix} c_{1,1}^{n+1} & c_{1,2}^{n+1} & \dots & c_{N_x,N_y}^{n+1} \end{pmatrix}$

(b) Implement the obtained numerical scheme. In particular, your implementation should make a good use of the Matlab `sparse` structure. Test your code using the following example:

Domain:	$\Omega = [-1; 3] \times [-1.5; 1.5]$,
Diffusivity:	$D = 0.7$,
Velocity field:	$v_x = -0.8$,
	$v_y = -0.4$,
Exact solution:	$c_{\text{exact}} = \sin(x)\cos(y)\exp(-t)$,

and where initial conditions $c_{\text{start}}(x,y)$, boundary conditions $c_{\text{bc}}(t,x,y)$ and $g(t,x,y)$ and source term $f(t,x,y)$ should be calculated from the given exact solution c_{exact}. Solve the advection-diffusion equation from $t_{\text{start}} = 0$ to $t_{\text{final}} = 1$ using grid resolutions $(N_x, N_y) = (20,15)$, $(40,30)$, $(80,60)$ and $(160,120)$ and time-step $\Delta t = 0.5\Delta x$. Calculate errors of numerical solutions as the maximum (among all grid points) absolute deviation from the exact solution at the moment of time $t = t_{\text{final}}$ and determine the order of accuracy of the numerical method.

(c) Use your code to simulate spreading of the oil in the ocean, that is, set parameters of the problem to

$$
\begin{aligned}
\text{Domain:} \quad & \Omega = [0;12] \times [0;3], \\
\text{Diffusivity:} \quad & D = 0.2, \\
\text{Velocity field:} \quad & v_x = -0.8, \\
& v_y = -0.4, \\
\text{Initial conditions:} \quad & c_{\text{start}}(t,x,y) = 0, \\
\text{Boundary conditions:} \quad & c_{\text{bc}}(t,x,y) = 0, \\
& g(t,x,y) = 0, \\
\text{Source term:} \quad & f(t,x,y) = \begin{cases} \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{(x-x_s)^2+y^2}-r_s}{\epsilon}\right)\right), & \text{if } t < 0.5, \\ 0, \quad \text{if } t > 0.5, \end{cases}
\end{aligned}
$$

where $x_s = 10$, $r_s = 0.1$ and $\epsilon = 0.1$.

Solve the advection-diffusion equation from $t_{\text{start}} = 0$ to $t_{\text{final}} = 10$ using $N_x = 160$ grid points in the x-direction and $N_y = 40$ points in the y-direction and time step $\Delta t = 0.1$. Plot the oil concentration at points $(x,y) = (4,0)$, $(6,0)$ and $(8,0)$ vs time and determine time periods at which each of the three beaches should be closed. Take snapshots of the oil concentration at moments of time $t = 1$, 4, 7.

What to turn in: your code and a professional report should be uploaded to Gauchospace.