Cutting γ-Liouville quantum gravity by Schramm-Loewner evolution for $\kappa \notin \{\gamma^2, 16/\gamma^2\}$

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Abstract

There are many deep and useful theorems relating Schramm-Loewner evolution (SLE_κ) and Liouville quantum gravity (γ-LQG) in the case when the parameters satisfy $\kappa \in \{\gamma^2, 16/\gamma^2\}$. Roughly speaking, these theorems say that the SLE_κ curve cuts the γ-LQG surface into two or more independent γ-LQG surfaces.

We extend these theorems to the case when $\kappa \notin \{\gamma^2, 16/\gamma^2\}$. Roughly speaking we show that if we have an appropriate variant of SLE_κ and an independent γ-LQG disk, then the SLE curve cuts the LQG disk into two or more γ-LQG surfaces which are *conditionally* independent given the values along the SLE curve of a certain collection of auxiliary imaginary geometry fields, viewed modulo conformal coordinate change. These fields are sampled independently from the SLE and the LQG and have the property that that the sum of the central charges associated with the SLE_κ curve, the γ-LQG surface, and the auxiliary fields is 26. This condition on the central charge is natural from the perspective of bosonic string theory. We also prove analogous statements when the SLE curve is replaced by, e.g., an LQG metric ball or a Brownian motion path. Statements of this type were conjectured by Sheffield and are continuum analogs of certain Markov properties of random planar maps decorated by two or more statistical physics models.

We include a substantial list of open problems.

Contents

1 Introduction

1.1 Overview

Schramm-Loewner evolution (SLE_κ) for $\kappa > 0$ is a one-parameter family of random fractal curves in the plane originally introduced by Schramm [Sch00]. SLE describes or is conjectured to describe the scaling limits of various discrete random curves which arise in statistical mechanics. See, e.g., [Law05, BN] for introductory expository works on SLE.

  Liouville quantum gravity (γ-LQG) for $\gamma \in (0,2]$ is a one-parameter family of random fractal surfaces (2d Riemannian manifolds) which arise, e.g., in string theory and conformal field theory [Pol81], and as the scaling limits of random planar maps. LQG surfaces are too rough to be Riemannian manifolds in the literal sense. Instead, a γ-LQG surface can be represented as a random metric measure space parametrized by a domain in $\mathbb{C}$ (or more generally a Riemann surface), viewed modulo a conformal change of coordinates rule. See Definition 1.1 for a precise definition and, e.g., [BP, Gwy20, She22] for introductory expository articles on LQG.

  Instead of the parameters κ and γ, it is often useful to instead describe SLE and LQG in terms of the **central charge** parameters, which are related to κ and γ by[1]

$$\mathbf{c}_{\mathrm{SLE}} = 1 - 6\left(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}\right)^2 \quad \text{and} \quad \mathbf{c}_{\mathrm{L}} = 1 + 6\left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)^2. \tag{1.1}$$

Each of SLE and LQG can be associated with conformal field theories with their respective central charges, see, e.g., [KM13, DKRV16].

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[1]Some works associate LQG with the **matter central charge** $\mathbf{c}_{\mathrm{M}}$ instead of our $\mathbf{c}_{\mathrm{L}}$. Our $\mathbf{c}_{\mathrm{L}}$ is the central charge of Liouville conformal field theory and is related to $\mathbf{c}_{\mathrm{M}}$ by $\mathbf{c}_{\mathrm{M}} = 26 - \mathbf{c}_{\mathrm{L}}$.
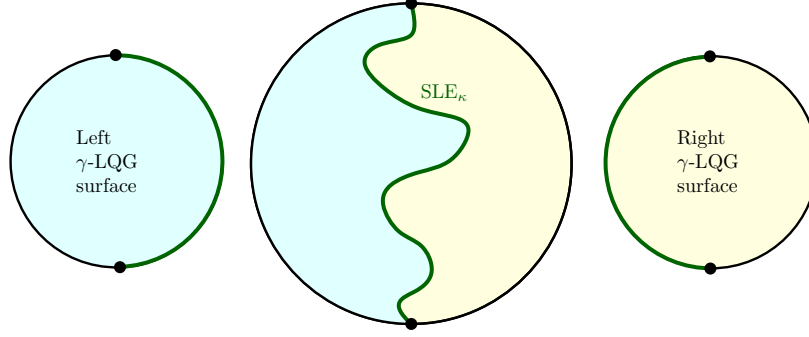
Figure 1: A Liouville quantum gravity surface parametrized by the disk together with an SLE_κ-type curve between two marked boundary points. Previous works (see in particular [She16, DMS21, AHS23]) have shown that if $\kappa = \gamma^2$ and the precise variant of SLE is chosen appropriately, then the sub-LQG surfaces parametrized by the regions to the left and right of the curve are conditionally independent given their LQG boundary lengths (see Definition 1.1 for a precise definition of LQG surfaces). In Theorem 1.9, we show that if $\kappa \neq \gamma^2$, then the sub-LQG surfaces parametrized by the left and right sides of the curve are conditionally independent given information about the values along the curve of certain auxiliary random fields, sampled independently from the SLE and the LQG. We also prove variants of this statement for other types of SLE_κ curves, including ones with multiple complementary connected components.

In this paper we will study the relationships between SLE and LQG. The first such relationship, called the **quantum zipper**, was established by Sheffield in [She16]. Roughly speaking, this result and its many extensions (including the **mating of trees** theorem [DMS21]) say the following. Suppose we have a certain SLE_κ-type curve and a certain γ-LQG surface, sampled independently from each other, and that the parameters are **matched** in the sense that

$$\kappa \in \left\{ \gamma^2, \frac{16}{\gamma^2} \right\}, \quad \text{equivalently} \quad \mathbf{c}_{\mathrm{SLE}} = 26 - \mathbf{c}_{\mathrm{L}}. \tag{1.2}$$

Then the sub-LQG surfaces parametrized by the complementary connected components of the SLE_κ curve are conditionally independent given the LQG lengths of their boundaries, and their laws can be described explicitly. See Figure 1. This independence property is the continuum analog of certain Markovian properties for random planar maps decorated by statistical physics models. See the survey article [GHS23] for more explanation.

Results of the above type have a huge number of applications, to such topics as SLE and LQG individually, conformal field theory (see, e.g., [ARS21]), the geometry of random planar maps (see, e.g., [GHS20]), random permutations (see, e.g., [Bor21]), and the moduli of random surfaces (see, e.g., [ARS22]). Particularly noteworthy consequences include the equivalence of $\gamma = \sqrt{8/3}$ LQG and the Brownian map [MS20] and convergences of conformally embedded random planar maps to LQG [GMS21, HS23]. Many of these applications are surveyed in [GHS23].

In this paper, we will establish the first relationships between SLE and LQG in the case when the parameters are **mismatched**, meaning that γ and κ are not related as in (1.2). Roughly speaking, we prove the following. Suppose we have an appropriate SLE_κ-type curve and a γ-LQG surface (specifically, an LQG disk), sampled independently from each other as above, but whose parameters do not satisfy (1.2). Then the sub-LQG surfaces parametrized by the complementary connected components of the SLE_κ curve are not independent, but they are *conditionally* independent if we

condition on certain extra information along the SLE_κ curve. The necessary extra information is described by one or more random generalized functions, sampled independently from the SLE and the LQG, with the property that $\mathbf{c}_{\mathrm{SLE}} + \mathbf{c}_{\mathrm{L}}$ plus the central charges associated with the extra generalized functions is equal to 26. These extra random generalized functions are described in terms of the theory of **imaginary geometry** [MS16a, MS16b, MS16c, MS17], which we review in Section 2.5. See Theorem 1.9 for a precise statement. Conditional independence statements of the above type were conjectured by Sheffield in private communication.

We also prove similar conditional independence statements when the SLE curve is replaced by other interesting random sets, such as a conformal loop ensemble gasket, a Brownian motion path, or an LQG metric ball. See Theorems 1.11, 1.13 and 1.14.

The condition that the total central charge should be 26 dates back to Polyakov's seminal path integral formulation of bosonic string theory. For the conformal field theory corresponding to LQG coupled with conformal matter, the central charge condition is equivalent to the Weyl invariance of the theory [Pol81, DP86], i.e., invariance when the underlying Riemannian manifold (Σ, g) is replaced by $(\Sigma, e^{2\sigma} g)$. This perspective, together with the ansatz that the conformal field theory should correspond to random planar maps decorated by statistical physics models (see the next paragraph), gives physical context and justification for the results of this paper. The relationships between SLE and LQG in the matched case (1.2) can be viewed as Markov properties for Liouville CFT with central charge $\mathbf{c}_{\mathrm{L}}$ coupled to a single matter field of central charge $26 - \mathbf{c}_{\mathrm{L}}$. This paper shows that one also has Markov properties when instead Liouville CFT is coupled to multiple matter fields with total central charge $26 - \mathbf{c}_{\mathrm{L}}$.

Analogously to the matched case, our results are continuum analogs of certain Markovian properties for random planar maps decorated by *multiple* statistical physics models. Roughly speaking, these properties say the following. Suppose we have a random planar map decorated by a two or more statistical physics models (e.g., a uniform spanning tree and two discrete Gaussian free fields) and we construct an interface from one of the models (e.g., a branch of the spanning tree). Then the planar maps in the complementary connected components of the interface are conditionally independent given the information about other statistical mechanics models along the interface (in our example, this corresponds to the restrictions of the discrete Gaussian free fields to the spanning tree branch). Similar Markovian properties also hold for related objects, such as uniform meanders. See Appendices A and B for further explanations.

Furthermore, just like in the matched case, our results have a large number of potential applications and extensions, some of which are discussed in Section 5.

The proofs in this paper involve several novel ideas which we expect to be useful elsewhere. These include a conditional independence statement for uniform meanders (Appendix B) and a "rotational invariance" property for the central charges associated with a collection of independent Gaussian free fields in the setting of imaginary geometry (Proposition 1.17). This rotational invariance property is the source of the "total central charge 26" condition in our results. See Remark 1.10 and Section 1.4 for further discussion of the main ideas of the proofs.

1.2 LQG and imaginary geometry surfaces

1.2.1 Liouville quantum gravity

For $\gamma \in (0, 2)$, γ**-Liouville quantum gravity (LQG)** is, heuristically speaking, the random geometry described by the random Riemannian metric tensor

$$e^{\gamma \Phi} \left(dx^2 + dy^2 \right)$$

where $dx^2 + dy^2$ denotes the Euclidean metric tensor on a domain $D \subset \mathbb{C}$ and Φ is a variant of the Gaussian free field (GFF) on D. This Riemannian metric tensor does not make literal sense since Φ is a generalized function. Following [DS11, She16, DMS21], we rigorously define LQG surfaces as equivalence classes of field/domain pairs modulo a conformal coordinate change formula depending on the parameter $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$. Equivalently,

$$Q(\mathbf{c}_{\mathrm{L}}) = \sqrt{\frac{\mathbf{c}_{\mathrm{L}} - 1}{6}} \quad \text{for } \mathbf{c}_{\mathrm{L}} > 25, \qquad\qquad \mathbf{c}_{\mathrm{L}}(Q) = 1 + 6Q^2 \quad \text{for } Q > 2.$$

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere.

Definition 1.1. Consider pairs (D, Φ) where $D \subset \mathbb{C}$ is a bounded open set and Φ is a distribution defined on D. For $\mathbf{c}_{\mathrm{L}} > 25$, a **(generalized) $\mathbf{c}_{\mathrm{L}}$-LQG surface** is an equivalence class of such pairs under the equivalence relation $\sim_{\mathbf{c}_{\mathrm{L}}}$ where $(D, \Phi) \sim_{\mathbf{c}_{\mathrm{L}}} (\widetilde{D}, \widetilde{\Phi})$ if there exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with $f(D) = \widetilde{D}$ that is conformal on D such that $\Phi = \widetilde{\Phi} \circ f + Q(\mathbf{c}_{\mathrm{L}}) \log |f'|$.

We emphasize that the domain D in Definition 1.1 need not be simply connected or even connected. Definition 1.1 differs slightly from the definitions of LQG surfaces found elsewhere in the literature (e.g., in [DMS21]), in that we require that D is bounded and we require that $f : \mathbb{C} \to \mathbb{C}$ instead of just $f : D \to \widetilde{D}$ or $f : \overline{D} \to \widetilde{\overline{D}}$. The reason for these modifications is that we will eventually consider LQG surfaces coupled to imaginary geometry fields, so we need to ensure that there is a well-defined notion of the argument of f'. See Section 2.1 for further discussion.

We call an equivalence class representative (D, Φ) an **embedding** of the LQG surface. When Φ is a variant of the Neumann (free-boundary) GFF on D, one can define the LQG area measure μ_Φ on D, which is a limit of regularized versions of $e^{\gamma \Phi(x+iy)} \, dx \, dy$ [DS11, Kah85, RV11, Ber17]. Similarly, one can define the LQG boundary length measure ν_Φ on ∂D and the LQG metric (distance function) $\mathfrak{d}_\Phi$ on $\overline{D}$ [DDDF20, GM21c] (see [HM22] for the extension of the metric from D to $\overline{D}$). See Section 2.4 for more background on the LQG metric. These are compatible with the LQG coordinate change, i.e., in Definition 1.1 we have $\mu_{\widetilde{\Phi}} = f_* \mu_\Phi$, $\nu_{\widetilde{\Phi}} = f_* \nu_\Phi$ and $\mathfrak{d}_{\widetilde{\Phi}} = f_* \mathfrak{d}_\Phi$, and are thus intrinsic to the LQG surface.

Remark 1.2. If (D, Φ) is an embedding of an LQG surface and $U \subset D$ is open, we will often slightly abuse notation by writing $(U, \Phi)/\sim_{\mathbf{c}_{\mathrm{L}}}$ instead of $(U, \Phi|_U)/\sim_{\mathbf{c}_{\mathrm{L}}}$ for the LQG surface obtained by restricting Φ to U. Moreover, if instead $D \subset \mathbb{C}$ is a bounded set, not necessarily open, then $(D, \Phi)/\sim_{\mathbf{c}_{\mathrm{L}}}$ refers to the LQG surface $(\mathrm{int}(D), \Phi)/\sim_{\mathbf{c}_{\mathrm{L}}}$. These conventions also apply for imaginary geometry surfaces.

1.2.2 Imaginary geometry

Imaginary geometry (IG) is, heuristically speaking, the random geometry described by the vector field

$$e^{i\Psi/\chi} \tag{1.3}$$

where $\chi > 0$ and Ψ is a variant of the Gaussian free field on a domain $D \subset \mathbb{C}$ [MS16a, MS16b, MS16c, MS17]. We associated IG with the central charge $\mathbf{c}_{\mathrm{M}} < 1$ which is related to χ by

$$\chi(\mathbf{c}_{\mathrm{M}}) = \sqrt{\frac{1 - \mathbf{c}_{\mathrm{M}}}{6}} \quad \text{for } \mathbf{c}_{\mathrm{M}} \leq 1, \qquad\qquad \mathbf{c}_{\mathrm{M}}(\chi) = 1 - 6\chi^2 \quad \text{for } \chi \geq 0. \tag{1.4}$$

IG surfaces with central charge $\mathbf{c}_{\mathrm{M}}$ play a central role in the study of SLE_κ when $\mathbf{c}_{\mathrm{SLE}}(\kappa) = \mathbf{c}_{\mathrm{M}}$. Roughly speaking, the flow lines of the vector field (1.3), i.e., the solutions to the formal differential

equation $\frac{d}{dt}\eta(t) = e^{i\Phi(\eta(t))/\chi}$, are $\mathrm{SLE}_\kappa(\rho)$ curves where $\kappa \in (0,4)$ satisfies $\mathbf{c}_{\mathrm{SLE}}(\kappa) = \mathbf{c}_{\mathrm{M}}$. If $\kappa' = 16/\kappa > 4$ is the other solution to $\mathbf{c}_{\mathrm{SLE}}(\kappa) = \mathbf{c}_{\mathrm{M}}$, then $\mathrm{SLE}_{\kappa'}(\rho')$ curves can instead be viewed as "counterflow lines" of this same vector field.

We will also consider IG surfaces in the case when $\chi = 0$, i.e., $\mathbf{c}_{\mathrm{M}} = 1$ and $\kappa = 4$. In this case, we view SLE_4 curves as level lines of the field Ψ (as in [SS13]), instead of as flow lines of the vector field (1.3). See Section 2.5 for background on $\mathrm{SLE}_\kappa(\rho)$, flow lines, counterflow lines, and level lines.

Similarly to the case of LQG, we define imaginary geometry surfaces rigorously in terms of a conformal coordinate change rule. Previous works using imaginary geometry have considered IG surfaces described by a single field. In this paper we will need to consider IG surfaces described by multiple fields, which are associated with a vector of central charge values.

Definition 1.3. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ and write $\underline{\mathbf{c}} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$. Consider tuples $(D, \Psi_1, \ldots, \Psi_n)$ where $D \subset \mathbb{C}$ is a bounded open set and $\Psi_1, \ldots, \Psi_n$ are distributions defined on D. A $\underline{\mathbf{c}}$-**IG surface** is an equivalence class of such tuples under the equivalence relation $\sim_{\underline{\mathbf{c}}}$ where $(D, \Psi_1, \ldots, \Psi_n) \sim_{\underline{\mathbf{c}}} (\widetilde{D}, \widetilde{\Psi}_1, \ldots, \widetilde{\Psi}_n)$ if there exists an integer m and a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with $f(D) = \widetilde{D}$ that is conformal on D such that $\Psi_i = \widetilde{\Psi}_i \circ f - \chi(\mathbf{c}_i) \arg f' + 2\pi\chi(\mathbf{c}_i)m$ for all i.

Here, when D is simply connected, $\arg f'$ is continuous version of the argument, so $\arg f'$ is uniquely specified up to an additive constant in $2\pi\mathbb{Z}$. For general D, $\arg f' : D \to \mathbb{R}$ can still be uniquely defined up to an element of $2\pi\mathbb{Z}$ because f is a homeomorphism of $\mathbb{C}$, see Section 2.7 for details. Definition 1.3 does not depend on the choice of $\arg f'$ since m can take any integer value.

Remark 1.4. The original definition of IG surface in [She16, MS16a] only considers $n = 1$ and makes the integer m implicit in the choice of the arg function. We make m explicit to clarify the $n > 1$ case. A single IG field can be understood as being defined modulo $2\pi\chi(\mathbf{c}_{\mathrm{M}})$, but this is no longer the case when there are multiple IG fields. For instance, the difference in boundary values for two IG fields with $\mathbf{c} = \mathbf{c}_{\mathrm{M}}$ is well defined as a real number and not just modulo $2\pi\chi(\mathbf{c}_{\mathrm{M}})$. See Section 2.1 for more discussion on differences between our Definitions 1.1 and 1.3 and other definitions in the literature.

1.2.3 LQG surfaces decorated by imaginary geometry fields

We now extend Definition 1.1 to the setting where a $\mathbf{c}_{\mathrm{L}}$-LQG surface is decorated by IG fields, that is, distributions that transform according to the IG coordinate change rule.

Definition 1.5. Let $\underline{\mathbf{c}} = (\mathbf{c}_{\mathrm{L}}, \mathbf{c}_1, \ldots, \mathbf{c}_n)$ where $\mathbf{c}_{\mathrm{L}} \geq 25$ and $\mathbf{c}_i \leq 1$ for each $i = 1, \ldots, n$. Consider tuples $(D, \Phi, \Psi_1, \ldots, \Psi_n)$ where $D \subset \mathbb{C}$ is a bounded open set and $\Phi, \Psi_1, \ldots, \Psi_n$ are distributions defined on D. We define an equivalence relation where $(D, \Phi, \Psi_1, \ldots, \Psi_n) \sim_{\underline{\mathbf{c}}} (\widetilde{D}, \widetilde{\Phi}, \widetilde{\Psi}_1, \ldots, \widetilde{\Psi}_n)$ if there exists an integer m and a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with $f(D) = \widetilde{D}$ that is conformal on D such that

$$\Phi = \widetilde{\Phi} \circ f + Q(\mathbf{c}_{\mathrm{L}}) \log|f'| \quad \text{and} \quad \Psi_i = \widetilde{\Psi}_i \circ f - \chi(\mathbf{c}_i) \arg f' + 2\pi\chi(\mathbf{c}_i)m, \quad \forall i = 1, \ldots, n. \quad (1.5)$$

We call an equivalence class under $\sim_{\underline{\mathbf{c}}}$ a **(generalized) $\mathbf{c}_{\mathrm{L}}$-LQG surface decorated by IG fields with central charges $\mathbf{c}_1, \ldots, \mathbf{c}_n$**.

An LQG surface decorated by n IG fields can be thought of as a continuum analog of a random planar map decorated by n statistical physics models. Indeed, for certain decorated random planar maps of this type, the LQG surface should describe the scaling limit of the underlying random planar map and the IG fields should describe the scaling limit of some notion of "height function"

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associated with the statistical physics models. See, e.g., [BLR20] for an example of a situation where an imaginary geometry field arises as the scaling limit of a height function.

We can extend Definition 1.5 to additionally keep track of sets and curves, by requiring that the homeomorphism identifies corresponding objects. Precisely, consider $n+4$-tuples $(D, \Phi, \Psi_1, \ldots, \Psi_n, (K_i)_{i \in I}, (\eta_j)_{j \in J})$ where $(D, \Phi, \Psi_1, \ldots, \Psi_n)$ is as in Definition 1.5, $(K_i)_{i \in I}$ is a collection of subsets of $\overline{D}$, and $(\eta_j)_{j \in J}$ is a collection of parametrized curves in $\overline{D}$. For two such $n + 4$-tuples, we say that

$$(D, \Phi, \Psi_1, \ldots, \Psi_n, (K_i)_{i \in I}, (\eta_j)_{j \in J}) \sim_{\underline{\mathbf{c}}} (\widetilde{D}, \widetilde{\Phi}, \widetilde{\Psi}_1, \ldots, \widetilde{\Psi}_n, (\widetilde{K}_i)_{i \in I}, (\widetilde{\eta}_j)_{j \in J})$$

if for some f and m as in Definition 1.5, we have that (1.5) holds and also

$$f(K_i) = \widetilde{K}_i, \quad \forall i \in I \quad \text{and} \quad f \circ \eta_j = \widetilde{\eta}_j, \quad \forall j \in J. \tag{1.6}$$

In the case where $K_i = \{z_i\}$ is a single point, we simply write z_i rather than $\{z_i\}$.

We will often omit the subscript in $\sim_{\mathbf{c}_{\mathrm{L}}}$ and $\sim_{\underline{\mathbf{c}}}$ and just write $\sim$ when the choice of $\mathbf{c}_{\mathrm{L}}$ or $\underline{\mathbf{c}}$ is clear from context.

The main LQG and IG surfaces we will work with are as follows.

Definition 1.6. For $\mathbf{c}_{\mathrm{L}} > 25$, let $\mathrm{UQD}_{\mathbf{c}_{\mathrm{L}}}$ be the law of the $\mathbf{c}_{\mathrm{L}}$-LQG disk $(\mathbb{D}, \Phi, -\mathbbm{i})/\sim_{\mathbf{c}_{\mathrm{L}}}$ conditioned to have unit boundary length and having one marked boundary point sampled from its boundary length measure. This LQG surface describes the scaling limit of planar maps with the disk topology. It was introduced in [DMS21, Section 4.5], and can alternatively be defined via Liouville CFT [HRV18, Cer21]; see Definition 2.2 for a precise description. Similarly, let $\mathrm{UQD}_{\mathbf{c}_{\mathrm{L}}}^{\bullet}$ be the law of the LQG surface $(\mathbb{D}, \Phi, -\mathbbm{i}, 0)/\sim_{\mathbf{c}_{\mathrm{L}}}$ where $(\mathbb{D}, \Phi, -\mathbbm{i})/\sim_{\mathbf{c}_{\mathrm{L}}}$ is sampled from the re-weighted probability measure $\frac{\mu_\Phi(\mathbb{D})}{\mathbb{E}[\mu_\Phi(\mathbb{D})]} \cdot \mathrm{UQD}_{\mathbf{c}_{\mathrm{L}}}$ (where μ_Φ is the LQG area measure) and 0 corresponds to a point sampled from μ_Φ, normalized to be a probability measure.

In this paper we will not need to work with the precise descriptions of $\mathrm{UQD}_{\mathbf{c}_{\mathrm{L}}}$ or $\mathrm{UQD}_{\mathbf{c}_{\mathrm{L}}}^{\bullet}$.

Let $\mathfrak{h} : \mathbb{D} \to \mathbb{R}$ be the harmonic function whose boundary value at $z \in \partial\mathbb{D}$ is $\arg(\mathbbm{i}z)$, where $\arg$ takes values in $[0, 2\pi)$. On the boundary, $\mathfrak{h}(z)$ equals the counterclockwise tangent direction at z, with a discontinuity at $-\mathbbm{i}$.

Definition 1.7. For $\mathbf{c}_{\mathrm{M}} \leq 1$, let $\mathrm{IG}_{\mathbf{c}_{\mathrm{M}}}$ denote the law of the $\mathbf{c}_{\mathrm{M}}$-IG surface $(\mathbb{D}, h + \chi(\mathbf{c}_{\mathrm{M}})\mathfrak{h}, -\mathbbm{i})$ on the one-pointed domain $(\mathbb{D}, -\mathbbm{i})$ where h is a zero boundary GFF on $\mathbb{D}$. Similarly, for $\underline{\mathbf{c}} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$, let $h_1, \ldots, h_n$ be n independent zero boundary GFFs on $\mathbb{D}$, and let $\mathrm{IG}_{\underline{\mathbf{c}}}$ be the law of the $\underline{\mathbf{c}}$-IG surface $(\mathbb{D}, h_1 + \chi(\mathbf{c}_1)\mathfrak{h}, \ldots, h_n + \chi(\mathbf{c}_n)\mathfrak{h}, -\mathbbm{i})/\sim_{\underline{\mathbf{c}}}$.

See Lemma 2.7 for a description of $\mathrm{IG}_{\mathbf{c}_{\mathrm{M}}}$ in other one-pointed domains. In particular, for the unbounded one-pointed domain $(\mathbb{H}, \infty)$ the IG surface corresponding to $\mathrm{IG}_{\underline{\mathbf{c}}}$ would be $(\mathbb{H}, h_1, \ldots, h_n, \infty)/\sim_{\underline{\mathbf{c}}}$ where the h_i are independent zero boundary GFFs, see Section 2.1 for details.

Remark 1.8. The zero boundary GFF, viewed modulo conformal maps, is the special case of $\mathrm{IG}_{\mathbf{c}_{\mathrm{M}}}$ where $\mathbf{c}_{\mathrm{M}} = 1$. Indeed, $\chi(1) = 0$ so the ($\mathbf{c}_{\mathrm{M}} = 1$)-IG coordinate change rule is $(D, \Psi) \sim_{\mathbf{c}_{\mathrm{M}}=1} (\widetilde{D}, \widetilde{\Psi})$ if $\Psi = \widetilde{\Psi} \circ f$ for some homeomorphism $f : \mathbb{C} \to \mathbb{C}$ which is conformal on D. Thus, for a Jordan domain D and $x \in \partial D$, if Ψ is a zero boundary GFF on D then the law of $(D, \Psi, x)/\sim_{\mathbf{c}_{\mathrm{M}}=1}$ is $\mathrm{IG}_{\mathbf{c}_{\mathrm{M}}=1}$.

1.3   Main results

The main results of this paper take the following form. Suppose we have an LQG disk decorated by a vector of imaginary geometry fields and a set K (which could be an SLE curve, an LQG metric

ball, etc.), as in Definition 1.5 and the discussion just after. If the sum of the central charges of all of these objects is 26, then the IG-decorated LQG surfaces obtained by restricting our fields to the complementary connected components of K are conditionally independent given, roughly speaking, the IG-decorated LQG surface obtained by restricting the fields to an infinitesimal neighborhood of $K \cup \partial \mathbb{D}$.

For concreteness, we only state our results for the LQG disk, but we expect that similar statements hold for LQG surfaces with other topologies; see Problem 5.3.

Our first main result gives a conditional independence statement of the above type when we cut by a chordal SLE curve. See Figures 1 and 2 (a). We first discuss the setup.

Suppose $\mathbf{c}_L > 25$ and $\mathbf{c}_{\mathrm{SLE}}, \mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ satisfy

$$\mathbf{c}_L + \mathbf{c}_{\mathrm{SLE}} + \mathbf{c}_1 + \cdots + \mathbf{c}_n = 26. \tag{1.7}$$

Let $(\mathbb{D}, \widetilde{\Phi}, -\mathtt{i})$ be an embedding of a central charge-$\mathbf{c}_L$ LQG disk with unit boundary length (i.e., a sample from $\mathrm{UQD}_{\mathbf{c}_L}$; see Definition 1.6), let $x \in \partial \mathbb{D}$ be the point such that the two boundary arcs from $-\mathtt{i}$ to x each have LQG length $\frac{1}{2}$, and apply a conformal map fixing $-\mathtt{i}$ and sending $x \mapsto \mathtt{i}$ to get an embedding $(\mathbb{D}, \Phi, -\mathtt{i})$ of the same LQG surface. For this embedding, the LQG lengths of the boundary arcs from $-\mathtt{i}$ to $\mathtt{i}$ are equal. Equivalently, $(\mathbb{D}, \Phi, -\mathtt{i}, \mathtt{i})$ is an embedding of a doubly marked LQG disk with left and right boundary lengths each equal to $1/2$.

Independently from Φ, let $(\mathbb{D}, \Psi_1, \ldots, \Psi_n, -\mathtt{i})$ be an embedding of an imaginary geometry surface sampled from $\mathrm{IG}_{(\mathbf{c}_1, \ldots, \mathbf{c}_n)}$ (Definition 1.3). Also let $\kappa > 0$ satisfy $\mathbf{c}_{\mathrm{SLE}}(\kappa) = \mathbf{c}_{\mathrm{SLE}}$ and let K be the trace of an independent $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ curve in $(\mathbb{D}, -\mathtt{i}, \mathtt{i})$ (with force points immediately to the left and right of its starting point) such that

$$\min(\rho_L, \rho_R) > -2 \quad \text{and} \quad \rho_L + \rho_R \in \{-2, \kappa - 6\}. \tag{1.8}$$

This constraint on ρ_L, ρ_R ensures that K can be realized as a flow line or counterflow line of an imaginary geometry surface as in Definition 1.7 embedded into $(\mathbb{D}, -\mathtt{i})$, see [MS16a, Theorem 1.1] or Section 2.5.

Recall that we want to condition on the IG-decorated LQG surface obtained by restricting $\Phi, \Psi_1, \ldots, \Psi_n$ to an infinitesimal neighborhood of $K \cup \partial \mathbb{D}$. To make sense of this, we will define the σ-algebra we want to condition on as the intersection of σ-algebras obtained by restricting our fields to neighborhoods of $K \cup \partial \mathbb{D}$. This is similar to how one defines the σ-algebra generated by the restriction of the GFF to a closed set when talking about local sets (see [SS13, Section 3.3]).

The LQG metric $\mathfrak{d}_\Phi$ provides a convenient way of defining neighborhoods of $K \cup \partial \mathbb{D}$ in a way which is intrinsic to the LQG surface. We write $\mathcal{B}_r(z; \mathfrak{d}_\Phi)$ to denote the $\mathfrak{d}_\Phi$-metric ball centered at z having radius r, and more generally, for a set $S \subset \mathbb{D}$, we write

$$\mathcal{B}_r(S; \mathfrak{d}_\Phi) = \{z \, : \, \mathfrak{d}_\Phi(z, S) < r\}. \tag{1.9}$$

We then define (using the notational convention of Remark 1.2) the σ-algebra

$$\mathcal{F} = \sigma((\mathcal{B}_{0+}(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n)/\sim) := \bigcap_{\varepsilon > 0} \sigma((\mathcal{B}_\varepsilon(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, K, -\mathtt{i})/\sim).$$
$$\tag{1.10}$$

See Section 2.2 for a precise definition of σ-algebras generated by decorated LQG surfaces. We use the LQG metric $\mathfrak{d}_\Phi$ to give a concise definition of $\mathcal{F}$ in (1.10), but there are also other equivalent definitions. For example, $\mathcal{F}$ can equivalently be defined using space-filling SLE segments rather than the LQG metric, see Lemma 3.14.

We emphasize that $\mathcal{F}$ contains much less information than $\sigma(K \cup \partial\mathbb{D}, \Phi|_K, \Psi_1|_K, \ldots, \Psi_n|_K)$ since we are viewing everything modulo conformal maps. For example, in the closely related case where $\kappa = \gamma^2 \in (2, 4)$, Φ is an embedding of the so-called weight 4 γ-LQG disk with left and right boundary lengths both equal to 1, K is an independent SLE$_\kappa$ curve, and we have no auxiliary imaginary geometry fields (i.e., $n = 0$), then it is possible to show[2] that $\mathcal{F} = \sigma((\mathcal{B}_{0+}(K \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi)/\sim)$ equals $\sigma(\nu_\Phi(K))$, where ν_Φ denotes γ-LQG length. We expect that similar statements also hold in other situations where $\kappa = \gamma^2$ and there are no auxiliary imaginary geometry fields. See also Problem 5.5.

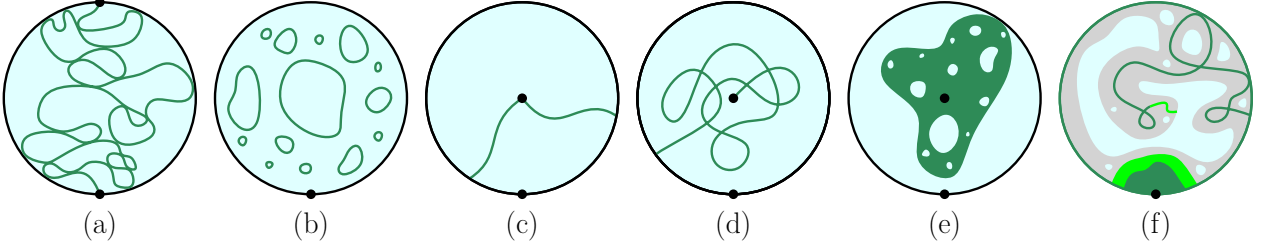(a)   (b)   (c)   (d)   (e)   (f)

Figure 2: For (a)–(e), K is the green set. Our main results state that conditioned on an infinitesimal LQG neighborhood of $K \cup \partial\mathbb{D}$ viewed as an LQG surface decorated by IG fields $\Psi_1, \ldots, \Psi_n$, the complementary LQG surfaces decorated by IG fields are conditionally independent. **(a)** K is an SLE$_\kappa$-type curve for $\kappa > 0$ (Theorem 1.9); in the figure we have $\kappa > 4$. **(b)** K is the gasket of a CLE$_\kappa$ for $\kappa \in [4, 8)$ (we have drawn simple loops for clarity, but in actuality CLE$_\kappa$ loops intersect themselves and each other for $\kappa \in (4, 8)$). **(c)** K is the union of a pair of SLE$_\kappa$-type flow lines of an IG field started at a bulk point where $\kappa < 4$ (Theorem 1.12). **(d)** K is the image of a Brownian motion run until it hits the boundary (Theorem 1.13). **(e)** K is an LQG metric ball grown until it hits $\partial\mathbb{D}$ (Theorem 1.14). **(f)** A conditional independence statement (Theorem 1.16) also holds for the LQG-IG peeling processes $(K_.)_{[0,T]}$ of Definition 1.15. For each $t, \delta > 0$, conditioned on $T \geq t$, let t' be the earlier of T and the time the process hits $\partial\mathcal{B}_\delta(K_t; \mathfrak{d}_\Phi)$. Here K_t is dark green, $K_{t'} \backslash K_t$ is light green, and $\mathcal{B}_\varepsilon(K_t; \mathfrak{d}_\Phi)$ is gray and green. For LQG-IG peeling processes, $(\mathcal{B}_\delta(K_t; \mathfrak{d}_\Phi), \Phi, (K_.)_{[0,t']})/\sim$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n)$ given $(\mathcal{B}_\delta(K_t; \mathfrak{d}_\Phi), \Phi, (K_.)_{[0,t]})/\sim$.

Theorem 1.9. *In the setting described just above, the IG-decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ parametrized by the connected components[3] U of $\mathbb{D}\backslash K$ (Definition 1.5) are conditionally independent given the σ-algebra $\mathcal{F}$ of (1.10).*

Remark 1.10. Conditional independence statements of the type proven in this paper have been expected to be true at least for several years, e.g., based on Markovian properties of decorated random planar maps (see Appendix A). We first heard about such properties from Sheffield in private communication.

However, proofs of such statements turned out to be rather elusive, in large part because it is not clear where the condition that the total central charge is 26 appears in the continuum setting (except in the matched case (1.2)). In this paper, the total central charge condition arises by combining the

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[2]This follows from two inputs. Firstly, conditioned on $\nu_\Phi(K)$, cutting by K gives two independent LQG disks with two marked boundary points and boundary arcs of lengths 1 and $\nu_\Phi(K)$ [AHS23]. Secondly, if $(\mathbb{D}, \widetilde{\Phi}, -\mathfrak{i})/\sim$ is an LQG disk, then $\sigma((\partial\mathbb{D}, \widetilde{\Phi})/\sim) = \sigma(\nu_{\widetilde{\Phi}}(\partial\mathbb{D}))$; this can be obtained from [MSW21, Theorem 5.1] by exploring a small neighborhood of $\partial\mathbb{D}$ using conformal percolation interfaces.

[3]To be precise, one assigns indices to the connected components U_i in some way measurable with respect to $\mathcal{F}$, for instance in decreasing order of their central charge-$\mathbf{c}_L$ LQG boundary lengths with respect Φ. Then Theorem 1.9 states that the IG-decorated LQG surfaces $(U_i, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ are conditionally independent given $\mathcal{F}$.

relationship between SLE and LQG in the matched case; a certain locality property for imaginary geometry fields with central charge $\mathbf{c}_M = 0$ (Theorem 3.10), which allows us to ignore such fields when proving conditional independence statements; and a "rotational invariance" property for the total central charge associated with a vector of IG fields (Proposition 1.17). See Section 1.4 for details.

Another way to see the central charge in the continuum, which may provide an alternative route to results of the type proven in this paper, is to weight the law of LQG surfaces by the Brownian loop soup partition function, in an appropriate regularized sense. See [APPS22] for details.

Instead of using SLE_κ, we can cut by a **conformal loop ensemble** (CLE_κ), a random countable collection of non-nested loops in a simply connected domain $D \subset \mathbb{C}$ which locally look like SLE_κ curves [She09, SW12]. See Figure 2 (b). The **gasket** of the CLE_κ is the closure of the union of the loops.

Theorem 1.11. *Theorem 1.9 holds if we replace K by the gasket of a CLE_κ in $\mathbb{D}$ for $\kappa \in [4, 8)$, sampled independently from everything else.*

The full parameter range for CLE_κ is $\frac{8}{3} < \kappa < 8$. We expect that Theorem 1.11 can also be proven for the remaining case $\kappa \in (\frac{8}{3}, 4)$ by similar arguments, see Remark 3.15.

Our next theorem applies to SLE started at the bulk marked point of $\text{UQD}^\bullet_{\mathbf{c}_L}$ which we sample via imaginary geometry, see Figure 2 (c). This setup is particularly interesting since the outer boundary of space-filling SLE run until it hits the marked point is a pair of flow lines started at that point [MS17, Section 1.2.3], see Section 2.5 for IG background.

Theorem 1.12. *Suppose $\kappa < 4$. Let $\mathbf{c}_L > 25$ and $\mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ satisfy $\mathbf{c}_L + \mathbf{c}_{\text{SLE}}(\kappa) + \mathbf{c}_1 + \cdots + \mathbf{c}_n = 26$. Let $(\mathbb{D}, \Phi, 0, -\mathrm{i})$ be an embedding of a unit boundary length LQG disk with a marked interior point, i.e., a sample from $\text{UQD}^\bullet_{\mathbf{c}_L}$ (Definition 1.6). Independently, let $(\mathbb{D}, \Psi_1, \ldots, \Psi_n, \Psi, -\mathrm{i})$ be an embedding of a sample from $\text{IG}_{(\mathbf{c}_1, \ldots, \mathbf{c}_n, \mathbf{c}_{\text{SLE}}(\kappa))}$ (Definition 1.7). Let K be the union of a pair of flow lines of Ψ started at 0 in $\mathbb{D}$ and run until they hit the boundary, with two distinct specified angles [MS17, Theorem 1.1]. Let $\mathcal{F} = \sigma((\mathcal{B}_{0+}(K \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, 0)/\sim)$ be defined as in (1.10) (with an extra marked point at 0). Then the decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ parametrized by the connected components U of $\mathbb{D}\backslash K$ are conditionally independent given $\mathcal{F}$.*

In Theorem 1.12 we exclude the $\kappa = 4$ case since the natural analog of K in this case is not a pair of curves, but rather a collection of nested "level loops". This is related to the fact that the $\kappa = 4$ analog of space-filling SLE_κ is not a curve [AHPS21]. Next, in the case when $\mathbf{c}_{\text{SLE}} = 0$, instead of cutting using an SLE_κ-type curve, we can use Brownian motion, see Figure 2 (d). We understand Brownian motion to have central charge 0, see Remark A.3 for details.

