Normal Model for Backward-Looking In-Arrears RFR Caplets

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I extend the lognormal Lyashenko and Mercurio (1) model for caplets on forward risk-free term rates (e.g., daily compounded SOFR, SONIA, ESTER, etc.) to the case of **normally** distributed forward term rates. This *Bachelier Model* has the advantage of being robust with respect to negative rates. As the legacy LIBOR caplet market quotes both lognormal (*Black*) and normal-model implied volatilities, it is quite possible market participants will adopt this model for, *e.g.*, SOFR caplets. Model Risk teams can employ this model as well as the *Displaced Lognormal Model* for benchmarking production caplet valuation models.

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The daily-compounded forward risk-free rate (RFR) is

$$R_t\big[T_{start}^{(ref)}, T_{end}^{(ref)}\big] = \frac{1}{\alpha\big[T_{start}^{(ref)}, T_{end}^{(ref)}\big]}\left[\prod_{i=1}^{n}\Big(1+\delta_{i-1}r_{t,i-1}\Big)-1\right] \tag{1}$$

where the daily-fixing overnight RFR rate rate $r_{t,i-1}$ over the period $[T_{i-1}, T_i)$ is fixed within the reference period bounded by the start date $T_{start}^{(ref)}$ and end date $T_{end}^{(ref)}$. The duration of the accrual period for the reference period is $\alpha[T_{start}^{(ref)}, T_{end}^{(ref)}]$ and δ_{i-1} is the overnight accrual period (*e.g.*, $\frac{1}{360}$ for $30/360$ day count convention). The payoff at option expiry, $T_{end}^{(ref)}$, of the backward-looking RFR caplet with strike K is

$$v(T_{end}^{(ref)}) = notional \times \alpha\big[T_{start}^{(ref)}, T_{end}^{(ref)}\big] \times \Big[R_{T_{end}^{(ref)}}\big[T_{start}^{(ref)}, T_{end}^{(ref)}\big] - K\Big]^{+} \tag{2}$$

Lyashenko and Mercurio (1) introduced the *Forward Market Model* to price the caplet. The convenient numeraire choice in FMM is the extended zero-coupon bond,

$$P^{(extended)}(t,T) = \begin{cases} \mathbb{E}_Q\left[e^{-\int_t^T r(u)du}\Big|\mathcal{F}_t\right] & \text{for } t \le T \\ e^{\int_T^t r(u)du} & \text{for } t > T \end{cases} \tag{3}$$

Within FMM the forward RFR rate is assumed a lognormal martingale

$$dR[T_{start}^{(ref)}, T_{end}^{(ref)}](t) = \sigma g(t) R[T_{start}^{(ref)}, T_{end}^{(ref)}](t)\mathbb{1}\big(t \le T_{end}^{(ref)}\big)dW(t) \tag{4}$$

where $g(t)$ is the linear volatility decay function

$$\begin{aligned} g(t) &= g[T_{start}^{(ref)}, T_{end}^{(ref)}](t) \\ &= \min\left[\frac{\big(T_{end}^{(ref)}-t\big)^{+}}{T_{end}^{(ref)}-T_{start}^{(ref)}}, 1\right] \end{aligned} \tag{5}$$

This function captures the fact that the volatility of the term forward decreases with each daily RFR fixing during the reference period. The volatility $\sigma = \sigma[T^{(ref)}_{start}, T^{(ref)}_{end}]$ is presently assumed constant but the model can be modified to accommodate a state-dependent volatility. The Brownian motion applies to the forward rate bounded by the reference interval, $dW(t) = dW[T^{(ref)}_{start}, T^{(ref)}_{end}](t)$.

Within FMM the discretely compounded forward rate Eq.(1) is approximated as a continuously compounded rate

$$R[T^{(ref)}_{start}, T^{(ref)}_{end}] = \frac{1}{\alpha R_t[T^{(ref)}_{start}, T^{(ref)}_{end}]}\left[e^{+\int_{T^{(ref)}_{start}}^{T^{(ref)}_{end}} r(u)du} - 1\right] \tag{6}$$

$$\tag{7}$$

In the **Bachelier Model** extension of the FMM model, the forward RFR rate is assumed a **normal** martingale

$$dR[T^{(ref)}_{start}, T^{(ref)}_{end}](t) = \sigma g(t)\mathbb{1}\big(t \le T^{(ref)}_{end}\big)dW(t) \tag{8}$$

and in the **Displaced Lognormal Model** it is a displaced lognormal martingale (with displacement parameter s)

$$dR[T^{(ref)}_{start}, T^{(ref)}_{end}](t) = \sigma_j g_j(t)\mathbb{1}\big(t \le T_j\big)\big(R[T^{(ref)}_{start}, T^{(ref)}_{end}](t) + s\big)dW(t) \tag{9}$$

Note this model has the disadvantage of an exogenous displacement parameter. As illustrated below this Displaced Lognormal model is materially sensitive to this parameter.

