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STAT184-F22

Assignment #0

Due: 11:59pm EST, Sept. 7, 2022

1. Policies

   1.1 List of Collaborators: I did not collaborate with anyone on this assignment

   1.2 List of Acknowledgements: None

   1.3 Certification of Reading of Instructions: I have read these policies

2. Website

   2.1 I have read the course policies on the website.

3. Bayes

   3.1 Let D be the event that I test positive and let P be the event that I test negative. That means that I have $P(D) = 0.0001$ and $P(P|D) = 0.99$.

   By applying Bayes Rule, we get the following:

$$P(D|P) = \frac{P(P|D)P(D)}{P(P)} = \frac{P(P|D)P(D)}{P(P|D)P(D) + P(P|D^C)P(D^C)}$$
$$= \frac{0.99 \cdot 0.0001}{0.99 \cdot 0.0001 + 0.001 \cdot 0.9999} \approx 0.0098 = \boxed{0.98\%}$$

4. Probability

   4.1 In the discrete case, we have

$$P(Z = z) = \sum_{z \in \mathbb{R}} P(X = x, Y = z - x)$$

   this expresses the probability that, given a value for x, y is exactly its difference to z. Since X and Y are independent, we have

$$P(Z = z) = \sum_{z \in \mathbb{R}} P(X = x)P(Y = z - x)$$

   In the continuous case, this translates to

$$h(z) = \int_{-\infty}^{\infty} f(x)g(x - z)dx$$

   since we are "summing" over all possibilities for x times their infinitesimal probabilities of occurring.

   4.2 (a) We have

$$h(z) = \int_{-\infty}^{\infty} f(x)g(x - z)dx$$

   $f(x) = 0$ for $x \notin [0, 1]$ and 1 otherwise:

$$h(z) = \int_0^1 g(x-z)dx$$

$g(x-z) = 1$ if $0 \leq x - z \leq 1$ and 0 otherwise. We have two cases for which $0 \leq x - z \leq 1$:

Case 1: $0 \leq z \leq 1$: In this case, $h(z) = z$ as the integral will be non-zero only for $0 \leq x \leq z$.

Case 2: $1 \leq z \leq 2$: In this case, $h(z) = 2 - z$. As the integral will be non-zero only for $z - 1 \leq x \leq 1$.

Hence, we have

$$\boxed{h(z) = \begin{cases} z & 0 \leq z \leq 1 \\ 2 - z & 1 \leq z \leq 2 \\ 0 & \text{otherwise} \end{cases}}$$

(b) We have

$$P(X \leq 1/2 | X + Y \geq 5/4) = P(X \leq 1/2 | Z \geq 5/4) = \frac{P(X \leq 1/2, Z \geq 5/4)}{P(Z \geq 5/4)}$$

We also have that the joint PDF of X, Z is

$$f_{X,Z}(x,z) = 1_{\{0 \leq x \leq 1, x \leq z \leq x+1\}}$$

Therefore, we have, in the numerator,

$$\begin{aligned}
&P(X \leq 1/2 | Z \geq 5/4) \\
&= \int_0^{1/2} \int_{5/4}^{2} f_{X,Z}(x,z) dz dx \\
&= \int_0^{1/2} \int_{5/4}^{2} 1_{\{0 \leq x \leq 1, x \leq z \leq x+1\}} dz dx \\
&= \int_0^{1/2} \int_{5/4}^{2} 1_{\{x \leq z \leq x+1\}} dz dx \\
&= \int_{1/4}^{1/2} \int_{5/4}^{3/2} 1_{\{x \leq z \leq x+1\}} dz dx \\
&= \int_{1/4}^{1/2} \int_{5/4}^{x+1} 1 \, dz dx \\
&= \int_{1/4}^{1/2} x + 1 - 5/4 \, dx \\
&= \int_{1/4}^{1/2} x + 1 - 5/4 \, dx \\
&= \int_{1/4}^{1/2} x - 1/4 \, dx
\end{aligned}$$

$$= P(X \leq 1/2 | Z \geq 5/4) = 1/32$$

In the denominator we have

$$P(Z \geq 5/4)$$
$$= \int_{5/4}^{2} h(z)dz$$
$$= \int_{5/4}^{2} 2 - zdz$$
$$= 9/32$$