Theorem 1.13. *Suppose $\mathbf{c}_L > 25$ and $\mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ satisfy $\mathbf{c}_L + \mathbf{c}_1 + \cdots + \mathbf{c}_n = 26$. Let $(\mathbb{D}, \Phi, 0, -\mathrm{i})$ be an embedding of a sample from $\text{UQD}^\bullet_{\mathbf{c}_L}$ (Definition 1.6). Independently, let $(\mathbb{D}, \Psi_1, \ldots, \Psi_n, -\mathrm{i})$ be an embedding of a sample from $\text{IG}_{(\mathbf{c}_1, \ldots, \mathbf{c}_n)}$ (Definition 1.7). Let K be the trace of a Brownian motion started at 0 and stopped when it hits $\partial\mathbb{D}$, sampled independently from everything else. Let $\mathcal{F} = \sigma((\mathcal{B}_{0+}(K \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, K, 0)/\sim)$ be defined as in (1.10). Then the decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ parametrized by the connected components U of $\mathbb{D}\backslash K$ are conditionally independent given $\mathcal{F}$.*

Our next result applies when we cut by an LQG metric ball, and is illustrated in Figure 2 (e).

Theorem 1.14. *Theorem 1.13 holds if we instead let K be the $\mathfrak{d}_\Phi$-metric ball centered at 0 grown until it hits $\partial\mathbb{D}$, i.e., $K = \overline{\mathcal{B}_{\mathfrak{d}_\Phi(0,\partial\mathbb{D})}(0; \mathfrak{d}_\Phi)}$.*

There are also analogs of Theorem 1.13 and 1.14 for other growth processes started from 0 and stopped upon hitting $\partial\mathbb{D}$ which depend on the LQG surface in a local manner. Some examples include **LQG harmonic balls** defined in [BG22b], which describe the scaling limit of internal DLA on random planar maps [BG22c]; and finite unions of Brownian motion paths, LQG metric balls, and harmonic balls. See Theorem 3.19 for a general statement.

Finally, we can define exploration processes that depend only on an infinitesimal neighborhood of the explored region; see Figure 2(f). Conditional independence results also hold for such processes.

Definition 1.15. Suppose $\mathbf{c}_L > 25$ and $\mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ satisfy $\mathbf{c}_L + \mathbf{c}_1 + \cdots + \mathbf{c}_n = 26$. Let $(\mathbb{D}, \Phi, -\mathbb{i})$ be an embedding of a sample from $\mathrm{UQD}_{\mathbf{c}_L}$ (Definition 2.2) and independently let $(\mathbb{D}, \Psi_1, \ldots, \Psi_n, -\mathbb{i})$ be an embedding of a sample from $\mathrm{IG}_{(\mathbf{c}_1, \ldots, \mathbf{c}_n)}$ (Definition 1.7). Let $(K_t)_{t \leq T}$ be an increasing process of compact connected subsets of $\overline{\mathbb{D}}$ with random duration $T < \infty$, such that $K_0 = \partial\mathbb{D}$, and the process is continuous with respect to the Hausdorff metric. We call $(K_t)_{t \leq T}$ an **LQG-IG peeling process** if the following holds.

Let $\delta > 0$. For $t \leq T$ define $\mathcal{F}_t^\delta = \sigma((\mathcal{B}_\delta(K_t \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, (K_s)_{s \leq t})/\sim)$. Let τ be any stopping time for $\mathcal{F}_t^\delta$. Let $\tau' = T \wedge \inf\{t \geq \tau : K_t \cap \partial\mathcal{B}_\delta(K_\tau; \mathfrak{d}_\Phi) \cap \mathbb{D} \neq \emptyset\}$. Conditioned on $\mathcal{F}_\tau^\delta$, the decorated LQG surface $(\mathcal{B}_\delta(K_\tau \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, (K_t)_{t \leq \tau'})/\sim$ is conditionally independent from $(\Phi, \Psi_1, \ldots, \Psi_n)$.

Roughly speaking, $(K_t)_{t \leq T}$ is an LQG-IG peeling process if the growth of K_t depends locally on $\Phi, \Psi_1, \ldots, \Psi_n$, modulo LQG / IG coordinate change. Examples of LQG-IG peeling processes include LQG metric balls started at boundary points (due to the locality property of the LQG metric, see Section 2.4), chordal SLE_6 curves sampled independently from everything else (due to the locality property of SLE_6 [Law05]), and flow lines or counterflow lines of $\Psi_1, \ldots, \Psi_n$ started from boundary points (since flow and counterflow lines depend on the IG field in a local way). The reason for the name "LQG-IG peeling process" is that these processes are in some sense a continuum analog of "peeling processes" for random planar maps [Ang03], except that the growth is allowed to depend on the IG fields $\Psi_1, \ldots, \Psi_n$, instead of just on the LQG field Φ.

Theorem 1.16. *In the setting of Definition 1.15, conditioned on $\mathcal{F} := \bigcap_{\delta > 0} \mathcal{F}_T^\delta$, the decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ parametrized by the connected components U of $\mathbb{D} \backslash K_T$ are conditionally independent.*

1.4 Main ideas of the proof

Theorems 1.9–1.16 are all special cases of Theorem 3.3 for $\mathrm{UQD}_{\mathbf{c}_L}$ and Theorem 3.19 for $\mathrm{UQD}_{\mathbf{c}_L}^\bullet$. The proof of Theorem 3.3 can be broken down into three steps, each generalizing the result of the previous. Theorem 3.19 is similarly shown. These steps are carried out in Sections 3.1, 3.2 and 3.3 respectively.

Step 1. We prove a discretized version of Theorem 3.3 with $n = 1$ (a single IG field with central charge $26 - \mathbf{c}_L$) using the seminal **mating-of-trees** theorem which describes a γ-LQG surface decorated by an independent space-filling $\mathrm{SLE}_{\kappa' = 16/\gamma^2}$ curve in terms of planar Brownian motion [DMS21]. The desired conditional independence is a consequence of the Markov property of Brownian motion and a combinatorial independence property for submaps of a planar map arising from discretizing LQG decorated by SLE (a minor variant of the **mated-CRT map**). The details of this part of the argument are given in Section 4. The combinatorial independence property is a generalization of a certain conditional independence property for uniform meanders, which is of independent interest and which we prove in Appendix B.

Step 2. We use the **locality** of $\mathrm{IG}_{\mathbf{c}=0}$ to extend to the setting of $n \geq 1$ IG fields with central charges $(\mathbf{c}_1, \ldots, \mathbf{c}_n) = (26 - \mathbf{c}_L, 0, \ldots, 0)$. This locality property, stated as Theorem 3.10, is due to [Dub09b] and can be viewed as an IG field variant of the locality of SLE_6.

Step 3. Suppose we have a vector of n independent IG fields with central charges $\mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ satisfying $\mathbf{c}_1 + \cdots + \mathbf{c}_n = 26 - \mathbf{c}_L$. We **rotate** this vector of IG fields to get another vector of n independent IG fields with central charges $(26 - \mathbf{c}_L, 0, \ldots, 0)$. We then apply the case treated in Step 2 to conclude.

The rotation is accomplished by means of the following elementary but extremely useful observation, which states that n independent IG fields can be linearly transformed into another collection of n independent IG fields with the same total central charge.

Proposition 1.17 (Rotation of IG fields). *Fix $n \geq 1$ and let A be an $n \times n$ orthogonal matrix. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ and let $\chi_i = \chi(\mathbf{c}_i)$. Suppose $(\mathbb{D}, \Psi_1, \ldots, \Psi_n, -\mathrm{i})$ is an embedding of a sample from $\mathrm{IG}_{(\mathbf{c}_1, \ldots, \mathbf{c}_n)}$ (Definition 1.7). Define*

$$\begin{pmatrix} \widehat{\Psi}_1 \\ \vdots \\ \widehat{\Psi}_n \end{pmatrix} = A \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}, \qquad \begin{pmatrix} \widehat{\chi}_1 \\ \vdots \\ \widehat{\chi}_n \end{pmatrix} = A \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} \tag{1.11}$$

and suppose $\widehat{\chi}_i \geq 0$ for all i. Let $\widehat{\mathbf{c}}_i = 1 - 6\widehat{\chi}_i^2$. Then $\sum_i \widehat{\mathbf{c}}_i = \sum_i \mathbf{c}_i$, the IG surface $(\mathbb{D}, \widehat{\Psi}_1, \ldots, \widehat{\Psi}_n, -\mathrm{i})/\sim_{(\widehat{\mathbf{c}}_1, \ldots, \widehat{\mathbf{c}}_n)}$ is well defined (independently of the choice of embedding $\Psi_1, \ldots, \Psi_n$) and its law is $\mathrm{IG}_{(\widehat{\mathbf{c}}_1, \ldots, \widehat{\mathbf{c}}_n)}$.

Proof. Since orthogonal matrices preserve the Euclidean norm, we have

$$\sum_{i=1}^{n} \widehat{\mathbf{c}}_i = n - 6 \sum_{i=1}^{n} \widehat{\chi}_i^2 = n - 6 \sum_{i=1}^{n} \chi_i^2 = \sum_{i=1}^{n} \mathbf{c}_i.$$

Next, we verify that multiplication by A as in (1.11) transforms a $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$-IG surface into a $(\widehat{\mathbf{c}}_1, \ldots, \widehat{\mathbf{c}}_n)$-IG surface; that is, multiplication by A is compatible with the coordinate change rule in Definition 1.3. If $(D, \Psi_1, \ldots, \Psi_n) \sim_{(\mathbf{c}_1, \ldots, \mathbf{c}_n)} (\widetilde{D}, \widetilde{\Psi}_1, \ldots, \widetilde{\Psi}_n)$ and f is a homeomorphism witnessing this equivalence with $f(D) = \widetilde{D}$, then

$$A \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix} = A \begin{pmatrix} \widetilde{\Psi}_1 \circ f - \chi_1(\arg f' - 2\pi m) \\ \vdots \\ \widetilde{\Psi}_n \circ f - \chi_n(\arg f' - 2\pi m) \end{pmatrix} = A \begin{pmatrix} \widetilde{\Psi}_1 \circ f \\ \vdots \\ \widetilde{\Psi}_n \circ f \end{pmatrix} - \begin{pmatrix} \widehat{\chi}_1(\arg f' - 2\pi m) \\ \vdots \\ \widehat{\chi}_n(\arg f' - 2\pi m) \end{pmatrix}.$$

Finally, we identify the law of $(\mathbb{D}, \widehat{\Psi}_1, \ldots, \widehat{\Psi}_n, -\mathrm{i})/\sim_{(\widehat{\mathbf{c}}_1, \ldots, \widehat{\mathbf{c}}_n)}$. Without loss of generality we may assume that $\Psi_i = h_i + \chi(\mathbf{c}_i)\mathfrak{h}$ for $i = 1, \ldots, n$ where $h_1, \ldots, h_n$ are independent zero boundary GFFs on $\mathbb{D}$ and $\mathfrak{h}$ is the harmonic function on $\mathbb{D}$ with boundary data given by $\arg(\mathrm{i}\cdot)$, as in Definition 1.7.

Let $(\widehat{h}_1, \ldots, \widehat{h}_n)$ be the vector of fields obtained by multiplying $(h_1, \ldots, h_n)$ by A, as in (1.11). We claim that $(\widehat{h}_1, \ldots, \widehat{h}_n)$ also has the law of n independent zero boundary GFFs. Indeed, a zero boundary GFF can be sampled as $\sum_{i=1}^{\infty} g_i f_i$ where (f_i) is an orthonormal basis of the Hilbert space completion of compactly supported functions on $\mathbb{D}$ with finite Dirichlet energy and g_i are independent standard Gaussians [She07]. If $\underline{g}$ is a standard Gaussian in $\mathbb{R}^n$ then $\underline{g} \overset{d}{=} A\underline{g}$. Applying this to the coefficients in the orthonormal basis expansions of $h_1, \ldots, h_n$ gives the claim.

From (1.11) we have $\widehat{\Psi}_i = \widehat{h}_i + \widehat{\chi}_i\mathfrak{h}$ for all i, so the law of $(\mathbb{D}, \widehat{\Psi}_1, \ldots, \widehat{\Psi}_n)/\sim_{(\widehat{\mathbf{c}}_1, \ldots, \widehat{\mathbf{c}}_n)}$ is $\mathrm{IG}_{(\widehat{\mathbf{c}}_1, \ldots, \widehat{\mathbf{c}}_n)}$. $\qquad\square$

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Remark 1.18. The idea of rotating a vector of fields via an orthogonal matrix to get a new vector of fields with the same total charge is also used in [AG23]. In that paper, a key idea is to rotate the field describing a critical ($\gamma = 2$) LQG disk and a zero-boundary GFF to get two LQG disks of central charges $\mathbf{c}_{\mathrm{L}}, 26 - \mathbf{c}_{\mathrm{L}} \in (1, 25)$. See also [AG23, Appendix A] for a rotational invariance property analogous to Proposition 1.17 for vectors of fields sampled from the infinite measure on LQG disks or spheres.

2 Preliminaries

2.1 Comments on conformal coordinate change

Our definitions of LQG and IG surfaces (Definitions 1.1, 1.3 and 1.5) differ slightly from definitions elsewhere in the literature [She16, MS16a, DMS21]. In this subsection we will explain the extent of the difference and the reason why we use a different definition. For concreteness we will focus on the case of IG surfaces, but a similar discussion also applies for LQG surfaces.

In most other works on imaginary geometry, one only considers simply connected IG surfaces, and defines an IG surface as an equivalence class of pairs (D, Ψ) where $D \subset \mathbb{C}$ is a *simply connected* and *possibly unbounded* open set and Ψ is a distribution on D, and $(D, \Psi) \sim'_{\mathbf{c}} (\widetilde{D}, \widetilde{\Psi})$ if there is a conformal map $f : D \to \widetilde{D}$ such that $\Psi = \widetilde{\Psi} \circ f - \chi(\mathbf{c}) \arg f' + 2\pi\chi(\mathbf{c})m$ for some integer m. Note that f *need not extend to a homeomorphism of* $\mathbb{C}$.

For the set of pairs (D, Ψ) where D is a Jordan domain, the relations $\sim_{\mathbf{c}}$ and $\sim'_{\mathbf{c}}$ are equivalent: $(D, \Psi) \sim_{\mathbf{c}} (\widetilde{D}, \widetilde{\Psi})$ if and only if $(D, \Psi) \sim'_{\mathbf{c}} (\widetilde{D}, \widetilde{\Psi})$. On the larger set of pairs (D, Ψ) where D is bounded and simply connected, $(D, \Psi) \sim_{\mathbf{c}} (\widetilde{D}, \widetilde{\Psi})$ implies $(D, \Psi) \sim'_{\mathbf{c}} (\widetilde{D}, \widetilde{\Psi})$ (but the reverse implication does not hold, since ∂D can have "exterior" self-intersections). Consequently, for any bounded simply connected D, any object determined by $(D, \Psi)/\sim'_{\mathbf{c}}$ is also determined by $(D, \Psi)/\sim_{\mathbf{c}}$. In particular, flow and counterflow lines of a simply connected IG surface $(D, \Psi)/\sim_{\mathbf{c}}$ are determined by $(D, \Psi)/\sim_{\mathbf{c}}$ (see Section 2.5). To discuss these objects, we will sometimes want to embed in the upper half-plane $\mathbb{H}$ via $\sim'_{\mathbf{c}}$ to match the convention of [MS16a].

We now explain why we use $\sim_{\mathbf{c}}$ rather than $\sim'_{\mathbf{c}}$. For non-simply connected domains, in order to define an IG surface as an equivalence class of pairs (D, Ψ), we need $\arg f'$ to be well-defined modulo a single global additive multiple of $2\pi\chi$. This is not necessarily true if we only require $f : D \to \widetilde{D}$ to be conformal, but as we see in Proposition 2.6 there is a canonical way to define $\arg f'$ (modulo additive multiple of 2π) when f extends to a homeomorphism of $\mathbb{C}$. This explains the condition on f in Definition 1.3. The homeomorphism condition distinguishes the point ∞ on the Riemann sphere, and we choose to work with bounded domains to avoid interacting with this point.

We similarly work with a modified definition of LQG surface, so as to make them compatible with IG surfaces (as in Definition 1.5).

Remark 2.1. An important configuration that arises in [MS16a, She16, DMS21] is the slitted domain $\mathbb{H}\backslash\eta$ where η is a simple curve from 0 to the interior of $\mathbb{H}$ arising as a segment of a flow line of an IG field Ψ. Crucially, one can embed the slitted surface in $\mathbb{H}$ using $\sim'_{\mathbf{c}}$, i.e., $(\mathbb{H}\backslash\eta, \Psi) \sim'_{\mathbf{c}} (\mathbb{H}, \widetilde{\Psi})$ for some $\widetilde{\Psi}$. This is not possible using our equivalence relation $\sim_{\mathbf{c}}$. We will not need to consider slitted domains in this paper since we will always cut our domains by curves which disconnect the domain (usually into connected components which are each Jordan domains), and we view the connected components as parametrizing separate LQG/IG surfaces.

2.2 σ-algebras generated by LQG surfaces

One can define a topology (and hence a σ-algebra) for LQG surfaces with specific conformal structures (e.g., simply connected LQG surfaces) by conformally mapping to a canonical reference domain, see, e.g., [GM21a, Section 2.2.5]. It is not obvious how to define a topology for, say, LQG surfaces parametrized by domains with infinitely many complementary connected components. However, it is straightforward to define the σ-algebras generated by such LQG surfaces, as we now explain.

Consider the set $\mathcal{X}$ of pairs (D, ϕ) where D is a bounded open set and ϕ is a generalized function belonging to the local Sobolev space $H_{\mathrm{loc}}^{-1}(D)$. We equip $\mathcal{X}$ with some reasonable topology. For concreteness, we use the topology whereby (D_n, ϕ_n) converges to (D, ϕ) if and only if $(\mathbb{C} \cup \{\infty\}) \setminus D_n \to (\mathbb{C} \cup \{\infty\}) \setminus D$ with respect to the Hausdorff distance for the spherical metric on $\mathbb{C} \cup \{\infty\}$; and, for each bounded open set U with $\overline{U} \subset D$, we have $\phi_n \to \phi$ with respect to the metric on $H^{-1}(U)$ (i.e., the one induced by the operator norm on the dual of $H^1(U)$). One can check that this topology is separable and metrizable.

Let $\mathcal{B}$ be the Borel σ-algebra for this topology on $\mathcal{X}$, and let $\mathcal{B}'$ be the sub-σ-algebra generated by Borel measurable functions $F : \mathcal{X} \to \mathbb{R}$ such that $F(D, \phi) = F(\widetilde{D}, \widetilde{\phi})$ whenever $(D, \phi) \sim_{\mathbf{c}_L} (\widetilde{D}, \widetilde{\phi})$, where $\sim_{\mathbf{c}_L}$ is as in Definition 1.1. For a random LQG surface $\mathcal{S}$, we define $\sigma(\mathcal{S})$ to be the collection of events of the form $\{(D, \Phi) \in E\}$ where $E \in \mathcal{B}'$ and (D, Φ) is any embedding of $\mathcal{S}$. Informally, the information carried by $\sigma(\mathcal{S})$ is precisely the values of the functions of (D, Φ) which are invariant under LQG coordinate change.

More generally, this same approach allows us to define the σ-algebras generated by LQG surfaces decorated by IG fields, sets, and/or curves as in (1.6).

2.3 Unit boundary length LQG disk

We will not need to use the precise definition of the law $\mathrm{UQD}_{\mathbf{c}_L}$ of the unit boundary length LQG disk, but we include one possible definition (which comes from [Cer21, Corollary 1.2]) for completeness. See [DMS21, HRV18, AHS21] for alternative definitions.

Definition 2.2. Let $\mathbf{c}_L > 25$ and let $\gamma \in (0, 2)$ satisfy $\mathbf{c}_L = 1 + 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$. Let h be a Neumann GFF on $\mathbb{D}$ normalized to have average zero on $\partial \mathbb{D}$ and let

$$\widehat{\Phi}_0 = h - \gamma \log|\cdot + \mathring{\imath}| - \gamma \log|\cdot - 1| - \gamma \log|\cdot + 1|.$$

Let $\widehat{\Phi}$ be sampled from the law of $\widehat{\Phi}_0$ weighted by $\nu_{\widehat{\Phi}_0}(\partial \mathbb{D})^{\frac{4}{\gamma^2} - 2} / \mathbb{E}[\nu_{\widehat{\Phi}_0}(\partial \mathbb{D})^{\frac{4}{\gamma^2} - 2}]$ where $\nu_{\widehat{\Phi}_0}$ is the γ-LQG boundary length measure, and let $\Phi = \widehat{\Phi} - \frac{2}{\gamma} \log \nu_{\widehat{\Phi}}(\partial \mathbb{D})$. Then $\mathrm{UQD}_{\mathbf{c}_L}$ is the law of the γ-LQG surface $(\mathbb{D}, \Phi, -\mathring{\imath})/\sim_{\mathbf{c}_L}$.

2.4 LQG metric

Let $D \subset \mathbb{C}$ be an open domain such that Brownian motion started at any point of D a.s. exits D in finite time. We say that a random generalized function Φ on D is a **GFF plus a continuous function on** D if there is a coupling of Φ with a zero-boundary GFF Φ^0 on D such that a.s. $\Phi - \Phi^0$ is a continuous function on D (with no assumption about the behavior of $\Phi - \Phi^0$ near ∂D). We say that Φ is a **free-boundary GFF plus a continuous function on** $\overline{D}$ if there is a coupling of Φ with a free-boundary GFF Φ^{Free} on D such that a.s. $\Phi - \Phi^{\mathrm{Free}}$ is a continuous function on $\overline{D}$. It is clear from Definition 2.2 that if Φ is an embedding of the LQG disk, then Φ is a free-boundary GFF plus a continuous function plus finitely many functions of the form $\gamma \log|\cdot - z|$, for $z \in \partial D$.

Let $D \subset \mathbb{C}$, and let Φ be a GFF plus a continuous function on D. For each $\mathbf{c}_L > 25$, there is a unique (up to deterministic multiplicative constant) metric $\mathfrak{d}_\Phi : D \times D \to [0, \infty)$ (i.e., a measurable assignment $\Phi \mapsto \mathfrak{d}_\Phi$) satisfying a list of natural axioms [DDDF20, GM21c], called the **LQG metric**. Heuristically, $\mathfrak{d}_\Phi(z, w)$ corresponds to the infimum over paths from z to w of "the integral of $e^{\xi(\mathbf{c}_L)\Phi}$ along the path", where $\xi(\mathbf{c}_L) > 0$ is a constant defined in terms of the fractal dimension of $\mathbf{c}_L$-LQG. See [DDG21] for a survey of the LQG metric and its properties. The LQG metric can also be defined for $\mathbf{c}_L \in (1, 25]$ [DG23a, DG23b], but we will not need this case here.

We will now review the basic properties of $\mathfrak{d}_\Phi$ which we will need in this paper.

Topology. Almost surely, the LQG metric $\mathfrak{d}_\Phi$ induces the Euclidean topology on D. Furthermore, a.s. $\mathfrak{d}_\Phi$ is a length metric, i.e., $\mathfrak{d}_\Phi(z, w)$ is the infimum of the $\mathfrak{d}_\Phi$-lengths of paths joining z and w [GM21c, Axiom I].

Coordinate change. If $f : D \to \widetilde{D}$ is a conformal map and $\Phi = \widetilde{\Phi} \circ f + Q \log |f'|$, then [GM21b] almost surely

$$\mathfrak{d}_{\widetilde{\Phi}}(f(z), f(w)) = \mathfrak{d}_\Phi(z, w), \quad \forall z, w \in D. \tag{2.1}$$

Locality. For any open set $U \subset D$, the internal metric $\mathfrak{d}_\Phi(\cdot, \cdot; U)$ is the metric on U defined by

$$\mathfrak{d}_\Phi(z, w; U) := \inf \operatorname{len}(P; \mathfrak{d}_\phi), \quad \forall z, w \in U \tag{2.2}$$

where the infimum is taken over paths P from z to w in U and $\operatorname{len}(P; \mathfrak{d}_\Phi)$ denotes the $\mathfrak{d}_\Phi$-length of P. Note that $\mathfrak{d}_\Phi(z, w; U) \geq \mathfrak{d}_\Phi(z, w)$. We will frequently use the locality property of $\mathfrak{d}_\Phi$ [GM21c, Axiom II]: for any deterministic open set $U \subset D$, the internal metric $\mathfrak{d}_\Phi(\cdot, \cdot; U)$ is a.s. given by a measurable function of $\Phi|_U$. In particular, for $z \in U$ and $r > 0$ the event $\{\mathcal{B}_r(z; \mathfrak{d}_\Phi) \subset U\}$ is measurable with respect to $\sigma(\Phi|_U)$, and on this event the set $\mathcal{B}_r(z; \mathfrak{d}_\Phi)$ is measurable with respect to $\sigma(\Phi|_U)$.

Boundary extension. When Φ is a free-boundary GFF plus a continuous function on $\overline{D}$, the LQG metric $\mathfrak{d}_\Phi$ extends by continuity to a metric $\mathfrak{d}_\Phi : \overline{D} \times \overline{D} \to [0, \infty)$ which induces the Euclidean topology on $\overline{D}$ [HM22, Proposition 1.6]. The same is true if we add finitely many functions of the form $\alpha \log |\cdot - z|$ for $\alpha < Q$ and distinct deterministic points $z \in \partial \mathbb{D}$. In particular, the LQG metric associated with an embedding of the quantum disk extends to a metric on $\overline{D}$ which induces the Euclidean topology on $\overline{D}$. It is immediate that for this extended definition of $\mathfrak{d}_\Phi$, the locality property stated above still holds for relatively open $U \subset \overline{D}$.

2.5 SLE and imaginary geometry

For $\rho_L, \rho_R > -2$, there is a natural variant of SLE_κ called $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ [LSW03, Dub09a, MS16a] which is locally absolutely continuous with respect to SLE_κ away from the boundaries. It keeps track of two force points, which we will always assume are immediately to the left and right of the starting point of the curve. $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ hits the left boundary if and only if $\rho_L < \frac{\kappa}{2} - 2$, and similarly for the right boundary.

Imaginary geometry [MS16a, MS16b, MS16c, MS17] studies SLE via its coupling with IG fields, building on [Dub09b]. We only discuss a few special cases of the general theory here. Let $\kappa \neq 4, \mathbf{c} = \mathbf{c}_{\mathrm{SLE}}(\kappa)$ and let $\rho_L, \rho_R > -2$. Let Ψ be a zero boundary GFF in $\mathbb{H}$ plus the harmonic function with boundary values $-\frac{\pi}{\sqrt{\kappa}}(1 + \rho_L)$ on $(-\infty, 0)$ and $\frac{\pi}{\sqrt{\kappa}}(1 + \rho_R)$ on $(0, \infty)$. Then there is a random curve η in $\overline{\mathbb{H}}$ measurable with respect to Ψ whose marginal law is $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$, such that conditioned on η, the restrictions of Ψ to the connected components U of $\mathbb{H} \backslash \eta$ are conditionally independent. The conditional law of $\Psi|_U$ is that of a zero boundary GFF plus a harmonic function

with boundary conditions on $\partial U \cap \mathbb{R}$ as above and boundary conditions on $\partial U \cap \eta$ given by a constant plus $-\chi(\mathbf{c}) \arg f'$ where $f : U \to \mathbb{H}$ is a suitable conformal map. The boundary conditions on η are measurable with respect to η and are compatible with $\mathbf{c}$-IG coordinate change (Definition 1.3).

In this coupling, η is a **local set** [SS13, Section 3.3] of Ψ: for any relatively open set $U \subset \overline{\mathbb{H}}$ containing neighborhoods of 0 and ∞, the event $\{\eta \subset U\}$ is measurable with respect to $\Psi|_U$, and on this event η is measurable with respect to $\Psi|_U$.

When $\kappa < 4$, the curve η is called a **flow line** of Ψ, and similarly we say an $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ curve is a flow line of angle θ if it is a flow line of $\Psi + \theta \chi(\mathbf{c})$. When $\kappa > 4$ these curves are instead called **counterflow lines**[4]. Flow lines can also be started at interior points $z \in \mathbb{H}$ and run until they hit $\partial \mathbb{H}$, and satisfy properties analogous to those stated above [MS17]. The case $\kappa = 4$ ($\chi = 0$) falls outside the scope of the imaginary geometry framework, but the above statements for chordal η still hold, and η is called a **level line** [SS13, WW17].

For $\kappa > 4$, given a zero boundary GFF Ψ in $\mathbb{H}$ there is a counterclockwise space-filling SLE_κ loop based at ∞ measurable with respect to Ψ defined as follows. For each point with rational coordinates $z \in \mathbb{Q}^2 \cap \mathbb{H}$, let $\eta^{z,L}$ and $\eta^{z,R}$ be the flow lines of $\Psi - \frac{\pi}{\sqrt{\kappa}}$ started at z with angles $\theta^L = -\frac{\pi}{2}$ and $\theta^R = \frac{\pi}{2}$ respectively. We define a total order $\preceq$ on $\mathbb{Q}^2 \cap \mathbb{H}$ by saying $z \preceq w$ if z lies in a connected component of $\mathbb{H} \backslash (\eta^{w,L} \cup \eta^{w,R})$ whose boundary is traced by the left side of $\eta^{w,L}$ and the right side of $\eta^{w,R}$. This ordering is well defined due to properties of interacting flow lines, and gives rise to a space-filling loop in $\mathbb{H}$. See [BG22a, Appendix A.3] for details. Since this coupling is constructed via the flow lines of Ψ viewed as a $\mathbf{c}_{\mathrm{SLE}}(\kappa)$-IG field, in any simply connected domain with one boundary point we have a coupling of the counterclockwise space-filling SLE_κ loop with $\mathrm{IG}_{\mathbf{c}_{\mathrm{SLE}}(\kappa)}$.

Let $\eta : [0, T] \to \overline{\mathbb{D}}$ be a counterclockwise space-filling SLE_κ loop. When $\kappa \geq 8$, a.s. the interior of $\eta([a, b])$ is simply connected for all $0 < a < b < T$. On the other hand, when $\kappa \in (4, 8)$, a.s. the interior of $\eta([a, b])$ has countably many connected components for all $0 < a < b < T$. See Figure 4.

2.6 Total curvature and winding

We recall the notion of total curvature, also known as winding in some probability literature such as [MS16a]. This will be used in Section 2.7 to make sense of LQG / IG coordinate change as in Definitions 1.3 and 1.5 for domains which are not necessarily connected, and in Section 4.5 for the main combinatorial argument of the paper (see also Appendix B).

Suppose $\eta : [0, 1] \to \mathbb{C}$ is a regular curve, meaning it is continuously differentiable with nonvanishing derivative. Let $\theta : [0, 1] \to \mathbb{R}$ be the unique continuous function such that $\theta(0) \in [0, 2\pi)$ and $e^{-i\theta(t)} \eta'(t) > 0$ for each $t \in [0, 1]$. The *total curvature* of η is defined to be $\theta(1) - \theta(0)$, and more generally we define the total curvature of $\eta|_{[t,t']}$ to be $\theta(t') - \theta(t)$. This name comes from the fact that when η is twice-differentiable, if $\widehat{\eta} : [0, T] \to \mathbb{C}$ is the parametrization of η according to arc length and $k(t) = \frac{\widehat{\eta}''(t)}{i\widehat{\eta}'(t)} \in \mathbb{R}$ is its signed curvature, then

$$\theta(1) - \theta(0) = \int_0^T k(t)\, dt.$$

A regular simple closed curve $\eta : [0, T] \to \mathbb{C}$ is a simple loop ($\eta(0) = \eta(T)$) with a continuous nonvanishing derivative such that $\eta'(0) = \eta'(T)$. The following classical fact is often called *Hopf's Umlaufsatz*, see e.g. [Tu17, Theorem 17.4].

Lemma 2.3. *The total curvature of a regular simple closed curve is $\pm 2\pi$, with the $+$ sign if the loop is oriented in the counterclockwise direction and $-$ sign otherwise.*

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[4] By convention, in the above coupling η is a counterflow line of $-\Psi$ rather than Ψ.

Lemma 2.4. *Suppose η_0 and η_1 are regular simple curves such that for some $\varepsilon > 0$ we have $\eta_0|_{[0,\varepsilon]} = \eta_1|_{[0,\varepsilon]}$ and $\eta_0|_{[1-\varepsilon,1]} = \eta_1|_{[1-\varepsilon,1]}$, and $\eta_0|_{[\varepsilon,1-\varepsilon]}$ and $\eta_1|_{[\varepsilon,1-\varepsilon]}$ are homotopic in the twice-slitted domain $\mathbb{C}\backslash(\eta_0([0,\varepsilon]) \cup \eta_0([1-\varepsilon,1]))$. Then the total curvatures of η_0 and η_1 agree.*

Proof. Let η be a simple regular curve from $\eta_0(1)$ to $\eta_0(0)$ which concatenates with η_0 to form a regular simple closed curve. By Lemma 2.3, the concatenation has total curvature $\pm 2\pi$ with sign depending on the orientation of the loop. Thus, if such an η can be chosen such that the loop obtained by concatenating η with η_1 is also simple, then η_0 and η_1 have the same total curvature, as needed.

We now address the general case where no such η can be chosen. Our homotopy hypothesis implies that there exists $\varepsilon > 0$ and a homotopy of *simple* curves $(\eta_t)_{[0,1]}$ from η_0 to η_1 such that $\eta_t|_{[0,\varepsilon]}$ and $\eta_t|_{[1-\varepsilon,1]}$ are continuously differentiable and do not depend on $t \in [0,1]$, see e.g. [FM11, Section 1.2.7] for details on why we can take the curves to be simple. Pick a finite set of times $0 = t_0 < \cdots < t_k = 1$ such that for each $j = 1, \ldots, k$ the points $\eta_0(0)$ and $\eta_0(1)$ lie on the boundary of the same connected component of $\mathbb{C}\backslash(\eta_{t_{j-1}}([0,1]) \cup \eta_{t_j}([0,1]))$. Let $\widetilde{\eta}_0 = \eta_0$ and $\widetilde{\eta}_k = \eta_1$. For $j = 1, \ldots, k-1$ let $\widetilde{\eta}_j$ be a simple regular curve such that $\widetilde{\eta}_j|_{[0,\varepsilon]} = \eta_0|_{[0,\varepsilon]}$ and $\widetilde{\eta}_j|_{[1-\varepsilon,1]} = \eta_0|_{[1-\varepsilon,1]}$ for all j, and $\widetilde{\eta}_j|_{[\varepsilon,1-\varepsilon]}$ is homotopic to $\eta_0|_{[\varepsilon,1-\varepsilon]}$ in $\mathbb{C}\backslash(\eta_0([0,\varepsilon]) \cup \eta_0([1-\varepsilon,1]))$. Moreover we assume $\eta_0(0)$ and $\eta_0(1)$ lie on the boundary of the same connected component of $\mathbb{C}\backslash(\widetilde{\eta}_{j-1}([0,1]) \cup \widetilde{\eta}_j([0,1]))$ for $j = 1, \ldots, k$. Such curves $\widetilde{\eta}_j$ can be chosen by picking simple regular curves that stay sufficiently close to η_{t_j} in the Hausdorff topology for each $j = 1, \ldots, k-1$. By the first paragraph, the total curvatures of $\widetilde{\eta}_{j-1}$ and $\widetilde{\eta}_j$ agree for $j = 1, \ldots, k$, and hence the total curvatures of $\widetilde{\eta}_0 = \eta_0$ and $\widetilde{\eta}_k = \eta_1$ agree, as needed. $\qquad\square$

By Lemma 2.4, the following definition makes sense.

Definition 2.5. Suppose $\eta : [0,1] \to \mathbb{C}$ is a (possibly non-smooth) simple curve such that $\eta|_{[0,\varepsilon]}$ and $\eta|_{[1-\varepsilon,1]}$ are regular for some $\varepsilon > 0$. Define the total curvature of η to be that of any regular simple curve $\eta^{\mathrm{reg}} : [0,1] \to \mathbb{C}$ such that $\eta|_{[0,\varepsilon]} = \eta^{\mathrm{reg}}|_{[0,\varepsilon]}$ and $\eta|_{[1-\varepsilon,1]} = \eta^{\mathrm{reg}}|_{[1-\varepsilon,1]}$, and $\eta|_{[\varepsilon,1-\varepsilon]}$ and $\eta^{\mathrm{reg}}|_{[\varepsilon,1-\varepsilon]}$ are homotopic in the twice-slitted domain $\mathbb{C}\backslash(\eta([0,\varepsilon]) \cup \eta([1-\varepsilon,1]))$.

2.7   IG coordinate change for general domains

Suppose D and $\widetilde{D}$ are bounded open subsets of $\mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism with $f(D) = \widetilde{D}$ which is conformal on D. When D is simply connected (which in particular means that it is connected), one can define $\arg f' : D \to \mathbb{R}$ uniquely up to an additive constant in $2\pi\mathbb{Z}$. In this section we extend the definition of $\arg f'$ to general D. Roughly speaking the condition that f is a homeomorphism rules out the possibility of "spinning" a connected component an arbitrary number of times. This ensures that the definition of an IG surface (Definition 1.3) makes sense even if D is not simply connected, or even connected.

Here is the extended definition of $\arg f' : D \to \mathbb{R}$. First, fix any point $z_0 \in D$ and define $(\arg f')(z_0)$ to be any real θ satisfying $f'(z_0)e^{-i\theta} > 0$; this specifies it up to an element of $2\pi\mathbb{Z}$. For $z \in D\backslash\{z_0\}$, pick a curve $\eta : [0,1] \to \mathbb{C}$ with $\eta(0) = z_0$ and $\eta(1) = z$ and which is regular in neighborhoods of its endpoints. Let $\widetilde{\eta} = f \circ \eta$, then we define

$$(\arg f')(z) = (\arg f')(z_0) + w(\widetilde{\eta}) - w(\eta), \tag{2.3}$$

where w denotes total curvature as in Definition 2.5. The following result states that this definition is valid. It uses Lemmas 2.8 and 2.9 which we state and prove at the end of this section.

Proposition 2.6. *The above definition of $\arg f' : D \to \mathbb{R}$ does not depend on the choices of z_0 and η, and agrees with the usual definition of $\arg f'$ when D is simply connected.*

Proof. Agreement with the usual definition in the simply connected setting follows from Lemma 2.8. The non-dependence on η is stated as Lemma 2.9. For different choices of base point z_0 and z_0', the consistency of the above definition can be shown by concatenating with a curve from z_0' to z_0. $\quad\square$

We now use (2.3) to explain how total curvature relates to IG boundary conditions.

Lemma 2.7. *Let* $\mathbf{c}_M \leq 1$. *Suppose* $D \subset \mathbb{C}$ *is a simply connected domain whose boundary is a regular simple closed curve, and let* $x \in \partial D$. *Let* $\theta_0 \in [0, 2\pi)$ *be the angle such that the tangent to D at x in the counterclockwise direction is parallel to* $e^{i\theta_0}$. *Let* $\mathfrak{h}_D : D \to \mathbb{R}$ *be the harmonic function whose boundary value at $y \neq x$ is the total curvature of the counterclockwise boundary arc from x to y. If h is a zero boundary GFF in D, then the law of* $(D, h + \chi(\mathbf{c}_M)(\mathfrak{h}_D + \theta_0), x)/\sim_{\mathbf{c}}$ *is* $IG_{\mathbf{c}_M}$.