The valuation formula for the caplet is calculated by taking the extended forward measure expectation of the numeraire-rebased terminal caplet payoff to obtain

$$v(t) = \alpha\big[T^{(ref)}_{start}, T^{(ref)}_{end}\big]P^{(extended)}(t, T^{(ref)}_{end})\Big\{(R(0)-K)\Phi\Big(\frac{R(0)-K}{\sqrt{\nu}}\Big) + \sqrt{\nu}\phi\Big(\frac{R(0)-K}{\sqrt{\nu}}\Big)\Big\} \tag{10}$$

for the *Normal Model* and

$$v(t) = \alpha\big[T^{(ref)}_{start}, T^{(ref)}_{end}\big]P^{(extended)}(t, T^{(ref)}_{end})\Bigg[\tilde{R}(0)\Phi\Big[\frac{\ln\big(\frac{\tilde{R}(0)}{\tilde{K}}\big) + \frac{1}{2}\nu}{\sqrt{\nu}}\Big] - \tilde{K}\Phi\Big[\frac{\ln\big(\frac{\tilde{R}(0)}{\tilde{K}}\big) - \frac{1}{2}\nu}{\sqrt{\nu}}\Big]\Bigg] \tag{11}$$

for the *Displaced Lognormal Model*. Φ and ϕ are the normal Gaussian CDF and PDF, respectively, $R(0) = R[T^{(ref)}_{start}, T^{(ref)}_{end}](0)$, $\tilde{R}(0) = R(0) + s$, and $\tilde{K} = K + s$. The variance of the process Eq. (8) is

$$\begin{aligned}\nu &= var^{Q^{T^{(ref)}_{end}}}\Big[R_j(t)\Big] \\ &= \sigma^2\Big[(T^{(ref)}_{start} - t)^+ + \frac{1}{3}\frac{\big(T^{(ref)}_{end} - (T^{(ref)}_{start} \vee t)\big)^3}{\big(T^{(ref)}_{end} - T^{(ref)}_{start}\big)^2}\Big]\end{aligned} \tag{12}$$

Fig. (1) is a plot of the value, Eqs. (10) and (11) for a two-year expiry caplet struck at -5bp for a three-month forward rate. The equivalent Bachelier volatility is implied from the Black volatility value 50bp. The Bachelier (Normal) value curve is bounded by the two

Displaced Lognormal curves (11bp and 50bp displacement, respectively). Fig. 2 shows the first order risk measure, $\Delta = \frac{\partial v}{\partial R(0)}$, for the same three models.

RG Risk Consulting can provide the derivative model benchmarking and testing expertise so your model risk and governance groups will stay ahead of the SOFR to LIBOR transition.

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References

(1) Lyashenko, Andrei and Fabio Mercurio, *Looking Forward to Backward-Looking Rates: A Modeling Framework for Term Rates Replacing LIBOR,* working paper (Feb. 2020).

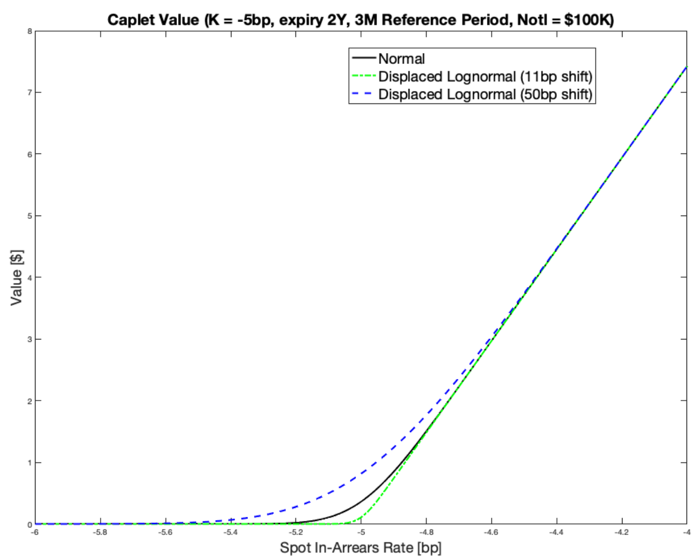

Figure 1: Caplet value as a function of underlying spot forward RFR rate for a two-year expiry option on a three-month term RFR forward rate with strike $K = -5bp$. The normal model, diplaced lognormal with shifts $11bp, 50bp$, respectively, are shown.

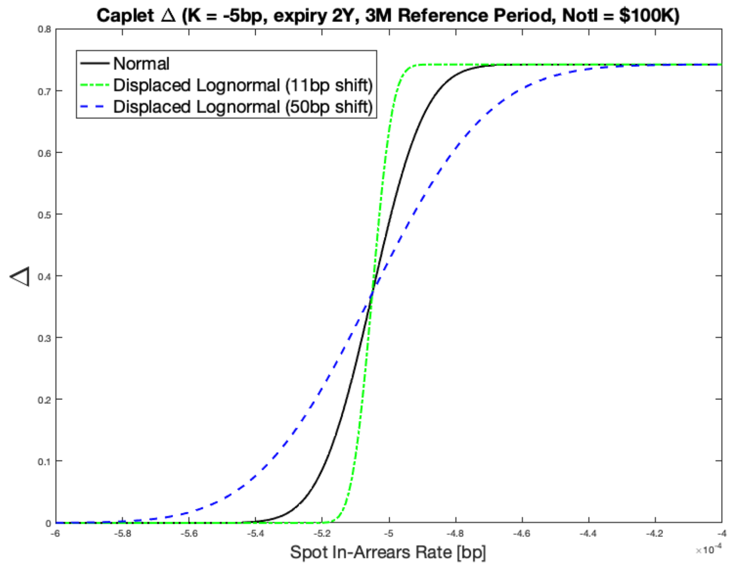

Figure 2: Caplet Delta ($\Delta = \frac{\partial v}{\partial R(0)}$) as a function of underlying spot forward RFR rate for a two-year expiry option on a three-month term RFR forward rate with strike $K = -5bp$. The normal model, diplaced lognormal with shifts $11bp, 50bp$, respectively, are shown.