Putting this together we have

$$P(X \leq 1/2 | X + Y \geq 5/4) = P(X \leq 1/2 | Z \geq 5/4) = \frac{P(X \leq 1/2, Z \geq 5/4)}{P(Z \geq 5/4)}$$
$$= \frac{1/32}{9/32}$$
$$\boxed{P(X \leq 1/2 | X + Y \geq 5/4) = 1/9}$$

4.3 Let G be the CDF of Y. Then we have

$$G(y) = P(Y \leq y) = P(aX + b \leq y) = P(X \leq \frac{y-b}{a}) = F(\frac{y-b}{a})$$

Through differentiation we get the PDF of Y:

$$g(y) = \frac{1}{a} f(\frac{y-b}{a})$$
$$= \frac{1}{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{\frac{y-b}{a} - \mu}{\sigma}\right)^2}$$
$$= \frac{1}{(a\sigma)\sqrt{2\pi}} e^{-1/2\left(\frac{y-(a\mu+b)}{(a\sigma)}\right)^2}$$

Which is the PDF of a random variable with distribution $\mathcal{N}(a\mu + b, a^2\sigma^2)$.

In order for $Y \sim \mathcal{N}(0,1)$, therefore, we need $a\mu + b = 0$ and $a^2\sigma^2 = 1$. This is accomplished with

$$\boxed{a = 1/\sigma} \text{ and } \boxed{b = -\mu/b}$$

4.4 (a) We have

$$E[XY] = E[E[XY|X = x]]$$
$$= E[E[xY|X = x]]$$
$$= \int_{-\infty}^{\infty} E[xY|X = x] f_X(x)$$
$$= \int_{-\infty}^{\infty} xE[Y|X = x] f_X(x)$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)$$
$$= E[X^2]$$

We also have

$$E[Y] = E[E[Y|x]]$$
$$= \int_{-\infty}^{\infty} E[Y|X = x] f_X(x) dx$$
$$= \int_{-\infty}^{\infty} x f_X(x) dx$$
$$= E[x]$$

Putting this together we have

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$
$$= E[X^2] - E[X]^2 = E[(X - E[X])^2]$$

(b) We have

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

If X, Y are independent then we have

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[X]E[Y] - E[X]E[Y] = 0$$

4.5 (a) We have

$$E[\hat{F}_n(x)] = E\left[\frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n} E\left[1_{\{X_i \leq x\}}\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n} F(x)$$
$$= F(x)$$

(b) We have

$$\text{Var}(\hat{F}_1(x)) = E[\hat{F}_1(x)^2] - E[\hat{F}_1(x)]^2$$
$$= E\left[\left(\frac{1}{n}\sum_{i=1}^{1} 1_{\{X_i \leq x\}}\right)^2\right] - F(x)^2$$
$$= E\left[1_{\{X_i \leq x\}}\right] - F(x)^2$$
$$= F(x) - F(x)^2$$
$$= F(x)(1 - F(x))$$

(c) We have

$$\begin{aligned}
\mathrm{Var}(\hat{F}_n(x)) &= E[\hat{F}_n(x)^2] - E[\hat{F}_n(x)]^2 \\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \le x\}}\right)^2\right] - F(x)^2 \\
&= \frac{1}{n^2} E\left[\sum_{i=1}^{n}\sum_{k=1}^{n} 1_{\{X_i \le x\}} 1_{\{X_k \le x\}}\right] - F(x)^2 \\
&= \frac{1}{n^2} E\left[\sum_{i=1}^{n}\left(1_{\{X_i \le x\}} 1_{\{X_i \le x\}} + \sum_{k \ne i}^{n} 1_{\{X_i \le x\}} 1_{\{X_k \le x\}}\right)\right] - F(x)^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\left(F(x) + \sum_{i \ne k}^{n} F(x)^2\right) - F(x)^2 \\
&= \frac{1}{n}F(x) + \frac{1}{n^2}n(n-1)F(x)^2 - F(x)^2 \\
&= \frac{1}{n}F(x)(1 - F(x))
\end{aligned}$$

