Notes on real analysis

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Chapter 1

Preface

First, let me be clear. I am not a mathematician. These notes are not intended as a manual, however I like to teach, explain science, and I am firm believer that the best way to learn something is to teach each. Richard Feynman famously said the best way to learn is to follow these steps:

1. Study: arguably the easiest part, whether you like to take notes on paper, tablet or your computer. Whether you like to sit on a table or on the couch. However, as many people who have come this far know, reading, taking notes or making exercises only take you so far.

2. Teach: this is where the fun begins, as you try to explain something you know (or think you know) to someone, you start being more aware of your limitations, of the gaps in the proofs you cannot explain, the unexpected questions that may appear lead you astray. Even without an audience, this is a nice thing to do as it forces you to be clear and think about how to explain something in a clear yet rigorous way.

3. Fill the gaps: now it is time to come back to studying, reading more, exploring new books, papers or what else. Once you have discovered your limitations on the previous step, you are once again in the position to learn and study, but now you know where to look.

4. Simplify: one of the greatest sins we commit is to get stuck with fancy proofs, delude ourselves in the beauty of math. Make it so that people will understand and enjoy what they are reading or listening.

In this document, I have written my learnings from studying Real Analysis. I hope to learn more while writing it.

Chapter 2

The real numbers

During high school math we are often given a simplified definition of the real numbers, one it may take a while to fully grasp how awkward it is: "The real numbers is the set which contains the rational numbers and the irrational numbers". Taking alone it may seem a reasonable statement. In fact, it is true. However, if we start with the natural numbers there is a very concise and clear way of writing it:

$$\mathbb{N} = \{1, 2, 3, ...\} \tag{2.1}$$

Taking one step further, the integers follow quite naturally:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \tag{2.2}$$

And even for the rationals, we can clearly write:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\} \tag{2.3}$$

Now, for the real numbers things are not so clear. So, we are stuck with our initial understanding of the big set which includes the rational and irrational numbers. So, let's start by looking more carefully at this similarly weird creature.

2.1 Irrational numbers

Before we proceed, let's take a minute to appreciate why we need irrational numbers. The following result will play an important role to distinguish the "holes" of the rational numbers when compared with the reals. We begin with a theorem.

Theorem 1. *There is no such number whose square root is* 2

Proof. As stated before, a rational number is one that can be written in the form p/q, with $q \neq 0$. Our approach here is what is called proof by contradiction. We will assume the opposite of what we want to prove, once we arrive at some absurd result we will conclude our initial assumption was wrong. Therefore, assume $\exists p, q \in \mathbb{Z} : (p/q)^2 = 2$, additionally, we take p and q with no common

factors, such that the fraction p/q is written in its simplest form.
If this is true, we can rearrange the relation into: $p^2 = 2q^2$. Which implies p^2 is an even number, since the square of any odd number is odd, p must also be even, *i.e.* $p = 2r$.
Now, replacing p on the previous equation yields $4r^2 = 2q^2 \Rightarrow q^2 = 2r^2$, which implies q^2 and so is q.
This directly contradicts our initial assumption, since p and q are both even from the result above.
Hence, our initial assumption must be wrong, and we conclude $\nexists p, q \in \mathbb{Z} : (p/q)^2 = 2$. $\square$

In order to deal with irrational numbers, the set of real numbers is the natural extension necessary. Before we deal with it in a rigorous way, will start with the necessary tools to help us on this journey.

2.2 Preliminaries

This section aims to define some basic definitions and results that will help us to deal with real numbers, and the other topics of interest.

2.2.1 Set theory

Definition 1 (Set). *A set is a collection of objects, called elements or members. An empty set is a set with no elements, denoted* $\varnothing$.

Usually, we write a set A as $A = \{a_1, a_2, ...\}$ where $a_1, a_2, ...$ are the elements of the set. Some important notations are:

- $a \in A$: meaning a is an element of A
- $a \notin A$: meaning a is not an element of A
- $\forall$: meaning 'for all'. For example, in mathematical notation the expression 'for all a which is an element of A' would be $\forall a \in B$
- $\exists$: meaning 'there exists'. The opposite would be $\nexists$
- $\Rightarrow$: implies
- $\Leftrightarrow$: if and only if

On a sidenote, the terms 'implies' and 'if and only if' have fundamental differences which will lead to different approaches in demonstrations and results. For instance, let's say proposition P_1 implies proposition P_2. In mathematical notation $P_1 \Rightarrow P_2$. This means that if P_1 is true, so is P_2, but it does not say anything about the opposite direction. That is, if P_2 is true, not necessarily P_1 is true. On the other hand when the relation is P_1 is true if, and only if, P_2 is true. Or, $P_1 \Leftrightarrow P_2$ then the relation works both ways: if P_1 is true, so is P_2 and if P_2 is true, so is P_1. During proofs, when we have a $Leftrightarrow$ relation, the result must be proven in both directions.