Proof. Recall the harmonic function $\mathfrak{h} : \mathbb{D} \to \mathbb{R}$ in Definition 1.7 whose boundary value at $y \in \partial\mathbb{D}$ is the total curvature of the counterclockwise boundary arc from $-i$ to y. Let $f : \mathbb{D} \to D$ be a conformal map sending $-i \mapsto x$. By the definition of an IG surface (Definition 1.3), we must show that

$$\chi(\mathbf{c}_M)\mathfrak{h}_D \circ f + \chi(\mathbf{c}_M)\theta_0 - \chi(\mathbf{c}_M)\arg f' = \chi(\mathbf{c}_M)(\mathfrak{h} + 2\pi m)$$

for some integer m. Since both functions are harmonic and $\arg f'(-i) - \theta_0 \in 2\pi\mathbb{Z}$, it suffices to show $\arg f'(y) = \arg f'(-i) + \mathfrak{h}_D(f(y)) - \mathfrak{h}(y)$ for all $y \in \partial\mathbb{D}$ with $y \neq -i$. Let η (resp. $\widetilde{\eta}$) be the counterclockwise boundary arc of $\mathbb{D}$ from $-i$ to y (resp. arc of D from x to $f(y)$). Since $\mathfrak{h}(y)$ (resp. $\mathfrak{h}_D(f(y))$) is the total curvature of η (resp. $\widetilde{\eta}$), the claim would follow from (2.3) if that equation applied to the curves η and $\widetilde{\eta}$. Instead, we approximate η and $\widetilde{\eta}$ by smooth curves in $\mathbb{D}$ and D, where the approximation is with respect to the supremum norm for the curve and its first derivative, apply (2.3), and take a limit to conclude. $\quad\square$

In the rest of this section, we state and prove the lemmas needed for the proof of Proposition 2.6 above.

Lemma 2.8. *Suppose D is a bounded simply connected open set and let* $\arg_0 f' : D \to \mathbb{R}$ *be a continuous function such that* $f'(z)e^{-i(\arg_0 f')(z)} > 0$ *for all $z \in D$. Let* $\eta : [0,1] \to D$ *be a regular curve (not necessarily simple) and* $\widetilde{\eta} = f \circ \eta$. *Then* $(\arg_0 f')(\eta(1)) = (\arg_0 f')(\eta(0)) + w(\widetilde{\eta}) - w(\eta)$.

Proof. Let $\theta : [0,1] \to \mathbb{R}$ be a continuous function such that $\eta'(t)e^{-i\theta(t)} > 0$ for all t, and similarly define $\widetilde{\theta}$ for $\widetilde{\eta}$. Since $\widetilde{\eta} = f \circ \eta$ we have $\widetilde{\eta}'(t) = f'(\eta(t))\eta'(t)$, so $\widetilde{\theta}(t)$ and $(\arg_0 f')(\eta(t)) + \theta(t)$ agree up to an additive integer multiple of 2π. The function $\widetilde{\theta}(t) - (\arg_0 f')(\eta(t)) - \theta(t)$ is continuous in t and takes values in $2\pi\mathbb{Z}$, thus is constant. Identifying its values at 0 and 1 gives the result. $\quad\square$

In Lemma 2.9 below we do not assume D is simply connected.

Lemma 2.9. *For $j = 1, 2$ let* $\eta_j : [0,1] \to \mathbb{C}$ *be a simple curve which is regular in neighborhoods of its endpoints. Suppose* $\eta_1(0) = \eta_2(0)$ *and* $\eta_2(1) = \eta_2(1)$, *and these endpoints lie in D. Let* $\widetilde{\eta}_j = f \circ \eta_j$ *for $j = 1, 2$. Then* $w(\widetilde{\eta}_1) - w(\eta_1) = w(\widetilde{\eta}_2) - w(\eta_2)$.

Proof. Call an intersection point of two curve segments *generic* if at that point a curve crosses from one side to the other side of the other curve. For η_1 and η_2 as in Lemma 2.9, we use the notation $\eta_1 \perp \eta_2$ to mean that η_1 and η_2 intersect only finitely many times, each intersection is generic, the tangent vectors of η_1 and η_2 at their starting point are not parallel, and similarly their tangent vectors at their ending point are not parallel.

We first prove the result under the assumption that $\eta_1 \perp \eta_2$. Let $p = \eta_j(0)$ and $q = \eta_j(1)$. Let η_p be a regular curve which intersects itself only at its endpoints $\eta_p(0) = \eta_p(1) = p$, is disjoint from η_1 and η_2, lies in a simply connected neighborhood of p in D, and satisfies $\eta'_{2,r}(1) = \eta'_p(0)$ and

$\eta_p'(1) = \eta_1'(0)$. Let $\eta_q : [0,1] \to \mathbb{C}$ be a regular curve which intersects itself only at its endpoints $\eta_q(0) = \eta_q(1) = q$, is disjoint from η_1 and η_2, lies in a simply connected neighborhood of q in D, and satisfies $\eta_1'(1) = \eta_q'(0)$ and $\eta_q'(1) = \eta_{2,r}'(0)$. See Figure 3.

Then $\eta : [0,4] \to \mathbb{C}$ obtained by concatenating $\eta_p, \eta_1, \eta_q, \eta_{2,r}$ is a loop which intersects itself finitely many times, all generically. By standard topological arguments, there exists a homeomorphism $g : \mathbb{C} \to \mathbb{C}$ fixing simply connected neighborhoods of η_p and η_q in D, which sends η to a regular loop $\eta^{\mathrm{reg}} = g \circ \eta$ which intersects itself finitely many times, each intersection being transversal. Decompose the loop η^{reg} into four segments $\eta_p, \eta_1^{\mathrm{reg}}, \eta_q, \eta_{2,r}^{\mathrm{reg}}$.

Let $\widetilde{\eta}_p = f \circ \eta_p, \widetilde{\eta}_q = f \circ \eta_q$, and $\widetilde{\eta} = f \circ \eta$. Let $\widetilde{g}$ be a homeomorphism of $\mathbb{C}$ fixing simply connected neighborhoods of $\widetilde{\eta}_p$ and $\widetilde{\eta}_q$ in $\widetilde{D}$, and sending $\widetilde{\eta}$ to a regular simple closed curve $\widetilde{\eta}^{\mathrm{reg}}$ with finitely many self-intersections, all of which are transversal. Decompose this closed curve into four segments $\widetilde{\eta}_p, \widetilde{\eta}_1^{\mathrm{reg}}, \widetilde{\eta}_q, \widetilde{\eta}_{2,r}^{\mathrm{reg}}$. Since the orientation-preserving homeomorphism $\widetilde{g} \circ f \circ g^{-1} : \mathbb{C} \to \mathbb{C}$ sends η^{reg} to $\widetilde{\eta}^{\mathrm{reg}}$, and these regular loops only intersect themselves finitely many times, each intersection being transversal and a double-point, by [Whi37] the total curvatures of $\widetilde{\eta}^{\mathrm{reg}}$ and η^{reg} agree[5]. In other words,

$$w(\widetilde{\eta}_p) + w(\widetilde{\eta}_1^{\mathrm{reg}}) + w(\widetilde{\eta}_q) + w(\widetilde{\eta}_{2,r}^{\mathrm{reg}}) = w(\eta_p) + w(\eta_1^{\mathrm{reg}}) + w(\eta_q) + w(\eta_{2,r}^{\mathrm{reg}}).$$

By Lemma 2.8 we have $w(\widetilde{\eta}_p) = w(\eta_p)$ and $w(\widetilde{\eta}_q) = w(\eta_q)$, so $w(\widetilde{\eta}_1^{\mathrm{reg}}) - w(\eta_1^{\mathrm{reg}}) = w(\eta_{2,r}^{\mathrm{reg}}) - w(\widetilde{\eta}_{2,r}^{\mathrm{reg}})$. Applying Definition 2.5 gives $w(\widetilde{\eta}_1) - w(\eta_1) = w(\eta_{2,r}) - w(\widetilde{\eta}_{2,r}) = w(\widetilde{\eta}_2) - w(\eta_2)$ as desired.

Now we address the case where we do not assume $\eta_1 \perp \eta_2$. By Lemma 2.10 just below, we can find a third curve $\eta : [0,1] \to \mathbb{C}$ such that $\eta_1 \perp \eta$ and $\eta_2 \perp \eta$. Then the previous paragraph implies $w(\widetilde{\eta}_1) - w(\eta_1) = w(f \circ \eta) - w(\eta) = w(\widetilde{\eta}_2) - w(\eta_2)$ as needed. $\qquad\square$
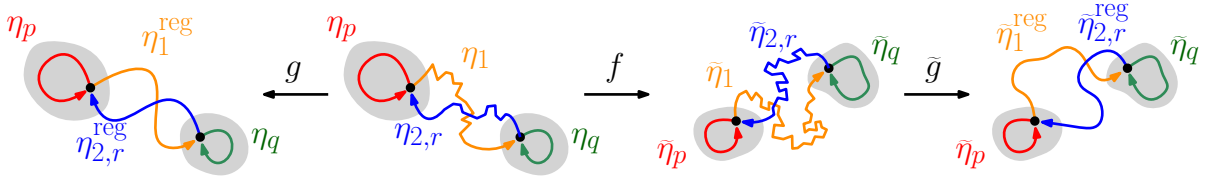

Figure 3: Figure for the proof of Lemma 2.9. We introduce curves η_p and η_q in simply connected neighborhoods of p and q (gray) which join with η_1 and $\eta_{2,r}$ to yield a loop η whose restriction to these neighborhoods is regular. Let $g : \mathbb{C} \to \mathbb{C}$ be a homeomorphism fixing these neighborhoods such that $g \circ \eta$ is regular. One can similarly choose $\widetilde{g}$ for the loop $f \circ \eta$.

Lemma 2.10. *Let $\perp$ be as defined in the proof of Lemma 2.9. For any curves η_1, η_2 as in Lemma 2.9, there exists a third curve η such that $\eta_1 \perp \eta$ and $\eta_2 \perp \eta$.*

Proof. By applying a suitable homeomorphism if necessary, we may assume that η_1 is a line segment.

We first choose the initial and ending segments of η. Let $\eta|_{[0,1/3]}$ be a line segment with $\eta(0) = \eta_1(0)$ such that $\eta_1'(0)$ and $\eta_2'(0)$ are not parallel to $\eta'(0)$, and $\eta((0,1/3))$ is disjoint from η_1 and η_2. This is possible since η_1 and η_2 are regular in neighborhoods of their starting points. Similarly, let $\eta|_{[2/3,1]}$ be a line segment ending at $\eta_1(1)$ such that $\eta_1'(1)$ and $\eta_2'(1)$ are not parallel to $\eta'(1)$, and $\eta([2/3,1))$ is disjoint from η_1 and η_2.

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[5] [Whi37, Theorem 2] gives a formula for the total curvature of a loop which is invariant under orientation-preserving homeomorphism, see e.g. [BP11, Theorem 2] for the particular formulation we use.

Next, we say an interval $[a, b]$ is *separating* if $\eta_2(a)$ and $\eta_2(b)$ each lie on η_1, $\eta_2((a, b))$ is disjoint from η_1, and the loop obtained as the union of $\eta_2([a, b])$ and the segment of η_1 joining $\eta_2(a)$ and $\eta_2(b)$ separates $\eta((0, 1/3])$ from $\eta([2/3, 1))$. If $[a, b]$ is separating, then $\operatorname{diam}(\eta_2([a, b])) > \min(\operatorname{diam}(\eta((0, 1/3])), \operatorname{diam}(\eta([2/3, 1))))$. Since $[0, 1]$ is compact and η_2 is continuous, the curve η_2 is uniformly continuous, thus the diameter lower bound implies there are at most finitely many separating intervals. The initial and ending segments of η can then be extended to a simple curve $\eta : [0, 1] \to \mathbb{C}$ such that $\eta((0, 1))$ is disjoint from η_1 and crosses η_2 once for each separating interval. This is the desired curve. □

3 Proofs of main theorems

In this section, we prove our most general result Theorem 3.3 on the LQG-IG local sets of Definition 3.1. The main theorems stated in Section 1.3 are all straightforward consequences of Theorem 3.3 and its minor variant Theorem 3.19.

First, we give the definition of an LQG-IG local set. This is an analog of the definition of a local set of the GFF [SS13, Section 3] in the setting of LQG surfaces decorated by imaginary geometry fields. Roughly speaking, a random closed set K is an LQG-IG local set of an IG-decorated LQG surface $\mathcal{S}$ if, for "any" IG-decorated LQG subsurface $\mathcal{S}'$ of $\mathcal{S}$ chosen conditionally independently of K given $\mathcal{S}$, the event $\{K$ lies in $\mathcal{S}'\}$ is conditionally independent of $\mathcal{S}$ given $\mathcal{S}'$, and further conditioned on this event, the decorated LQG surface $\mathcal{S}'$ further decorated by K is conditionally independent of $\mathcal{S}$. To make sense of a "random LQG sub-surface" $\mathcal{S}'$, we will discretize our setup at scale ε and define ε-LQG-IG local sets.

Let $\mathbf{c}_L > 25$, $n \geq 1$ and $\mathbf{c}_1, \ldots, \mathbf{c}_n \leq 1$ satisfy $\mathbf{c}_L + \mathbf{c}_1 + \cdots + \mathbf{c}_n = 26$. Let $(\mathbb{D}, \Phi, -\mathrm{i})$ be an embedding of a sample from $\mathrm{UQD}_{\mathbf{c}_L}$ (Definition 1.6), and independently let $(\mathbb{D}, \Psi_1, \ldots, \Psi_n, -\mathrm{i})$ be an embedding of a sample from $\mathrm{IG}_{(\mathbf{c}_1, \ldots, \mathbf{c}_n)}$ (Definition 1.7). Let η' be a counterclockwise space-filling SLE with central charge $(26 - \mathbf{c}_L)$ measurable with respect to $\Psi_1 \ldots, \Psi_n$ in the following way. Let $\widehat{\mathbf{c}}_1 = 26 - \mathbf{c}_L$ and $\widehat{\mathbf{c}}_2 = \cdots = \widehat{\mathbf{c}}_n = 0$, and let $\chi_i = \chi(\mathbf{c}_i)$ and $\widehat{\chi}_i = \chi(\widehat{\mathbf{c}}_i)$ for $i = 1, \ldots, n$. Let A be an orthogonal matrix which maps $(\chi_1, \ldots, \chi_n)$ to $(\widehat{\chi}_1, \ldots, \widehat{\chi}_n)$, and let $(\widehat{\Psi}_1, \ldots, \widehat{\Psi}_n)$ be the image of $(\Psi_1, \ldots, \Psi_n)$ under A as in (1.11), so by Proposition 1.17 the law of $(\widehat{\Psi}_1, \ldots, \widehat{\Psi}_n)$ is $\mathrm{IG}_{(\widehat{\mathbf{c}}_1, \ldots, \widehat{\mathbf{c}}_n)}$. Let η' be the counterclockwise space-filling SLE curve coupled with $\widehat{\Psi}_1$ as in Section 2.

In Definition 3.1, we will discretize the above setup using a finite set of points $\Lambda \subset \mathbb{D}$ independent of η', chosen using the LQG area measure μ_Φ. The curve η' is split by the points of Λ into $|\Lambda| + 1$ curve segments which we call $\eta_0', \ldots, \eta_{|\Lambda|}'$; each η_i' is defined on a time interval $[0, t_i]$. For $i = 0, \ldots, |\Lambda|$, let $D_i = \eta_i'([0, t_i])$ be the region traced out by η_i', let $a_i = \eta_i'(0)$ and $d_i = \eta_i'(t_i)$ be the start and end points of η_i' respectively, and let b_i (resp. c_i) be the furthest point on the clockwise (resp. counterclockwise) arc[6] of ∂D_i from a_i to d_i such that the boundary arc from a_i to b_i (resp. c_i) is a subset of $\overline{(\partial \mathbb{D} \cup \bigcup_{j \leq i} D_j)}$. See Figure 4.

Definition 3.1. Let $\varepsilon > 0$. An ε-**LQG-IG local set** $K \subset \overline{\mathbb{D}}$ is a random compact set coupled with $(\Phi, \Psi_1, \ldots, \Psi_n, \eta')$ such that $K \cup \partial \mathbb{D}$ is a.s. connected, and the following holds. See Figure 5 (left).

Independently of $(\Psi_1, \ldots, \Psi_n, K, \eta')$, sample a Poisson point process $\Lambda \subset \mathbb{D}$ with intensity measure $\varepsilon^{-1}\mu_\Phi$ and define $D_i, \eta_i', a_i, b_i, c_i, d_i$ as above. Independently of everything else sample a nonnegative integer $T \geq 0$ with $\mathbb{P}[T = t] = 2^{-t-1}$. Let I_0 be the set of indices i such that $D_i \cap \partial \mathbb{D} \neq \emptyset$, and for $j \geq 0$ inductively define I_{j+1} from I_j by independently sampling a point $p_{j+1} \in (\partial \bigcup_{i \in I_j} D_i) \setminus \partial \mathbb{D}$ from the quantum length measure, then setting $I_{j+1} = I_j \cup \{i : p_{j+1} \in \partial D_i\}$.

---

[6]For $\kappa' \geq 8$ the boundary of ∂D_i is simple. For $\kappa' \in (4, 8)$, the clockwise arc of ∂D_i from a_i to d_i is a nonsimple curve, constructed by ordering the points of ∂D_i hit by the left side of η_i' according to the last time they are hit.
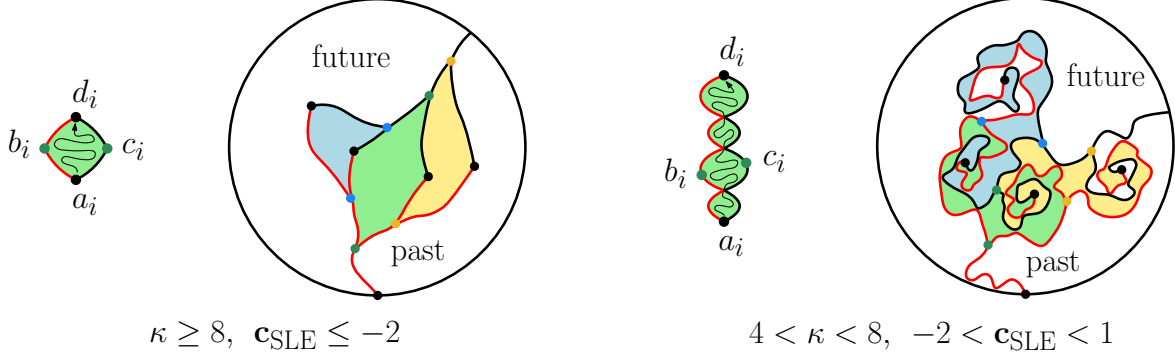
Figure 4: Let $\kappa > 4$ satisfy $\mathbf{c}_{\mathrm{SLE}}(\kappa) = 26 - \mathbf{c}_{\mathrm{L}}$. In each figure the space-filling SLE_κ loop η' traces three D_i regions in the order yellow, green, blue. The points of Λ are shown in black. In particular a_i and d_i are black, whereas b_i and c_i are shown in color. The left boundaries of the D_i (clockwise boundary arcs from a_i to d_i) are red, while the right boundaries are black. **Left:** When $\kappa \geq 8$, each of the D_i is simply connected. **Right:** When $\kappa \in (4,8)$, each D_i has countably many components, and arises from a sequence of simply-connected regions ordered by the time they are traced by η'.

We stop the process either at time T or at the time t when $\bigcup_{i \in I_t} D_i = \overline{\overline{\mathbb{D}}}$, whichever is earlier; call this time τ.

Conditioned on $(\bigcup_{i \in I_\tau} D_i, \Phi, \Psi_1, \ldots, \Psi_n, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}, \Lambda \cap \bigcup_{i \in I_\tau} D_i)/\sim$, the event $\{K \subset \bigcup_{i \in I_\tau} D_i\}$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \eta', \Lambda)$. Moreover, further conditioning on $\{K \subset \bigcup_{i \in I_\tau} D_i\}$, the decorated LQG surface $(\bigcup_{i \in I_\tau} D_i, \Phi, K)/\sim$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \eta', \Lambda)$.

The procedure of Definition 3.1 can be viewed as a semi-continuous analog of *peeling* for random planar maps (see, e.g., [Ang05]). Indeed, let $\mathcal{M}_t$ be the decorated LQG sub-surface parametrized by $\bigcup_{i \in I_t} D_t$, then if $t < \tau$, we explore from a point on $\partial \mathcal{M}_t$ chosen in a way only depending on $\mathcal{M}_t$ to obtain $\mathcal{M}_{t+1}$.

In Definition 3.1, we could equivalently replace $(\bigcup_{i \in I_\tau} D_i, \Phi, K)/\sim$ with

$$(\bigcup_{i \in I_\tau} D_i, \Phi, \Psi_1, \ldots, \Psi_n, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}, \Lambda \cap \bigcup_{i \in I_\tau} D_i, K)/\sim,$$

since the latter decorated LQG surface is determined by $(\bigcup_{i \in I_\tau} D_i, \Phi, K)/\sim$ and

$$(\bigcup_{i \in I_\tau} D_i, \Phi, \Psi_1, \ldots, \Psi_n, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}, \Lambda \cap \bigcup_{i \in I_\tau} D_i)/\sim.$$

Definition 3.1 is closely related to the definition of local sets of the GFF given in [SS13, Lemma 3.9] but more complicated since we consider LQG sub-surfaces rather than planar domains. Roughly speaking, replacing the deterministic open sets in their definition of GFF local sets with the randomly chosen LQG surface $(\bigcup_{i \in I_\tau} D_i, \Phi, \Psi_1, \ldots, \Psi_n, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}, \Lambda)/\sim$ gives Definition 3.1.

We emphasize that in the above definition, for the decorated LQG surface we condition on, $\{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}$ is an *unordered* set of tuples, and each tuple is not labelled by its index i.

Definition 3.2. A random compact set $K \subset \overline{\mathbb{D}}$ coupled with $(\Phi, \Psi_1, \ldots, \Psi_n, \eta')$ is called an **LQG-IG local set** if it is an ε-LQG-IG local set for all $\varepsilon > 0$.
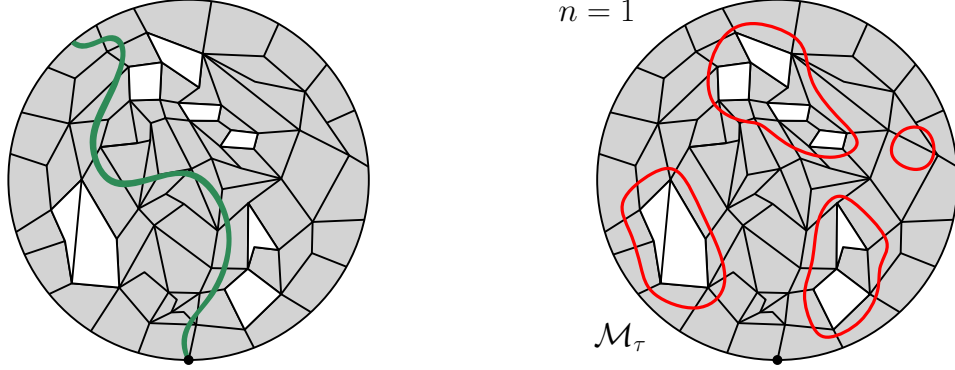
Figure 5: Combinatorially, the planar map structure depicted here cannot arise in our setup, but see Definition 4.7 for a description of the possible planar maps and Figures 8 and 9 (left) for examples. **Left.** Figure for Definition 3.1. Using η' and Λ, the disk $\mathbb{D}$ is divided into regions D_i. A random domain $\bigcup_{i \in I_\tau} D_i$ (gray) is explored independently of K (green). We say K is an ε-LQG-IG local set if, roughly speaking, the event $\{K \subset \bigcup_{i \in I_\tau} D_i\}$ is conditionally independent of the unexplored decorated LQG surfaces (white) given the explored decorated LQG surface (gray, not decorated by K), and further conditioning on $\{K \subset \bigcup_{i \in I_\tau} D_i\}$, the explored decorated LQG surface further decorated by K is conditionally independent of the unexplored decorated LQG surfaces. **Right.** Figure for Proposition 3.5. The explored LQG surface $\mathcal{M}_\tau$ is depicted in gray. Assuming there is only one IG field, and the loops L_i (red) on $\mathcal{M}_\tau$ are chosen in a way measurable with respect to $\mathcal{M}_\tau$, Proposition 3.5 says the encircled IG field-decorated LQG surfaces are conditionally independent given $\mathcal{M}_\tau$.

Examples of LQG-IG local sets include flow lines of Ψ_1, SLE_6, Brownian motion, and $\mathfrak{d}_\Phi$-metric balls run until stopping times intrinsic to the decorated LQG surface. See the proofs of Theorems 1.9, 1.13 and 1.14 in Section 3.4 for details.

The following conditional independence result is the main result of this section.

Theorem 3.3. *Suppose* $(\Phi, \Psi_1, \ldots, \Psi_n, \eta', K)$ *are as in Definition 3.2. Let* $\mathcal{F} = \bigcap_{\varepsilon > 0} \sigma((\mathcal{B}_\varepsilon(K \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, K, -\mathfrak{i})/\sim)$. *Then the decorated LQG surfaces* $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ *parametrized by the connected components* U *of* $\mathbb{D} \backslash K$ *are conditionally independent given* $\mathcal{F}$.

There is a variant of Theorem 3.3 for $\mathrm{UQD}^\bullet_{\mathbf{c}_L}$ (Theorem 3.19) whose statement we defer to Section 3.4 to avoid notational confusion. Its proof is essentially the same as that of Theorem 3.3.

Remark 3.4. Definition 3.1 gives a discretization of SLE-decorated LQG via a Poisson point process of intensity $\varepsilon^{-1}\mu_\Phi$. A related discretization where η' is segmented such that each D_i has fixed LQG area is used to motivate and analyze the *mated-CRT map* [DMS21, GHS19, GMS21]. The Poissonian formulation simplifies our arguments due to its locality property: the restrictions of a Poisson point process to fixed subdomains are independent.

In Sections 3.1–3.3 we carry out the three steps described in the introduction to prove Theorem 3.3. In Section 3.1 we state Proposition 3.5 which is a special case of Theorem 3.3 where $n = 1$ and K is a particular set defined in terms of mating of trees; its proof is deferred to Section 4. In Section 3.2 we use the locality of $\mathrm{IG}_{\mathbf{c}=0}$ fields to extend to the $n \geq 1$ setting where all but one IG field has central charge 0. Finally, in Section 3.3 we use Proposition 1.17 to generalize to the setting of arbitrary central charges, completing the proof of Theorem 3.3. In Section 3.4 we prove Theorem 3.19, and deduce the results stated in the introduction from Theorems 3.3 and 3.19.

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3.1 An $n = 1$ special case via mating-of-trees

In Definition 3.1 with $n = 1$, we independently sample an LQG surface from $\mathrm{UQD}_{\mathbf{c_L}}$ and an IG field from $\mathrm{IG}_{26-\mathbf{c_L}}$, discretize this IG-decorated LQG surface in a Poissonian way, and discover a random decorated LQG sub-surface via a peeling process. Proposition 3.5 below states that given this explored LQG sub-surface, its complementary LQG surfaces are conditionally independent, where "complementary LQG surface" is appropriately defined. See Figure 5 (right).

Proposition 3.5. *Consider the setting of Definition 3.1 with $n = 1$. Let $L_1, \ldots, L_m$ be simple loops in the interior of $\bigcup_{i \in I_\tau} D_i$ and for $k = 1, \ldots, m$ let O_k be the bounded connected component of $\mathbb{C} \backslash L_k$. Suppose $(\bigcup_{i \in I_\tau} D_i, \Phi, L_1, \ldots, L_m)/\sim$ is measurable with respect to $\mathcal{M}_\tau := (\bigcup_{i \in I_\tau} D_i, \Phi, \Psi_1, \{\eta_i', a_i, b_i, c_i, d_i) : i \in I_\tau\}, \Lambda)/\sim$, and the O_k are pairwise disjoint. Then conditioned on $\mathcal{M}_\tau$, the IG-decorated LQG surfaces $(O_1, \Phi, \Psi)/\sim, \ldots, (O_m, \Phi, \Psi)/\sim$ are conditionally independent.*

Remark 3.6. For $\gamma \in (0, \sqrt{2}]$, the proposition still holds if we simply define the O_i to be the complementary connected components of $\mathbb{D} \backslash M_\tau$. For $\gamma \in (\sqrt{2}, 2)$, this simpler formulation runs into topological issues because the interior of each D_i has multiple components, hence D_i might intersect multiple connected components of $\mathbb{D} \backslash M_\tau$. While a suitable modification of this formulation is true, we prefer to avoid this issue altogether by using the loops L_i.

We defer the proof of Proposition 3.5 to Section 4. The starting point is the mating-of-trees theorem for the LQG disk [DMS21, AG21], which describes the independent coupling of $\mathrm{UQD}_{\mathbf{c_L}}$ and $\mathrm{IG}_{26-\mathbf{c_L}}$ in terms of a pair of correlated one-dimensional Brownian motions. See Section 4.1. A statement like Proposition 3.5 should then seem plausible due to the independent increments of Brownian motion; see Proposition 4.3 for the the factorized description of the LQG surfaces $(D_i, \Phi, \eta_i', a_i, b_i, c_i, d_i)/\sim$ we will use.

On the other hand, consider the planar map whose faces correspond to the D_i from Definition 3.1; its precise definition is given in Section 4.3 (see Figure 9 (left)). This random planar map has the global condition that the faces must "chain together" to form a *single* path, corresponding to the order they are traced by the space-filling SLE loop. Further, adjacent faces (not necessarily consecutive) must intersect in a way which is compatible with their relative ordering in this path. See Definition 4.7 for details. As such, one could a priori expect complicated dependencies between disjoint regions of the map. To deal with this difficulty, we show that conditioned on the random submap corresponding to $\mathcal{M}_\tau$ in Proposition 3.5, the set of possible complementary submaps factorizes as a product (Proposition 4.12), i.e. the possible realizations of each submap do not depend on the other complementary submaps. We prove this in Section 4.5 via a topological/combinatorial argument involving *total curvature*, a.k.a. *winding* (Section 2.6). In the context of imaginary geometry, winding is natural since it is related to the boundary conditions of flow and counterflow lines. We prove that the global conditions described above are equivalent to local conditions related to winding, and hence obtain the combinatorial factorization Proposition 4.12.

Proposition 3.5 is proved by combining the probabilistic input of Proposition 4.3 with the combinatorial input of Proposition 4.12.

A simpler version of the argument of Section 4.5 leads to a conditional independence statement for the "upper" and "lower" halves of a uniform meander given its associated winding function. We record this result and its proof in Appendix B, both to motivate the argument of Section 4.5 and because the result is of independent interest.

3.2 Extending to $(\mathbf{c}_1, \ldots, \mathbf{c}_n) = (26 - \mathbf{c}_L, 0, \ldots, 0)$ via locality of $\mathbf{c} = 0$ IG fields

In this section we build on Proposition 3.5 to prove Proposition 3.8, a special case of Theorem 3.3 where all but one of the IG fields has central charge 0 and we approximate K by the union of the space-filling SLE segments D_i which it intersects rather than by $\mathcal{B}_\varepsilon(K; \mathfrak{d}_\Phi)$. We first state the more general Proposition 3.7 whose proof we defer to Section 3.3.

Proposition 3.7. *Let $\varepsilon > 0$. In the setting of Definition 3.1, let K be an ε-LQG-IG local set coupled with $(\Phi, \Psi_1, \ldots, \Psi_n, \eta')$. Sample a Poisson point process $\Lambda \subset \mathbb{D}$ with intensity measure $\varepsilon^{-1}\mu_\Phi$ independently of $(\Psi_1, \ldots, \Psi_n, \eta', K)$ and define $D_i, \eta_i', a_i, b_i, c_i, d_i$ as in Definition 3.1. Then the decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ parametrized by the connected components U of $\mathbb{D}\backslash K$ are conditionally independent given $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$, where*

$$\mathcal{F}_\varepsilon^{\mathrm{MOT}} := \sigma\big((K_\varepsilon, \Phi, \Psi_1, \ldots, \Psi_n, \{(\eta_i', a_i, b_i, c_i, d_i) \ : \ i \in I\}, K)/\sim\big), \tag{3.1}$$

$I = \{i \ : \ D_i \cap K \neq \emptyset\}$, and $K_\varepsilon = \bigcup_{i \in I} D_i$.

In this section we only prove the following special case of Proposition 3.7. The general case is proven in Section 3.3 using Proposition 1.17.

Proposition 3.8. *Proposition 3.7 holds when $(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n) = (26 - \mathbf{c}_L, 0, \ldots, 0)$ and η' is the counterclockwise space-filling SLE coupled with Ψ_1.*

Proposition 3.8 follows from Proposition 3.5 and a *locality* property for $\mathbf{c} = 0$ IG fields. In Section 3.2.1 we state the locality property for planar domains due to Dubédat [Dub09b]. In Section 3.2.2 we extend locality to the setting of IG surfaces. We use this locality property to prove Proposition 3.8 in Section 3.2.3.

3.2.1 Locality in planar domains

In this section we state Theorem 3.10 on the locality of $\mathrm{IG}_{\mathbf{c}=0}$ in planar domains, due to Dubédat [Dub09b]. For the reader's convenience we include the argument in Appendix C.

For $i, j \in \{1, 2\}$ let D_{ij} be a bounded simply-connected domain with smooth boundary. Suppose that for the D_{ij}, there exists a pair of common open boundary segments and a pair of common smooth disjoint cross-cuts η_ℓ, η_r joining them; call the region they bound $B \subset D_{ij}$, and let $x \in \partial B \cap D_{ij}$ be a common boundary point. We call the connected component of $D_{ij}\backslash B$ immediately clockwise of x the *left subdomain* of D_{ij}, and similarly define the *right subdomain*. For $i, j \in \{1, 2\}$ we suppose the left subdomains of D_{i1} and D_{i2} agree and the right subdomains of D_{1j} and D_{2j} agree. See Figure 6.
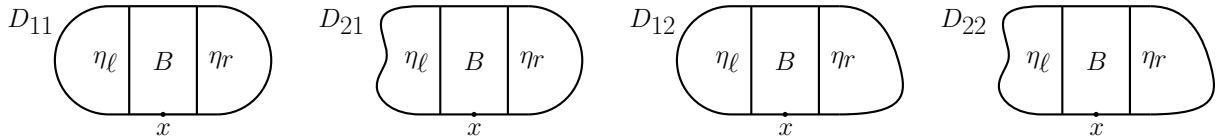

Figure 6: Setup for Theorem 3.10.

For an IG surface $(D_{ij}, \Psi, x)/\sim_{\mathbf{c}}$ sampled from $\mathrm{IG}_{\mathbf{c}}$ on the one pointed domain (Definition 1.7), let $\mathrm{IG}_{\mathbf{c}}^{B,ij}$ denote the law of the decorated IG surface $(B, \Psi, \eta_\ell, \eta_r, x)/\sim_{\mathbf{c}}$.

Lemma 3.9. *The measures $\mathrm{IG}_{\mathbf{c}}^{B,ij}$ are mutually absolutely continuous for all $i, j \in \{1, 2\}$.*

Proof. Let $\theta_0 \in [0, 2\pi)$ be the angle such that the counterclockwise tangent to B at x is parallel to $e^{i\theta_0}$, and let $\mathfrak{h}_{ij}$ be the harmonic function on D_{ij} whose boundary value at $y \neq x$ equals the total curvature of the counterclockwise boundary arc from x to y. By Lemma 2.7, if h_{ij} is a zero boundary GFF on D_{ij}, then the law of $(D_{ij}, h_{ij} + \chi(\mathbf{c})(\mathfrak{h}_{ij} + \theta_0), x)/\sim_{\mathbf{c}}$ is $\mathrm{IG}_{\mathbf{c}}$. By Lemma 2.4, the values of $\mathfrak{h}_{ij}$ on $\partial B \cap \partial D_{ij}$ (and hence on a neighborhood of $\partial B \cap \partial D_{ij}$ in ∂D_{ij}) agree for all i, j. Therefore the laws of $(h_{ij} + \chi(\mathbf{c})(\mathfrak{h}_{ij} + \theta_0))|_B$ are mutually absolutely continuous for all i, j [MS16a, Proposition 3.4]. $\square$

Theorem 3.10 ([Dub09b]).

$$\frac{d\mathrm{IG}_{\mathbf{c}=0}^{B,11}}{d\mathrm{IG}_{\mathbf{c}=0}^{B,21}}(\mathcal{S}) = \frac{d\mathrm{IG}_{\mathbf{c}=0}^{B,12}}{d\mathrm{IG}_{\mathbf{c}=0}^{B,22}}(\mathcal{S}) \qquad \text{for } \mathrm{IG}_{\mathbf{c}=0}^{B,ij}\text{-a.e. IG surface } \mathcal{S}. \tag{3.2}$$

The left hand side of (3.2) describes the relative likelihood of observing $\mathcal{S}$ when we vary the left subdomain. Equation (3.2) states that this relative likelihood does not depend on the right subdomain. In this way Theorem 3.10 is a locality result for $\mathrm{IG}_{\mathbf{c}=0}$.

3.2.2   Locality for $\underline{c} = (0, \ldots, 0)$ IG surfaces

In this section we adapt Theorem 3.10 to the setting of IG surfaces.

We work in the domain $(\mathbb{D}, x)$ where $x = -\mathbf{i}$. See Figure 7 (left). For $i, j \in \{1, 2\}$, consider two disjoint cross-cuts $\eta_\ell^{ij}, \eta_r^{ij}$ which do not contain x. Suppose the connected component B^{ij} of $\mathbb{D} \backslash (\eta_\ell^{ij} \cup \eta_r^{ij})$ having x on its boundary also has η_ℓ^{ij} and η_r^{ij} on its boundary, and $x, \eta_\ell^{ij}, \eta_r^{ij}$ lie on ∂B^{ij} in clockwise order. Let L^{ij} be the connected component of $\mathbb{D} \backslash B^{ij}$ having η_ℓ^{ij} on its boundary, let R^{ij} be the other connected component, and let V^{ij} be a simply connected subdomain of $\mathbb{D}$ such that $B^{ij} \subset V^{ij}$ and $\partial V^{ij} \setminus \partial \mathbb{D}$ lies at positive distance from B^{ij}.

Suppose for each $i = 1, 2$ the curve-decorated domains $(L^{ij} \cup V^{ij}, \eta_\ell^{ij}, \eta_r^{ij})$ for $j = 1, 2$ are conformally equivalent, and suppose for each $j = 1, 2$ that $(V^{ij} \cup R^{ij}, \eta_\ell^{ij}, \eta_r^{ij})$ for $i = 1, 2$ are conformally equivalent. One should think of this as saying that modulo conformal equivalence, the four curve-decorated domains arise from choosing among two possible left subdomains and two possible right subdomains. In particular, unlike in Theorem 3.10, we do not assume $L^{1j} = L^{2j}$.

For $(\Psi_1, \ldots, \Psi_n)$ sampled from $\mathrm{IG}_{\underline{c}}$ on $(\mathbb{D}, x)$, let $\mathrm{IG}_{\underline{c}}^{ij}$ denote the law of the curve-decorated $\underline{c}$-IG surface $(B^{ij}, \Psi_1, \ldots, \Psi_n, \eta_\ell^{ij}, \eta_r^{ij})/\sim$.

Proposition 3.11. *Let $\underline{c}$ be a vector with all entries equal to 0. The four laws $\mathrm{IG}_{\underline{c}}^{ij}$ are mutually absolutely continuous, and*

$$\frac{d\mathrm{IG}_{\underline{c}}^{11}}{d\mathrm{IG}_{\underline{c}}^{21}}(\mathcal{S}) = \frac{d\mathrm{IG}_{\underline{c}}^{12}}{d\mathrm{IG}_{\underline{c}}^{22}}(\mathcal{S}) \qquad \text{for } \mathrm{IG}_{\underline{c}}^{ij}\text{-a.e. IG surface } \mathcal{S}.$$

Proof. From our conformal equivalence assumptions, it is immediate that $(V^{ij}, \eta_\ell^{ij}, \eta_r^{ij})$ for $i, j \in \{1, 2\}$ are all conformally equivalent. Pick a simply connected domain $\widetilde{V}^{11}$ with smooth boundary such that $B^{11} \subset \widetilde{V}^{11} \subset V^{11}$, and B^{11} (resp. $\widetilde{V}^{11}$) lies at positive distance from $\partial\widetilde{V}^{11}\backslash\partial\mathbb{D}$ (resp. $\partial V^{11}\backslash\partial\mathbb{D}$). Define $\widetilde{V}^{ij}$ for $i, j \in \{1, 2\}$ using the above conformal equivalence. See Figure 7 (middle).