(d) We have

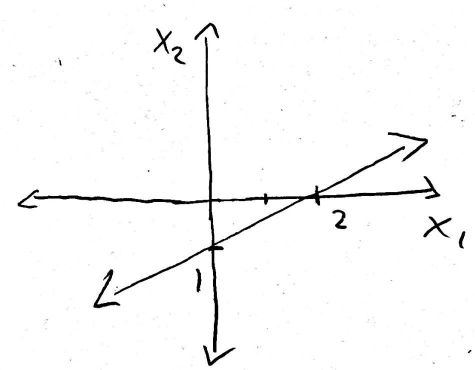
$$\mathrm{Var}(\hat{F}_n(x)) = \frac{1}{n}F(x)(1 - F(x))$$

Since $0 \le F(x) \le 1$ we have $F(x)(1 - F(x)) \le 1/4$ and therefore

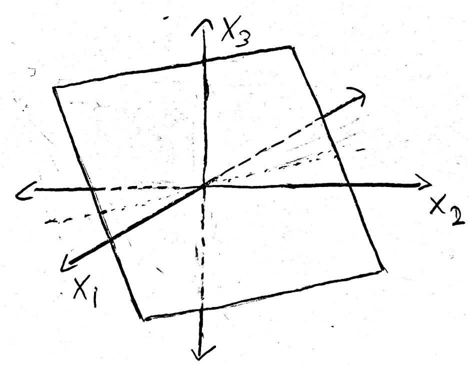
$$\begin{aligned}
\mathrm{Var}(\hat{F}_n(x)) &= \frac{1}{n}F(x)(1 - F(x)) \\
&\le \frac{1}{n} \cdot \frac{1}{4} = \frac{1}{4n}
\end{aligned}$$

5. Geometry and Linear Algebra

5.1 (a) The drawing is the following



(b) The drawing is the following



(c) The problem can be solved using Lagrange Multipliers. The Lagrangian function is

$$L(x, \lambda) = ||x_0 - x||^2 - \lambda(w^\top x + b)$$

and the corresponding partial derivatives are

$$(1) \quad \frac{\partial}{\partial x} L(x, \lambda) = 2(x_0 - x) - \lambda w$$

$$(2) \quad \frac{\partial}{\partial \lambda} L(x, \lambda) = w^\top x + b$$

solving for $\Delta L(x, \lambda) = 0$ yields

$$(3) \quad 2(x_0 - x) - \lambda w \implies x = x_0 - \frac{1}{2}\lambda w$$

$$(4) \quad w^\top x + b = 0$$

$$\implies \lambda = \frac{2w^\top x_0 + b}{w^\top w}$$

finally, our shortest distance will be

$$\begin{aligned} ||x_0 - x|| &= ||x_0 - (x_0 - \frac{1}{2}\lambda w)|| \\ &= ||x_0 - x_0 + \frac{1}{2}\lambda w|| \\ &= \frac{1}{2}\lambda ||w|| \\ &= \frac{1}{2}\frac{2w^\top x_0 + b}{w^\top w}||w|| \\ &= \frac{w^\top x_0 + b}{||w||} \end{aligned}$$

and this means that the minimum squared distance will be

$$\boxed{\left(\frac{w^\top x_0 + b}{||w||}\right)^2}$$

5.2 (a) The rank of the matrix is 2, given that the first column is a linear combination of the other two and that that the other two are linearly independent of each other.