Definition 2 (Subset). *A is a subset of B if, every element of A is also an element of B. Notation:* $A \subseteq B$. *Equivalently, if* B *is a superset of* A*, it is denoted* $B \supseteq A$.

Informally, we understand that two sets are equal if every element of one set is also an element of the other, and vice-versa. On mathematical notation:

Definition 3 (Equal sets). *Two sets, A and B, are equal if $A \subseteq B$ and $B \subseteq A$. Hence, $A = B$.*

Definition 4 (Proper subset). *A set A is a proper subset of B if $A \subseteq B$ and $A \neq B$. Notation: $A \subsetneq B$.*

Now, tow (or more) sets can be combined by operations. We define:

- Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- Difference: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$
- Complement: $A^C = \{x \notin A\}$

Definition 5 (Disjoint sets). *Two sets, A and B, are disjoint if $A \cap B = \varnothing$.*

Some important from set theory are the so-called De Morgan's laws:

Theorem 2 (De Morgan's Laws). *If A, B and C are sets, then:*

- $(B \cup C)^C = B^C \cap C^C$
- $(B \cap C)^C = B^C \cup C^C$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Following the proof of the first result is shown, the other results can be derived similarly.

Proof. Two sets X and Y are equal if $X \subseteq Y$ and $Y \subseteq X$. Our goal is to show that $(B \cup C)^C \subseteq B^C \cap C^C$ and $(B \cup C)^C \supseteq B^C \cap C^C$.
Let $x \in (B \cup C)^C$. It follows that $x \notin (B \cup C)$ then $x \notin B$ and $x \notin C$. So, $x \in B^C \cap C^C$ and we have $(B \cup C)^C \subseteq B^C \cap C^C$.
From the opposite direction, let $x \in B^C \cap C^C$. Then $x \in B^C$ and $x \in C^C$ which means $x \notin B$ and $x \notin C$. So $x \notin (B \cup C) \Rightarrow x \in (B \cup C)^C$. So $B^C \cap C^C \subseteq (B \cup C)^C$. Since $(B \cup C)^C \subseteq B^C \cap C^C$ and $B^C \cap C^C \subseteq (B \cup C)^C$, we have $(B \cup C)^C \subseteq B^C \cap C^C$. □

Fields

Definition 6 (Field). *A set F is a field if satisfies the following properties:*

- *For addition*
  1. *If $x, y \in F \Rightarrow x + y \in F$*
  2. *Commutativity: $\forall x, y \in F : x + y = y + x$*

3. *Associativity:* $\forall x, y, z \in F : (x + y) + z = x + (y + z)$
4. *Additive identity:* $\exists 0 \in F : 0 + x = x, \forall x \in F$
5. *Additive inverse:* $\exists -x \in F : x + (-x) = 0, \forall x \in F$

- *For multiplication*
  1. *If* $x, y \in F \Rightarrow x \cdot y \in F$
  2. *Commutativity:* $\forall x, y \in F : x \cdot y = y \cdot x$
  3. *Associativity:* $\forall x, y, z \in F : (x \cdot y)z = x(y \cdot z)$
  4. *Multiplicative identity:* $\exists 1 \in F : x \cdot 1 = x, \forall x \in F$
  5. *Multiplicative inverse:* $\exists x^{-1} \in F : x \cdot x^{-1} = 1, \forall x \in F$

Theorem 3. *If F is a field,* $\forall x \in F : x \cdot 0 = 0$.

Proof. If $x \in F$ then $0x \in F$ so $0 = 0x + (-0x) = 0x + 0x + (-0x) = 0x$. □

Definition 7 (Ordered field)**.** *An ordered field F is a field which satisfies $\forall x, y, z \in F$:*

1. *If $x < y$ then $x + z < y + z$*
2. *If $x > 0$ and $y > 0$ then $xy > 0$*

Bounds

Definition 8 (Bounds)**.** *Let $A \subseteq B$. Then,*

1. *If $\exists u \in B : u \geq a, \forall a \in A$ then A is bounded above and u is an upper bound for A.*
2. *If $\exists l \in B : l \leq a, \forall a \in A$ then A is bounded below and l is a lower bound for A.*

Example Consider the set $B = \mathbb{R}$ and $A = [0, 1]$. Then, $2, 2.5, \pi$ are all upper bounds for A. Similarly, $-1, 0, -\pi$ are all lower bounds for A.