For this paragraph we fix the choice $i = j = 1$. Let L_0 (resp. R_0) be the connected component of $\widetilde{V}^{11}\backslash B^{11}$ immediately clockwise (resp. counterclockwise) of x, and write $L_1 = L^{11}$ and $R_1 = R^{11}$. For $i', j' \in \{0, 1\}$ let $D_{i'j'} = \overline{L_{i'} \cup B^{11} \cup R_{j'}}$, see Figure 7 (middle). Let n be the number of entries of $\underline{c}$. Recall $\mathrm{IG}_{\underline{c}}^{11}$ is the law of the $\underline{c}$-IG surface $(B^{11}, \Psi_1|_{B^{11}}, \ldots, \Psi_n|_{B^{11}}, \eta_\ell^{11}, \eta_r^{11})/\sim$ where $\Psi_1, \ldots, \Psi_n$ are independently sampled from $\mathrm{IG}_{\mathbf{c}=0}$ on (D_{11}, x) with additive constant fixed such that the boundary
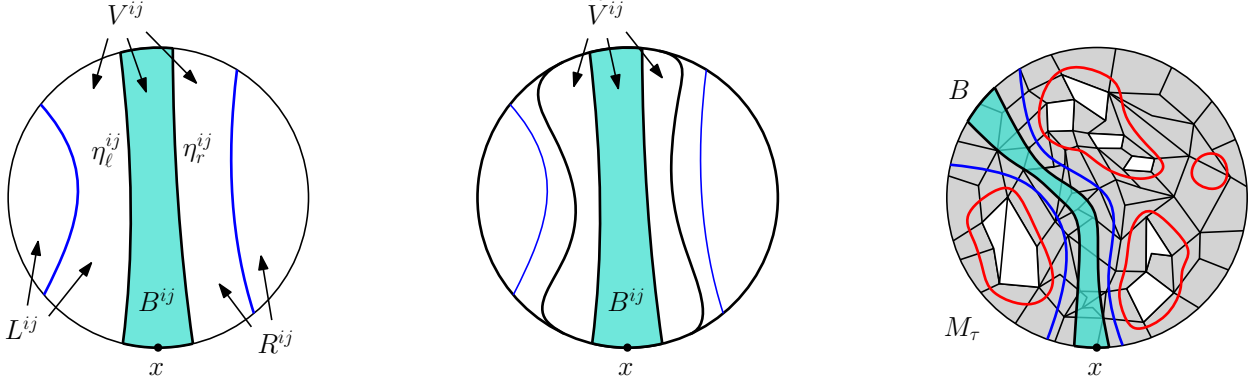
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Figure 7: **Left.** Proposition 3.11 gives an identity for Radon-Nikodym derivatives of the turquoise **c**-IG surface when i and j take values in $\{1,2\}$. **Middle.** In the proof of Proposition 3.11, we define a domain $\widetilde{V}^{ij}$ with smooth boundary. Four possible domains (boundaries shown in bold) can be obtained by gluing to B^{ij} a "left piece" and "right piece" chosen from the connected components of $\widetilde{V}^{ij}\backslash B^{ij}$ and $\mathbb{D}\backslash B^{ij}$. We apply Theorem 3.10 to these four domains. **Right.** Setup for the proof of Proposition 3.12. The shaded region is $M_\tau = \bigcup_{i\in I_\tau} D_i \subset \mathbb{D}$, and $L_1,\ldots,L_m$ are the red loops. We choose a region B (turquoise) bounded by a pair of cross-cuts separating L_1 from the other loops such that $x \in \partial B$. Let L and R be the two connected components of $\mathbb{D}\backslash B$ (each colored gray and white), and let $V \subset M_\tau$ be a simply connected neighborhood of B (between blue curves).

value infinitesimally counterclockwise of $x = -\mathfrak{i}$ in $\partial\mathbb{D}$ is 0. Similarly, we define $\mathrm{IG}_{\underline{\mathbf{c}}}^{00}$, $\mathrm{IG}_{\underline{\mathbf{c}}}^{10}$ and $\mathrm{IG}_{\underline{\mathbf{c}}}^{01}$ by replacing D_{11} with D_{00}, D_{10} and D_{01} respectively. This give four measures, which are mutually absolutely continuous by the sentence above Theorem 3.10. Applying Theorem 3.10 n times gives

$$\frac{d\mathrm{IG}_{\underline{\mathbf{c}}}^{00}}{d\mathrm{IG}_{\underline{\mathbf{c}}}^{10}}\frac{d\mathrm{IG}_{\underline{\mathbf{c}}}^{11}}{d\mathrm{IG}_{\underline{\mathbf{c}}}^{01}} = 1 \qquad \text{for } \mathrm{IG}_{\underline{\mathbf{c}}}^{00}\text{-a.e. IG surface.}$$

For each $i,j \in \{1,2\}$ we repeat the preceding paragraph to define three measures $\mathrm{IG}_{\underline{\mathbf{c}}}^{00}, \mathrm{IG}_{\underline{\mathbf{c}}}^{i0}, \mathrm{IG}_{\underline{\mathbf{c}}}^{0j}$. Some measures are multiply defined (for instance $\mathrm{IG}_{\underline{\mathbf{c}}}^{10}$ is defined twice, for $i=1, j=1,2$), but these definitions are consistent by conformal equivalence of the domains. In total this gives nine mutually absolutely continuous measures satisfying

$$\frac{d\mathrm{IG}_{\underline{\mathbf{c}}}^{00}}{d\mathrm{IG}_{\underline{\mathbf{c}}}^{i0}}\frac{d\mathrm{IG}_{\underline{\mathbf{c}}}^{ij}}{d\mathrm{IG}_{\underline{\mathbf{c}}}^{0j}} = 1 \qquad \text{for } \mathrm{IG}_{\underline{\mathbf{c}}}^{00}\text{-a.e. IG surface, for } i,j \in \{1,2\}. \tag{3.3}$$

Multiplying the left hand sides of (3.3) for $(i,j) \in \{(1,1),(2,2)\}$, and dividing by the left hand sides for $(i,j) \in \{(1,2),(2,1)\}$, we obtain $\frac{d\mathrm{IG}_{\underline{\mathbf{c}}}^{11}}{d\mathrm{IG}_{\underline{\mathbf{c}}}^{21}}\frac{d\mathrm{IG}_{\underline{\mathbf{c}}}^{22}}{d\mathrm{IG}_{\underline{\mathbf{c}}}^{12}} = 1$ for $\mathrm{IG}_{\underline{\mathbf{c}}}^{ij}$-a.e. IG surface, as desired. $\square$

3.2.3 LQG-IG local sets of $(\mathbf{c}_1,\ldots,\mathbf{c}_n) = (26 - \mathbf{c}_{\mathrm{L}}, 0, \ldots, 0)$ IG-decorated LQG surfaces

In this section we prove Proposition 3.8. We work in the setting of Proposition 3.8, so $(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n) = (26 - \mathbf{c}_{\mathrm{L}}, 0, \ldots, 0)$, K is an ε-LQG-IG local set coupled with $(\Phi, \Psi_1, \ldots, \Psi_n, \eta')$, and $\Lambda \subset \mathbb{D}$ is a Poisson point process with intensity measure $\varepsilon^{-1}\mu_\Phi$ independent of $(\Psi_1, \ldots, \Psi_n, \eta', K)$. As before the curve η' is segmented by the points Λ into curve segments $\eta'_0, \ldots, \eta'_{|\Lambda|}$, and each η'_i traces out a domain D_i with boundary points a_i, b_i, c_i, d_i.

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Recall that in Definition 3.1, we sample a random set of indices I_τ conditionally independent of K given $(\Phi, \Psi_1, \ldots, \Psi_n, \eta', \Lambda)$. We first show an $n \geq 1$ generalization of Proposition 3.5 via locality of $\mathbf{c} = 0$ IG fields.

Proposition 3.12. *Assume $\mathbf{c}_{\mathrm{L}} > 25$ and $(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n) = (26 - \mathbf{c}_{\mathrm{L}}, 0, \ldots, 0)$. In the setting of Definition 3.1, let $L_1, \ldots, L_m$ be disjoint loops in M_τ which bound disjoint regions $O_1, \ldots, O_m$ and let $\widehat{\mathcal{M}}_\tau = (M_\tau, \Phi, \Psi_1, \ldots, \Psi_n, \{\eta'_i, a_i, b_i, c_i, d_i \ : \ i \in I_\tau\})/\sim$. Suppose $(M_\tau, \Phi, L_1, \ldots, L_m)/\sim$ is measurable with respect to $\sigma(\widehat{\mathcal{M}}_\tau)$, i.e., the loops on $\widehat{\mathcal{M}}_\tau$ are chosen in a way depending only on $\widehat{\mathcal{M}}_\tau$. Then almost surely, given $\widehat{\mathcal{M}}_\tau$ the IG-decorated LQG surfaces $(O_i, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ for $i = 1, \ldots, m$ are conditionally independent.*

Proof. We begin with a proof outline. Let $\mathcal{M}_\tau := (M_\tau, \Phi, \Psi_1, \{(\eta'_i, a_i, b_i, c_i, d_i) \ : \ i \in I_\tau\})/\sim$ be $\widehat{\mathcal{M}}_\tau$ with the $\mathbf{c} = 0$ fields forgotten. We split $\mathbb{D}$ into three regions L, B, R with $B \subset M_\tau$ such that $(M_\tau, \Phi, B)/\sim$ is measurable with respect to $\mathcal{M}_\tau$ (see Figure 7, right). By our $n = 1$ result (Proposition 3.5) the IG-decorated LQG surfaces $(L \cup B, \Phi, \Psi_1)/\sim$ and $(R \cup B, \Phi, \Psi_1)/\sim$ are conditionally independent given $\mathcal{M}_\tau$. Using the $\mathbf{c} = 0$ IG locality property (Proposition 3.11), we can show this conditional independence still holds when we instead condition on $\mathcal{M}_\tau$ further decorated by $\Psi_2|_B, \ldots, \Psi_n|_B$. The Markov property of the GFF then implies the conditional independence of $(L \cup B, \Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)/\sim$ and $(R \cup B, \Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)/\sim$ under this conditioning; an easy argument then shows this still holds when conditioning on $\widehat{\mathcal{M}}_\tau$ instead. Varying our choice of B gives the desired result.

We now turn to the proof proper. Draw a pair of cross-cuts η_ℓ, η_r in $\mathbb{D}$ which stay in M_τ and do not hit $x := -\mathfrak{i}$, such that the connected component B of $\mathbb{D}\backslash(\eta_\ell \cup \eta_r)$ having x on its boundary also has η_ℓ and η_r on its boundary, and x, η_ℓ, η_r lie on ∂B in clockwise order. Let L be the connected component of $\mathbb{D}\backslash B$ having η_ℓ on its boundary, and let R be the other connected component. Choose a simply connected domain V such that $B \subset V \subset M_\tau$, and B (resp. V) lies at positive distance from $\partial V\backslash\partial\mathbb{D}$ (resp. $\partial M_\tau\backslash\partial\mathbb{D}$). We do this in such a way that $(M_\tau, \Phi, B, V)/\sim$ is measurable with respect to $\mathcal{M}_\tau$, and $O_1 \subset L$ while $O_2, \ldots, O_m \subset R$. See Figure 7 (right).

Let $\mathcal{S}_\ell = (L \cup B, \Phi, \Psi_1)/\sim$ and $\mathcal{S}_r = (R \cup B, \Phi, \Psi_1)/\sim$. Note that $\mathcal{S}_\ell$ is measurable with respect to $\mathcal{M}_\tau$ and $(O_1, \Phi, \Psi_1)/\sim$, and $\mathcal{S}_r$ is measurable with respect to $\mathcal{M}_\tau$ and $(O_i, \Phi, \Psi_1)/\sim$ for $i = 2, \ldots, m$. Thus, by Proposition 3.5, conditioned on $\mathcal{M}_\tau$, the decorated LQG surfaces $\mathcal{S}_\ell, \mathcal{S}_r$ are conditionally independent. Let

$$\mathcal{M}^*_\tau = (M_\tau, \Phi, \Psi_1, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}, \Psi_2|_B, \ldots, \Psi_n|_B)/\sim.$$

Using the locality of $\mathbf{c} = 0$ IG fields, we will prove that conditioned on $\mathcal{M}^*_\tau$ the decorated LQG surfaces $\mathcal{S}_\ell, \mathcal{S}_r$ are still conditionally independent.

Let $P_{\mathcal{M}_\tau}$ be the conditional law of of $\mathcal{M}^*_\tau$ given $\mathcal{M}_\tau$. Similarly let $P_{\mathcal{S}_\ell, \mathcal{S}_r}$ be the conditional law of $\mathcal{M}^*_\tau$ given $(\mathcal{M}_\tau, \mathcal{S}_\ell, \mathcal{S}_r)$. Let $\mathcal{L}_{\mathcal{M}_\tau}$ (resp. $\mathcal{L}_{\mathcal{M}^*_\tau}$) be the conditional law of $(\mathcal{S}_\ell, \mathcal{S}_r)$ given $\mathcal{M}_\tau$ (resp. $\mathcal{M}^*_\tau$). By the aforementioned conditional independence we have $\mathcal{L}_{\mathcal{M}_\tau}(d\mathcal{S}_\ell, d\mathcal{S}_r) = \mathcal{L}^\ell_{\mathcal{M}_\tau}(d\mathcal{S}_\ell)\mathcal{L}^r_{\mathcal{M}_\tau}(d\mathcal{S}_r)$ for some probability measures $\mathcal{L}^\ell_{\mathcal{M}_\tau}, \mathcal{L}^r_{\mathcal{M}_\tau}$ which are measurable functions of $\mathcal{M}_\tau$. We will need the following lemma.

Lemma 3.13. *Condition on $\mathcal{M}_\tau$ and fix $\mathcal{S}^1_\ell, \mathcal{S}^2_\ell, \mathcal{S}^1_r, \mathcal{S}^2_r$ (where the $\mathcal{S}^i_\ell$ and $\mathcal{S}^j_r$ denote possible realizations of $\mathcal{S}_\ell$ and $\mathcal{S}_r$). Then the probability measures $P_{\mathcal{S}^i_\ell, \mathcal{S}^j_r}$ are mutually absolutely continuous for $i, j \in \{1, 2\}$. Furthermore, for a.e. $\mathcal{M}^*_\tau$ we have $\dfrac{dP_{\mathcal{S}^1_\ell, \mathcal{S}^1_r}}{dP_{\mathcal{S}^2_\ell, \mathcal{S}^1_r}}(\mathcal{M}^*_\tau) = \dfrac{dP_{\mathcal{S}^1_\ell, \mathcal{S}^2_r}}{dP_{\mathcal{S}^2_\ell, \mathcal{S}^2_r}}(\mathcal{M}^*_\tau).$*

Proof. Throughout this proof we condition on $\mathcal{M}_\tau$. For each $i, j \in \{1, 2\}$, we have a copy of the setup described above, where we have fields $\Phi^{ij}, \Psi^{ij}_1, \ldots, \Psi^{ij}_n$, a Poisson point process Λ^{ij}, and

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domains $B^{ij} \subset V^{ij} \subset M^{ij}$. Fixing $\mathcal{S}_\ell^i$ and $\mathcal{S}_r^j$ corresponds to conditioning on all of the above (modulo conformal coordinate change) except $\Psi_2^{ij}, \ldots, \Psi_n^{ij}$. Proposition 3.11 gives mutual absolute continuity of the conditional laws of the IG surface $\mathcal{B} = (B^{ij}, \Psi_2, \ldots, \Psi_n, \eta_\ell^{ij}, \eta_r^{ij})/\sim$ for $i, j \in \{1, 2\}$, and further gives an identity for Radon-Nikodym derivatives of these conditional laws. Since $\mathcal{M}_\tau^*$ determines and is determined by $(\mathcal{B}, \mathcal{M}_\tau)$, we obtain the desired Lemma 3.13. $\qquad\square$

Condition on $\mathcal{M}_\tau$. By definition we have

$$P_{\mathcal{M}_\tau} = \iint P_{\mathcal{S}_\ell', \mathcal{S}_r'} \mathcal{L}_{\mathcal{M}_\tau}^\ell(d\mathcal{S}_\ell') \mathcal{L}_{\mathcal{M}_\tau}^r(d\mathcal{S}_r') \tag{3.4}$$

where $\mathcal{S}_\ell', \mathcal{S}_r'$ denote possible realizations of $\mathcal{S}_\ell$ and $\mathcal{S}_r$. The mutual absolute continuity claim of Lemma 3.13 applied to (3.4) implies that a.s. $P_{\mathcal{S}_\ell, \mathcal{S}_r}$ and $P_{\mathcal{M}_\tau}$ are mutually absolutely continuous. We can thus take Radon-Nikodym derivatives in (3.4) to get

$$\frac{dP_{\mathcal{M}_\tau}}{dP_{\mathcal{S}_\ell, \mathcal{S}_r}}(\mathcal{M}_\tau^*) = \iint \frac{dP_{\mathcal{S}_\ell', \mathcal{S}_r'}}{dP_{\mathcal{S}_\ell, \mathcal{S}_r'}}(\mathcal{M}_\tau^*) \frac{dP_{\mathcal{S}_\ell, \mathcal{S}_r'}}{dP_{\mathcal{S}_\ell, \mathcal{S}_r}}(\mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^\ell(d\mathcal{S}_\ell') \mathcal{L}_{\mathcal{M}_\tau}^r(d\mathcal{S}_r'). \tag{3.5}$$

Lemma 3.13 implies that $\frac{dP_{\mathcal{S}_\ell', \mathcal{S}_r'}}{dP_{\mathcal{S}_\ell, \mathcal{S}_r'}}(\mathcal{M}_\tau^*)$ does not depend on $\mathcal{S}_r'$, i.e., it equals $f_\ell(\mathcal{S}_\ell, \mathcal{S}_\ell', \mathcal{M}_\tau^*)$ for some function f_ℓ. Likewise $\frac{dP_{\mathcal{S}_\ell, \mathcal{S}_r'}}{dP_{\mathcal{S}_\ell, \mathcal{S}_r}}(\mathcal{M}_\tau^*) = f_r(\mathcal{S}_r, \mathcal{S}_r', \mathcal{M}_\tau^*)$ for some function f_r. Thus (3.5) gives

$$\frac{dP_{\mathcal{M}_\tau}}{dP_{\mathcal{S}_\ell, \mathcal{S}_r}}(\mathcal{M}_\tau^*) = \int f_\ell(\mathcal{S}_\ell, \mathcal{S}_\ell', \mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^\ell(d\mathcal{S}_\ell') \int f_r(\mathcal{S}_r, \mathcal{S}_r', \mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^r(d\mathcal{S}_r'),$$

so $\frac{dP_{\mathcal{S}_\ell, \mathcal{S}_r}}{dP_{\mathcal{M}_\tau}} = F_\ell(\mathcal{S}_\ell, \mathcal{M}_\tau^*) F_r(\mathcal{S}_r, \mathcal{M}_\tau^*)$ for some functions F_ℓ, F_r.

Recall that $\mathcal{L}_{\mathcal{M}_\tau^*}$ denotes the conditional law of $(\mathcal{S}_\ell, \mathcal{S}_r)$ given $\mathcal{M}_\tau^*$. By considering two ways of sampling from the joint law of $(\mathcal{M}_\tau^*, \mathcal{S}_\ell, \mathcal{S}_r)$ given $\mathcal{M}_\tau$, we have

$$\mathcal{L}_{\mathcal{M}_\tau^*}(d\mathcal{S}_\ell, d\mathcal{S}_r) P_{\mathcal{M}_\tau}(\mathcal{M}_\tau^*) = P_{\mathcal{S}_\ell, \mathcal{S}_r}(d\mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^\ell(d\mathcal{S}_\ell) \mathcal{L}_{\mathcal{M}_\tau}^r(d\mathcal{S}_r)$$
$$= F_\ell(\mathcal{S}_\ell, \mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^\ell(d\mathcal{S}_\ell) F_r(\mathcal{S}_r, \mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^r(d\mathcal{S}_r) P_{\mathcal{M}_\tau}(d\mathcal{M}_\tau^*).$$

Thus, for a.e. $\mathcal{M}_\tau^*$ the measure $\mathcal{L}_{\mathcal{M}_\tau^*}$ factorizes as $F_\ell(\mathcal{S}_\ell, \mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^\ell(d\mathcal{S}_\ell) F_r(\mathcal{S}_r, \mathcal{M}_\tau^*) \mathcal{L}_{\mathcal{M}_\tau}^r(d\mathcal{S}_r)$, or in other words, a.s. conditioned on $\mathcal{M}_\tau^*$ the decorated LQG surfaces $\mathcal{S}_\ell$ and $\mathcal{S}_r$ are conditionally independent.

Now, let $\widehat{\mathcal{S}}_\ell = (L \cup B, \Phi, \Psi_1, \eta_\ell, \Psi_2, \ldots, \Psi_n)/\sim$ be $\mathcal{S}_\ell$ decorated by the additional $\mathbf{c} = 0$ IG fields, and likewise define $\widehat{\mathcal{S}}_r$. By the above conditional independence and the domain Markov property of the GFF, $\widehat{\mathcal{S}}_\ell$ and $\widehat{\mathcal{S}}_r$ are conditionally independent given $\mathcal{M}_\tau^*$. Next, let $\mathcal{M}_\tau^{*, \ell}$ be $\mathcal{M}_\tau^*$ decorated by the fields $\Psi_i|_{(L \cup B) \cap M_\tau}$ rather than $\Psi_i|_B$ for $i = 2, \ldots, n$. It is measurable with respect to $\mathcal{M}_\tau^*$ and $\widehat{\mathcal{S}}_\ell$, so $\widehat{\mathcal{S}}_\ell, \widehat{\mathcal{S}}_r$ are conditionally independent given $\mathcal{M}_\tau^{*, \ell}$. Finally, $\widehat{\mathcal{M}}_\tau$ is $\mathcal{M}_\tau^{*, \ell}$ decorated by the fields $\Psi_i|_{M_\tau}$ rather than $\Psi_i|_{(L \cup B) \cap M_\tau}$ for $i = 2, \ldots, n$, hence is measurable with respect to $\mathcal{M}_\tau^{*, \ell}$ and $\widehat{\mathcal{S}}_r$. We conclude $\widehat{\mathcal{S}}_\ell, \widehat{\mathcal{S}}_r$ are conditionally independent given $\widehat{\mathcal{M}}_\tau$.

Thus, given $\widehat{\mathcal{M}}_\tau$, the decorated LQG surface $(O_1, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ is conditionally independent of $(O_2, \Phi, \Psi_1, \ldots, \Psi_n)/\sim, \ldots, (O_m, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$. Repeating the argument for the other $m - 1$ indices, we obtain the mutual independence of the $(O_i, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ given $\widehat{\mathcal{M}}_\tau$. $\qquad\square$

Proof of Proposition 3.8. Let $U_1, \ldots, U_m$ be a finite collection of connected components of $\mathbb{D} \backslash K$ chosen such that $(K_\varepsilon, \Phi, \partial U_1, \ldots, \partial U_m)/\sim$ is measurable with respect to $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$, and let $L_1, \ldots, L_m$

be disjoint simple loops in K_ε such that L_i is homotopic[7] to ∂U_i in $U_i \cap K_\varepsilon$, and $(K_\varepsilon, \Phi, L_1, \ldots, L_m)/\sim$ is measurable with respect to $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$. Note that even if $\overline{U}_i \cap \overline{U}_j \neq \emptyset$ for some $i \neq j$ (e.g., when K has "pinch points"), by construction K_ε contains a neighborhood of K, so the loops $L_1, \ldots, L_m$ can always be chosen

Let $\widehat{\mathcal{M}}_\tau$ be defined as in Proposition 3.12. Let $\widehat{\mathcal{M}} = (K_\varepsilon, \Phi, \Psi_1, \ldots, \Psi_n, \{\eta_i', a_i, b_i, c_i, d_i : i \in I\})/\sim$, so that $\mathcal{F}_\varepsilon^{\mathrm{MOT}} = \sigma(\widehat{\mathcal{M}}, (K_\varepsilon, \Phi, K)/\sim)$. By Definition 3.1, conditioned on $\widehat{\mathcal{M}}_\tau$, the event $\{\widehat{\mathcal{M}}_\tau = \widehat{\mathcal{M}}\}$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \eta', \Lambda)$, and further conditioned on $\{\widehat{\mathcal{M}}_\tau = \widehat{\mathcal{M}}\}$, the decorated LQG surface $(K_\varepsilon, \Phi, K)/\sim$ (and hence $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$) is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \eta', \Lambda)$. Our loops are chosen in a way which is determined by $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$, so by Proposition 3.12, conditioned on $\widehat{\mathcal{M}}_\tau$, $\{\widehat{\mathcal{M}}_\tau = \widehat{\mathcal{M}}\}$ and $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$, the decorated LQG surfaces bounded by the L_i are conditionally independent. Consequently, with the same conditioning, the $(U_i, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ are conditionally independent.

Notice that on $\{\widehat{\mathcal{M}}_\tau = \widehat{\mathcal{M}}\}$, the decorated LQG surface $\widehat{\mathcal{M}}_\tau$ is measurable with respect to $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$. Therefore the $(U_i, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ are conditionally independent given $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$ and $\{\widehat{\mathcal{M}}_\tau = \widehat{\mathcal{M}}\}$. But conditioned on $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$, the event $\{\widehat{\mathcal{M}}_\tau = \widehat{\mathcal{M}}\}$ is conditionally independent of $(\Phi, \Lambda, \Psi, \Psi_1, \ldots, \Psi_n)$, since the event corresponds, in the inductive definition of $\widehat{\mathcal{M}}_\tau$, to always selecting the point $p_j \in \partial(\bigcup_{i \in I_j} D_i)$ to lie in $\widehat{\mathcal{M}}$ at every step j and stopping when the whole $\widehat{\mathcal{M}}$ is explored. Thus, conditioned just on $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$, the m complementary LQG surfaces $(U_i, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ are conditionally independent. Since m is arbitrary, this gives the result. $\qquad\square$

3.3 General $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ by rotation: proof of Theorem 3.3

We now use Proposition 1.17 to deduce Proposition 3.7 from Proposition 3.8, after which Theorem 3.3 is straightforward.

Proof of Proposition 3.7. Let $\underline{\mathbf{c}} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ and $\hat{\underline{\mathbf{c}}} = (26 - \mathbf{c}_{\mathrm{L}}, 0, \ldots, 0)$. Recall from the discussion at the beginning of Section 3 the vector of IG fields $(\widehat{\Psi}_1, \ldots, \widehat{\Psi}_n)$ with central charges $(26 - \mathbf{c}_{\mathrm{L}}, 0, \ldots, 0)$ and the $n \times n$ orthogonal matrix A taking $(\Psi_1, \ldots, \Psi_n)$ to $(\widehat{\Psi}_1, \ldots, \hat{\Psi}_n)$. By Proposition 1.17, for any D the IG-decorated LQG surfaces $(D, \Phi, \Psi_1, \ldots, \Psi_n)/\sim_{\underline{\mathbf{c}}}$ and $(D, \Phi, \widehat{\Psi}_1, \ldots, \widehat{\Psi}_n)/\sim_{\hat{\underline{\mathbf{c}}}}$ determine each other (by applying A or A^{-1}). Thus, applying Proposition 3.8 to $(\Phi, \widehat{\Psi}_1, \ldots, \widehat{\Psi}_n, \eta', K)$, the decorated LQG surfaces $(U_i, \Phi, \Psi_1, \ldots, \Psi_n)/\sim_{\underline{\mathbf{c}}}$ are conditionally independent given $\mathcal{F}_\varepsilon^{\mathrm{MOT}}$. $\qquad\square$

Proof of Theorem 3.3. Let $\varepsilon > 0$, and define

$$\mathcal{F}_\varepsilon = \sigma((\mathcal{B}_\varepsilon(K \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, K)/\sim),$$

so $\mathcal{F} = \bigcap_{\varepsilon > 0} \mathcal{F}_\varepsilon$. It suffices to show that the decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ parametrized by the connected components U of $\mathbb{D} \setminus K$ are conditionally independent given $\mathcal{F}_\varepsilon$. Indeed, conditional independence is preserved under decreasing limits of σ-algebras by the backward martingale convergence theorem.

Sample a Poisson point process $\Lambda^* \subset \mathbb{D} \times \mathbb{R}_+$ with intensity measure $\mu_\Phi \times \mathrm{Leb}_{\mathbb{R}_+}$ which, given Φ, is conditionally independent of everything else. For $\varepsilon' > 0$ define $\Lambda_{\varepsilon'} := \{z : (z, t) \in \Lambda^* \text{ and } t < (\varepsilon')^{-1}\}$, so that $\Lambda_{\varepsilon'}$ is a Poisson point process with intensity measure $(\varepsilon')^{-1}\mu_\Phi$. Then, segment η' using $\Lambda_{\varepsilon'}$ and define $D_i(\varepsilon'), \eta_i'(\varepsilon'), a_i(\varepsilon'), b_i(\varepsilon'), c_i(\varepsilon'), d_i(\varepsilon')$ for $i = 0, \ldots, |\Lambda_{\varepsilon'}|$ as in Definition 3.1. Let

$$I(\varepsilon') := \{i : K \cap D_i(\varepsilon') \neq \emptyset\} \quad \text{and} \quad K_{\varepsilon'} = \bigcup_{i \in I(\varepsilon')} D_i(\varepsilon').$$

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[7]More precisely, if $f : U_i \to \mathbb{D}$ is a conformal map then $f(L_i)$ is homotopic to $\partial\mathbb{D}$ in $f(U_i \cap K_\varepsilon)$.

For $S \subset \overline{\mathbb{D}}$, write $\Lambda^*|_S := \{(z, t) \in \Lambda^* : z \in S\}$. Define

$$\mathcal{F}_{\varepsilon'}^{\mathrm{MOT}} := \sigma\big((K_{\varepsilon'}, \Phi, \Psi_1, \ldots, \Psi_n, \{(\eta_i'(\varepsilon'), a_i(\varepsilon'), b_i(\varepsilon'), c_i(\varepsilon'), d_i(\varepsilon')) : i \in I(\varepsilon')\}, K, \Lambda^*|_{K_{\varepsilon'}})/\sim\big).$$

This is the same σ-algebra as defined in Proposition 3.7 except that we further keep track of Λ^*. The same argument still applies so Proposition 3.7 holds in this setting.

For $\tilde{\varepsilon}' < \varepsilon'$, the set $K_{\tilde{\varepsilon}'} \subset K_{\varepsilon'}$ is measurable with respect to $K_{\varepsilon'}$ and $\Psi_1|_{K_{\varepsilon'}}, \ldots, \Psi_n|_{K_{\varepsilon'}}$. Indeed, the segments of η' contained in $K_{\varepsilon'}$ are locally determined by $\widehat{\Psi}_1|_{K_{\varepsilon'}}$ [GMS19, Lemma 2.4] and hence by the $\Psi_i|_{K_{\varepsilon'}}$. Since η' is defined via flow lines, which are intrinsic to the IG surface, we conclude that the σ-algebra $\mathcal{F}_{\varepsilon'}^{\mathrm{MOT}}$ is decreasing. Let $\mathcal{F}_{0+}^{\mathrm{MOT}} = \bigcap_{\varepsilon' > 0} \mathcal{F}_{\varepsilon'}^{\mathrm{MOT}}$. As $K_{\varepsilon'} \downarrow K$ as $\varepsilon' \to 0$, and $\mathfrak{d}_\Phi$ induces the Euclidean topology [HM22, Proposition 1.7], we see that a.s. $K_{\varepsilon'} \subset \mathcal{B}_\varepsilon(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi)$ for all sufficiently small ε'. Since η' is locally determined by $\widehat{\Psi}_1$ and hence by $\Psi_1, \ldots, \Psi_n$, for all sufficiently small ε' the set $K_{\varepsilon'}$ is determined by $\mathcal{B}_\varepsilon(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi)$ and the restrictions of $\Psi_1, \ldots, \Psi_n, \Lambda^*$ to this domain. As before, it follows that $\mathcal{F}_{0+}^{\mathrm{MOT}} \subset \widetilde{\mathcal{F}}_\varepsilon$ where

$$\widetilde{\mathcal{F}}_\varepsilon := \sigma((\mathcal{B}_\varepsilon(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, K, \Lambda^*|_{\mathcal{B}_\varepsilon(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi)})/\sim).$$

For each connected component U of $\mathbb{D} \setminus K$ let $U_\varepsilon = \mathcal{B}_\varepsilon(\partial U; \mathfrak{d}_\Phi) \cap U$. Let $\mathcal{U}_\varepsilon$ be the collection of all decorated LQG surfaces $(U_\varepsilon, \Phi, \Psi_1, \ldots, \Psi_n, \Lambda^*|_{U_\varepsilon})/\sim$ as U varies. We know $\widetilde{\mathcal{F}}_\varepsilon$ is measurable with respect to $\mathcal{F}_{\varepsilon'}^{\mathrm{MOT}}$ and $\sigma(\mathcal{U}_\varepsilon)$ for all $\varepsilon' > 0$ because $\mathcal{B}_\varepsilon(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi) \subset K_{\varepsilon'} \cup \bigcup_U U_\varepsilon$. Hence $\widetilde{\mathcal{F}}_\varepsilon$ is measurable with respect to $\mathcal{F}_{0+}^{\mathrm{MOT}}$ and $\sigma(\mathcal{U}_\varepsilon)$. On the other hand, $\mathcal{F}_{0+}^{\mathrm{MOT}} \subset \widetilde{\mathcal{F}}_\varepsilon$, and $\sigma(\mathcal{U}_\varepsilon) \subset \widetilde{\mathcal{F}}_\varepsilon$. We conclude $\widetilde{\mathcal{F}}_\varepsilon = \sigma(\mathcal{F}_{0+}^{\mathrm{MOT}}, \mathcal{U}_\varepsilon)$.

By Proposition 3.7 and the backward martingale convergence theorem, the collection of decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n, \Lambda^*|_U)/\sim$ indexed by the connected components U of $\mathbb{D} \setminus K$ are conditionally independent given $\mathcal{F}_{0+}^{\mathrm{MOT}}$. Since each $(U_\varepsilon, \Phi, \Psi_1, \ldots, \Psi_n, \Lambda^*|_{U_\varepsilon})/\sim$ is measurable with respect to $(U, \Phi, \Psi_1, \ldots, \Psi_n, \Lambda^*|_U)/\sim$ by the locality and conformal covariance of the LQG metric (see Section 2.4), the conditional independence still holds if we further condition on $\mathcal{U}_\varepsilon$. We conclude that the decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n, \Lambda^*|_U)/\sim$ are conditionally independent given $\widetilde{\mathcal{F}}_\varepsilon = \sigma(\mathcal{F}_{0+}^{\mathrm{MOT}}, \mathcal{U}_\varepsilon)$. Since the restrictions of a Poisson point process to disjoint domains are independent Poisson point processes in these domains, the decorated LQG surfaces $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ are conditionally independent from $\widetilde{\mathcal{F}}_\varepsilon$ given $\mathcal{F}_\varepsilon$, hence are conditionally independent given $\mathcal{F}_\varepsilon$. This completes the proof. $\qquad\square$

In Lemma 3.14 below we give a σ-algebra equivalent to $\mathcal{F}$ in Theorem 3.3 which does not use the LQG metric. We will not need to use Lemma 3.14 but state it to give context. Recall that for a measure space $(\Sigma, \mathcal{G}, \mu)$, the *completion* of $\mathcal{G}$ is $\sigma(\mathcal{G}, \mathcal{N}_\mu)$ where $\mathcal{N}_\mu$ is the collection of all subsets of μ-null sets in $\mathcal{G}$.

Lemma 3.14. *In the setting of Theorem 3.3, sample a Poisson point process $\Lambda^* \subset \mathbb{D} \times \mathbb{R}_+$ with intensity measure $\mu_\Phi \times \mathrm{Leb}_{\mathbb{R}_+}$ which, given Φ, is conditionally independent of everything else. For $\varepsilon > 0$ let $\Lambda_\varepsilon = \{z \in \mathbb{D} : (z, t) \in \Lambda^* \text{ and } t \le \varepsilon^{-1}\}$. The curve η' is split by the points of $\Lambda_\varepsilon \setminus K$ into a collection of curve segments; let $\{D_i'(\varepsilon)\}$ be the traces of these curve segments. Define*

$$I'(\varepsilon) := \{i : K \cap D_i'(\varepsilon) \neq \emptyset\}, \qquad K_\varepsilon' = \bigcup_{i \in I'(\varepsilon)} D_i'(\varepsilon),$$

$$\mathcal{F}_\varepsilon' = \sigma((K_\varepsilon', \Phi, \Psi_1, \ldots, \Psi_n, K, \{(z, t) \in \Lambda^* : z \in K_\varepsilon' \setminus K\})/\sim), \qquad \mathcal{F}' = \bigcap_{\varepsilon > 0} \mathcal{F}_\varepsilon'.$$

Then $\mathcal{F}$ and $\mathcal{F}'$ agree up to null sets, i.e., they have the same completion.

Proof. By the arguments in the proof of Theorem 3.3 immediately above, $\mathcal{F}'$ is equal to

$$\mathcal{F}'' := \bigcap_{\varepsilon > 0} \sigma((\mathcal{B}_\varepsilon(K \cup \partial\mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \dots, \Psi_n, K, \{(z,t) \in \Lambda^* \; : \; z \in \mathcal{B}_\varepsilon(K \cup \partial\mathbb{D}; \mathfrak{d}_\Phi)\backslash K\})/\sim).$$

By definition $\mathcal{F} \subset \mathcal{F}''$, and by Kolmogorov's zero-one law, conditioned on $\mathcal{F}$, every event in $\mathcal{F}''$ has conditional probability either 0 or 1. Therefore $\mathcal{F}$ and $\mathcal{F}''$ have the same completion, as needed. $\quad\square$

3.4 Proofs of remaining theorems

Proof of Theorem 1.9. We explain the $\kappa < 4$ case by constructing η as a flow line of an $\mathrm{IG}_{\mathbf{c}_{\mathrm{SLE}}}$ field; the $\kappa > 4$ and $\kappa = 4$ cases are the same except η is a counterflow line or level line of the IG field, respectively.

Let $(\mathbb{D}, \Phi, -\mathrm{i})$ be an embedding of a sample from $\mathrm{UQD}_{\mathbf{c}_L}$ and independently let $(\mathbb{D}, \Psi_1, \dots, \Psi_n, \Psi, -\mathrm{i})$ be an embedding of a sample from $\mathrm{IG}_{(\mathbf{c}_1, \dots, \mathbf{c}_n, \mathbf{c}_{\mathrm{SLE}})}$. For concreteness we use the embedding given right below Definition 1.7.

Suppose $\rho_L + \rho_R = \kappa - 6$. Note that if $f : \mathbb{H} \to \mathbb{D}$ is a conformal map with $f(0) = -\mathrm{i}$, then $\Psi \circ f - \chi \arg f'$ has the law of a zero boundary GFF plus the harmonic function $\mathfrak{h}$ such that $\mathfrak{h}(x) = 2\pi\chi$ for $x \in (-\infty, 0)$ and $\mathfrak{h}(x) = 0$ for $x \in (0, \infty)$, where $\chi = \chi(\mathbf{c}_{\mathrm{SLE}})$ as defined in (1.4). Thus, the flow line of angle $\theta = \frac{\pi}{\chi\sqrt{\kappa}}(1 + \rho_R)$ is an $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ curve from $-\mathrm{i}$ to i; see Section 2.5 for details. Let K be this angle θ flow line of Ψ. Since K is a local set of Ψ, in the setting of Definition 3.1, conditioned on $\bigcup_{i \in I_\tau} D_i$ the event $\{K \subset \bigcup_{i \in I_\tau} D_i\}$ is measurable with respect to $\Psi|_{\bigcup_{i \in I_\tau} D_i}$, and on this event K is measurable with respect to $\Psi|_{\bigcup_{i \in I_\tau} D_i}$. Consequently, the event $\{K \subset \bigcup_{i \in I_\tau} D_i\}$ is measurable with respect to $(\bigcup_{i \in I_\tau} D_i, \Psi)/\sim$, and on this event $(\bigcup_{i \in I_\tau} D_i, \Psi, K)/\sim$ is measurable with respect to $(\bigcup_{i \in I_\tau} D_i, \Psi)/\sim$. This follows from the fact that flow lines are intrinsic to IG surfaces, plus absolute continuity considerations; see the discussion after the statement of [MS16a, Proposition 3.4] for details. Therefore, K is an ε-LQG-IG local set for all $\varepsilon > 0$, and hence an LQG-IG local set.

