LISTA 1: Métodos Computacionais Para a Física C

① Estabilidade numérica do método de Lax:

Lax:

$$f_j^{n+1} = \frac{1}{2}\left(f_{j+1}^{n} + f_{j-1}^{n}\right) - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^{n} - f_{j-1}^{n}\right)$$

$$f_j^n = A^n e^{iqj\Delta x}$$

$$A^{n+1} e^{iqj\Delta x} = \frac{1}{2}\left(A^n e^{iq\Delta x(j+1)} + A^n e^{iq\Delta x(j-1)}\right) - \frac{v\Delta t}{2\Delta x}\left(A^n e^{iq\Delta x(j+1)} - A^n e^{iq\Delta x(j-1)}\right)$$

$$\frac{A^{n+1}}{A^n} = \frac{1}{2}\left(1 - \frac{v\Delta t}{\Delta x}\right) e^{iq\Delta x} + \frac{1}{2}\left(1 + \frac{v\Delta t}{\Delta x}\right) e^{-iq\Delta x} \qquad K = \frac{v\Delta t}{\Delta x}$$

$$\frac{A_{n+1}}{A_n} = \frac{e^{iq\Delta x} + e^{-iq\Delta x}}{2} + K\,\frac{\left(e^{-iq\Delta x} - e^{iq\Delta x}\right)}{2}$$

$$\frac{A_{n+1}}{A_n} = \cos(q\Delta x) - iK\,\mathrm{sen}(q\Delta x)$$

$$\left|\frac{A_{n+1}}{A_n}\right|^2 = \left[\cos(q\Delta x) + iK\,\mathrm{sen}(q\Delta x)\right]\left[\cos(q\Delta x) - iK\,\mathrm{sen}(q\Delta x)\right]$$

$$\left|\frac{A_{n+1}}{A_n}\right|^2 = \cos^2(q\Delta x) - i^2K^2\,\mathrm{sen}^2(q\Delta x)$$

$$\left|\frac{A_{n+1}}{A_n}\right|^2 = \cos^2(q\Delta x) + K^2\,\mathrm{sen}^2(q\Delta x)$$

$$\left|\frac{A_{n+1}}{A_n}\right|^2 = 1 - \mathrm{sen}^2(q\Delta x) + K^2\mathrm{sen}^2(q\Delta x) = 1 + (K^2 - 1)\,\mathrm{sen}^2(q\Delta x)$$

o valor máximo de $\mathrm{sen}^2(q\Delta x) = 1$ para $q\Delta x = 2n+1$ $(n \in \mathbb{Z})$

$$\left[(e^{iq\Delta x}+e^{-iq\Delta x})+(e^{iq\Delta x}-e^{-iq\Delta x})\frac{v\Delta t}{2\Delta x}\right]A^{n+1}=$$

$$\left[(e^{iq\Delta x}+e^{-iq\Delta x})+(e^{-iq\Delta x}-e^{iq\Delta x})\frac{v\Delta t}{2\Delta x}\right]A^{n}$$

$$k=\frac{v\Delta t}{2\Delta x}$$

$$\left[\cos(q\Delta x)+\frac{iv\Delta t}{2\Delta x}\operatorname{sen}(q\Delta x)\right]A^{n+1}=\left[\cos(q\Delta x)-\frac{iv\Delta t}{2\Delta x}\operatorname{sen}(q\Delta x)\right]A^{n}$$

$$\downarrow |x|^2$$

$$\left[\cos^2(q\Delta x)+k^2\operatorname{sen}^2(q\Delta x)\right]|A^{n+1}|^2=\left[\cos^2(q\Delta x)+k^2\operatorname{sen}^2(q\Delta x)\right]|A^{n}|^2$$

$$\left|\frac{A^{n+1}}{A^n}\right|^2=\frac{[\cos^2(q\Delta x)+k^2\operatorname{sen}^2(q\Delta x)]}{[\cos^2(q\Delta x)+k^2\operatorname{sen}^2(q\Delta x)]}=1$$

?