(b) A minimal size basis of the column span is $\{[2,0,1]^\top, [1,3,2]^\top\}$ (the latter two columns of the matrix; the only two columns that are linearly independent of each other).

5.3 (a) We have

$$Ac = [6,8,7]^\top$$

(b) Through the calculation of the Reduced Row Echelon Form of the combined matrix $[A : b]$ we get $x = [-2, 1, -1]^\top$

5.4 We have

$$\begin{aligned} x^\top Ax &= x^\top \left[\sum_j A_{1j}x_j, ..., \sum_j A_{nj}x_j \right]^\top \\ &= \sum_i \sum_j A_{ij}x_i x_j \end{aligned}$$

and similarly

$$y^\top Bx = \sum_i \sum_j B_{ij} y_i x_j$$

hence we have

$$\boxed{f(x,y) = \sum_i \sum_j A_{ij}x_i x_j + \sum_i \sum_j B_{ij} y_i x_j + c}$$

6. Programming

6.1 The computed values are the following

```
A^-1 is
 [[ 0.125 -0.625  0.75 ]
 [-0.25   0.75  -0.5  ]
 [ 0.375 -0.375  0.25 ]]

A^-1 b is
 [[-2.]
 [ 1.]
 [-1.]]

A c is
 [[6]
 [8]
 [7]]
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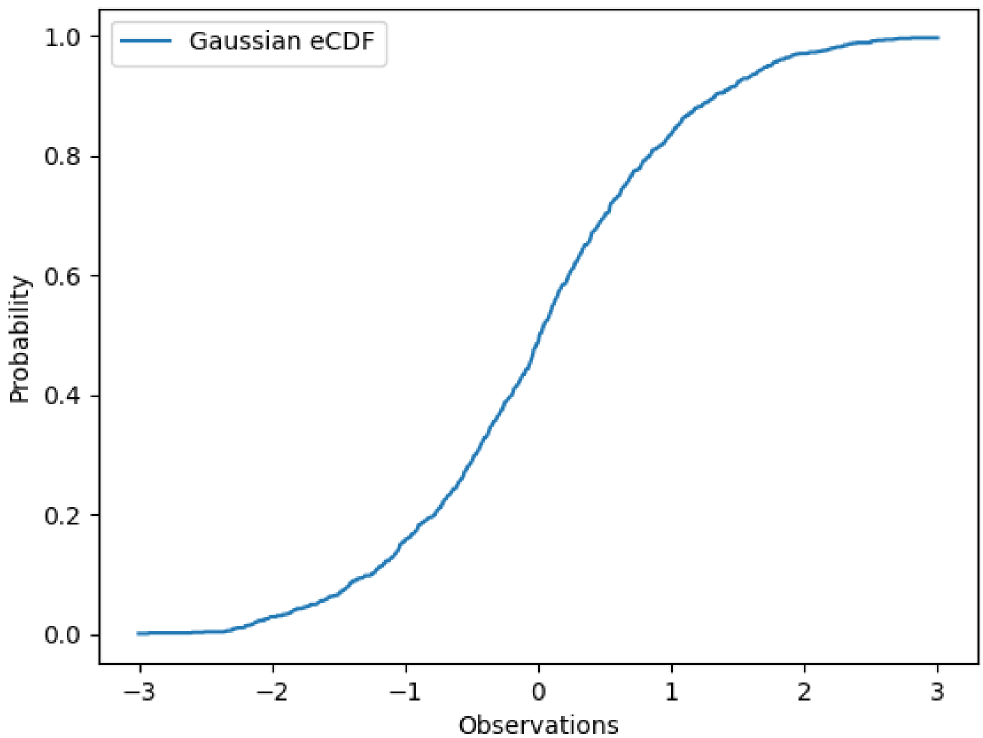
6.2 I chose $n = 1000$. The resulting graph is



6.3 The resulting graph is