By Theorem 3.3, given $\mathcal{F}' = \sigma((\mathcal{B}_{0+}(K \cup \partial D; \mathfrak{d}_\Phi), \Phi, \Psi_1, \dots, \Psi_n, \Psi)/\sim)$ defined as in (1.10) (i.e., $\mathcal{F}$ where we additionally keep track of Ψ in an infinitesimal neighborhood of $K \cup \partial\mathbb{D}$), the complementary IG-decorated LQG surfaces are conditionally independent. In the imaginary geometry coupling of K and Ψ, the conditional law of Ψ given K is that of the the zero boundary GFF plus a harmonic function with boundary values measurable with respect to K (see Section 2). The information carried by Ψ in an infinitesimal neighborhood of $K \cup \partial\mathbb{D}$ is thus just the boundary values of the harmonic function, and so is measurable with respect to K. We conclude $\mathcal{F}' = \mathcal{F}$, finishing the proof.

Now instead assume $\rho_L + \rho_R = -2$. Let K be an angle θ flow line of Ψ in $\mathbb{D}$ from i to $-\mathrm{i}$; varying θ gives $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ for any $\rho_L, \rho_R > -2$ with sum -2. Proceeding as before completes the argument. $\quad\square$

Proof of Theorem 1.11. Suppose $\kappa \in (4, 8)$. Let $(\mathbb{D}, \Phi, -\mathrm{i})$ be an embedding of a sample from $\mathrm{UQD}_{\mathbf{c}_L}$ and independently let $(\mathbb{D}, \Psi_1, \dots, \Psi_n, \Psi, -\mathrm{i})$ be an embedding of a sample from $\mathrm{IG}_{\mathbf{c}_1, \dots, \mathbf{c}_n, \mathbf{c}_{\mathrm{SLE}}}$.

As in the proof of Theorem 1.9, the counterflow line of Ψ started at $-\mathrm{i}$ targeted at i with an appropriately chosen angle θ is an $\mathrm{SLE}_\kappa(\kappa - 6)$ curve. This counterflow line is a local set of Ψ. In fact, by choosing a countable dense set of target points, one can realize a *branching* $\mathrm{SLE}_\kappa(\kappa - 6)$ *process* as a local set of Ψ, and thus obtain the CLE_κ gasket as a local set of Ψ [MS17, Section 1.2.3]. Now the same argument as in the proof of Theorem 1.9 directly above applies.

If $\kappa = 4$, the same argument holds since CLE_4 can be realized as the level loops of a zero boundary GFF, and in this coupling the CLE_4 gasket is a local set of this GFF [MS] (see [ASW19, Proposition 1] for a published construction). $\quad\square$

Remark 3.15. When $\kappa \in (\frac{8}{3}, 4)$, if we assume the CLE_κ gasket can be constructed via flow and counterflow lines of a sample from $\mathrm{IG}_{\mathbf{c}_{\mathrm{SLE}}}$, then Theorem 1.11 holds for $\kappa \in (\frac{8}{3}, 4)$ by the same proof. We expect that the desired imaginary geometry construction of CLE follows from the arguments of [MSW17]. Indeed [MSW17, Theorem 7.8] iteratively constructs CLE_κ via the so-called *boundary CLE*, which arises from branching flow and counterflow lines for Ψ [MSW17, Table 1].

We now turn to Theorem 1.16. It suffices to show that if $(K_t)_{t \leq T}$ is an LQG-IG peeling process (Definition 1.15), then K_T is an LQG-IG local set.

Lemma 3.16. *If $(K_t)_{t \leq T}$ is an LQG-IG peeling process for $(\Phi, \Psi_1, \ldots, \Psi_n)$ and η' is coupled with $(\Phi, \Psi_1, \ldots, \Psi_n)$ as in Definition 3.2, then K_T is an LQG-IG local set for $(\Phi, \Psi_1, \ldots, \Psi_n, \eta')$.*

Proof. As in Definition 3.1, let $\varepsilon > 0$, and let $\Lambda \subset \mathbb{D}$ be a Poisson point process with intensity measure $\varepsilon^{-1}\mu_\Phi$ conditionally independent of everything else given Φ, and define $D_i, \eta_i', a_i, b_i, c_i, d_i$ as explained above Definition 3.1. Recall that I_τ is the set of indices i of the regions D_i discovered by the random inductive peeling process of Definition 3.1. To lighten notation define

$$M_\tau = \bigcup_{i \in I_\tau} D_i.$$

We first check that $\overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)}$ is an ε-LQG-IG local set. Let $t_0 = 0$ and inductively define the stopping times $t_{j+1} = T \wedge \inf\{s \geq t_j \ : \ K_s \cap \partial \mathcal{B}_\delta(K_{t_j}; \mathfrak{d}_\Phi) \cap \mathbb{D} \neq \emptyset\}$ for the filtration $\mathcal{F}_t^\delta$ defined in Definition 1.15. For each $j \geq 0$ let Claim_j be the following claim: "Conditioned on the decorated LQG surface

$$(M_\tau, \Phi, \Lambda, \Psi_1, \ldots, \Psi_n, \{(\eta_i', a_i, b_i, c_i, d_i) \ : \ i \in I_\tau\}, \Lambda \cap M_\tau)/\sim, \tag{3.6}$$

the event $\{\overline{\mathcal{B}_\delta(K_{t_j}; \mathfrak{d}_\Phi)} \subset M_\tau\}$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \Lambda)$. Moreover, further conditioning on $\{\overline{\mathcal{B}_\delta(K_{t_j}; \mathfrak{d}_\Phi)} \subset M_\tau\}$, the decorated LQG surface $(M_\tau, \Phi, (K_s)_{s \leq t_j})/\sim$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \Lambda)$." We will inductively prove that Claim_j holds for all j.

By definition $K_{t_0} = \partial \mathbb{D}$. By the locality of the LQG metric and the fact that it is invariant under LQG coordinate change (Section 2.4) the event $\{\overline{\mathcal{B}_\delta(K_{t_0}; \mathfrak{d}_\Phi)} \subset M_\tau\}$ is measurable with respect to (3.6). Conditioning on this event and on (3.6), by the definition of LQG-IG peeling process, the decorated LQG surface $(M_\tau, \Phi, (K_s)_{s \leq t_1})/\sim$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \Lambda)$. Thus Claim_0 holds. Now, assuming Claim_j, we will show Claim_{j+1}. Combining Claim_j with the fact that $(K_t)_{t \leq T}$ is an LQG-IG peeling process yields the following: Conditioned on (3.6), $\{\overline{\mathcal{B}_\delta(K_{t_j}; \mathfrak{d}_\Phi)} \subset M_\tau\}$ and $(M_\tau, \Phi, (K_s)_{s \leq t_j})/\sim$, the decorated LQG surface $(M_\tau, \Phi, (K_s)_{s \leq t_{j+1}})/\sim$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \Lambda)$. By the locality and conformal covariance of the LQG metric, with the above conditioning, the event $\{\overline{\mathcal{B}_\delta(K_{t_{j+1}}; \mathfrak{d}_\Phi)} \subset M_\tau\}$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \Lambda)$, and further conditioning on this event, $(M_\tau, \Phi, (K_s)_{s \leq t_{j+1}})/\sim$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \Lambda)$. We have thus obtained a version of Claim_{j+1} where we not only condition on (3.6), but additionally on $\{\overline{\mathcal{B}_\delta(K_{t_j}; \mathfrak{d}_\Phi)} \subset M_\tau\}$ and $(M_\tau, \Phi, (K_s)_{s \leq t_j})/\sim$. Since $\{\overline{\mathcal{B}_\delta(K_{t_j}; \mathfrak{d}_\Phi)} \subset M_\tau\} \subset \{\overline{\mathcal{B}_\delta(K_{t_{j+1}}; \mathfrak{d}_\Phi)} \subset M_\tau\}$ and $(M_\tau, \Phi, (K_s)_{s \leq t_{j+1}})/\sim$ determines $(M_\tau, \Phi, (K_s)_{s \leq t_j})/\sim$, by Claim_j, the various independence statements still hold if we do not condition on the additional terms above. Therefore Claim_{j+1} holds.

We claim that $t_j = T$ for all sufficiently large j. Suppose for the sake of contradiction this is false. Arbitrarily pick a point $p_j \in K_{t_j} \backslash \mathcal{B}_\delta(K_{t_{j-1}}; \mathfrak{d}_\Phi)$ for each $j > 0$. By definition $\mathfrak{d}_\Phi(p_i, p_j) > \delta$ for every $i < j$, contradicting the fact that $(\overline{\mathbb{D}}, \mathfrak{d}_\Phi)$ has the Euclidean topology and thus is compact. Thus $t_j = T$ for all large j. By Claim_j, we conclude $\overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)}$ is an ε-LQG-IG set.

Now, we need to justify that K_T itself is an ε-LQG-IG local set. Let $I^\delta = \{i : D_i \cap \overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)} \neq \emptyset\}$ and let $I = \{i \ : \ D_i \cap K_T \neq \emptyset\}$. Given (3.6), the event $\{I^\delta = I_\tau\}$ is conditionally independent

of $(\Phi, \Lambda, \Psi_1, \ldots, \Psi_n)$. Indeed, $\overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)}$ is ε-LQG-IG local, and $\{I^\delta = I_\tau\}$ is measurable with respect to (3.6) and $(\overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)}, \Phi, (K_t)_{s \leq T})/\sim$. We conclude that $1_{I = I_\tau} = \lim_{\delta \to 0} 1_{I^\delta = I_\tau}$ is conditionally independent of $(\Phi, \Lambda, \Psi_1, \ldots, \Psi_n)$ given (3.6), fulfilling the first condition of Definition 3.1. Next, since $\overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)}$ is ε-LQG-IG local, given (3.6) and $\{I^\delta = I_\tau\}$ the decorated LQG surface $(M_\tau, \Phi, \overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)})/\sim$ is conditionally independent of $(\Phi, \Lambda, \Psi, \Psi_1, \ldots, \Psi_n)$. Since $\overline{\mathcal{B}_\delta(K_T; \mathfrak{d}_\Phi)} \downarrow K_T$ and $1_{I = I_\tau} = \lim_{\delta \to 0} 1_{I^\delta = I_\tau}$, we obtain the second condition of Definition 3.1. Thus K_T is an ε-LQG-IG local set for all $\varepsilon > 0$, and hence is an LQG-IG local set. $\qquad\square$

Proof of Theorem 1.16. This follows from Theorem 3.3 and Lemma 3.16. $\qquad\square$

We next address the unit boundary length LQG disk with a marked interior point, whose law is denoted $\mathrm{UQD}^\bullet_{\mathbf{c_L}}$ (Definition 1.6). The main result is Theorem 3.19 for the LQG-IG local sets defined below, from which all the other theorems follow.

Definition 3.17. Consider the setting of Definition 3.1 except that $(\mathbb{D}, \Phi, 0, -\mathfrak{i})$ is an embedding of a sample from $\mathrm{UQD}^\bullet_{\mathbf{c_L}}$ instead. An ε-**LQG-IG local set** $K \subset \overline{\mathbb{D}}$ is a random compact set coupled with $(\Phi, \Psi_1, \ldots, \Psi_n, \eta')$ such that $K \cup \partial\mathbb{D}$ is a.s. connected, almost surely $0 \in K$, and the following holds.

Independently of $(\Psi_1, \ldots, \Psi_n, K, \eta')$, sample a Poisson point process $\Lambda \subset \mathbb{D}$ with intensity measure $\varepsilon^{-1}\mu_\Phi$ and define $D_i, \eta'_i, a_i, b_i, c_i, d_i$ as in Definition 3.1. Independently of everything else sample a nonnegative integer $T \geq 0$ with $\mathbb{P}[T = t] = 2^{-t-1}$, let I_0 be the set of indices i such that $D_i(\varepsilon) \cap \partial\mathbb{D} \neq \emptyset$, and for $j \geq 0$ inductively define I_{j+1} from I_j by independently sampling a point $p_{j+1} \in (\partial \bigcup_{i \in I_j} D_i)\backslash\partial\mathbb{D}$ from the LQG length measure, then setting $I_{j+1} = I_j \cup \{i : p_{j+1} \in \partial D_i\}$. We stop the process either at time T or at the time t when $I_t = \{0, \ldots, N+1\}$, whichever is earlier; call this time τ.

Conditioned on $\{0 \in \bigcup_{i \in I_\tau} D_i\}$ and on $(\bigcup_{i \in I_\tau} D_i, \Phi, \Psi_1, \ldots, \Psi_n, \{\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}, 0, \Lambda \cap \bigcup_{i \in I_\tau} D_i)/\sim$, the event $\{K \subset \bigcup_{i \in I_\tau} D_i\}$ is conditionally independent of $(\Phi, \Psi_1, \ldots, \Psi_n, \Lambda)$. Moreover, further conditioning on $\{K \subset \bigcup_{i \in I_\tau} D_i\}$, the decorated LQG surface $(\bigcup_{i \in I_\tau} D_i, \Phi, K)/\sim$ is conditionally independent of $(\Phi, \Psi, \Psi_1, \ldots, \Psi_n, \eta', \Lambda)$.

K is called a **LQG-IG local set** if it is an ε-LQG-IG local set for all $\varepsilon > 0$.

Definition 3.17 for $\mathrm{UQD}^\bullet_{\mathbf{c_L}}$ is the same as Definitions 3.1 and 3.2 for $\mathrm{UQD}_{\mathbf{c_L}}$, except that we require that K contains the marked bulk point 0, and in our random exploration of the disk we condition on 0 lying in the explored region. The following lemma is an analog of Proposition 3.5 for $\mathrm{UQD}^\bullet_{\mathbf{c_L}}$.

Lemma 3.18. *Consider the setting of Definition 3.1 with $n = 1$. Weight by $\mu_\Phi(\mathbb{D})$, sample a point $z \in \mathbb{D}$ from $\mu_\Phi/\mu_\Phi(\mathbb{D})$ conditionally independently of everything except Φ, and condition on $\{z \in \bigcup_{i \in I_\tau} D_i\}$. Let $L_1, \ldots, L_m$ be simple loops in the interior of $\bigcup_{i \in I_\tau} D_i$ and let O_k be the bounded connected component of $\mathbb{C}\backslash L_k$. Suppose $(\bigcup_{i \in I_\tau} D_i, \Phi, L_1, \ldots, L_m)/\sim$ is measurable with respect to $\mathcal{M}^\bullet_\tau := (\bigcup_{i \in I_\tau} D_i, \Phi, \Psi_1, \{\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\}, \Lambda, z)/\sim$, and the O_k are pairwise disjoint. Then conditioned on $\mathcal{M}^\bullet_\tau$, the IG-decorated LQG surfaces $(O_1, \Phi, \Psi)/\sim, \ldots, (O_m, \Phi, \Psi)/\sim$ are conditionally independent.*

Proof. For a finite measure M let $M^\# = M/|M|$ be the corresponding probability measure. In the setting of Definition 3.1, weighting by $\mu_\Phi(\mathbb{D})$, sampling $z \sim \mu_\Phi^\#$ and restricting to $\{z \in \bigcup_{i \in I_\tau} D_i\}$ is equivalent to weighting by $\mu_\Phi(\bigcup_{i \in I_\tau} D_i)$ then sampling $z \sim (\mu_\Phi|_{\bigcup_{i \in I_\tau} D_i})^\#$. Thus, the present setup can be obtained starting with the setup of Proposition 3.5, weighting $\mathcal{M}_\tau$ by its LQG area and sampling a point in $\mathcal{M}_\tau$ from the probability measure proportional to its area measure to get $\mathcal{M}^\bullet_\tau$. This weighting and sampling does not affect the conditional law of the $(U_i, \Phi, \Psi)/\sim$ given $\mathcal{M}_\tau$, so Proposition 3.5 gives the desired conditional independence. $\qquad\square$

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Now we state and prove Theorem 3.19, which is the $\text{UQD}_{\mathbf{c}_L}^{\bullet}$ variant of Theorem 3.3.

Theorem 3.19. *Suppose* $(\Phi, \Psi_1, \ldots, \Psi_n, \eta', K)$ *are as in Definition 3.17. Let* $\mathcal{F} = \bigcap_{\varepsilon > 0} \sigma((\mathcal{B}_\varepsilon(K \cup \partial \mathbb{D}; \mathfrak{d}_\Phi), \Phi, \Psi_1, \ldots, \Psi_n, K, 0)/\sim)$. *Then the decorated LQG surfaces* $(U, \Phi, \Psi_1, \ldots, \Psi_n)/\sim$ *parametrized by the connected components* U *of* $\mathbb{D}\backslash K$ *are conditionally independent given* $\mathcal{F}$.

Proof. Using Lemma 3.18 in place of Proposition 3.5, we can prove the $\text{UQD}_{\mathbf{c}_L}^{\bullet}$ analogs of Proposition 3.8, Proposition 3.7 and Theorem 3.3 by the same arguments. The last of these analogs is the desired result. $\qquad\square$

Proof of Theorem 1.12. By the same argument as in Theorem 1.9 we see that K is an LQG-IG local set so Theorem 3.19 applies. $\qquad\square$

Proof of Theorem 1.13. Brownian motion is conformally invariant, and local in the sense that for any domains $D_1 \subset D_2$, Brownian motion in D_1 started at $z \in D_1$ and run until it exits D_1 agrees in law with Brownian motion in D_2 started at z and run until it exits D_1. Consequently K is an LQG-IG local set, so Theorem 3.19 gives the result. $\qquad\square$

Proof of Theorem 1.14. The LQG metric ball K is an LQG-IG local set by the locality property of the LQG metric. Thus the claim follows from Theorem 3.19. $\qquad\square$

4 Independence of unexplored regions of the discretized LQG disk

The goal of this section is to prove Proposition 3.5 which we used earlier to prove Theorem 3.3. To that end we first rephrase it in a more convenient way, as Theorem 4.1.

Fix $\varepsilon > 0$ and $\mathbf{c}_L > 25$, let $(\mathbb{D}, \Phi, -\mathbf{i})$ be an embedding of a sample from $\text{UQD}_{\mathbf{c}_L}$, and let $\Lambda \subset \mathbb{D}$ be a Poisson point process with intensity measure $\varepsilon^{-1}\mu_\Phi$. Independently of (Φ, Λ), let $(\mathbb{D}, \Psi, -\mathbf{i})$ be an embedding of a sample from $\text{IG}_{26-\mathbf{c}_L}$ and let η' be the counterclockwise space-filling $\text{SLE}_{\kappa'}$ measurable with respect to Ψ. Here, the LQG parameter $\gamma \in (0, 2)$ and SLE parameter $\kappa' > 4$ satisfy $\kappa' = \frac{16}{\gamma^2}$ since $\mathbf{c}_L + (26 - \mathbf{c}_L) = 26$. Condition on $|\Lambda| > 0$ and write $N = |\Lambda| - 1$. The curve η' is split by the points Λ to give domains D_i decorated by curve segments $\eta_i' : [0, t_i] \to \overline{\mathbb{D}}$ and boundary points a_i, b_i, c_i, d_i for $0 \le i \le N + 1$, as in Definition 3.1. Recall $a_i = \eta_i'(0)$ and $d_i = \eta_i'(t_i)$, while b_i (resp. c_i) is the furthest point on the clockwise (resp. counterclockwise) arc of ∂D_i from a_i to d_i such that the boundary arc from a_i to b_i (resp. c_i) is a subset of $\overline{(\partial \mathbb{D} \cup \bigcup_{j \le i} D_i)}$. See Figure 4.

Independently of $(\Phi, \Psi, \eta', \Lambda)$, sample a nonnegative integer $T \ge 0$ with $\mathbb{P}[T = t] = 2^{-t-1}$, let I_0 be the set of indices i such that $D_i \cap \partial \mathbb{D} \ne \emptyset$, and for $j \ge 0$ inductively define I_{j+1} from I_j by independently sampling a point $p_{j+1} \in (\partial \bigcup_{i \in I_j} D_i) \backslash \partial \mathbb{D}$ from the LQG length measure, then setting $I_{j+1} = I_j \cup \{i : p_{j+1} \in \partial D_i\}$. We stop the process either at time T or at the time t when $\bigcup_{i \in I_t} D_i = \overline{\mathbb{D}}$, whichever is earlier; call this time τ. Let

$$M_\tau := \bigcup_{i \in I_\tau} D_i \quad \text{and} \quad \mathcal{M}_\tau = (M_\tau, \Phi, \Psi, \{(\eta_i', a_i, b_i, c_i, d_i) : i \in I_\tau\})/\sim. \tag{4.1}$$

See Figure 5 (right).

Theorem 4.1. *In the setting immediately above, let* $L_1, \ldots, L_m$ *be simple loops in the interior of* M_τ *and let* O_k *be the bounded connected component of* $\mathbb{C}\backslash L_k$. *Suppose* $(M_\tau, \Phi, L_1, \ldots, L_m)/\sim$ *is measurable with respect to* $\mathcal{M}_\tau$, *and the* O_k *are pairwise disjoint. Then conditioned on* $\mathcal{M}_\tau$, *the IG-decorated LQG surfaces* $(O_1, \Phi, \Psi)/\sim, \ldots, (O_m, \Phi, \Psi)/\sim$ *are conditionally independent.*

Proof of Proposition 3.5. When $|\Lambda| > 0$ the claim is equivalent to Theorem 4.1. When $|\Lambda| = 0$ the claim is vacuously true. $\qquad\square$

Our proof of Theorem 4.1 involves non-probability measures. In the setting of probability theory, independent random variables X and Y have a joint law that factorizes as a product measure $\mu_X \times \mu_Y$. In the non-probability setting, the notion of independence is no longer well defined, but factorization still makes sense. For instance, Lebesgue measure on $\mathbb{R}^2$ factorizes as a product of Lebesgue measures on $\mathbb{R}$. Our argument involves showing that certain measures, densities, and combinatorial structures factorize.

In Section 4.1 we explain the mating-of-trees theorem which identifies space-filling $\mathrm{SLE}_{\kappa'}$-decorated γ-LQG disks with correlated 2D Brownian motion excursions. In Section 4.2 we use the Markov property of Brownian motion to decompose the decorated γ-LQG disk into "cells" which are conditionally independent given their boundary lengths. We also obtain a factorized form for the joint density of the boundary lengths. In Section 4.3 we define decorated planar maps which we call "quilts" whose faces correspond to cells and whose edges are labelled by the corresponding LQG lengths, and we obtain a factorization for the law of the edge lengths for a given planar map. In Section 4.4 we prove that conditioned on the existence of a specified subquilt, the complementary connected subquilts are conditionally independent. The inputs are the factorization from the previous section, a factorization of uniform measures, and a combinatorial factorization of planar maps whose proof is deferred to Section 4.5. This last result is a generalization of Proposition B.1 on the independence of arc diagrams of a uniform meander given its winding function. Finally in Section 4.6 we prove Theorem 4.1.

4.1 Mating-of-trees for the LQG disk

In this section, we review the construction of Brownian excursions in $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_+^2$, then recall in Proposition 4.2 the mating-of-trees theorem which identifies an SLE-decorated LQG disk with a Brownian excursion in $\mathbb{R}_+^2$.

We first recall Brownian motion in the cone. Let $\mathrm{a}^2 = 2/\sin(\frac{\pi\gamma^2}{4})$ be the mating-of-trees variance computed in [ARS21, Theorem 1.3]. Consider Brownian motion (L_t, R_t) with $\mathrm{Var}(L_t) = \mathrm{Var}(R_t) = \mathrm{a}^2 t$ and $\mathrm{Cov}(L_t, R_t) = -\cos(\frac{\pi\gamma^2}{4})\mathrm{a}^2 t$. Let $\mu(t; z)$ be the law of Brownian motion started at z and run for time t, and let $\mu(t; z, w)$ be the disintegration of $\mu(t; z)$ over the endpoint of the Brownian motion, i.e. $\mu(t; z) = \int_{\mathbb{C}} \mu(t; z, w)\, dw$ and the measure $\mu(t; z, w)$ is supported on trajectories ending at w. We emphasize that $|\mu(t; z)| = 1$ (that is, $\mu(t; z)$ is a probability measure) but $|\mu(t; z, w)|$ is typically not 1 (rather, $|\mu(t; z, \cdot)|$ is the density of a two-dimensional Gaussian random variable with mean z and covariance matrix $t\,\mathrm{Id}$). The Markov property of Brownian motion implies that for any $t_1, t_2 > 0$ and $z_1, z_2 \in \mathbb{C}$ we have $\mu(t_1 + t_2; z_1, z_2) = \int_{\mathbb{C}} \mu(t_1; z_1, w)\mu(t_2; w, z_2)\, dw$ where the equality means that when we sample a pair of paths from the right hand side, concatenating them gives a sample from the left hand side.

We now recall the construction of non-probability measures which correspond to Brownian motion started at a vertex of a cone and restricted to staying in the cone. Let $E_{\mathbb{R}_+ \times \mathbb{R}}$ and $E_{\mathbb{R}_+^2}$ be the events that a Brownian motion trajectory stays in $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_+^2$ respectively. [Shi85] allows us to define the Brownian excursion in $\mathbb{R}_+ \times \mathbb{R}$ and Brownian excursion in $\mathbb{R}_+^2$ by

$$\mu_{\mathbb{R}_+ \times \mathbb{R}}(t; 0) = \lim_{\delta \to 0} \delta^{-1}(1_{E_{\mathbb{R}_+ \times \mathbb{R}}} \mu(t; \delta)), \qquad \mu_{\mathbb{R}_+^2}(t; 0) = \lim_{\delta \to 0} \delta^{-\frac{4}{\gamma^2}}(1_{E_{\mathbb{R}_+^2}} \mu(t; \delta e^{i\pi/4})).$$

Here the limits are in the sense of weak convergence on $C([0, t], \mathbb{C})$ equipped with the uniform topology. Indeed, for the first limit, the existence of $\lim_{\delta \to 0} \delta^{-1}|1_{E_{\mathbb{R}_+ \times \mathbb{R}}} \mu(t; \delta)|$ is stated in [Shi85, (4.1)], and

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the conditional law of $\mu(t;\delta)$ given $E_{\mathbb{R}_+\times\mathbb{R}}$ has a limit as $\delta\to 0$ by [Shi85, Theorem 2]. The second limit is likewise justified. Define by disintegration the measures $\mu_{\mathbb{R}_+\times\mathbb{R}}(t;0,z)$ and $\mu_{\mathbb{R}_+^2}(t;0,z)$ on the space of paths from 0 to z. [Shi85, Theorem 2] gives the following Markov properties:

$$\mu_{\mathbb{R}_+\times\mathbb{R}}(t_1+t_2;0,z)=1_{E_{\mathbb{R}_+\times\mathbb{R}}}\int_{\mathbb{R}_+\times\mathbb{R}}\mu_{\mathbb{R}_+\times\mathbb{R}}(t_1;0,w)\mu(t_2;w,z)\,dw,$$
$$\mu_{\mathbb{R}_+^2}(t_1+t_2;0,z)=1_{E_{\mathbb{R}_+^2}}\int_{\mathbb{R}_+^2}\mu_{\mathbb{R}_+^2}(t_1;0,w)\mu(t_2;w,z)\,dw. \qquad (4.2)$$

Here, on the right hand sides, the restriction to $E_{\mathbb{R}_+\times\mathbb{R}}$ or $E_{\mathbb{R}_+^2}$ applies to the path from 0 to z obtained by *concatenating* the pair of paths. Note that the non-probability measures in these identities induce weightings. For instance, if we sample (L_t,R_t) from $\mu_{\mathbb{R}_+^2}(t_1+t_2;0,z)$, then the law of $(L_t,R_t)|_{[0,t_1]}$ is $\int_{\mathbb{R}_+^2}\mu_{\mathbb{R}_+^2}(t_1;0,w)\,dw$ weighted by $\mu(t_2;L_{t_1}+\mathrm{i}R_{t_1},z)[E_{\mathbb{R}_+^2}]$.

Finally, we define the Brownian excursion in $\mathbb{R}_+^2$ from i to 0 via

$$\mu_{\mathbb{R}_+^2}(t;\mathrm{i},0):=1_{E_{\mathbb{R}_+^2}}\int_{\mathbb{R}_+^2}\mu_{\mathbb{R}_+\times\mathbb{R}}\left(\frac{t}{2};\mathrm{i},z\right)\times\mu_{\mathbb{R}_+^2}\left(\frac{t}{2};z,0\right)dz,$$

where we define $\mu_{\mathbb{R}_+\times\mathbb{R}}(t/2;\mathrm{i},z)$ from $\mu_{\mathbb{R}_+\times\mathbb{R}}(t/2;0,z-\mathrm{i})$ by translation, and define $\mu_{\mathbb{R}_+^2}(t/2;z,0)$ from $\mu_{\mathbb{R}_+^2}(t/2;0,z)$ by time-reversal. By its definition, the measure $\mu_{\mathbb{R}_+^2}(t;\mathrm{i},0)$ inherits an analogous Markov property from (4.2).

We can now state the mating-of-trees theorem for the LQG disk, see Figure 8 (left). Recall that for a finite measure M the probability measure $M^{\#}$ is defined to be $M/|M|$. The following mating-of-trees theorem was first shown by [DMS21] for the restricted range $\gamma\in(\sqrt{2},2)$, then proved in full by [AG21].

Proposition 4.2 ([AG21, Theorem 1.1]). *Let $\gamma\in(0,2)$ and $\kappa'=\frac{16}{\gamma^2}$. Let $(\mathbb{D},\Phi,-\mathrm{i})$ be an embedding of a γ-LQG disk conditioned to have unit boundary length, and let η' be an independent counterclockwise space-filling $\mathrm{SLE}_{\kappa'}$ in $(\mathbb{D},-\mathrm{i})$. Parametrize η' by μ_Φ-area, and let L_t (resp. R_t) be the LQG length of the counterclockwise (resp. clockwise) boundary arc of $\mathbb{D}\backslash\eta'([0,t])$ from $-\mathrm{i}$ to $\eta'(t)$. Then the law of the process $t\mapsto(L_t,R_t)$ is $(\int_0^\infty\mu_{\mathbb{R}_+^2}(t;\mathrm{i},0)\,dt)^{\#}$. See Figure 8 (left).*

4.2 Decomposition into cells and factorization

In the setting of Theorem 4.1, for $\gamma\in(0,\sqrt{2}]$ let $\mathcal{C}_i:=(D_i,\Phi,\eta_i',a_i,b_i,c_i,d_i)/\sim$ be the ith (simply connected) curve-decorated LQG surface explored by the curve; a similar definition is given below for $\gamma\in(\sqrt{2},2)$. The goal of this section is to prove that the side lengths of $\mathcal{C}_0,\ldots,\mathcal{C}_{N+1}$ have a joint probability density function that factors, and given these side lengths the $\mathcal{C}_i$ are conditionally independent (Proposition 4.3). In brief, this is a consequence of Proposition 4.2 and the Markov property of Brownian motion.

For $\gamma\in(\sqrt{2},2)$ the definition of $\mathcal{C}_i$ is more complicated for topological reasons: the curve η_i' fills out a countable chain of simply-connected domains, each parametrizing a curve-decorated LQG surface. See Figure 4 (right). Let $\mathcal{C}_i$ be the decorated generalized LQG surface given by the chain of curve-decorated LQG surfaces, further decorated by boundary points a_i,b_i,c_i,d_i. Note that $(D_i,\Phi,\eta_i',a_i,b_i,c_i,d_i)/\sim$ is obtained from $\mathcal{C}_i$ by identifying pairs of boundary points of $\mathcal{C}_i$.

The decorated LQG surfaces $\mathcal{C}_1,\ldots,\mathcal{C}_N$ each have four marked points, $\mathcal{C}_0$ has three marked points (since $a_0=b_0$), and $\mathcal{C}_{N+1}$ has two marked points (since $b_{N+1}=c_{N+1}=d_{N+1}$). Given $\mathcal{C}_0,\ldots,\mathcal{C}_{N+1}$,
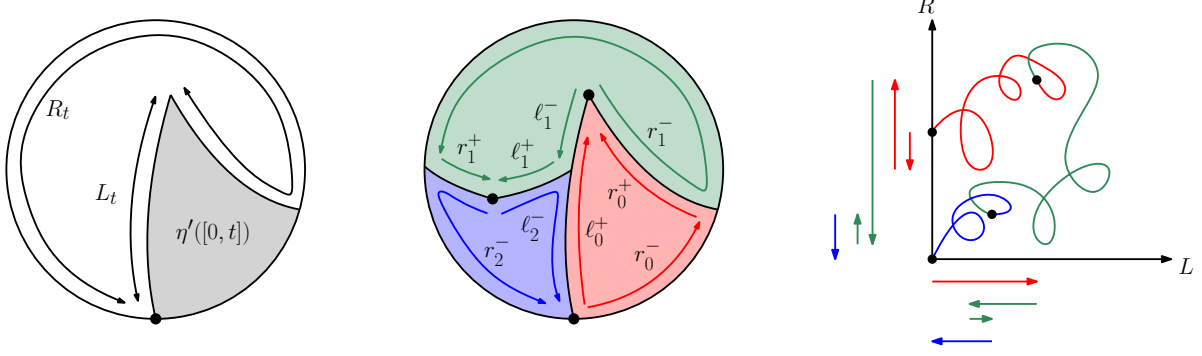
Figure 8: **Left:** A γ-LQG disk decorated by an independent counter-clockwise space-filling $SLE_{\kappa'}$ curve η' where $\kappa' = \frac{16}{\gamma^2}$. Parametrizing η' by LQG area, L_t and R_t are the left and right LQG boundary lengths of the region not yet explored by η' at time t. **Middle:** A Poisson point process with intensity given by the LQG area measure. Here it has two points, cutting η' into three segments, which in order of traversal are red, green and blue. We mark out some LQG lengths for each region. Note that when $\gamma \in (\sqrt{2}, 2)$ these segments are not simply connected, see Figure 4 (right). **Right:** The two points cut the Brownian motion into three paths. The lengths (ℓ_0^+, r_0^-, r_0^+), $(\ell_1^-, \ell_1^+, r_1^-, r_1^+)$ and (ℓ_2^-, r_2^-) correspond to the lengths marked by arrows.

we can recover $(\mathbb{D}, \Phi, -i, \eta', \Lambda_\varepsilon)/\sim$ by conformal welding. See for instance the discussion on conformal welding in [DMS21, Section 3.5].

Let (ℓ_0^+, r_0^+, r_0^-) be the boundary arc lengths of $\mathcal{C}_0$ ordered clockwise from a_0, for $1 \leq i \leq N$ let $(\ell_i^-, \ell_i^+, r_i^+, r_i^-)$ be the boundary arc lengths of $\mathcal{C}_i$ ordered clockwise from a_i, and let $(\ell_{N+1}^-, r_{N+1}^-)$ be the boundary arc lengths of $\mathcal{C}_{N+1}$ ordered clockwise from a_{N+1}. See Figure 8 (middle). These lengths can be expressed explicitly in terms of the corresponding Brownian motion increments, see (4.9) below.

Since the Brownian motion remains in the cone, restricting to $\{N = n\}$, we have the constraints

$$\ell_0^+ + \sum_{j=1}^{i-1}(\ell_j^+ - \ell_j^-) - \ell_i^- > 0 \quad \text{for } 1 \leq i \leq n, \qquad 1 + \sum_{j=0}^{i-1}(r_j^+ - r_j^-) - r_i^- > 0 \quad \text{for } 0 \leq i \leq n, \quad (4.3)$$

$$\ell_{n+1}^- = \ell_0^+ + \sum_{i=1}^{n}(\ell_i^+ - \ell_i^-), \qquad r_{n+1}^- = 1 + \sum_{i=0}^{n}(r_i^+ - r_i^-). \qquad (4.4)$$

Let $S_n \subset \mathbb{R}_+^3 \times (\mathbb{R}_+^4)^n$ be the set of lengths $((\ell_0^+, r_0^-, r_0^+), (\ell_i^-, \ell_i^+, r_i^-, r_i^+)_{i \leq n})$ satisfying (4.3). Let Leb_{S_n} be Lebesgue measure on S_n. The goal of this section is to prove the following factorized description of the joint law of these lengths, and the conditional independence of the $\mathcal{C}_i$ given these lengths.

Proposition 4.3. *There exists a constant $Z_\varepsilon > 0$ and functions $P_{\text{init}}^\varepsilon, P^\varepsilon, P_{\text{end}}^\varepsilon$ such that the law of $((\ell_0^+, r_0^-, r_0^+), (\ell_i^-, \ell_i^+, r_i^-, r_i^+)_{1 \leq i \leq N})$ is $\mathcal{L}^\varepsilon := \sum_{n \geq 0} \mathcal{L}_n^\varepsilon$, where, defining ℓ_{n+1}^- and r_{n+1}^- via (4.4),*

$$\mathcal{L}_n^\varepsilon := Z_\varepsilon^{-1} P_{\text{init}}^\varepsilon(\ell_0^+, r_0^-, r_0^+) \times \prod_{i=1}^{n} P^\varepsilon(\ell_i^-, \ell_i^+, r_i^-, r_i^+) \times P_{\text{end}}^\varepsilon(\ell_{n+1}^-, r_{n+1}^-) \, \text{Leb}_{S_n}. \qquad (4.5)$$

Conditioned on $\{N = n\}$ and on these $4n+3$ lengths, the LQG surfaces $\mathcal{C}_0, \ldots, \mathcal{C}_{n+1}$ are conditionally independent, and the conditional law of $\mathcal{C}_i$ depends only on $(\ell_i^-, \ell_i^+, r_i^-, r_i^+)$ (where we write $\ell_0^- = \ell_{n+1}^+ = r_{n+1}^+ = 0$).

In our proof, we will obtain explicit descriptions of $P_{\text{init}}^{\varepsilon}, P^{\varepsilon}$ and $P_{\text{end}}^{\varepsilon}$ in terms of Brownian motion, but these are not important for later arguments. The rest of this section is devoted to the proof of Proposition 4.3.

First, the Markov property of Brownian motion gives the following. Let $n \geq 0$ and $t_0, \dots, t_{n+1} > 0$, then

$$\mu_{\mathbb{R}_+^2}\left(\sum_{i=0}^{n+1} t_i; \dot{\mathbf{i}}, 0\right) = 1_{E_{\mathbb{R}_+^2}} \int_{\mathbb{C}^{n+1}} \mu_{\mathbb{R}_+ \times \mathbb{R}}(t_0; \dot{\mathbf{i}}, z_1)\left(\prod_{i=1}^{n} \mu(t_i; z_i, z_{i+1})\right) \mu_{\mathbb{R}_+^2}(t_{n+1}; z_{n+1}, 0) \prod_{i=1}^{n+1} dz_i.$$

Notice that the measures $\mu(t_i; z_i, z_{i+1})$ and $\mu_{\mathbb{R}_+ \times \mathbb{R}}(t_0; \dot{\mathbf{i}}, z_1)$ are not supported on the space of paths in $\mathbb{R}_+^2$, but the indicator $1_{E_{\mathbb{R}_+^2}}$ enforces that we only consider concatenated paths that stay in $\mathbb{R}_+^2$. We now rephrase in terms of the increments of the Brownian motion. Setting $w_0 = z_1 - \dot{\mathbf{i}}$ and $w_i = z_{i+1} - z_i$ for $1 \leq i \leq n$,

$$\mu_{\mathbb{R}_+^2}\left(\sum_{i=0}^{n+1} t_i; \dot{\mathbf{i}}, 0\right) = 1_{E_{\mathbb{R}_+^2}} \int_{\mathbb{C}} \mu_{\mathbb{R}_+ \times \mathbb{R}}(t_0; 0, w_0)\, dw_0 \prod_{i=1}^{n}\left(\int_{\mathbb{C}} \mu(t_i; 0, w_i)\, dw_i\right) \mu_{\mathbb{R}_+^2}\left(t_{n+1}; \sum_{j=0}^{n} w_j + \dot{\mathbf{i}}, 0\right). \tag{4.6}$$

That is, if we translate and concatenate the $(n+2)$ paths of a sample from the right hand side, we get a single path from $\dot{\mathbf{i}}$ to 0 whose law is $\mu_{\mathbb{R}_+^2}(\sum_{i=0}^{n+1} t_i; \dot{\mathbf{i}}, 0)$.

Equation (4.6) explains how to decompose a Brownian excursion of duration $t = \sum_{i=0}^{n+1} t_i$ into $n+2$ trajectories of durations $t_0, \dots, t_{n+1}$. Next, we will choose t randomly, then partition $[0, t]$ using a Poisson point process. Let $\text{Exp}_\lambda(dx) = 1_{x>0}\lambda e^{-\lambda x}$ be the law of the exponential random variable with intensity λ.