(CONTINUA AQUI)

$$\frac{f_j^{n+1}-f_j^n}{\Delta t}=\frac{-v}{2}\left[\frac{f_{j+1}^{n+1}-f_{j-1}^{n+1}}{2\Delta x}+\frac{f_{j+1}^{n}-f_{j-1}^{n}}{2\Delta x}\right]=\frac{-v}{4\Delta x}\left[f_{j+1}^{n+1}-f_{j-1}^{n+1}+f_{j+1}^{n}-f_{j-1}^{n}\right]$$

$$f_j^{n+1}+\frac{v\Delta t}{4\Delta x}f_{j+1}^{n+1}-\frac{v\Delta t}{4\Delta x}f_{j-1}^{n+1}=f_j^n-\frac{v\Delta t}{4\Delta x}f_{j+1}^{n}+\frac{v\Delta t}{4\Delta x}f_{j-1}^{n}$$

$$A^{n+1}e^{iqj\Delta x}+\frac{v\Delta t}{4\Delta x}\left(e^{iq\Delta x(j+1)}-e^{iq\Delta x(j-1)}\right)A^{n+1}=A^n e^{iqj\Delta x}+\frac{v\Delta t}{4\Delta x}\left(e^{iq\Delta x(j-1)}-e^{iq\Delta x(j+1)}\right)A^n$$

$$\frac{A^{n+1}}{A^n}\left[1+\frac{k}{2}(e^{iq\Delta x}-e^{-iq\Delta x})\right]=1+\frac{k}{2}(e^{-iq\Delta x}-e^{iq\Delta x})\qquad k=\frac{v\Delta t}{2\Delta x}$$

$$\frac{A^{n+1}}{A^n}\left[1+ik\operatorname{sen}(q\Delta x)\right]=1-ik\operatorname{sen}(q\Delta x)$$

$$\boxed{|a+ib|=\sqrt{(a)^2+(b)^2}}$$

$$\left|\frac{A^{n+1}}{A^n}\right|=\frac{|1-ik\operatorname{sen}(q\Delta x)|}{|1+ik\operatorname{sen}(q\Delta x)|}=\frac{\sqrt{1+(-k\operatorname{sen}(q\Delta x))^2}}{\sqrt{1+(k\operatorname{sen}(q\Delta x))^2}}<1$$

$$\sqrt{1+(-k\operatorname{sen}(q\Delta x))^2}<\sqrt{1+(k\operatorname{sen}(q\Delta x))^2}$$

$$-k\operatorname{sen}(q\Delta x)<k\operatorname{sen}(q\Delta x)$$

$$2k\operatorname{sen}(q\Delta x)>0\longrightarrow\boxed{k>0}$$

Logo,

Critério de Estabilidade:

$$\left|\frac{A_{n+1}}{A_n}\right|^2 = 1 + k^2 - 1 \rightarrow \left|\frac{A_{n+1}}{A_n}\right| = k \qquad \left|\frac{A_{n+1}}{A_n}\right| \leq 1$$

$$k \leq 1 \rightarrow \boxed{1 \geq \frac{v\Delta t}{\Delta x}}$$

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(2) Deriva Crank-Nicholson

Encontrar critério de estabilidade

$$\frac{df}{dt} = -v\frac{df}{dx}$$

Método de Crank-Nicholson faz a média de aplicações de Euler explícito e implícito

$$\frac{df}{dt} = L_r f(r,t) + s(r) \quad ; \quad \text{No caso:} \quad L_r = -v\frac{\partial}{\partial x} \quad ; \quad s(x) = 0$$

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{1}{2}\left[L_r f_j^{n+1} + L_r f_j^n\right] + s(r)$$

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -\frac{v}{2}\left[\frac{f_{j+1}^{n+1} - f_{j-1}^{n+1}}{2\Delta x} + \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}\right] + 0 \qquad f_j^n = \frac{f_{j+1}^n + f_{j-1}^n}{2}$$

$$\frac{(f_{j+1}^{n+1} + f_{j-1}^{n+1})}{2} - \frac{(f_{j+1}^n + f_{j-1}^n)}{2} = -\frac{v\Delta t}{4\Delta x}\left[f_{j+1}^{n+1} - f_{j-1}^{n+1} + f_{j+1}^n - f_{j-1}^n\right]$$

$$f_j^n = A^n e^{iqj\Delta x}$$

$$\left(1 + \frac{v\Delta t}{2\Delta x}\right)f_{j+1}^{n+1} + \left(1 - \frac{v\Delta t}{2\Delta x}\right)f_{j-1}^{n+1} = \left(1 + \frac{v\Delta t}{2\Delta x}\right)f_{j-1}^n + \left(1 - \frac{v\Delta t}{2\Delta x}\right)f_{j+1}^n$$

$$\left(1 + \frac{v\Delta t}{2\Delta x}\right)A^{n+1}e^{iq(j+1)\Delta x} + \left(1 - \frac{v\Delta t}{2\Delta x}\right)A^{n+1}e^{iq(j-1)\Delta x} = \left(1 + \frac{v\Delta t}{2\Delta x}\right)A^n e^{iq(j-1)\Delta x} + \left(1 - \frac{v\Delta t}{2\Delta x}\right)A^n e^{iq(j+1)\Delta x}$$

$$\left[\left(1 + \frac{v\Delta t}{2\Delta x}\right)e^{iq\Delta x} + \left(1 - \frac{v\Delta t}{2\Delta x}\right)e^{-iq\Delta x}\right]A^{n+1} = \left[\left(1 + \frac{v\Delta t}{2\Delta x}\right)e^{-iq\Delta x} + \left(1 - \frac{v\Delta t}{2\Delta x}\right)e^{iq\Delta x}\right]A^n$$