Lemma 4.4. *Let $\varepsilon > 0$. Sample $t \sim \text{Leb}_{\mathbb{R}_+}$, let Λ be a Poisson point process on $[0, t]$ with intensity $\varepsilon^{-1}\text{Leb}_{[0,t]}$, and let $t_0, \dots, t_m$ be the successive lengths of the intervals comprising $[0, t]\backslash\Lambda$. Then the law of $(t_0, \dots, t_m)$ is $\varepsilon \sum_{m \geq 0} \prod_{i=0}^{m} \text{Exp}_{1/\varepsilon}(dt_i)$.*

Proof. Let P denote the law of a Poisson point process on $\mathbb{R}_+$ with intensity $\varepsilon^{-1}\text{Leb}_{\mathbb{R}_+}$. Palm's theorem for Poisson point processes [Kal17, Page 5] implies the following two procedures give the same law of pairs (Λ, t) where $\Lambda \subset \mathbb{R}_+$ is countable and $t \in \Lambda$:

- Sample $(t, \Lambda_0) \sim (\varepsilon^{-1}\text{Leb}_{\mathbb{R}_+}) \times P$, and let $\Lambda = \Lambda_0 \cup \{t\}$.

- Sample $(M, \Lambda) \sim \text{Count}_{\mathbb{Z}_+} \times P$, and let t be the Mth smallest element of Λ. Here, $\text{Count}_{\mathbb{Z}_+}$ denotes the counting measure on $\mathbb{Z}_+ = \{1, 2, \dots\}$.

Note that if $\Lambda \sim P$, the law of the lengths of successive intervals comprising $\mathbb{R}_+\backslash\Lambda$ is $(\text{Exp}_{1/\varepsilon})^{\mathbb{Z}_+}$. Thus, multiplying both laws by ε and looking at the laws of the intervals of $[0, t]\backslash\Lambda$ gives the result. $\square$

Write $\mu^\varepsilon(z, w) := \int_0^\infty \mu(t; z, w)\text{Exp}_{1/\varepsilon}(dt)$ and similarly define $\mu_{\mathbb{R}_+^2}^\varepsilon$ and $\mu_{\mathbb{R}_+ \times \mathbb{R}}^\varepsilon$. Using Lemma 4.4 then (4.6),

$$\int_0^\infty \mu_{\mathbb{R}_+^2}(t; \dot{\mathbf{i}}, 0)\, dt = \varepsilon \sum_{m \geq 0} \int_{\mathbb{R}_+^{m+1}} \mu_{\mathbb{R}_+^2}\left(\sum_{i=0}^{m} t_i; \dot{\mathbf{i}}, 0\right) \prod_{i=0}^{m} \text{Exp}_{1/\varepsilon}(dt_i) \tag{4.7}$$

$$= \varepsilon\mu_{\mathbb{R}_+^2}^\varepsilon(\dot{\mathbf{i}}, 0) + \varepsilon \sum_{n \geq 0} 1_{E_{\mathbb{R}_+^2}} \int_{\mathbb{C}} \mu_{\mathbb{R}_+ \times \mathbb{R}}^\varepsilon(0, w_0)\, dw_0 \prod_{i=1}^{n}\left(\int_{\mathbb{C}} \mu^\varepsilon(0, w_i)\, dw_i\right) \mu_{\mathbb{R}_+^2}^\varepsilon\left(\sum_{i=0}^{n} w_j + \dot{\mathbf{i}}, 0\right).$$

Equation (4.7) says that if we take the total time duration t to be "sampled" from Lebesgue measure on $\mathbb{R}_+$, then the Brownian excursion measure $\mu_{\mathbb{R}_+^2}(t; \mathbbm{i}, 0)$ can be written as a countable sum of measures indexed by the total number of points $m \geq 0$ in the Poisson point process. For each of these measures for $m \geq 1$ ($n = m - 1 \geq 0$), the Brownian excursion can be decomposed as the union of $n + 2$ sub-paths. We can sample these $n + 2$ sub-paths as Brownian bridges with exponential durations, restricted to the event that they stay in the cone.

Proposition 4.5. *In the setting of Theorem 4.1, let L_t (resp. R_t) be the LQG length of the counterclockwise (resp. clockwise) boundary arc of $\mathbb{D} \backslash \eta'([0, t])$ from $-\mathbbm{i}$ to $\eta'(t)$. Writing $T_i = \sum_{j < i} t_j$, for $0 \leq i \leq N$ define $(L_t^i, R_t^i) = (L_{T_i + t} - L_{T_i}, R_{T_i + t} - R_{T_i})$ for $t \in [0, t_i]$, and define $(L_t^{N+1}, R_t^{N+1}) = (L_{T_{N+1} + t} - L_{T_{N+1} + t_{N+1}}, R_{T_{N+1} + t} - R_{T_{N+1} + t_{N+1}})$ for $t \in [0, t_{N+1}]$. Then the joint law of $(L_t^0, R_t^0), \ldots, (L_t^{N+1}, R_t^{N+1})$ equals, for some constant Z_ε,*

$$Z_\varepsilon^{-1} \sum_{n \geq 0} 1_{E_{\mathbb{R}_+^2}} \left(\int_{\mathbb{C}} \mu_{\mathbb{R}_+ \times \mathbb{R}}^\varepsilon (0, w_0) \, dw_0 \right) \prod_{i=1}^n \left(\int_{\mathbb{C}} \mu^\varepsilon(0, w_i) dw_i \right) \mu_{\mathbb{R}_+^2}^\varepsilon \left(\sum_{j=0}^n w_j + \mathbbm{i}, 0 \right). \tag{4.8}$$

Proof. We use Proposition 4.2. Let $t = \mu_h(\mathbb{D})$. Since η' is parametrized by LQG area, it is a measure-preserving transform from $([0, t], \mathrm{Leb}_{[0,t]})$ to $(\mathbb{D}, \mu_h)$, so if $\Lambda \subset \mathbb{D}$ is a Poisson point process with intensity $\varepsilon^{-1} \mu_h$, then $\eta^{-1}(\Lambda)$ is a Poisson point process on $[0, t]$ with intensity $\varepsilon^{-1} \mathrm{Leb}_{[0,t]}$. Restricting to the event that there are at least two cells corresponds to discarding the term $\varepsilon \mu_{\mathbb{R}_+^2}^\varepsilon(\mathbbm{i}, 0)$ in (4.7). This gives the result with Z_ε the normalization constant making (4.8) a probability measure. $\square$

We now define the functions $P_{\mathrm{init}}^\varepsilon, P^\varepsilon, P_{\mathrm{end}}^\varepsilon$, then Proposition 4.3 is essentially an immediate consequence of Proposition 4.5. For a Brownian excursion (L_t, R_t) sampled from $\int_{\mathbb{C}} \mu^\varepsilon(0, w) \, dw$ whose (random) duration we call T, let P^ε be the density function for $(- \inf_{t \in [0,T]} L_t, L_T - \inf_{t \in [0,T]} L_t, - \inf_{t \in [0,T]} R_t, R_T - \inf_{t \in [0,T]} R_t)$. For (L_t, R_t) sampled from $\int_{\mathbb{R}_+ \times \mathbb{R}} \mu_{\mathbb{R}_+ \times \mathbb{R}}^\varepsilon(0, w) \, dw$ with random duration T, let $P_{\mathrm{init}}^\varepsilon$ be the density function for $(L_T, - \inf_{[0,t]} R_t, R_T - \inf_{[0,t]} R_t)$. Let $P_{\mathrm{end}}^\varepsilon(\ell^-, r^-) = |\mu_{\mathbb{R}_+^2}^\varepsilon(\ell^- + \mathbbm{i} r^-, 0)|$ for $\ell^-, r^- > 0$.

Proof of Proposition 4.3. Recall the processes $(L_t^0, R_t^0), \ldots, (L_t^{N+1}, R_t^{N+1})$ defined in Proposition 4.5. Restrict to the event $\{N = n\}$. We can describe the lengths $\ell_i^\pm, r_i^\pm$ in terms of these processes:

$$\ell_0^+ = L_{t_0}^0, \quad r_0^- = - \inf_{t \leq t_0} R_t^0, \quad r_0^+ = R_{t_0}^0 + r_0^-, \quad \ell_{n+1}^- = L_0^{n+1}, \quad r_{n+1}^- = R_0^{n+1},$$
$$\ell_i^- = - \inf_{t \leq t_i} L_t^i, \quad \ell_i^+ = L_{t_i}^i + \ell_i^-, \quad r_i^- = - \inf_{t \leq t_i} R_t^i, \quad r_i^+ = R_{t_i}^i + r_i^- \quad \text{for } i = 1, \ldots, n. \tag{4.9}$$

Now, since the Brownian motion event $E_{\mathbb{R}_+^2}$ corresponds to the lengths lying in S_n, by Proposition 4.5 the law of the lengths $(\ell_0^+, r_0^-, r_0^+), (\ell_i^-, \ell_i^+, r_i^-, r_i^+)_{i \leq n}$ is (4.5).

Next, we study the conditional law of the $\mathcal{C}_i$ given the lengths. Let $\ell^\pm, r^\pm > 0$. Let $\mathcal{P}_{\mathrm{init}}^{\ell^+, r^-, r^+}$ be the law of $(L_t, R_t)_{[0,T]}$ sampled from $\int_{\mathbb{C}} \mu_{\mathbb{R}_+ \times \mathbb{R}}^\varepsilon(0, w) \, dw$ and conditioned on $\{\ell^+ = L_T, r^- = - \inf_{t \leq T} R_t, r^+ = R_T + r^-\}$. Let $\mathcal{P}^{\ell^-, \ell^+, r^-, r^+}$ be the law of $(L_t, R_t)_{[0,T]}$ sampled from $\int_{\mathbb{C}} \mu^\varepsilon(0, w) \, dw$ and conditioned on $\{\ell^- = - \inf_{t \leq T} L_t, \ell^+ = L_T + \ell^-, r^- = - \inf_{t \leq T} R_t, r^+ = R_T + r^-\}$. Let $\mathcal{P}_{\mathrm{end}}^{\ell^-, r^-}$ be the probability measure proportional to $\mu_{\mathbb{R}_+^2}^\varepsilon(\ell^- + \mathbbm{i} r^-, 0)$.

By Proposition 4.5, conditioned on $N = n$ and on the lengths $(\ell_0^+, r_0^-, r_0^+), (\ell_i^-, \ell_i^+, r_i^-, r_i^+)_{1 \leq i \leq n}, (\ell_{n+1}^-, r_{n+1}^-)$, the processes $(L_t^0, R_t^0), \ldots, (L_t^{n+1}, R_t^{n+1})$ have the conditional law $\mathcal{P}_{\mathrm{init}}^{\ell_0^+, r_0^-, r_0^+} \times \prod_{i=1}^n \mathcal{P}^{\ell_n^-, \ell_n^+, r_n^-, r_n^+} \times \mathcal{P}_{\mathrm{end}}^{\ell_{n+1}^-, r_{n+1}^-}$. By, e.g., [AG21, Lemma 2.17], $\mathcal{C}_i$ is measurable with respect to (L_t^i, R_t^i) for each $i = 0, \ldots, n+1$. This completes the proof. $\square$

4.3 The planar map structure of the discretized LQG disk

In this section, we introduce *templates* and *quilts* to describe the decorated planar map structure of the discretized LQG disk, see Figure 9. The main result of this section is a factorization result for quilts (Lemma 4.9), which arises from our factorization for sequences of cells (Proposition 4.3).

Consider planar maps with the sphere topology, with a possibly empty subset of faces marked as holes; in what follows we will only use the term "face" to refer to non-hole faces. A *template T* is such a planar map such that on each face we mark one or more vertices, one of which is distinguished as the root vertex for the face. Note that if v is a marked vertex for one face, it need not be a marked vertex for other faces. We call a face F of T a *k-gon* if it has k marked vertices, and we call a collection of edges between consecutive marked vertices of F a *side* of F. We denote the edge set of the planar map T by E_T. A *quilt* is obtained from a template by assigning a positive length to each edge of E_T; the set of all quilts with template T can can be identified with $\mathbb{R}_+^{E_T}$.
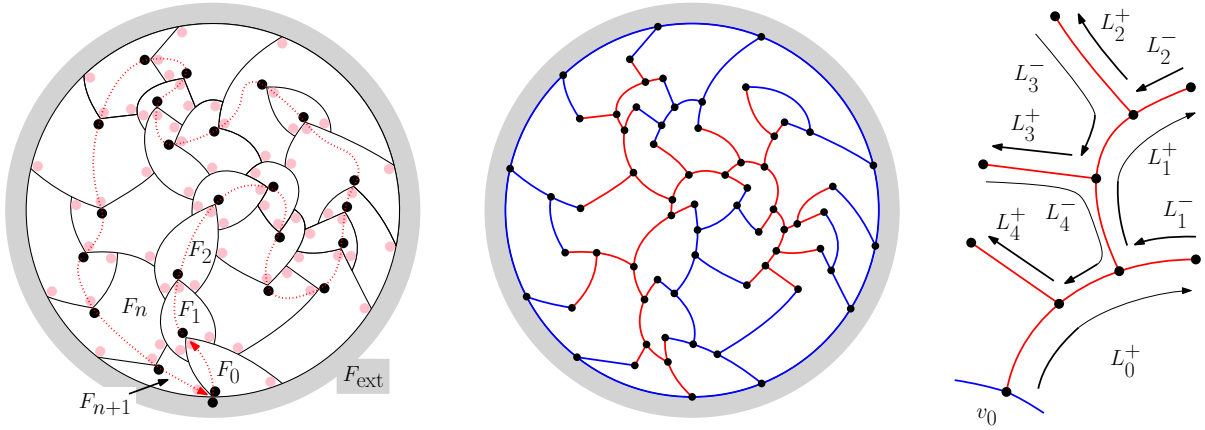

Figure 9: **Left:** Picture of a template with one 1-gon (shaded), one 2-gon, one 3-gon and $n = 21$ 4-gons. Each face has a root vertex (black) and possibly other marked vertices (pink). For $T \in \mathcal{T}$ we can order the polygons (red curve passing through $F_0, F_1, \ldots, F_n, F_{n+1}$). **Middle:** The template is a planar map with vertices shown in black. Its edge set is called E_T. A quilt can be obtained from the template by assigning a positive length to each edge. If $T \in \mathcal{T}$ we can color the left (resp. right) boundary arcs of each face red (resp. blue) to get a tree and a cycle-rooted forest. **Right:** Figure for Lemma 4.9. Order E_T^ℓ (red edges) by the first time an edge is traversed in a counter-clockwise contour around the tree. Each black arrow corresponds to a left side of a face of T. Each arrow traverses one or more edges, the earliest of which is bolded. The key to proving Lemma 4.9 is the observation that each edge is bolded in exactly one arrow.

In the setting of Theorem 4.1, the curve-decorated LQG disk is split into decorated LQG surfaces $\mathcal{C}_0, \ldots, \mathcal{C}_{N+1}$ (defined in Section 4.2) parametrized by the regions D_i traced out by the space-filling SLE between times when it hits the Poisson point process. We introduce templates and quilts to describe the planar map underlying this discretization, see Figure 8 (middle). We will construct a template with faces $F_0, \ldots, F_{n+1}$ corresponding to the LQG surfaces $\mathcal{C}_0, \ldots, \mathcal{C}_{n+1}$ respectively, and a final exterior face F_{ext}. The marked points of $\mathcal{C}_i$ correspond to the marked vertices of F_i, so the faces are a 2-gon, a 3-gon, and n 4-gons. For a 3-gon, we call the vertex clockwise of the root vertex the terminal vertex, and for a 2-gon or 4-gon we call the vertex opposite the root vertex the terminal vertex. The root and terminal vertices of F_i correspond to the starting and ending points of the SLE curve in $\mathcal{C}_i$.

When $\gamma \in (0, \sqrt{2}]$, this quilt is obtained as follows. Define the faces $(F_{\text{ext}}, F_0, \ldots, F_{N+1}) = (\hat{\mathbb{C}} \backslash \mathbb{D}, D_0, \ldots, D_{N+1})$ and the vertex set $\bigcup_{i=0}^{N+1} \{a_i, b_i, c_i, d_i\}$, where the points $a_i, b_i, c_i, d_i \in \partial D_i$ are as in Theorem 4.1. The edges are the connected components of $(\partial \mathbb{D} \cup \bigcup_{i=0}^{N+1} \partial D_i) \backslash (\bigcup_{i=0}^{N+1} \{a_i, b_i, c_i, d_i\})$, and each is labeled with its LQG length. Each F_i is given the marked vertices $a_i, b_i, c_i, d_i \in \partial D_i$ as defined in Theorem 4.1, the root vertex of F_i is a_i, and F_{ext} is a 1-gon with root vertex a_0. See Figure 9.

We now give an equivalent iterative definition which applies for all $\gamma \in (0, 2)$. Let $a_0 = b_0 = -\mathrm{i}$, $c_0 = 1$ and $d_0 = 0$. Draw radii from d_0 to b_0 and c_0 to cut $\mathbb{D}$ into a quarter-circle F_0 and a three-quarter-circle U_1. Consider the quilt with three faces: the 1-gon $F_{\text{ext}} = \hat{\mathbb{C}} \backslash \mathbb{D}$ with root vertex a_0, the 3-gon F_0 with root vertex a_0 and remaining marked vertices c_0, d_0, and the 2-gon U_1 with root vertex d_0 and other marked vertex $-\mathrm{i}$. Assign lengths to the edges such that the side lengths of ∂F_0 correspond to the side lengths of $\mathcal{C}_0$, and the sum of lengths along $\partial \mathbb{D}$ is 1. For $j = 1, \ldots, N$, we iteratively do the following. Let $\ell_j^-, \ell_j^+, r_j^-, r_j^+$ be the boundary lengths of $\mathcal{C}_j$ as defined in Section 4.2. Let $a_j = d_{j-1}$. Add a vertex b_j on the clockwise arc of ∂U_j from a_j to $-\mathrm{i}$ such that, if the previous vertex on this arc is v and the next vertex is v', the sum of lengths from a_j to v is less than ℓ_j^- and the sum of lengths from a_j to v' is greater than ℓ_j^-. Split the edge between v and v' at b_j, with lengths assigned such that the sum of lengths from a_j to b_j is ℓ_j^-. Likewise add a vertex c_j on the counterclockwise arc from a_j to $-\mathrm{i}$ such that the sum of lengths from a_i to c_i is r_j^-. Finally, mark a bulk point $d_j \in U_j$, add edges from d_j to b_j, c_j, and give them lengths ℓ_j^+ and r_j^+ respectively. This splits U_j into two faces; let U_{j+1} be the face having $-\mathrm{i}$ on its boundary and F_j the other face. Let F_j have root a_j and remaining marked vertices b_j, c_j, d_j, and let U_{j+1} have root vertex d_j and other marked vertex $-\mathrm{i}$. After carrying this procedure out for $j = 1, \ldots, N$, we are left with a quilt with faces $F_{\text{ext}}, F_0, \ldots, F_N, U_{N+1}$; rename U_{N+1} to F_{N+1} to complete the definition.

Remark 4.6 (Relationship with mated-CRT map)**.** An equivalent way to define the random quilt arising in the setting of Theorem 4.1 is as follows. Our setting is closely related to that of the mated-CRT map (see [GHS20]), except we discretize via a Poisson point process rather than at equal intervals (see Remark 3.4). We can construct a variant of the mated-CRT map from the discretization in exactly the same way; in this map, each D_i corresponds to a vertex. Take the dual of this mated-CRT map variant, and for each edge between a pair of consecutive faces (i.e. faces traced successively by the space-filling curve), split it by adding a degree 2 vertex. Marked vertices and lengths can be assigned to obtain the quilt.

We now describe the set of templates $\mathcal{T}$ that can arise from the discretized LQG disk by the above correspondence. See Figure 9.

Definition 4.7. Let $\mathcal{T}$ be the collection of templates T with the following desciption: T has no holes and the face set of T can be called $F_{\text{ext}}, F_0, F_1, \ldots, F_n, F_{n+1}$ for some $n \geq 0$. The face F_{ext} is a 1-gon, F_0 is a 3-gon, the faces $F_1, \ldots, F_n$ are 4-gons, and F_{n+1} is a 2-gon. Moreover,

(a) The root vertices of F_{ext} and F_0 agree, and the side of F_0 counterclockwise of the root vertex lies on ∂F_{ext}.

(b) For $i = 0, 1, \ldots, n$ the terminal vertex of F_i is the root vertex of F_{i+1}, and the terminal vertex of F_{n+1} is the root vertex of F_0. (This corresponds to the endpoint of the SLE curve in $\mathcal{C}_i$ agreeing with the starting point of the curve in $\mathcal{C}_{i+1}$.)

(c) For $i = 1, \ldots, n+1$ the root vertex of F_i has degree 2, and all other marked vertices of F_i have degree 3. (This corresponds to the non-consecutive $\mathcal{C}_i$ intersecting generically almost surely, see e.g. [GMS19, Figure 1].)

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(d) For $i = 1, \ldots, n$, the two sides of F_i adjacent to its root vertex both lie on $\partial(F_{\text{ext}} \cup \bigcup_{j=0}^{i-1} F_j)$, while the other two sides of F_i only intersect $\partial(F_{\text{ext}} \cup \bigcup_{j=0}^{i-1} F_j)$ at endpoints. (This corresponds to the definition of the marked points of $\mathcal{C}_i$.)

For $T \in \mathcal{T}$ let $\mathcal{Q}_T$ be the set of all quilts with template T.

By the correspondences described in Definition 4.7 between (b)-(d) and the properties of $(\mathcal{C}_0, \ldots, \mathcal{C}_{N+1})$, there is a.s. a unique template $T \in \mathcal{T}$ describing the discretized LQG disk via the correspondence stated right above Definition 4.7. Thus, there is a.s. a unique quilt $Q \in \mathcal{Q}_T$ such that the side lengths of F_i agree with those of $\mathcal{C}_i$ for $0 \leq i \leq N + 1$. Let $\mathcal{L}^\varepsilon$ denote the law of the random quilt Q arising from the discretization of the LQG disk, so $\mathcal{L}^\varepsilon$ is a probability measure on $\bigcup_{T \in \mathcal{T}} \mathcal{Q}_T$. (This is a slight abuse of notation since Proposition 4.3 defines $\mathcal{L}^\varepsilon$ to be the law of the side lengths of $\mathcal{C}_0, \ldots, \mathcal{C}_{N+1}$, but these lengths encode the same information as Q.)

For each $T \in \mathcal{T}$ we denote the restriction of $\mathcal{L}^\varepsilon$ to $\mathcal{Q}_T$ by $\mathcal{L}^\varepsilon|_{\mathcal{Q}_T}$; this is a finite non-probability measure. We will also need the following notation.

Definition 4.8. Let Unif_T be the infinite measure on $\mathcal{Q}_T$ given by Lebesgue measure on $\mathbb{R}_+^{E_T}$.

We give a factorization for quilts with template T.

Lemma 4.9. *Fix a template $T \in \mathcal{T}$ with $n + 3$ faces $F_{\text{ext}}, F_0, \ldots, F_{n+1}$. For a quilt $Q \in \mathcal{Q}_T$, let C_i be the tuple of side lengths of face F_i, and define $w(Q) := Z_\varepsilon^{-1} P_{\text{init}}^\varepsilon(C_0) \times \prod_{i=1}^n P^\varepsilon(C_i) \times P_{\text{end}}^\varepsilon(C_{n+1})$ with $Z_\varepsilon, P_{\text{init}}^\varepsilon, P^\varepsilon, P_{\text{end}}^\varepsilon$ as in (4.5). Then*

$$\mathcal{L}^\varepsilon|_{\mathcal{Q}_T}(dQ) = w(Q)\text{Unif}_T(dQ).$$

Proof. Recall S_n from (4.5) which parametrizes the space of quilts from $\mathcal{L}^\varepsilon$ with $n + 3$ faces via the lengths $(\ell_0^+, r_0^-, r_0^+), (\ell_i^-, \ell_i^+, r_i^-, r_i^+)_{1 \leq i \leq n}$. Let $S_T \subset S_n$ be the set of lengths such that the resulting quilt has template T. Recall that E_T is the edge set of T and that a quilt with template T is identified with its collection of edge lengths $(x_e)_{e \in E_T} \in \mathbb{R}_+^{E_T}$. Consider the affine bijection $f : \mathbb{R}_+^{E_T} \to S_T$ sending $(x_e)_{e \in E_T} \mapsto (\ell_0^+, r_0^-, r_0^+), (\ell_i^-, \ell_i^+, r_i^-, r_i^+)_{i \leq n}$. We will give a combinatorial argument that shows the Jacobian matrix of f has unit determinant, so $f_*^{-1}\text{Leb}_{S_T} = \text{Unif}_T$; combining with (4.5) then gives the desired result.

For $1 \leq i \leq n$, for the 4-gon F_i let L_i^- be the side clockwise of the root vertex and L_i^+ the side clockwise of L_i^-. (Under the correspondence described in Definition 4.7, L_i^- and L_i^+ correspond to the lower left and upper left marked boundary arcs of the ith space-filling SLE segment, respectively.) Let L_0^+ be the side of F_0 clockwise of its root vertex. For $(i, \pm) \in \{1, \ldots, n\} \times \{+, -\}$ and $(i, \pm) = (0, +)$, let $E_i^\pm \subset E_T$ be the set of edges that lie on the side $L_i^\pm$. Let $E_T^\ell = \bigcup_{i, \pm} E_i^\pm$. See Figure 9 (right). By definition $\ell_i^\pm = \sum_{e \in E_i^\pm} x_e$; this gives the affine bijection between $(\ell_0^+, \ell_1^-, \ell_1^+, \ldots, \ell_n^-, \ell_n^+)$ and $(x_e)_{e \in E_T^\ell}$. We note for later use that this bijection implies $|E_T^\ell| = 2n + 1$.

The edges E_T^ℓ form a tree rooted at the root vertex v_0 of F_0. (This tree corresponds to the tree of flow lines forming the left boundaries of the mated-CRT map cells.) Consider the counterclockwise contour around this tree starting at v_0, i.e., the depth-first exploration starting at v_0. This contour traverses each edge of E_T^ℓ twice, once in each of the two possible directions. Furthermore, this contour passes through the faces in increasing order, see Figure 9 (left) and (middle). This follows from property (b) of Definition 4.7. Thus, it traverses the sides $L_0^+, L_1^-, L_1^+, \ldots, L_n^+$ in order (tracing the edges on each $+/-$ side in the direction of increasing/decreasing distance from v_0 in the tree), then traces the left side of F_{n+1} to return to v_0, see Figure 9 (right). We assign an ordering $\prec$ to E_T^ℓ by the time an edge is *first* visited by the contour.

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We claim that for each $e \in E_T^\ell$, there is some $(i, \pm)$ such that $e \in E_i^\pm$ and e is the $\prec$-earliest edge in $E_i^\pm$, see Figure 9 (right). Let j be the index such that $e \in E_j^+$. If e is the $\prec$-earliest edge of E_j^+ (i.e. the edge of L_j^+ closest to v_0) we are done. Otherwise, let v be the endpoint of e closer to v_0. By (c), every internal vertex of the tree has degree 3, and, in our counterclockwise traversal, after tracing L_j^+ we have only visited two of the edges adjacent to v. Let $k > j$ be the index such that the side L_k^+ is traversed starting from v. Then v is the endpoint of L_k^-, so e is the $\prec$-earliest edge of E_k^-. Thus the claim holds. Further, since $|E_T^\ell| = 2n + 1 = |\{L_0^+, L_1^-, \ldots, L_n^+\}|$, each e is the $\prec$-earliest edge of exactly one $L_i^\pm$.

Consequently, if we order E_T^ℓ by $\prec$ and order the sides $L_i^\pm$ by the $\prec$-earliest edge they contain, the zero-one matrix sending $(x_e)_{e \in E_T^\ell}$ to $(\ell_0^+, \ell_1^-, \ell_1^+, \ldots, \ell_n^-, \ell_n^+)$ is upper triangular with ones on the diagonal, and so has determinant 1. A similar argument shows the zero-one matrix sending $(x_e)_{e \in E_T \setminus E_T^\ell}$ to $(r_0^-, r_0^+, r_1^-, r_1^+, \ldots, r_n^-, r_n^+)$ has unit determinant. Thus f has unit determinant as needed. $\square$

4.4 Independence of complementary subquilts

Recall $\mathcal{L}^\varepsilon$ is the probability measure on quilts arising from discretized LQG disks (see Lemma 4.9). The goal of this section is to show Proposition 4.10: informally, a quilt sampled from $\mathcal{L}^\varepsilon$ conditioned on having a specific subquilt has conditionally independent complementary subquilts. The proof uses our earlier factorization for quilts with a specified template (Lemma 4.9), a combinatorial factorization for the set of templates having a fixed subtemplate (Proposition 4.12) whose proof we defer to Section 4.5, and a factorization of Lebesgue measures (Lemma 4.13).

Let T^* be a template in $\mathcal{T}$ with a connected subset of faces marked. The marked faces give rise to another template as follows: say two faces are adjacent if their boundaries share an edge, replace each connected component of the non-marked faces with a hole, then delete every vertex except marked vertices of the faces and vertices adjacent to multiple faces. We call this the marked *subtemplate* of T^*. Each edge of the marked subtemplate corresponds to the union of one or more edges of the quilt (since deleting vertices amounts to merging edges).

Given a quilt $Q \in \bigcup_{T \in \mathcal{T}} \mathcal{Q}_T$, if we mark a connected subset of its faces, the marked *subquilt* is the quilt whose template is the marked subtemplate and whose lengths are inherited from Q. We say that a quilt M is a subquilt of Q if one can mark a subset of faces of Q so that the marked subquilt agrees with M. See Figure 10 (a),(b).

Fix a connected template T_{sub} with $b \geq 1$ holes labelled $1, \ldots, b$, whose faces are all 4-gons except for a 1-gon, 2-gon and 3-gon. Suppose further that T_{sub} can be realized as a marked subtemplate of a template in $\mathcal{T}$, see Figure 10 (a)-(c), so in particular its holes share no common boundary edges. Let $\mathcal{Q}_{T_{\mathrm{sub}}}$ be the set of quilts having template T_{sub} and whose 1-gon has length 1. The goal of this section is the following.

Proposition 4.10. *Recall the probability measure $\mathcal{L}^\varepsilon$ on quilts arising from discretized LQG disks in Lemma 4.9. For fixed T_{sub}, the following holds for all $M \in \mathcal{Q}_{T_{\mathrm{sub}}}$. Sample a quilt Q from $\mathcal{L}^\varepsilon$ conditioned on $\{Q \text{ has } M \text{ as a subquilt}\}$, and let Q_i be the subquilt of Q obtained by marking the faces of Q in the ith hole of M. Then $Q_1, \ldots, Q_b$ are conditionally independent.*

Implicitly, the proposition statement uses the fact that one can make sense of conditioning on the measure zero event of having a specific subquilt; this will be shown in the proof.

Equip $\mathcal{Q}_{T_{\mathrm{sub}}}$ with the metric $d(M^1, M^2) = \sum_e |x_e^1 - x_e^2|$, where the summation is taken over edges of T_{sub} and x_e^i is the length of e in the quilt M^i. Let $B_\delta(M^0)$ denote the radius δ ball with respect to this metric.

Proposition 4.11. *For fixed T_{sub} as above, there is a family of finite measures $\{m_M^\varepsilon\}_{M \in \mathcal{Q}_{T_{\mathrm{sub}}}}$ on $\bigcup_{\mathcal{T}} \mathcal{Q}_T$ such that the following holds.*

- *The measure m_M^ε is supported on the subset of quilts having M as a subquilt. Moreover, for all $M^0 \in \mathcal{Q}_{T_{\mathrm{sub}}}$ and $\delta > 0$ sufficiently small in terms of M^0, letting F_{δ,M^0} be the event that a quilt has a subquilt lying in $B_\delta(M^0)$,*

$$\mathcal{L}^\varepsilon|_{F_{\delta,M^0}} = \int_{B_\delta(M^0)} m_M^\varepsilon \mathrm{Unif}_{T_{\mathrm{sub}}}(dM). \tag{4.10}$$

- *For a sample from $m_M^\varepsilon/|m_M^\varepsilon|$, the b complementary subquilts of M are independent.*

In (4.10) we want δ sufficiently small to avoid the situation that Q has multiple subquilts lying in $B_\delta(M^0)$. If instead $\delta \in (0, \infty]$ is arbitrary, the left hand side of (4.10) needs to be weighted by the number of subquilts lying in $B_\delta(M^0)$.

The desired Proposition 4.10 is an immediate consequence.

Proof of Proposition 4.10. The existence of the conditional law of Q given $\{Q$ has M as a subquilt$\}$ follows from the first claim of Proposition 4.11, and the conditional independence of the Q_i follows from the second claim. $\qquad \square$

We now turn to the proof of Proposition 4.11. The key ingredients are the factorization of quilts of a given template (Lemma 4.9), a template factorization result (Proposition 4.12), and a Lebesgue measure factorization statement (Lemma 4.13).

Let $\mathcal{T}^*$ be the collection of all T^* having T_{sub} as its marked subtemplate. That is, $\mathcal{T}^*$ contains all templates of $\mathcal{T}$ obtained by filling in the holes of T_{sub} and marking the faces corresponding to T_{sub}. Recall the holes of T_{sub} are labelled $1, \ldots, b$. For $T^* \in \mathcal{T}^*$ and $i \leq b$, let T_i^* be the marked template with $b-1$ holes obtained from T^* by replacing the jth connected component of the unmarked faces with a hole for $j \neq i$, see Figure 10 (c),(d). Let $\mathcal{T}_i^*$ be the collection of all possible T_i^* as T^* varies.

Proposition 4.12. *The map $\mathcal{T}^* \to \prod_{i=1}^b \mathcal{T}_i^*$ sending $T^* \mapsto (T_1^*, \ldots, T_b^*)$ is a bijection.*

Proposition 4.12 is an independence statement about templates: no matter how we fill the holes indexed by $\{1, \ldots, b\} \setminus \{i\}$, the choices for how to fill the ith hole are the same, namely $\mathcal{T}_i^*$. See Figure 10 (b),(c). We defer the proof of Proposition 4.12 to Section 4.5.

Recall that Unif_{T^*} is the measure on marked quilts having template T^* given by Lebesgue measure on $\mathbb{R}_+^{E_{T^*}}$. For Q sampled from Unif_{T^*}, let $\widetilde{\mathrm{Unif}}_{T^*}$ be the law of $(M, Q_1, \ldots, Q_b)$ where M and $Q_1, \ldots, Q_b$ are the marked subquilt and complementary quilts of Q. For $T_i^* \in \mathcal{T}_i^*$, let $J_{T_i^*} \subset E_{T_i^*}$ be the edges on the boundaries of the unmarked faces, see Figure 10 (d). For any marked quilt $Q \in \mathcal{Q}_{T^*}$, the subquilt Q_i is specified by the lengths of the edges in $J_{T_i^*}$, so we may identify Q_i with an element of $\mathbb{R}_+^{J_{T_i^*}}$.

We now give a factorization for uniform measures on templates, which follows from the product structure of Lebesgue measure on Euclidean space, and the fact that the holes of T_{sub} do not share any common boundary edges. It is essentially just a linear change of variables.

Lemma 4.13. *There is a collection $\{\mathrm{Unif}_{J_{T_i^*}}^M\}$ of measures on $\mathbb{R}_+^{J_{T_i^*}}$ for $M \in \mathcal{Q}_{T_{\mathrm{sub}}}$, $1 \leq i \leq b$ and $T_i^* \in \mathcal{T}_i^*$ such that*

$$\widetilde{\mathrm{Unif}}_{T^*}(dM, dQ_1, \ldots, dQ_b) = \left(\prod_{i=1}^b \mathrm{Unif}_{J_{T_i^*}}^M(dQ_i)\right) \mathrm{Unif}_{E_{T_{\mathrm{sub}}}}(dM) \qquad \text{for all } T^* \in \mathcal{T}^*,$$

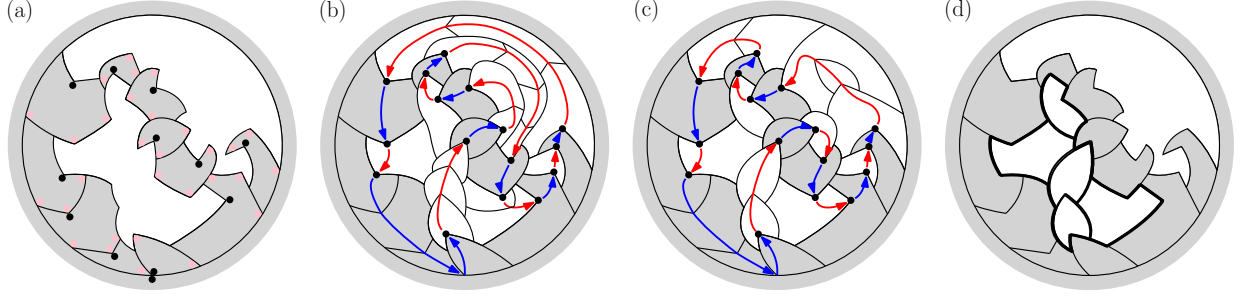*where $T_1^*, \ldots, T_b^*$ are defined via T^* as in Proposition 4.12.*

Figure 10: **(a)** A template T_{sub}. The white regions are holes of T_{sub}. **(b)**, **(c)** Two different $T^* \in \mathcal{T}^*$ (vertices of polygons not shown). The marked gray region correspond to T_{sub}, and the arrows depict the orderings on faces of T^*. Notice that the blue arrows are traversed in a different order in the two pictures. Perhaps surprisingly, regardless of how one hole is filled, the choices for filling the other hole are the same (Proposition 4.12). **(d)** For the marked template T^* depicted in (c), the marked template T_i^* is obtained by replacing all but the ith connected component of non-marked faces (bolded) with holes. In the figure T_i^* is the template with $b-1$ holes (white, not bolded). In Lemma 4.13 $J_{T_i^*} \subset E_{T_i^*}$ is the set of bolded edges, and the subquilt Q_i (bolded) can be identified with a tuple of edge lengths (in $\mathbb{R}_+^{J_{T_i^*}}$).

Proof. Consider the disjoint union $E_{T_{\text{sub}}} = E_0 \cup \bigcup_{i=1}^b E_i$ where E_i is the set of edges lying on the boundary of the ith hole of T_{sub} and E_0 the set of edges not lying on the boundary of any hole. By the identification $\mathcal{Q}_{T_{\text{sub}}} \leftrightarrow \mathbb{R}_+^{E_{T_{\text{sub}}}}$ we can identify M in $\mathcal{Q}_{T_{\text{sub}}}$ with a tuple $(M_0, M_1, \ldots, M_b)$ in $\mathbb{R}_+^{E_0} \times \prod_{i=1}^b \mathbb{R}_+^{E_i}$. Likewise identify a quilt in $\mathcal{Q}_{T^*}$ with a tuple $(M_0, Q_1, \ldots, Q_b)$ in $\mathbb{R}_+^{E_0} \times \prod_{i=1}^b \mathbb{R}_+^{J_{T_i^*}}$. Let Unif_{E_i} (resp. $\text{Unif}_{J_{T_i^*}}$) denote Lebesgue measure on $\mathbb{R}_+^{E_i}$ (resp. $\mathbb{R}_+^{J_{T_i^*}}$), then

$$\text{Unif}_{T^*}(dM_0, dQ_1, \ldots, dQ_b) = \text{Unif}_{E_0}(dM_0) \times \prod_{i=1}^b \text{Unif}_{J_{T_i^*}}(dQ_i). \qquad (4.11)$$

Since T_{sub} is the marked subtemplate of T^*, each edge $e \in E_i$ is a concatenation of one or more edges $e' \in J_{T_i^*}$. Define M_i as a function of Q_i by assigning each edge $e \in E_i$ the sum of lengths of the corresponding $e' \in J_{T_i^*}$. Push forward the measures in (4.11) under the map $(M_0, Q_1, \ldots, Q_b) \mapsto (M_0, Q_1, M_1, \ldots, Q_b, M_b)$ and denote the new measures with tildes:

$$\widetilde{\text{Unif}}_{T^*}(dM, dQ_1, \ldots, dQ_b) = \text{Unif}_{E_0}(dM_0) \times \prod_{i=1}^b \widetilde{\text{Unif}}_{J_{T_i^*}}(dQ_i, dM_i). \qquad (4.12)$$

Finally, we can disintegrate each $\widetilde{\text{Unif}}_{J_{T_i^*}}$ to obtain a family of measures[8] $\{\text{Unif}_{J_{T_i^*}}^{M_i}\}$ for $M_i \in \mathbb{R}_+^{E_i}$ such that $\widetilde{\text{Unif}}_{J_{T_i^*}}(dQ_i, dM_i) = \text{Unif}_{J_{T_i^*}}^{M_i}(dQ_i)\text{Unif}_{E_i}(dM_i)$. Since M_i is a function of M we may define $\text{Unif}_{J_{T_i^*}}^M = \text{Unif}_{J_{T_i^*}}^{M_i}$, then (4.12) equals $\left(\prod_{i=1}^b \text{Unif}_{J_{T_i^*}}^M(dQ_i)\right)\text{Unif}_{E_0}(dM_0)\prod_{i=1}^b \text{Unif}_{E_i}(dM_i)$. The dM_0 and dM_i terms factorize to give $\text{Unif}_{E_{T_{\text{sub}}}}(dM)$, as desired. $\qquad \square$

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[8]The general theory of disintegrations only specifies this family up to measure zero sets of M_i, but we are just applying a linear change of variables to a Lebesgue measure, so $\text{Unif}_{J_{T_i^*}}^{M_i}$ is well defined for every M_i.