③ Difusão por Crank-Nicholson

Critério de estabilidade

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \qquad L_r = D\frac{d^2}{dx}$$

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{1}{2}\left[L_r f_j^{n+1} + L_r f_j^n\right]$$

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{2\Delta x^2}\left[\left(f_{j+1}^{n+1} + f_{j-1}^{n+1} - 2f_j^{n+1}\right) + \left(f_{j+1}^n + f_{j-1}^n - 2f_j^n\right)\right] \qquad K = \frac{D\Delta t}{\Delta x^2}$$

$$f_j^{n+1}\left(1 + \frac{D\Delta t}{\Delta x^2}\right) - \frac{D\Delta t}{2\Delta x^2}\left(f_{j+1}^{n+1} + f_{j-1}^{n+1}\right) = f_j^n\left(1 - \frac{D\Delta t}{\Delta x^2}\right) + \frac{D\Delta t}{2\Delta x^2}\left(f_{j+1}^n + f_{j-1}^n\right)$$

$$A^{n+1}\left[\left(1+\frac{D\Delta t}{\Delta x^2}\right)e^{iq\Delta x j} - \frac{D\Delta t}{2\Delta x^2}\left(e^{iq\Delta x(j+1)} + e^{iq\Delta x(j-1)}\right)\right] = A^n\left[\left(1-\frac{D\Delta t}{\Delta x^2}\right)e^{iq\Delta x j} + \frac{D\Delta t}{2\Delta x^2}\left(e^{iq\Delta x(j+1)} + e^{iq\Delta x(j-1)}\right)\right]$$

$$\left(\frac{A^{n+1}}{A^n}\right)\left[(1+K) - \frac{K}{2}\left(e^{iq\Delta x} + e^{-iq\Delta x}\right)\right] = (1-K) + \frac{K}{2}\left(e^{iq\Delta x} + e^{-iq\Delta x}\right)$$

$$\left|\frac{A^{n+1}}{A^n}\right| = \frac{|1-K+K\cos(q\Delta x)|}{|1+K-K\cos(q\Delta x)|} \rightarrow \frac{\sqrt{(1-K+K\cos(q\Delta x))^2}}{\sqrt{(1+K-K\cos(q\Delta x))^2}} \leq 1$$

$$\sqrt{(1-K+K\cos(q\Delta x))^2} \leq \sqrt{(1+K-K\cos(q\Delta x))^2}$$

$$(1-K+K\cos(q\Delta x)) \leq (1+K-K\cos(q\Delta x))$$

$$0 \leq 2K\cos(q\Delta x) \leq 2K$$

$$\boxed{\cos(q\Delta x) \leq 1}$$

Válido para qualquer $(q\Delta x)$ real

Logo, é estável para qualquer K:

$$\boxed{\left|\frac{A^{n+1}}{A^n}\right| = \left|\frac{1-K(1-\cos(q\Delta x))}{1+K(1-\cos(q\Delta x))}\right| \leq 1 \ \forall K}$$

Exemplo 3.1

$f(x=0,t) = 1$

$f(L,t) = 0$

$f(x>0, t=0) = 0$

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}$$

(FTCS)

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{\Delta x^2}\left[f_{j+1}^n + f_{j-1}^n - 2 f_j^n\right]$$

$$f_j^{n+1} = f_j^n + \frac{D\Delta t}{\Delta x^2}\left[f_{j+1}^n + f_{j-1}^n - 2 f_j^n\right] \rightarrow \text{Integrar.}$$

ALGORITMO:

1) Define constantes: $D, \Delta t, \Delta x, L, t_{max}, \kappa$, etc ...
2) Define array para espaço: e.g. x = np.arrange(0, L)
3) Define condição inicial: $f(x,0) = 0$
4) Define condições de contorno: $f(0,t) = 1$; $f(L,t) = 0$
5) Define array auxiliar: fa = copy.deepcopy(f)
6) Evolui com loço temporal:

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while t<tmax:

    f[1:L-1] = f[1:L-1] + K(f[2:L] + f[0:L-2] - 2f[1:L-1])
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7) Plota.

Exemplo 3.2