Proof of Proposition 4.11. We will derive an expression for the law of $\mathcal{L}^\varepsilon|_{F_{\delta,M^0}}$, then write down a choice of m_M^ε for which (4.10) holds, and which satisfies the remaining conditions of finiteness and independent subquilts.

Fix $M^0 \in \mathcal{Q}_{T_{\text{sub}}}$. By definition, the quilts $Q \in \bigcup_{T \in \mathcal{T}} \mathcal{Q}_T$ satisfying F_{δ,M^0} are those which have a subquilt in $B_\delta(M^0)$; this subquilt is a.s. unique when $\delta > 0$ is sufficiently small since edge lengths are a.s. distinct. Thus, viewing a sample of $\mathcal{L}^\varepsilon|_{F_{\delta,M^0}}$ as being a *marked* quilt where the subquilt in $B_\delta(M^0)$ is marked, we have

$$\mathcal{L}^\varepsilon|_{F_{\delta,M^0}} = \sum_{T^* \in \mathcal{T}^*} \mathcal{L}^\varepsilon|_{\mathcal{Q}_{T^*} \cap F_{\delta,M^0}}.$$

Identify a marked quilt Q with the tuple $(M, Q_1, \ldots, Q_b)$ where M is the marked subquilt and Q_i are the complementary subquilts of M. Write F_{δ,M^0}^* for the event that the marked subquilt lies in $B_\delta(M^0)$. Lemma 4.9 gives $\mathcal{L}^\varepsilon|_{\mathcal{Q}_{T^*} \cap F_{\delta,M^0}} = 1_{F_{\delta,M^0}^*} w(Q) \mathrm{Unif}_{T^*}(dQ)$ where $w(Q) = Z_\varepsilon^{-1} P_{\text{init}}^\varepsilon(C_0) \times \prod_{i=1}^n P^\varepsilon(C_i) \times P_{\text{end}}^\varepsilon(C_{n+1})$ and $C_0, \ldots, C_{n+1}$ are the tuples of side lengths of the faces of Q. Since $M, Q_1, \ldots, Q_b$ partition the faces of Q, we can group the terms to get $w(Q) = Z_\varepsilon^{-1} \widetilde{w}(M) \prod_{i=1}^b \widetilde{w}(Q_i)$. Applying Lemma 4.13 thus gives

$$\mathcal{L}^\varepsilon|_{\mathcal{Q}_{T^*} \cap F_{\delta,M^0}}(dQ) = 1_{F_{\delta,M^0}^*} w(Q) \mathrm{Unif}_{T^*}(dQ) = 1_{F_{\delta,M_0}^*} Z_\varepsilon^{-1} \left(\prod_{i=1}^b \widetilde{w}(Q_i) \mathrm{Unif}_{J_{T_i^*}}^M(dQ_i) \right) \widetilde{w}(M) \mathrm{Unif}_{E_{T_{\text{sub}}}}(dM).$$

By Proposition 4.12,

$$\mathcal{L}^\varepsilon|_{F_{\delta,M^0}} = \sum_{T^* \in \mathcal{T}^*} \mathcal{L}^\varepsilon|_{\mathcal{Q}_{T^*} \cap F_{\delta,M^0}}(dQ) = 1_{F_{\delta,M^0}^*} Z_\varepsilon^{-1} \left(\prod_{i=1}^b \left(\sum_{T_i^* \in \mathcal{T}_i^*} \widetilde{w}(Q_i) \mathrm{Unif}_{J_{T_i^*}}^M(dQ_i) \right) \right) \widetilde{w}(M) \mathrm{Unif}_{E_{T_{\text{sub}}}}(dM).$$

Define for each $M \in \mathcal{Q}_{T_{\text{sub}}}$ the measure

$$m_M^\varepsilon := Z_\varepsilon^{-1} \widetilde{w}(M) \prod_{i=1}^b \left(\sum_{T_i^* \in \mathcal{T}_i^*} \widetilde{w}(Q_i) \mathrm{Unif}_{J_{T_i^*}}^M(dQ_i) \right), \tag{4.13}$$

and view m_M^ε as a measure on the set of quilts having M as a subquilt. The above gives (4.10). Finiteness of the measures follows from (4.10) since $|\mathcal{L}^\varepsilon| = 1 < \infty$, and independence of the complementary subquilts of a sample from $m_M^\varepsilon/|m_M^\varepsilon|$ follows from the product structure of (4.13). □

4.5 Combinatorial independence of complementary subtemplates

The goal of this section is to prove Proposition 4.12. The arguments are topological in nature and essentially depend on Hopf's Umlaufsatz: the total curvature/winding in a simple loop is $\pm 2\pi$ (Lemma 2.3). See Section 2.6 for details on total curvature/winding. The reader may want to read Appendix B before reading this section to motivate the arguments involved.

Throughout this section we work in the setting of Proposition 4.12 and use the notation of that proposition. It is notationally convenient for us to slightly modify $\mathcal{T}^*$ and T_{sub} by replacing the 1-gon face F_{ext} with a hole H_{ext}.

The proof makes repeated use of the following observation. We say two faces of a template are *adjacent* if their boundaries intersect on an edge.

Lemma 4.14. *Suppose $T \in \mathcal{T}$ has external hole H_{ext} and faces $F_0, \ldots, F_{n+1}$. For each 4-gon F_i, let $\partial_- F_i$ be the union of the two sides adjacent to the root vertex of F_i. Let $\partial_- F_0$ be the side counterclockwise of the root vertex of F_0, and let $\partial_- F_{n+1} = \partial F_{n+1}$. Let $\partial_+ F_i = \partial F_i \backslash \partial_- F_i$ for all i. If F_i and F_j are adjacent, then $i < j$ if and only if $\partial_+ F_i$ intersects $\partial_- F_j$ on an edge.*

Proof. This is immediate from (d) in Definition 4.7. $\qquad\square$

Recall the holes of T_{sub} are called $H_{\text{ext}}, H_1, \ldots, H_b$. We embed T_{sub} in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that H_{ext} contains ∞. Let $x \in \partial H_{\text{ext}}$ be the root vertex of the 3-gon. For $i = 1, \ldots, b$ let $V_i \subset \partial H_i$ be the set of all points on the ith boundary loop which are a root vertex or a terminal vertex of some face.

In each face F of T_{sub} we draw a smooth curve η_F from its root vertex to its terminal vertex; we call η_F the curve of F. We do this in a way such that if the curves of two faces meet at an endpoint, they can be smoothly concatenated. This gives us an embedded directed graph G in $\hat{\mathbb{C}} \backslash (H_{\text{ext}} \cup \bigcup_{i=1}^b H_i)$ whose vertex set is $\{x\} \cup \bigcup_{i=1}^b V_i$ and whose edges are the maximal concatenations of the curves, see Figure 11 (left) for an embedding of G for the T_{sub} depicted in Figure 10 (a).

Suppose $T^* \in \mathcal{T}^*$, so T_{sub} is the marked subtemplate of T^*. Embed T^* in $\hat{\mathbb{C}}$ such that T_{sub} has the same embedding as before; this corresponds to drawing faces to fill the holes $H_1, \ldots, H_b$. We draw curves in each added face from its root vertex to its terminal vertex, and so link up the previous smooth curves to get a single simple smooth curve from x to x. This curve visits the faces of T_{sub} in the order induced by the face ordering of T^*. Recall the notion of *total curvature* from Section 2.6. Let $\theta(x) \in [0, 2\pi)$ be the angle such that the initial tangent to the curve at x is parallel to $e^{i\theta(x)}$, and for each point p on the curve, let $\theta(p)$ equal $\theta(x)$ plus the total curvature of the curve from x to p. We label each $v \in \bigcup_i V_i$ with the value $\theta(v)$. See Figure 11 (right).

We will show in Lemma 4.16 that the winding on $\bigcup V_i$ does not depend on T^*. We first need the following lemma. For each face F, let $\partial_\ell F \subset \partial F$ (resp. $\partial_r F \subset \partial F$) be the clockwise (resp. counterclockwise) boundary arc of F from its root vertex to its terminal vertex.

Lemma 4.15. *Let F and F' be faces of T_{sub}, and let the root and terminal vertices of F (resp. F') be v_- and v_+ (resp. v'_- and v'_+). Suppose F and F' are adjacent (have a common boundary edge), F preceeds F' in the ordering of faces, and F and F' are not consecutive faces ($v_+ \neq v'_-$). Suppose $\partial_r F$ intersects $\partial_r F'$ along an edge and let $\eta_{FF'}$ be the clockwise boundary arc of $\partial(F \cup F')$ from v_+ to v'_-. Then the curve obtained by concatenating η_F, $\eta_{FF'}$ and $\eta_{F'}$ has total curvature $\theta(v'_+) - \theta(v_-)$, in the sense of Definition 2.5. See Figure 12 (left).*

The statement also holds if we replace ∂_r with ∂_ℓ and clockwise with counterclockwise.

Proof. We address the first claim, the other case follows the same argument. Let η_1 (resp. η_2) be the concatenations of the curves of all faces of T^* preceding F (resp. between F and F'). Then η_1 is a curve from x to v_-, and η_2 is a curve from v_+ to v'_-. See Figure 12 (left). The connected components of $\mathbb{C} \backslash \overline{(\eta_1 \cup F \cup F' \cup H_{\text{ext}})}$ are simply connected. Let D be the component containing η_2, then $\overline{D}$ also contains $\eta_{FF'}$. Since $\overline{D}$ is simply connected, η_2 and $\eta_{FF'}$ are homotopic in $\mathbb{C} \backslash (\eta_F \cup \eta_{F'})$, so by Definition 2.5 the total curvature of the concatenation of $\eta_F, \eta_{FF'}$ and $\eta_{F'}$ agrees with that of the concatenation of $\eta_F, \eta_2, \eta_{F'}$, which by definition is $\theta(v'_+) - \theta(v_-)$. $\qquad\square$

Lemma 4.16. *The function $\theta(v)$ on $\bigcup_i V_i$ defined above depends on the embedding of T_{sub} and the curves of the faces of T_{sub}, but does not depend on the choice of $T^* \in \mathcal{T}^*$ or the embedding of T^*.*

Proof. We claim that if F and F' are adjacent faces of T_{sub} (i.e. have a common boundary edge), then the values of θ on the curve of F determine its values on the curve of F', and vice versa. Each $T^* \in \mathcal{T}^*$ induces an ordering on the faces of T_{sub}, and Lemma 4.14 implies the relative order

of F and F' does not depend on T^* (since F and F' are adjacent); without loss of generality F precedes F'. If the curve passes directly from F to F', then the claim is immediate. Otherwise, by Lemma 4.15 $\theta(v'_+) - \theta(v_-)$ equals the total curvature of a curve that does not depend on T^*, so the claim also holds in this case.

The set of faces of T_{sub} is connected under the adjacency relation, and the values of θ on the curve segment in the 3-gon are known by definition, so repeatedly using the claim gives the result. $\square$
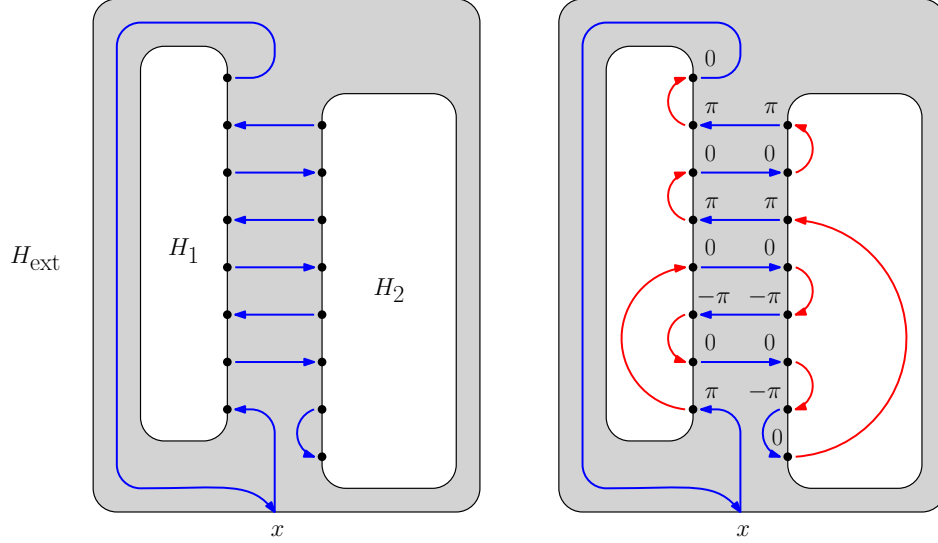

Figure 11: **Left:** We represent T_{sub} in Figure 10 (a) by a graph embedded in the plane. Each edge is a smooth curve (blue arrow) corresponding to a sequence of faces of T_{sub} (blue arrows in Figure 10 (b), (c)). **Right:** We add red edges according to the template T^* shown in Figure 10 (c). We define $\theta(v)$ to be the total curvature of the concatenated curve from x to v. This function θ (labelled values) turns out not to depend on the choice of T^*.

In our embedded directed graph G, the degree of each vertex in $\bigcup_{i=1}^b V_i$ is one, each curve lies in $\hat{\mathbb{C}} \setminus (H_{\text{ext}} \cup \bigcup_{i=1}^b H_i)$, and the in-degree and out-degree of x is one. Let A_i be the finite collection of ways of adding directed edges between vertices in V_i such that the edges lie in H_i, the resulting graph is still planar, the indegree and outdegree of each vertex in V_i is one, and the following compatibility relation holds: any added edge e from v to w, drawn as a simple smooth curve in H_i which joins smoothly with the edges of G at v and w, has total curvature equal to $\theta(w) - \theta(v)$. This compatibility condition makes sense since $\theta(u), \theta(v)$ are well defined (Lemma 4.16), and every embedding of e as a simple smooth curve in H_i that joins smoothly with edges of G has the same total curvature (Lemma 2.4). See Figure 11 (right) where for each hole H_i the collection of red arrows represents an element of A_i.

Lemma 4.17. *Suppose $T_i^* \in \mathcal{T}_i^*$. In each unmarked face of T_i^* draw a curve from its root vertex to its terminal vertex. Concatenating these gives a collection of curves from V_i to V_i; view each curve as an edge from V_i to V_i and let α_i be the set of these edges. Then $\alpha_i \in A_i$.*

Proof. This follows from Lemma 4.16 since T_i^* is a subtemplate of some marked template in $\mathcal{T}^*$. $\square$

Lemma 4.18. *For any $(\alpha_1, \ldots, \alpha_n) \in \prod_{i=1}^n A_i$, adding the edges of each of the α_i to G gives a Hamiltonian cycle, i.e. a cycle that visits every vertex.*

Proof. In the new graph every vertex has indegree and outdegree one. Thus the edges form a collection of cycles. Suppose for the sake of contradiction that there is a cycle not containing x. Since for any edge from w to v the total curvature is $\theta(v) - \theta(w)$, the total curvature of the cycle is 0, contradicting the fact that simple smooth closed curves (loops) have total curvature $\pm 2\pi$ (Hopf's Umlaufsatz). □

By definition, each $T_i^* \in \mathcal{T}_i^*$ is a subtemplate of some $T^* \in \mathcal{T}^*$; this T^* defines an ordering on the faces of T_i^*. Lemma 4.14 says the relative ordering of adjacent faces of T_i^* is determined by T_i^*, so this relative ordering does not depend on the choice of $(\alpha_j)_{j \neq i}$ arising from the choice of T^*. More strongly, as we see next, *every* $(\alpha_j)_{j \neq i} \in \prod_{j \neq i} A_j$ gives the same partial ordering on adjacent faces, not just those coming from some T^*. (We expect, but do not prove, that every $(\alpha_j)_{j \leq b} \in \prod_{j \leq b} A_j$ arises from some choice of T^*.)

Lemma 4.19. *Fix $i \leq b$ and $T_i^* \in \mathcal{T}_i^*$. Suppose $\alpha_j \in A_j$ for each $j \neq i$. Draw a curve in each face of T_i^* from the root vertex to the terminal vertex, and draw curves in each H_j according to α_j, to get a simple curve from x to x visiting each face exactly once (Lemma 4.18). For the induced ordering on faces of T_i^*, the relative order of adjacent faces does not depend on $(\alpha_j)_{j \neq i}$.*

Proof. Let F, F' be adjacent faces of T_i^*, and let v_-, v_+ (resp. v'_-, v'_+) be the root and terminal vertices of F (resp. F'). If $\{v_-, v_+\} \cap \{v'_-, v'_+\} \neq \emptyset$, then one face's terminal vertex is the other's root vertex, and the former face precedes the latter regardless of the choice of $(\alpha_j)_{j \neq i}$.

Otherwise, suppose for the sake of contradiction that there exists $\{\alpha_j^{FF'}\}_{j \neq i}$ for which F precedes F', and $\{\alpha_j^{F'F}\}_{j \neq i}$ for which F' precedes F. Assume $\partial_r F$ intersects $\partial_r F'$ on an edge; the other case is similar. Let $\eta_{FF'}$ be the clockwise boundary arc of $\partial(F \cup F')$ from v_+ to v'_-, and let $\eta_{F'F}$ be the clockwise boundary arc of $\partial(F \cup F')$ from v'_+ to v_-, see Figure 12 (middle). By precisely the same argument of Lemma 4.15 with the curve from T^* replaced by the curve arising from $\{\alpha_j^{FF'}\}_{j \neq i}$, the total curvature of the concatenation of η_F, $\eta_{FF'}$ and $\eta_{F'}$ is $\theta(v'_+) - \theta(v_-)$. By instead using the curve arising from $\{\alpha_j^{F'F}\}_{j \neq i}$, the total curvature of the concatenation of $\eta_{F'}, \eta_{F'F}, \eta_F$ is $\theta(v_+) - \theta(v'_-)$. Let η be a simple loop obtained by concatenating $\eta_F, \eta_{FF'}, \eta_{F'}, \eta_{F'F}$ and having its starting and ending point be a point on the interior of η_F, so its total curvature is well-defined by Definition 2.5. By combining the previous two total curvature values, and the fact that the total curvatures of η_F and $\eta_{F'}$ are $\theta(v_+) - \theta(v_-)$ and $\theta(v'_+) - \theta(v'_-)$, we conclude that the total curvature of η is 0. On the other hand, η is a simple loop so it should have total curvature $\pm 2\pi$ (Lemma 2.3), a contradiction. □

We now turn to the main proof.

Proof of Proposition 4.12. Clearly the map is injective. We will show it is surjective. Suppose $(T_1^*, \ldots, T_b^*) \in \prod_{i=1}^b \mathcal{T}_i^*$. Let T^* be the marked template obtained by filling in the ith hole of T_{sub} in the same way as in T_i^*. We will show that $T^* \in \mathcal{T}^*$. Since T^* has T_{sub} as its marked subtemplate, we need to show the four conditions (a)-(d) defining $\mathcal{T}$ in Definition 4.7 hold for T^*. (a) is automatic since T^* has T_{sub} as a subtemplate. By Lemmas 4.17 and 4.18 the faces of T^* can be ordered as $F_0, F_1, \ldots, F_{n+1}$ so that (b) holds. Next (c) holds for each T_i^* since T_i^* is a subtemplate for some element of $\mathcal{T}^*$, so (c) also holds for T^*. Finally we address (d). First suppose F is an unmarked face of T^*, i.e. F lies in the ith hole of T_{sub} for some i. We know T_i^* is a subtemplate of some $\hat{T}^* \in \mathcal{T}^*$, and $\hat{T}^*$ satisfies (d) for the face F. By Lemma 4.19, the relative order of F and any adjacent face is the same in T^* and $\hat{T}^*$, so T^* also satisfies (d) for the face F. A similar argument applies if F is a marked face of T^*, so (d) holds. Thus, T satisfies all four conditions so $T \in \mathcal{T}$, and so $T^* \in \mathcal{T}^*$. □
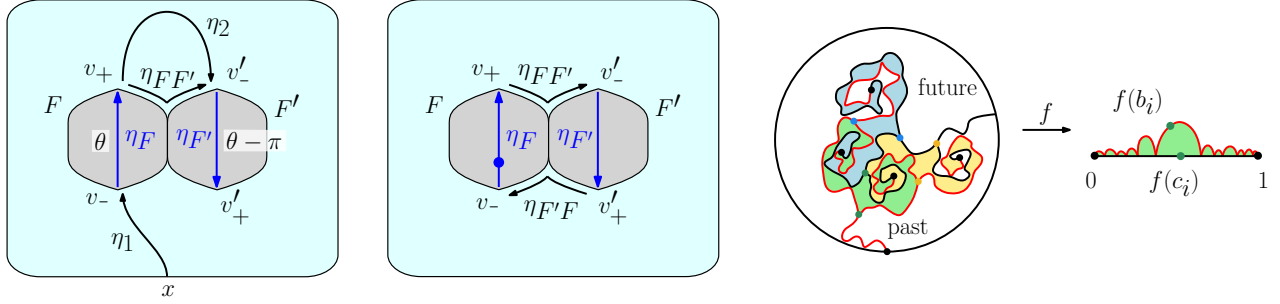
Figure 12: **Left:** Figure for Lemma 4.15. Suppose F preceeds F' but are not consecutive faces, and suppose $\partial_r F$ intersects $\partial_r F'$. Then $\theta(v'_+) - \theta(v_-)$ is the total curvature of the concatenation of $\eta_F, \eta_{FF'}$ and $\eta_{F'}$. In this figure, the curves of F and F' are line segments, and the winding in $\eta_{F'}$ is π less that that in η_F. **Middle:** Figure for Lemma 4.19. If F preceeds F' for some $\{\alpha_j^{FF'}\}_{j\neq i}$ but F' preceeds F for some $\{\alpha_j^{F'F}\}_{j\neq i}$, then the the loop obtained by concatenating $\eta_F, \eta_{FF'}, \eta_{F'}, \eta_{F'F}$ and rooting at a point in the interior of η_F has total curvature 0 (in the sense of Definition 2.5). This contradicts the fact that this loop has total curvature $\pm 2\pi$. **Right:** Figure for Lemma 4.20 for $\gamma \in (\sqrt{2}, 2)$, where the interior of D_i (green, left) is conformally mapped into a subset of $\mathbb{H}$ such that the right boundary of D_i is sent to $[0, 1]$.

4.6   Proof of Theorem 4.1

In order to prove Theorem 4.1, we will show that the IG field is measurable with respect to the LQG surface decorated by space-filling SLE curve segments. We do this first for each D_i (Lemma 4.20), then for the whole explored region (Lemma 4.21).

Lemma 4.20. *There is a measurable function F such that the following holds. In the setting described just above Theorem 4.1, a.s. $(D_i, \Phi, \Psi, (\eta'_i, a_i, b_i, c_i, d_i))/\sim = F((D_i, \Phi, \eta'_i, a_i, b_i, c_i, d_i)/\sim)$ for each $i = 0, \ldots, N+1$.*

Proof. Let $\widehat{\Psi}$ be the distribution Ψ viewed modulo additive constant in $2\pi\mathbb{Z}$. We will show that $\widehat{\Psi}$ is measurable with respect to $(D_i, \Phi, \eta'_i, a_i, b_i, c_i, d_i)$. Since the arguments we will use do not depend on the choice of embedding of the decorated LQG surface, this implies the lemma.

By [MS17, Theorem 1.10], for each connected component U of $\mathrm{int}(D_i)$ (the interior of D_i), the field $\Psi|_U$ is determined up to additive constant in $2\pi\mathbb{Z}$. This completes the proof for $\gamma \in (0, \sqrt{2}]$ since $\mathrm{int}(D)$ is connected in that regime. For $\gamma \in (\sqrt{2}, 2)$, it remains to show that fixing one of these additive constants fixes all the remaining constants. Basically, the reason for this is that ∂D_i is the union of four flow lines of Ψ, and the values of Ψ along a flow line are well-defined up to adding a single integer multiple of $2\pi\chi$.

Recall that D_i was defined to be the region traced by the space-filling curve η' between hitting two points. Therefore, ∂D_i comprises segments of flow lines of Ψ with specified angles. For each connected component U of D_i let $\partial^p U$ denote the prime end boundary of D_i, let $\partial_r^p U$ be the right boundary arc of U with orientation chosen such that its starting and ending points agree with those of η in U, and let $\partial_r^p D_i$ be the concatenation of the $\partial_r^p U$ ordered by the time U is filled out by η.

Let $f : \mathrm{int}(D_i) \to \mathbb{H}$ be a map such that $f : \mathrm{int}(D_i) \to f(\mathrm{int}(D_i))$ is conformal, and whose extension to $\partial_r^p D_i$ is a homeomorphism from $\partial_r^p D_i$ to $[0, 1] \subset \mathbb{H}$. See Figure 12 (right). Let $\widetilde{\Psi}$ be the distribution on $F(\mathrm{int}(D_i))$ given by $\widetilde{\Psi} = \Psi \circ f^{-1} - \chi(26 - \mathbf{c}_{\mathrm{L}}) \arg((f^{-1})')$, where $\arg((f^{-1})')$ is as defined in (2.3). By the definition of boundary conditions of interior flow lines [MS17, Theorem 1.1],

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and the consistency of (2.3) with the usual definition of $\arg((f^{-1})')$ for simply connected domains (Proposition 2.6), $\widetilde{\Psi}$ is constant on $[0, f(c_i))$ and on $(f(c_i), 1]$. Furthermore, the point c_i lies on the boundary of one of the connected components of the interior of D_i. Consequently, fixing the additive constant of one of the $\Psi|_U$ fixes the additive constants of $\Psi|_{U'}$ for all U', as needed. $\quad\square$

Lemma 4.21. *In the setting of Theorem 4.1 let $\mathcal{M}'_\tau = (M_\tau, \Phi, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\})/\sim$ be the decorated LQG surface obtained from $\mathcal{M}_\tau$ by forgetting the IG field Ψ. Then $\mathcal{M}_\tau$ is measurable with respect to $\mathcal{M}'_\tau$.*

Proof. Let $\widehat{\Psi}$ be Ψ viewed modulo additive constant in $2\pi\mathbb{Z}$. We will show that $\widehat{\Psi}|_{M_\tau}$ is measurable with respect to $(M_\tau, \Phi, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\})$, i.e., we prove that the desired result holds when LQG surfaces are replaced by their embeddings in M_τ. None of the arguments we will use depend on the choice of embedding, so this implies the desired result.

Condition on $(M_\tau, \Phi, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\})$. By Lemma 4.20, $\Psi|_{D_i}$ is determined modulo additive constant in $2\pi\mathbb{Z}$ for each i. Next, if D_i intersects $D_{i'}$ on an interval η, then since η is a segment of a flow line, the additive constant for $\Psi|_{D_i}$ fixes the additive constant for $\Psi|_{D_{i'}}$ by flow line boundary conditions [MS17, Theorem 1.1]. Finally, for any $i, i' \in I_\tau$ it is possible to find a sequence $i_1, \ldots, i_j$ with $i_0 = i$ and $i_j = i'$ such that D_{i_k} and $D_{i_{k+1}}$ intersect on an interval for all k. We conclude that under this conditioning $\widehat{\Psi}|_{M_\tau}$ is deterministic, i.e., $\widehat{\Psi}|_{M_\tau}$ is measurable with respect to $(M_\tau, \Phi, \{(\eta'_i, a_i, b_i, c_i, d_i) : i \in I_\tau\})$. $\quad\square$

Finally, we show that for all quilts having a certain subtemplate, the peeling procedure of Theorem 4.1 chooses this subtemplate with probability not depending on the quilt.

Lemma 4.22. *Fix $M \in \mathcal{Q}_{T_{\text{sub}}}$ for some T_{sub} as in Section 4.4, and let Q be any quilt having M as a subtemplate. Carry out the peeling procedure on Q analogous to that of Theorem 4.1. That is, sample $T \geq 0$ with $\mathbb{P}[T = t] = 2^{-t-1}$, explore all faces adjacent to the external face at step 0, at each subsequent step explore the faces adjacent to a boundary point chosen uniformly at random according to length, and stop after step T or when all faces are explored, whichever is earlier. Then the probability the explored subtemplate is M does not depend on the choice of Q.*

Proof. Given Q, the event {explored subtemplate is M} occurs when T equals the number of non-boundary-touching faces of M, and during each of the T peeling steps we pick a point that lies in M. The probability of this event depends only on M since M determines all the lengths involved in the probability computation for the T peeling steps. $\quad\square$

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We can sample the ε-discretized LQG disk in two steps. First, we sample a quilt $Q \in \bigcup_{T \in \mathcal{T}} \mathcal{Q}_T$ from $\mathcal{L}^\varepsilon$. Then conditioned on Q, for each face F we sample an independent LQG cell $\mathcal{C}_F$ conditioned on having boundary lengths given by the side lengths of F, and conformally weld the $\mathcal{C}_F$ according to Q. By Proposition 4.3 this gives an ε-discretized LQG disk.

The peeling process described can already be applied after the first step of the above sampling procedure, namely to the quilt Q (see Lemma 4.22). Mark the faces of Q discovered during the peeling process, and let M be the marked subquilt. By Bayes' theorem, we may flip this procedure around: sample M from its marginal law, then sample $Q \sim \mathcal{L}^\varepsilon$ conditioned on having M as a subquilt and weighted by the probability of marking M given Q. By Lemma 4.22 this probability does not depend on Q, so there is in fact no weighting. Thus, by Proposition 4.10, conditioned on M the complementary subquilts $Q_1, \ldots, Q_b$ of M are conditionally independent. Conditioned on $M, Q_1, \ldots, Q_b$, sample independent collections of LQG cells $(\mathcal{C}_F)_{F \in M}, (\mathcal{C}_F)_{F \in Q_1}, \ldots, (\mathcal{C}_F)_{F \in Q_b}$.

Suppose $\gamma \in (0, \sqrt{2}]$, and for $1 \le i \le b$ let $\mathcal{D}_i$ be the decorated LQG surface arising from conformally welding $(\mathcal{C}_F)_{F \in Q_j}$ according to Q_j. Then $\mathcal{D}_1, \ldots, \mathcal{D}_b$ are conditionally independent given $\sigma(M, (\mathcal{C}_F)_{F \in M})$. Let $\widetilde{\mathcal{M}}_\tau$ be the conformal welding of $(\mathcal{C}_F)_{F \in M}$ according to M, and $\mathcal{M}'_\tau$ the decorated LQG surface $\mathcal{M}_\tau$ after forgetting the IG field. Since $\widetilde{\mathcal{M}}_\tau = \mathcal{M}'_\tau$, and using Lemma 4.21, we have $\sigma(M, (\mathcal{C}_F)_{F \in M}) = \sigma(\widetilde{\mathcal{M}}_\tau) = \sigma(\mathcal{M}'_\tau) = \sigma(\mathcal{M}_\tau)$. Thus $\mathcal{D}_1, \ldots, \mathcal{D}_b$ are conditionally independent given $\mathcal{M}_\tau$. Finally, the loops $\eta_1, \ldots, \eta_m$ partition $\{\mathcal{D}_1, \ldots, \mathcal{D}_b\}$, and each $(O_k, \Phi, \Psi)/\sim$ is measurable with respect to $\mathcal{M}_\tau$ and the kth subset of the partition (Lemma 4.21), so $(O_1, \Phi, \Psi)/\sim, \ldots, (O_m, \Phi, \Psi)/\sim$ are conditionally independent given $\mathcal{M}_\tau$.

When $\gamma \in (\sqrt{2}, 2)$, we do not necessarily have $\widetilde{\mathcal{M}}_\tau = \mathcal{M}'_\tau$. Each $\mathcal{C}_F$ has cut points, so even though the planar maps Q_i each have the disk topology, the LQG surfaces $\mathcal{D}_i$ may have cut points. Thus $\mathcal{M}'_\tau$ arises from $\widetilde{\mathcal{M}}_\tau$ by identifying pairs of boundary points of $\widetilde{\mathcal{M}}_\tau$ according to the cut points of $\mathcal{D}_1, \ldots, \mathcal{D}_b$. Since the identifications on the ith boundary component of $\widetilde{\mathcal{M}}_\tau$ are a function of $\widetilde{\mathcal{M}}_\tau$ and $\mathcal{D}_i$, and $\mathcal{D}_1, \ldots, \mathcal{D}_b$ are conditionally independent given $\widetilde{\mathcal{M}}_\tau$, we conclude $\mathcal{D}_1, \ldots, \mathcal{D}_b$ are conditionally independent given $\mathcal{M}'_\tau$, and thus conditionally independent given $\mathcal{M}_\tau$ (Lemma 4.21). The rest of the argument is identical to the $\gamma \in (0, \sqrt{2}]$ case. $\qquad\square$

5 Open problems

The results of this paper are only proven for LQG in the subcritical phase, i.e., when $\mathbf{c}_{\mathrm{L}} > 25$ or equivalently $\gamma \in (0, 2)$. The reason for this is that our proofs rely on the mating of trees theorem [DMS21, AG21] in order to get a decomposition of the LQG disk into small pieces with nice independence properties. An analog of the mating of trees theorem in the critical case $\mathbf{c}_{\mathrm{L}} = 25$, $\gamma = 2$ is proven in [AHPS21], but the analog of space-filling SLE is not a continuous curve, so this theorem does not give a decomposition of the LQG disk into *small* independent pieces.

Problem 5.1. Extend the theorems from Section 1.3 to the case of critical LQG ($\mathbf{c}_{\mathrm{L}} = 25$, $\gamma = 2$).

One possible approach to Problem 5.1 is to take a limit of the objects involved in our theorem statements as $\gamma \to 2^-$.

The recent paper [AG23] defines an analog of the LQG disk in the supercritical phase $\mathbf{c}_{\mathrm{L}} \in (1, 25)$. The paper also constructs a coupling of the supercritical LQG disk with CLE$_4$, which has similar properties to the relationships between LQG with $\mathbf{c}_{\mathrm{L}} \ge 25$ and independent SLE with central charge $\mathbf{c}_{\mathrm{M}} = 26 - \mathbf{c}_{\mathrm{L}}$. It is natural to ask whether there are also "mismatched" coupling statements for LQG with $\mathbf{c}_{\mathrm{L}} \ge 25$.

Problem 5.2. Extend the theorems from Section 1.3 to the case of supercritical LQG ($\mathbf{c}_{\mathrm{L}} \in (1, 25)$, $\gamma \in \mathbb{C}$ with $|\gamma| = 2$).

An especially interesting special case of Problem 5.2 is when we have a supercritical LQG surface of central charge $\mathbf{c}_{\mathrm{L}} = 26 - k \in \{2, \ldots, 24\}$ decorated by k independent Gaussian free fields (i.e., $\mathbf{c}_1 = \cdots = \mathbf{c}_k = 1$). The reason is that, from the work of Polyakov [Pol81], this object is expected to be related to bosonic string theory in $\mathbb{R}^k$, see [AG23, Remark 1.4] for explanations.

In the matched case $\kappa \in \{\gamma^2, 16/\gamma^2\}$, there are relationships between SLE and LQG for a wide variety of different LQG surfaces (see, e.g., [DMS21, MS19, AG21, AHS21, ASY22]). For simplicity, in this paper we focused exclusively on the case of the LQG disk, but we expect that analogs of our results are true for other types of LQG surfaces.

Problem 5.3. Prove analogs of the theorems from Section 1.3 for other LQG surfaces besides the LQG disk, such as LQG disks with boundary insertions, LQG triangles, LQG spheres, LQG wedges, LQG cones, and non-simply connected LQG surfaces.

It is of interest to consider variants of the results of this paper where the imaginary geometry fields are replaced by other types of "matter fields" (random generalized functions satisfying a conformal covariance rule). Some examples of possible matter fields to consider are conformal loop ensemble nesting fields [MWW15], Ising spin fields [CGN15], *winding fields* and *layering fields* associated with Brownian loop soups [CGK16, vdBCL18, LJ10, CGPR21], and the conformally invariant fields associated with Brownian loop soups constructed in [JLQ23].

Problem 5.4. Are there analogs of the conditional independence results of this paper where the auxiliary imaginary geometry fields are replaced by fields of the types mentioned just above?

Our main results give exact conditional independence statements for LQG surfaces cut by various types of sets, but, in contrast to the matched case, we do not have an exact description of either the law of the information we are conditioning on or the conditional laws of the complementary LQG surfaces given this information.

Problem 5.5. Assume that we are in the setting of Theorem 1.9. Is there an explicit random process which generates the σ-algebra $\mathcal{F}$ of (1.10)? Can the law of such a random process can be described explicitly? What about the analogous questions in the settings of the other main results from Section 1.3?

Problem 5.6. In the setting of Theorem 1.9, is there an explicit description of the conditional laws of the IG-decorated LQG surfaces parametrized by the complementary connected components of the SLE curve K given the σ-algebra $\mathcal{F}$? What about in the settings of the other main results from Section 1.3?

Here is a possible approach for Problem 5.5. For simplicity, assume that K is the image of an $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ curve η with $\rho_L \geq \kappa/2 - 4$, so that η does not hit the clockwise arc of $\partial\mathbb{D}$ from its starting point to its ending point [Dub09a, Lemma 15]. Let ν_Φ be the γ-LQG length measure on η, which should be defined as a Gaussian multiplicative chaos measure with respect to the Euclidean Minkowski content measure on η [LR15]. Assume that η is parametrized so that $\nu_\Phi(\eta[0, t]) = t$ for each $t \in [0, T]$, where $T = \nu_\Phi(\eta)$.

Let f be a conformal map from the connected component of $\mathbb{D} \setminus \eta$ which lies to the left of η to the upper half-plane $\mathbb{H}$, taking the starting and ending points of η to 0 and ∞, respectively. For $j = 1, \ldots, n$, define

$$F_j(t) = \int_0^t \left(\Psi_j - \chi(\mathbf{c}_j) \arg f'\right)(\eta(s)) \, ds, \quad \forall t \in [0, T], \tag{5.1}$$

where we recall that $(\Psi_1, \ldots, \Psi_n)$ are the auxiliary imaginary geometry fields of central charges $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$. We conjecture that the process $(F_1, \ldots, F_n)$ generates the σ-algebra (1.10). It would be very interesting to find an explicit description of the law of $(F_1, \ldots, F_n)$.

One of the most important relationships between SLE and LQG in the matched case is the **mating of trees** theorem [DMS21, Theorem 1.9]. This theorem says the following: suppose we have a γ-LQG cone together with an independent space-filling $\mathrm{SLE}_{\kappa'}$ curve η' for $\kappa' = 16/\gamma^2$. Parametrize η' so that it traverses one unit of γ-LQG mass in one unit of time. Let L_t (resp. R_t) denote the net change in the γ-LQG length of the left (resp. right) outer boundary of η' relative to time 0. Then the process (L, R) is a two-dimensional Brownian motion with correlation $\mathrm{Corr}(L_t, R_t) = -\cos(\pi\gamma^2/4)$ (note that the correlation is zero if $\kappa' = 8$). This theorem is proven building on the quantum zipper theorem in [She16]. Our results can be viewed as a mismatched analog of the quantum zipper theorem. It is therefore natural to ask of there is also a mismatched mating of trees theorem.

Problem 5.7. Suppose we have an appropriate γ-LQG surface together with a space-filling $SLE_{\kappa'}$ curve for $\kappa' \neq 16/\gamma^2 > 4$, sampled independently from the LQG surface and then parametrized by LQG mass. Define the (mismatched) left/right γ-LQG boundary length process (L, R) as above. Is there a variant of our main results which says that the joint conditional law of L and R given certain auxiliary information along the curve η' is particularly simple? Are L and R conditionally independent given this information for certain special values of γ and κ'?

In a similar vein to (5.1), a possible choice of auxiliary information to consider in Problem 5.7 is the following. Let Ψ_1 be the whole-plane GFF coupled with η' in the sense of imaginary geometry and let Ψ_2 be another imaginary geometry field (independent from everything else) such that the central charges associated with η', the γ-LQG surface, and Ψ_2 sum to 26. Let $a, b \in [-1, 1]$ with $a^2 + b^2 = 1$ be chosen so that $a\Psi_1 + b\Psi_2$ is a field of central charge 1 (i.e., with $\chi = 0$). Then, consider the σ-algebra generated by the function

$$t \mapsto \int_0^t (a\Psi_1 + b\Psi_2)(\eta(s)) \, ds. \tag{5.2}$$

In the special case when our LQG surface is an LQG sphere with $\mathbf{c}_L = 30$, $\kappa' = 8$, and Ψ_2 is an imaginary geometry field of central charge -2, we expect that L and R are conditionally independent given the σ-algebra generated by (5.2). Indeed, such a conditional independence statement would be a continuum analog of the conditional independence statement for meanders given in Proposition B.1; c.f. [BGS22, Conjecture 1.3].

Next, let Ψ_1 and Ψ_2 be independent imaginary geometry fields on $\mathbb{D}$ of central charges $\mathbf{c}_1$ and $\mathbf{c}_2$, respectively. By Proposition 1.17, if $a, b \in [-1, 1]$ with $a^2 + b^2 = 1$, then $\widehat{\Psi} := a\Psi_1 + b\Psi_2$ is another imaginary geometry field with a possibly different central charge. It is natural to wonder how the flow lines of Ψ_1, Ψ_2, and $\widehat{\Psi}$ are related.

Problem 5.8. Is there a direct way to construct the flow lines of $\widehat{\Psi}$ from the flow lines of Ψ_1 and Ψ_2?

Here is a possible approach to Problem 5.8. Let $q > 0$ and $\varepsilon > 0$. Grow a flow line of Ψ_1 (parametrized by Minkowski content, say) up to time ε. Then, grow a flow line of Ψ_2 started from the tip of this first flow line, stopped at time $q\varepsilon$. Continue alternating ε-increments of flow lines of Ψ_1 and $q\varepsilon$-increments of flow lines of Ψ_2 in this manner to construct a curve η_ε. It is reasonable to guess that if we choose $q = q(a, b)$ appropriately, then η_ε should converge to a flow line of $\widehat{\Psi}$ as $\varepsilon \to 0$.

One motivation for Problem 5.8 comes from lattice models. To explain this, we follow the exposition in [Dub09b, Section 1]. Let $\varepsilon > 0$ and let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two independent copies of the dimer model on $\varepsilon\mathbb{Z}^2 \cap \mathbb{D}$. By the Temperlyan bijection, $\mathcal{D}_1$ gives rise to a height function ψ_1 on the dual graph of $\varepsilon\mathbb{Z}^2 \cap \mathbb{D}$, and similarly for $\mathcal{D}_2$. These same height functions can also be obtained as height functions of two independent uniform spanning trees T_1, T_2 on $\varepsilon\mathbb{Z}^2 \cap \mathbb{D}$. The height functions ψ_1 and ψ_2 converge as $\varepsilon \to 0$ to two independent copies of the $\kappa = 2$ ($\mathbf{c} = -2$) imaginary geometry field on the disk [Ken01]. Moreover, the peano curves associated with T_1 and T_2 converge to two independent SLE_8s [LSW04] which should be counterflow lines of these two imaginary geometry fields.

Now, consider the double dimer model $\mathcal{D}_1 \cup \mathcal{D}_2$. It is believed (but not proven) that the set of boundaries of connected components of $\mathcal{D}_1 \cup \mathcal{D}_2$ converges as $\varepsilon \to 0$ to CLE_4. As explained in [Dub09b, Section 1.2], the height function associated with $\mathcal{D}_1 \cup \mathcal{D}_2$ is the difference $\widehat{\psi} = \psi_1 - \psi_2$. This height function should converge as $\varepsilon \to 0$ to a GFF on $\mathbb{D}$ whose level lines are the CLE_4 loops. Thus, the continuum limit of the double dimer model gives us two $\mathbf{c} = -2$ imaginary geometry fields on $\mathbb{D}$ whose difference is a GFF.

Problem 5.9. Are there any other collections of lattice models where one naturally sees linear combinations of fields which converge to imaginary geometry fields? Can the solution of Problem 5.8 be useful in analyzing these discrete models?

A   Planar maps decorated by multiple statistical physics models

Theorem 1.9 is the continuum analog of certain Markovian properties for random planar maps with boundary decorated by more than one statistical physics model. The general principle is as follows. Suppose that one of our statistical physics models gives rise to an "interface" η, viewed as a curve on the planar map M (or on a closely related planar, such as the dual map). We can view the connected components of $M \setminus \eta$ as submaps of M. Then in many cases these submaps are conditionally independent given the appropriate information about the restrictions to η of the *other* statistical physics models (besides the one used to generate η). In a similar vein, our other main results can be seen as discrete analogs of Markov properties for, e.g., random walk or graph distance balls on random planar maps decorated by one or more statistical physics models.

For concreteness, we now fully explain this principle in one special case: planar maps decorated by two instances of the Gaussian free field and one spanning tree. This case is particularly nice since the underlying random planar map is exactly uniform. In general, the marginal law of the underlying random planar map may not admit a simple description.

For a graph G, write $\mathcal{V}(G)$ and $\mathcal{E}(G)$ for its sets of vertices and edges, respectively. For $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the set of triangulations with the disk topology with simple boundary of perimeter n (i.e., triangulations with a marked exterior face whose boundary is a simple cycle of length n). For concreteness, we allow multiple edges but not self-loops. We note that everything in this section also works with other types of planar maps, e.g., triangulations with no multiple edges, quadrangulations, or planar maps with mixed face degrees.

Let $\mathcal{M}_n^\bullet$ be the set of 4-tuples (M, T, ϕ_1, ϕ_2) where $M \in \mathcal{M}_n$, T is a spanning tree on M with wired boundary condition, and $\phi_1, \phi_2 : \mathcal{V}(M) \to \mathbb{R}$ are functions on the vertex set of M which vanish at each boundary vertex.

To define a probability measure on $\mathcal{M}_n^\bullet$, let $\beta > 0$ be a universal constant to be chosen later. For $M \in \mathcal{M}_n$ write λ^M for Lebesgue measure on the space of functions $\phi : \mathcal{V}(M) \to \mathbb{R}$ which vanish on the boundary. Also write μ for counting measure on pairs (M, T) where $M \in \mathcal{M}_n$ and T is a spanning tree on M with wired boundary condition. We define a probability measure on $\mathcal{M}_n^\bullet$ by

$$\frac{1}{Z} e^{-\beta \# \mathcal{E}(M)} \exp\left(- \sum_{i \in \{1,2\}} \sum_{\{x,y\} \in \mathcal{E}(M)} |\phi_i(x) - \phi_i(y)|^2 \right) d\lambda^M(\phi_1)\, d\lambda^M(\phi_2)\, d\mu(M,T) \qquad \text{(A.1)}$$

where Z is a normalizing constant.

Under this probability measure, the conditional law of (T, ϕ_1, ϕ_2) given M is that of an independent triple consisting of a uniform spanning tree on M with wired boundary conditions and two discrete zero-boundary Gaussian free fields (GFFs) on M. Moreover, the marginal law of M is given by the counting measure on $\mathcal{M}_n$ weighted by (a quantity proportional to)

$$e^{-\beta \# \mathcal{E}(M)} \left[\int \exp\left(- \sum_{\{x,y\} \in \mathcal{E}(M)} |\phi(x) - \phi(y)|^2 \right) d\lambda^M(\phi) \right]^2 \times \#\{\text{spanning trees of } M\}. \qquad \text{(A.2)}$$

By a standard computation for the discrete GFF (see, e.g., [Dub09b, Section 4.1]), the integral with respect to λ^M in (A.2) is equal to an exponential function of $\#\mathcal{V}(M)$ times $(\det \Delta_M)^{-1/2}$,

where Δ_M is the discrete Laplacian determinant on M (restricted to functions which vanish on the boundary). Furthermore, by the Kirchoff matrix-tree theorem, the number of spanning trees of M is equal to $\det \Delta_M$. Hence, by canceling two factors of $\det \Delta_M$, we get that the quantity (A.2) depends only on $\#\mathcal{V}(M)$. Therefore, the marginal law of M under (A.1) is given by the counting measure on $\mathcal{M}_n$ weighted by an exponential function of $\#\mathcal{V}(M)$. Since we are looking at triangulations, the Euler characteristic formula shows that this is the same as the counting measure on $\mathcal{M}_n$ weighted by an exponential function of $\#\mathcal{E}(M)$.

If we choose the constant β appropriately, then under the above marginal law of M the law of $\#\mathcal{E}(M)$ has a polynomial tail (see, e.g., [Cur16, Section 3.2] for a more general statement). Henceforth fix this choice of β. For this β, the triangulation M converges in law $n \to \infty$ in the Gromov-Hausdorff sense to the so-called **free area Brownian disk** [BM17, GM19, AHS20], which is in turn equivalent (as a metric measure space) to the unit boundary length $\sqrt{8/3}$-LQG disk [MS21].

Due to the convergence of the discrete GFF to the continuum GFF and the convergence of the peano curve of the uniform spanning tree on a fixed lattice to SLE_8 [LSW04], the above convergence leads naturally to the following conjecture.

Conjecture A.1. *Let $(M, T, \phi_1, \phi_2) \in \mathcal{M}_n^\bullet$ be sampled from (A.1) and let P be the path in M which traverses T in contour (depth-first) order starting and ending at a marked boundary vertex. The 4-tuple (M, P, ϕ_1, ϕ_2) converges in law under an appropriate scaling limit as $n \to \infty$ to a unit boundary length $\sqrt{8/3}$-LQG disk together with an SLE_8 curve starting and ending at the same marked boundary point and two independent zero-boundary GFFs.*

Possible topologies of convergence in Conjecture A.1 include generalizations of the Gromov-Hausdorff topology for metric spaces decorated by a curve and a pair of generalized functions (see, e.g., [GM17, Khe18]) or convergence under an appropriate embedding of M into the disk (e.g., the Tutte embedding [GMS21] or the Cardy embedding [HS23]).

Let us now describe a Markov property for (M, T, ϕ_1, ϕ_2). See Figure 13 for an illustration. Let M^* be the dual map of M and let T^* be the dual spanning tree of T, which consists of all edges of M^* which do not cross edges of T. Let $k \leq n - 2$ and let $e_1, e_2 \in \partial M$ be two edges with the property that the two boundary arcs separating them have lengths k and $n - k - 2$, respectively. There is a unique path η in T^* from the vertex of M^* corresponding to the face of M incident to e_1, to the vertex of M^* corresponding to the face of M incident to e_2. The path η forms the outer boundary of the contour path P of Conjecture A.1 if we start P at the leftmost vertex of e_1 and stop P when it first hits a vertex of e_2. By, e.g., [AG21, Lemma 2.6], Conjecture A.1 suggests that the scaling limit of η should be a chordal $\mathrm{SLE}_2(-1, -1)$ curve between two boundary points of the disk (provided we choose $k = k(n)$ so that $k(n)/n$ converges to a number in $(0, 1)$).

Let H be the sub-map of M consisting the faces of M traversed by η and the vertices and edges of M on the boundaries of these faces. We view H as a planar map with boundary equipped with two marked edges, namely e_1 and e_2. We count each edge of H which is not crossed by η as a boundary edge (even if this edge is not part of the external face of H). See Figure 13.

Let M_L (resp. M_R) be the submap of M consisting of the vertices, edges, and faces of M which lie strictly to the left (resp. right) of η (not including e_1, e_2, or the edges and faces which are crossed by η). Each of M_L and M_R is a triangulation with (not necessarily simple) boundary. We view M_L (resp. M_R) as being equipped with two marked vertices, namely the endpoints of e_1 and e_2. Let T_L (resp. T_R) be the subgraph of T induced by the set of edges of T which lie on M_L (resp. M_R).

The following is a discrete analog of Theorem 1.9.

Proposition A.2. *The left and right decorated random planar maps $(M_L, T_L, \phi_1|_{M_L}, \phi_2|_{M_L})$ and $(M_R, T_R, \phi_1|_{M_R}, \phi_2|_{M_R})$ are conditionally independent given the middle decorated planar map $(H, \phi_1|_H, \phi_2|_H)$.*
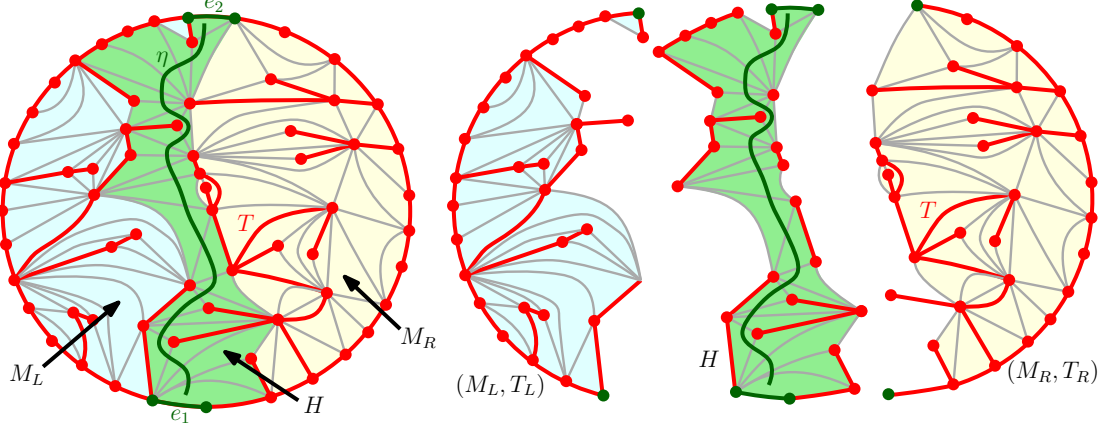
Figure 13: **Left:** A triangulation with boundary M (red, gray, and green edges) decorated by a spanning tree T with wired boundary condition (red). The path η is the branch of the dual tree between the marked boundary edges e_1, e_2, and H (light green) is the submap of M induced by the set of faces traversed by η. **Right:** The decorated planar maps (M_L, T_L) and (M_R, T_R) lying to the left and right of H (light blue and light yellow) are connected but do not necessarily have simple boundary. Identifying one boundary arc of each of these maps with the boundary arcs of H to the left and right of η, respectively, gives us back (M, T). If we sample the tree / discrete GFF decorated triangulation (M, T, ϕ_1, ϕ_2) from the probability measure (A.1), then Proposition A.2 tells us that $(M_L, T_L, \phi_1|_{M_L}, \phi_2|_{M_L})$ and $(M_R, T_R, \phi_1|_{M_R}, \phi_2|_{M_R})$ are conditionally independent given $(H, \phi_1|_H, \phi_2|_H)$.

Proof. We claim that the probability measure (A.1) admits a factorization of the form

$$d\nu_{H,L}(M_L, T_L, \phi_1|_{M_L}, \phi_2|_{M_L}) \, d\nu_{H,R}(M_L, T_L, \phi_1|_{M_L}, \phi_2|_{M_L}) \, d\nu_0(H, \phi_1|_H, \phi_2|_H) \qquad \text{(A.3)}$$

where ν_0 is the law of $(H, \phi_1|_H, \phi_2|_H)$ and $\nu_{H,L}$ and $\nu_{H,R}$ are probability measures that depend only on H. This immediately implies the desired conditional independence statement. We will now check (A.3) via elementary manipulations of (A.2).

Let us first note that by construction,

$$\#\mathcal{E}(M) = \#\mathcal{E}(H \setminus \partial H) + \#\mathcal{E}(M_L) + \#\mathcal{E}(M_R). \qquad \text{(A.4)}$$

Next, note that the only edges of M which join a vertex of M_L and a vertex of M_R are the marked boundary edges e_1, e_2. Since ϕ_1 and ϕ_2 vanish on ∂M, for $i \in \{1, 2\}$ we have

$$\exp\left(-\sum_{\{x,y\} \in \mathcal{E}(M)} |\phi_i(x) - \phi_i(y)|^2\right) = \prod_{A \in \{M_L, M_R, H\}} \exp\left(-\sum_{\{x,y\} \in \mathcal{E}(A)} |\phi_i(x) - \phi_i(y)|^2\right) \qquad \text{(A.5)}$$

Finally, we deal with the counting measure term in (A.1). We first claim that if we see the decorated planar maps (M_L, T_L), (M_R, T_R), and H, then there is a unique way of gluing them together to recover (M, T). Indeed, exactly one of the two boundary arcs of M_L separating its two marked points includes an edge which is not in T_L. We identify the edges of this boundary arc in chronological order with the right boundary arc of H between its two marked edges, and similarly for M_R. This allows us to recover M. By definition, η crosses no edges of T, so $T_L \cup T_R = T \setminus \{e_1, e_2\}$.

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One easily sees that the set of possibilities for (M_L, T_L) depends only on H (not on (M_R, T_R)) and similarly with L and R interchanged. This allows us to decompose the counting measure μ on pairs (M, T) as a product measure

$$d\mu(M, T) = d\mu_{H,L}(M_L, T_L)\, d\mu_{H,R}(M_R, T_R)\, d\mu_0(H) \tag{A.6}$$

where μ_0 is the law of H and $\mu_{H,L}$ and $\mu_{H,R}$ are counting measures on H-dependent sets of tree-decorated maps. Plugging (A.4), (A.5), and (A.6) into (A.1) gives (A.3). $\qquad\square$

Finally, we make some general comments on the central charges associated to statistical physics models.

Remark A.3. If M is a planar map and S is a statistical mechanics model on M, then the central charge associated with S is the number $\mathbf{c}$ such that when M is large, the partition function of S is "close" (in a sense that we deliberately leave vague) to an exponential function of $\#\mathcal{E}(M)$ or $\#\mathcal{V}(M)$ times $(\det \Delta_M)^{-\mathbf{c}/2}$, where Δ_M is the discrete Laplacian. By the discussion immediately after (A.2), spanning trees have central charge -2 and the GFF has central charge 1. As another example, random walks have central charge 0. Indeed, let $\mathfrak{M}$ be a finite collection of planar maps with boundary decorated by a bulk marked point, and consider all pairs (M, P) where $M \in \mathfrak{M}$ and $P = (v_1, \ldots, v_n)$ is a path in M started at the marked point and terminated when it hits the boundary. Each pair (M, P) is given a weight $\prod_{i=1}^{n-1} d_i^{-1}$ where d_i is the degree of v_i. If we sample (M, P) proportionately to its weight, then the marginal law of M is the uniform measure on $\mathfrak{M}$; this lack of weighting means random walks have central charge 0. Brownian motion, as the scaling limit of random walk, should also be viewed as having central charge 0.

B Conditional independence for uniform meanders

In this appendix we prove a conditional independence property for uniform meanders (Proposition B.1 below). This property is the inspiration for the arguments in Section 4.5 and can also be viewed as a discrete analog of certain variants of our main results.

An *(open) meander* of size m is a self-avoiding loop from ∞ to ∞ in $\mathbb{C} \cup \{\infty\}$ which intersects $\mathbb{R}$ at points $1, 2, \ldots, 2m - 1$, crossing it transversally at each of those points, and viewed modulo homeomorphisms $f : \mathbb{C} \to \mathbb{C}$ with $f(\mathbb{H}) = \mathbb{H}$ and $f(i) = i$ for $i \in \{1, \ldots, 2m - 1\}$. Meanders have been studied at least since the work of Poincaré [Poi12], and are connected to many different topics in math and physics, but are notoriously difficult to analyze mathematically. See [Zvo21, La 03] for expository works on meanders. A meander of size m can be viewed as a planar map decorated by two Hamiltonian paths (corresponding to the loop and the real line). It is conjectured in [BGS22], building on [DFGG00], that uniform random meanders of size m converge in the scaling limit to the LQG sphere with central charge $\mathbf{c}_\mathrm{L} = 30$ together with two independent SLE_8 curves.

A meander can be encoded by a pair of *arc diagrams* (non-crossing perfect matchings) on $\{1, \ldots, 2m - 1\} \cup \{\infty\}$, one drawn above and the other drawn below $\mathbb{R}$. Orient the meander curve so it starts at ∞ in the lower half-plane and ends at ∞ in the upper half-plane. Then each arc inherits an orientation from the meander.

We can define a *winding function* $\theta : \{1, \ldots, 2m - 1\} \to \mathbb{R}$ as follows. With the above orientation of the curve, let it intersect $\mathbb{R}$ at points $v_1, \ldots, v_{2m-1}$ in order (so $v_1, \ldots, v_{2m-1}$ is a permutation of $1, \ldots, 2m - 1$). Let $\theta(v_1) = 0$, and inductively, we set $\theta(v_{i+1}) = \theta(v_i) + \pi$ if the arc from v_i to v_{i+1} is traversed in the counterclockwise direction and otherwise set $\theta(v_{i+1}) = \theta(v_i) - \pi$. See Figure 14. Note that if we draw the arcs as semicircles and rays as in the figure, then $\theta(v) + \frac{\pi}{2}$ is the restriction to $\{1, \ldots, 2m - 1\}$ of a winding function of the curve (defined in Section 2.6).
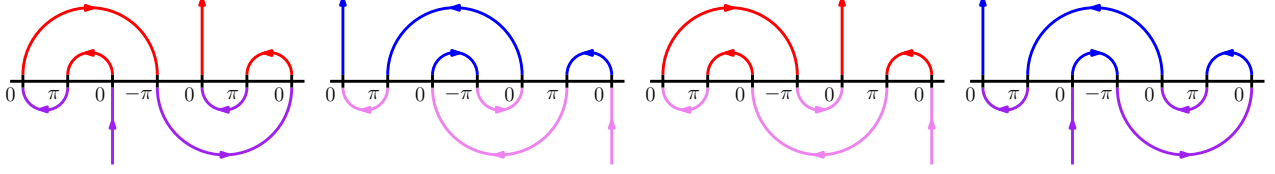
Figure 14: The four meanders with winding function given by $(\theta(1), \ldots, \theta(7)) = (0, \pi, 0, -\pi, 0, \pi, 0)$. In the leftmost diagram, we have $(v_1, \ldots, v_7) = (3, 2, 1, 4, 7, 6, 5)$. In the proof of Proposition B.1, we have $A_\theta^+ = \{\text{red, blue}\}$ and $A_\theta^- = \{\text{purple, pink}\}$, and the set of all meanders with winding function θ corresponds to $A_\theta^+ \times A_\theta^-$.

We have the following conditional independence statement for meanders, whose proof is a simpler version of the argument of Section 4.5.

Proposition B.1. *For $m \geq 1$, sample a meander uniformly at random from the set of all meanders of size m. Conditioned on its winding function $\theta : \{1, \ldots, 2m-1\} \to \mathbb{R}$, the (oriented) arc diagrams above and below $\mathbb{R}$ are conditionally independent.*

Proof. Fix θ, and let A_θ^+ (resp. A_θ^-) be the set of all arc diagrams on $\{1, \ldots, 2m-1\} \cup \{\infty\}$ drawn above $\mathbb{R}$ (resp. below $\mathbb{R}$), such that for every arc with endpoints $1 \leq x < y \leq 2m-1$ we have $\theta(y) = \theta(x) - \pi$ (resp. $\theta(y) = \theta(x) + \pi$). For each arc diagram in A_θ^+ (resp. A_θ^-), we give the arc hitting ∞ the orientation pointing towards (resp. away from) ∞, then orient all other arcs such that along $\mathbb{R}$ the arcs alternately enter and exit $\mathbb{R}$. By the definition of θ, every meander with winding function θ decomposes into a pair of arc diagrams in $A_\theta^+ \times A_\theta^-$, see Figure 14. We claim that conversely, every pair of arc diagrams in $A_\theta^+ \times A_\theta^-$ gives a meander with winding function θ. Indeed, any pair of arc diagrams in $A_\theta^+ \times A_\theta^-$ form a collection of one or more loops in $\mathbb{C} \cup \{\infty\}$; suppose for the sake of contradiction there is more than one loop. Pick a loop not containing ∞. By the definitions of A_θ^+, A_θ^-, when tracing the loop, each clockwise (resp. counterclockwise) arc should decrease (resp. increase) the winding function by π, so since the winding function at the start and end of the traversal is the same, there must be an equal number of clockwise and counterclockwise half-turns. This is topologically impossible: Hopf's Umlaufsatz states that the total curvature of a simple smooth loop is $\pm 2\pi$, meaning #clockwise turns $-$ #counterclockwise turns $= \pm 2$. Thus, every pair in $A_\theta^+ \times A_\theta^-$ gives a single loop from ∞ to ∞. Each such loop has winding function θ by the definition of $A_\theta^\pm$.

We conclude that the set of all meanders with winding function θ is in bijection with $A_\theta^+ \times A_\theta^-$, giving the desired conditional independence. $\square$

Remark B.2. A *meandric system* is defined in a similar manner to a meander, except that one allows there to be multiple loops. For a meandric system, the arc diagrams above and below the real line are exactly independent, rather than just conditionally independent. It is conjectured in [BGP23] that uniform meandric systems converge in the scaling limit to the LQG sphere with central charge $\mathbf{c}_{\mathrm{L}} = 28$ together with an SLE$_8$ curve and an independent CLE$_6$. Since $\mathbf{c}(\kappa = 8) = -2 = 26 - 28$, the LQG and the SLE$_8$ curve have "matched" parameter values. The fact that we have conditional independence, rather than exact independence, in Proposition B.1 is analogous to the fact that we have conditional independence, rather than exact independence, in the main results of this paper, as stated in Section 1.3.

We expect that it is possible to extend the results of this paper to get a continuum analog of Proposition B.1. See Problem 5.7 and the discussion just after. We plan to explore discrete analogs

of the results of this paper in the setting of meanders and meandric systems further in future work.

C Dubédat's proof of locality

In this appendix we explain the proof of Theorem 3.10. It is a special case of Theorem C.1 below, which was proved by Dubédat who used it to establish his seminal SLE-GFF coupling [Dub09b]. It is not stated as a theorem in [Dub09b], but its discrete analog is stated in the last equation of Section 4.1 there.

In the setting of Section 3.2.1, we have four simply connected domains D_{ij} having a common sub-domain B bounded by a pair of cross cuts and common boundary point $x \in \partial B \cap \partial D_{ij}$. Recall that $\mathrm{IG}_{\mathbf{c}}$ in (D_{ij}, x) as defined in Definition 1.7 can be viewed as a probability measure on fields in D_{ij} modulo $2\pi\chi(\mathbf{c})$; let Ψ be a sample with its additive constant fixed by requiring that its boundary value infinitesimally counterclockwise of x in ∂D_{ij} lies in $[0, 2\pi\chi(\mathbf{c}))$. Let $\mathrm{IG}_{\mathbf{c}}^{B,ij}$ be the law of $\psi = \Psi|_B$. Note that here, $\mathrm{IG}_{\mathbf{c}}^{B,ij}$ is a measure on the space of generalized functions and not $\mathbf{c}$-IG fields; this is a slight change in notation from Theorem 3.10.

Theorem C.1 ([Dub09b]). *We have*

$$\left(\frac{\det_\zeta(\Delta_{D_{11}})\det_\zeta(\Delta_{D_{22}})}{\det_\zeta(\Delta_{D_{21}})\det_\zeta(\Delta_{D_{12}})} \right)^{-\mathbf{c}/2} \frac{d\mathrm{IG}_{\mathbf{c}}^{B,11}}{d\mathrm{IG}_{\mathbf{c}}^{B,21}}(\psi) \frac{d\mathrm{IG}_{\mathbf{c}}^{B,22}}{d\mathrm{IG}_{\mathbf{c}}^{B,12}}(\psi) = 1 \qquad \textit{for } \mathrm{IG}_{\mathbf{c}}^{B,ij}\textit{-a.e. } \psi \qquad (\text{C.1})$$

where $\det_\zeta(\Delta_D)$ is the zeta regularized determinant of the Dirichlet Laplacian.

We note that Theorem C.1 suggests the partition function of $\mathrm{IG}_{\mathbf{c}}$ in a domain D should be $\det_\zeta(\Delta_D)^{-\mathbf{c}/2}$.

Proof of Theorem 3.10. Set $\mathbf{c} = 0$ in Theorem C.1. $\qquad\qquad\square$

Let D be one of the domains D_{ij} and let $m = m_{ij}$ be the harmonic function on D_{ij} whose boundary values agree with those of a sample from $\mathrm{IG}_{\mathbf{c}}$ in (D_{ij}, x) with additive constant fixed as before. Note that m_{ij} is continuous on ∂D_{ij} except at x where there is a jump of size $2\pi\chi(\mathbf{c})$. Let $\delta = \partial B \cap D_{ij}$ be the union of the pair of crosscuts bounding B. Let μ be the law of $\Psi|_\delta$ where Ψ is a GFF with boundary conditions given by m, and μ^0 the law of $\Psi^0|_\delta$ where Ψ^0 is a zero boundary GFF on D. As explained in [Dub09b, Section 4.3], μ and μ^0 are probability measures on the negative Sobolev space $H^{-s}(\delta)$ for any $s > 0$. We will write w to denote a field on δ. Let Pw be the harmonic extension of w to $D\backslash\delta$ with zero boundary conditions on ∂D.

For a function f on D and a point $z \in \delta$ let $\partial_\ell f(z) := \lim_{\varepsilon\to 0} \varepsilon^{-1}(f(z + \varepsilon n_\ell) - f(z))$ where n_ℓ is the left-pointing normal unit vector on δ at w, and define ∂_r the same way with left replaced by right. Thus $\partial_\ell f$ and $\partial_r f$ are functions with domain δ, and $(\partial_\ell + \partial_r)f = 0$ at points where f is smooth.

Let $m' = m - P(m|_\delta)$. This is the function on D such that $m'|_\delta = 0, m'|_{\partial D} = m|_{\partial D}$, and m' is harmonic on $D\backslash\delta$. The following computation is carried out in [Dub09b, Proof of Lemma 6.3]. We repeat the proof for the reader's convenience.

Lemma C.2. *Define the regularized Dirichlet energy*

$$(m, m)_\nabla^{\mathrm{reg}} := \frac{1}{2\pi} \lim_{\varepsilon\to 0} \left(\frac{(2\pi\chi(\mathbf{c}))^2}{\pi} \log\varepsilon + \int_{D_{ij}\backslash B_\varepsilon(x)} \nabla m \cdot \nabla m \right).$$

With the following integral performed with respect to arc length on δ, we have

$$\frac{d\mu}{d\mu^0}(w) = \exp\left(-\frac{1}{2\pi} \int_\delta w(\partial_\ell + \partial_r)m' - \frac{1}{2}(m', m')_\nabla^{\mathrm{reg}} + \frac{1}{2}(m, m)_\nabla^{\mathrm{reg}} \right).$$

Proof. Let $m'' = P(m|_\delta)$, so $m''|_\delta = m|_\delta$ and $m''|_{\partial D} = 0$. Let Ψ^0 be a zero boundary GFF on D. By Girsanov's theorem (see e.g. [MS17, Proposition 2.9]), the Radon-Nikodym derivative of the law of $\Psi^0 + m''$ with respect to that of Ψ^0 is

$$\exp((\Psi^0, m'')_\nabla - \frac{1}{2}(m'', m'')_\nabla).$$

By Stoke's theorem in each of the components of $D\backslash\delta$ and since $\Delta m''|_{D\backslash\delta} = 0$ and $\Phi^0|_{\partial D} = 0$,

$$(\Psi^0, m'')_\nabla = \frac{1}{2\pi}\int_\delta \Psi^0(\partial_\ell + \partial_r)m'' = -\frac{1}{2\pi}\int_\delta \Psi^0(\partial_\ell + \partial_r)m'.$$

Here, the second equality follows from $m = m' + m''$ and $(\partial_\ell + \partial_r)m = 0$. Thus,

$$\frac{d\mu}{d\mu^0}(w) = \exp(-\frac{1}{2\pi}\int_\delta w(\partial_\ell + \partial_r)m' - \frac{1}{2}(m'', m'')_\nabla). \tag{C.2}$$

Finally, we need to show that $(m'', m'')_\nabla = (m', m')_\nabla^{\text{reg}} - (m, m)_\nabla^{\text{reg}}$. Let $\varepsilon > 0$ and let m_ε be a harmonic function on D with continuous boundary data. With $m''_\varepsilon = P(m_\varepsilon|_\delta)$ and $m'_\varepsilon = m_\varepsilon - m''_\varepsilon$, Stoke's theorem gives $2\pi(m_\varepsilon, m''_\varepsilon)_\nabla = -\int_{\partial D} m''_\varepsilon \partial_n m_\varepsilon + \int_D m''_\varepsilon \Delta m_\varepsilon$. This is zero because $m''_\varepsilon|_{\partial D} = 0$ and $\Delta m_\varepsilon = 0$, so expanding $(m_\varepsilon, m''_\varepsilon)_\nabla = 0$ gives $(m_\varepsilon, m_\varepsilon)_\nabla = (m_\varepsilon, m'_\varepsilon)_\nabla$. This implies $(m''_\varepsilon, m''_\varepsilon)_\nabla = (m'_\varepsilon, m'_\varepsilon)_\nabla - (m_\varepsilon, m_\varepsilon)_\nabla$.

We will now choose m_ε to approximate m then send $\varepsilon \to 0$. Assume the boundary data of m_ε agrees with that of m in $\partial D\backslash B_{\varepsilon^2}(x)$, and assume this boundary data is uniformly bounded for all ε. Since $m_\varepsilon|_\delta$ converges uniformly to $m|_\delta$ we have $(m''_\varepsilon, m''_\varepsilon)_\nabla \to (m'', m'')_\nabla$. Next, since the harmonic measure of $\partial D \cap B_{\varepsilon^2}(x)$ is $o_\varepsilon(1)$ uniformly for all $z \in D\backslash B_\varepsilon(x)$, we have $\left|\int_{D\backslash B_\varepsilon(x)} \nabla m_\varepsilon \cdot \nabla m_\varepsilon - \nabla m \cdot \nabla m\right| = o_\varepsilon(1)$, and similarly $\left|\int_{D\backslash B_\varepsilon(x)} \nabla m'_\varepsilon \cdot \nabla m'_\varepsilon - \nabla m' \cdot \nabla m'\right| = o_\varepsilon(1)$. Moreover, since Brownian motion started in $D\cap B_\varepsilon(x)$ hits ∂D before $B_{\sqrt{\varepsilon}}(x)$ with probability $1 - o_\varepsilon(1)$ uniformly in the starting point, and the harmonic functions m_ε in D and m'_ε in $D\backslash\delta$ have the same boundary data on $\partial D \cap B_{\sqrt{\varepsilon}}(x)$, we conclude $\left|\int_{D\cap B_\varepsilon(x)} \nabla m_\varepsilon \cdot \nabla m_\varepsilon - \nabla m'_\varepsilon \cdot \nabla m'_\varepsilon\right| = o_\varepsilon(1)$. To summarize, $(m'_\varepsilon, m'_\varepsilon)_\nabla - (m_\varepsilon, m_\varepsilon)_\nabla \to (m', m')_\nabla^{\text{reg}} - (m, m)_\nabla^{\text{reg}}$. Therefore $(m'', m'')_\nabla = (m', m')_\nabla^{\text{reg}} - (m, m)_\nabla^{\text{reg}}$, which with (C.2) completes the proof. $\square$

The following identity follows from setting $(a, b) = (0, \chi(\mathbf{c})/\sqrt{2\pi})$ in [Dub09b, Proposition 5.2].

Lemma C.3 ([Dub09b, Proposition 5.2]). *There is a constant λ such that*

$$e^{-\frac{1}{2}(m,m)_\nabla^{\text{reg}}} = \lambda\det_\zeta(\Delta_D)^{3\chi(\mathbf{c})^2}.$$

Now we state two other results from [Dub09b]. Let $m^l(D,; K_1, K_2)$ be the mass of Brownian loops in D intersecting both K_1 and K_2, as introduced in [LW04].

Lemma C.4 ([Dub09b, Proposition 2.1]). *Let C be a collar neighborhood of δ in D, that is, a neighborhood of δ which is diffeomorphic to $\delta \times (-1, 1)$ via a diffeomoephism which takes the intersection of the neighborhood with ∂D to $(\partial\delta) \times (-1, 1)$. Then*

$$\exp(-m^l(D; \delta, D\backslash C)) = \frac{\det_\zeta(\Delta_D)\det_\zeta(\Delta_{C\backslash\delta})}{\det_\zeta(\Delta_{D\backslash\delta})\det_\zeta(\Delta_C)}.$$

For $i, j, \in \{1, 2\}$, let μ_{ij} (resp. μ_{ij}^0) be the law of h sampled from $\text{IG}_{\mathbf{c}}^{K,ij}$ (resp. zero boundary GFF on D_{ij}) restricted to δ. Let $P_{ij}w$ be the function on D_{ij} which is harmonic on $D_{ij}\backslash\delta$, equals w on δ and equals zero on ∂D_{ij}.

Lemma C.5 ([Dub09b, Lemma 4.4]). *Let C be any common collar neighborhood of δ for D_{ij} and $D_{i'j'}$. Then*

$$\frac{d\mu^0_{i'j'}}{d\mu^0_{ij}}(w) = \exp\left(\pi((P_{ij}w, P_{ij}w)_\nabla - (P_{i'j'}w, P_{i'j'}w)_\nabla) + \frac{1}{2}(m^l(D_{ij}; \delta, D_{ij}\backslash C) - m^l(D_{i'j'}; \delta, D_{i'j'}\backslash C))\right).$$

We turn to the proof of Theorem C.1. It is essentially done in [Dub09b, Proof of Lemma 6.3].

Proof of Theorem C.1. All Radon-Nikodym derivative identities in this proof hold for μ_{ij}-a.e. w. We have

$$\frac{d\mu_{11}}{d\mu_{21}} \times \frac{d\mu_{22}}{d\mu_{12}} = \frac{d\mu_{11}}{d\mu^0_{11}}\frac{d\mu^0_{11}}{d\mu^0_{21}}\frac{d\mu^0_{21}}{d\mu_{21}} \times \frac{d\mu_{22}}{d\mu^0_{22}}\frac{d\mu^0_{22}}{d\mu^0_{12}}\frac{d\mu^0_{12}}{d\mu_{12}}.$$

Let $P_{ij}w$ denote the harmonic function on D_{ij} with boundary conditions of w on δ and zero boundary conditions on ∂D_{ij}. Then $m'_{ij} := m_{ij} - P_{ij}(m_{ij}|_\delta)$ is the harmonic function on $D_{ij}\backslash\delta$ with boundary conditions on ∂D_{ij} agreeing with those of m_{ij}, and zero boundary conditions on δ. Notice that the boundary values of m_{ij} in the left subdomain of D_{ij} do not depend on j since the boundary values near x do not depend on j and the boundary values along $\partial\mathbb{D}$ in the clockwise direction from x vary according to the total curvature (winding) of the boundary. Thus m'_{ij} restricted to the left subdomain of D_{ij} does not depend on j, and likewise m'_{ij} restricted to the right subdomain does not depend on i. Thus, by Lemma C.2

$$\frac{d\mu_{11}}{d\mu^0_{11}}\frac{d\mu^0_{21}}{d\mu_{21}}\frac{d\mu_{22}}{d\mu^0_{22}}\frac{d\mu^0_{12}}{d\mu_{12}} = \exp(\frac{1}{2}((m_{11}, m_{11})^{\text{reg}}_\nabla - (m_{21}, m_{21})^{\text{reg}}_\nabla + (m_{22}, m_{22})^{\text{reg}}_\nabla - (m_{12}, m_{12})^{\text{reg}}_\nabla)),$$

since all terms involving m'_{ij} cancel. Next, since $P_{ij}w$ restricted to the left (resp. right) subdomain of D_{ij} does not depend on j (resp. i), we have

$$(P_{11}w, P_{11}w)_\nabla - (P_{21}w, P_{21}w)_\nabla + (P_{22}w, P_{22}w)_\nabla - (P_{12}w, P_{12}w)_\nabla = 0.$$

Thus, by Lemmas C.4 and C.5, we have

$$\frac{d\mu^0_{11}}{d\mu^0_{21}}\frac{d\mu^0_{22}}{d\mu^0_{12}} = \sqrt{\frac{\det_\zeta(\Delta_{D_{11}})\det_\zeta(\Delta_{D_{22}})}{\det_\zeta(\Delta_{D_{21}})\det_\zeta(\Delta_{D_{12}})}}.$$

Combining the above identities and Lemma C.3, and using $-\frac{1}{2} + 3\chi(\mathbf{c})^2 = -\frac{\mathbf{c}}{2}$, we conclude

$$\left(\frac{\det_\zeta(\Delta_{D_{11}})\det_\zeta(\Delta_{D_{22}})}{\det_\zeta(\Delta_{D_{21}})\det_\zeta(\Delta_{D_{12}})}\right)^{-\mathbf{c}/2}\frac{d\mu_{11}}{d\mu_{21}}\frac{d\mu_{22}}{d\mu_{12}} = 1.$$

By the domain Markov property of the Gaussian free field, this implies (C.1). □

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