Definition 9 (Supremum)**.** *Let $A \subseteq B$ with A bounded above. Then s is the least upper bound (or supremum) if:*

1. *s is an upper bound for A, and*
2. *If u is another upper bound for A then $s \leq u$.*

Mathematically, we write $s = \sup A$.

Definition 10 (Infimum)**.** *Let $A \subseteq B$, with A bounded below. Then, i is the greatest lower bound (or infimum) of A if:*

1. *i is a lower bound for A, and*
2. *If l is another lower bound for A then $i \geq l$.*

Mathematically, we write $i = \inf A$.

Example Consider $B = \mathbb{R}$ and $A = (0, 1) \subseteq A$. Then $1, \pi, 10$ are all upper bounds for A but 1 is the least upper bound (or infimum). On the other hand, $-10, -1, 0$ are all lower bounds for A, but only 0 is the greatest lower bound (or infimum) of A.

The previous example shows an important characteristic of the supremum (or infimum). In this case $0 \notin A$ and $1 \notin B$. We can also define:

Definition 11 (Maximum)**.** *Let A be a set bounded above, then M is the maximum of A if $M \in A$ and $M \geq a, \forall a \in A$.*

Definition 12 (Minimum)**.** *Let A be a set bounded below, then m is the minimum of A if $m \in A$ and $m \leq a, \forall a \in A$.*

Notice that a set may have an infimum and not a minimum, as the previous example, since $0 \notin A$. The same result is valid for the supremum and maximum. On the other hand, if a set A has a maximum, then it necessarily has a supremum. An equivalent result holds for the infimum and minimum.

2.2.2 Function

The formal definition of function is the following:

Definition 13 (Function)**.** *Given a set A and a set B, a function is a mapping rule which takes as an argument an element $a \in A$ and associates it with an element of B. We write $f : A \to B$. $f(a)$ is used to express the element of B, $f(a) \in B$, associated with the element $a \in A$. A is called the domain of the function, while B is its codmoain. The image of f is not necessarily equal to B, but refers to $\{b \in B : b = f(a) \text{ for some } a \in A\} \subseteq B$.*

It is worth noting how this definition liberates math from the usual 'formula' understanding of a function. In particular, this definition is closer to Dirichlet's definition, and it allows math to deal with more interesting and complex functions, such as:

Example - Dirichlet's function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases} \tag{2.4}$$

This broader definition of function will lead to interesting results and test some limits in math. But more on that latter.

Classification

Definition 14. *The function $f : A \to B$ is called 1-1 or injective if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. Equivalently, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$.*

Definition 15. *The function $f : A \to B$ is called onto or surjective if $\forall b \in B, \exists a \in A$ such that $f(a) = b$.*

Definition 16. *A function that is both injective and surjective is called bijective.*

Composition and inverse

Definition 17 (Composite function)**.** *If $f : A \to B$ and $g : B \to C$, then $f \circ g : A \to C$ is defined by $(f \circ g)(x) = g(f(x))$.*

Definition 18 (Inverse function)**.** *Consider $f : A \to B$ a bijective function. Then the inverse function $f^{-1} : B \to A$ is defined by: if $b \in B$ then $f^{-1}(b) \in A$ is the unique element $f^{-1}(b)$ such that $f(f^{-1}(b)) = b$.*

2.2.3 The absolute function

The absolute function plays an important role in the proofs and arguments that are to come. First, it is defined as:

Definition 19 (Absolute function)**.** *The absolute function $f(x) : \mathbb{R} \to \mathbb{R}_+$ is defined as:*

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$

It leads to a very important result, called the triangle inequality:

Theorem 4 (Triangle inequality)**.** *$\forall a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$*

Proof. Let $x, y \in \mathbb{R}$. Then, $x + y \leq |x| + |y|$ and

$$(-x) + (-y) \leq |-x| + |-y| = |x| + |y|$$

Hence, $-(|x| + |y|) \leq x + y \leq |x| + |y|$ and we obtain

$$|x + y| \leq |x| + |y|$$

□

2.2.4 Induction

The natural numbers have a property which leads to very important applications. This can be enunciated as:

Well ordering property of $\mathbb{N}$

If $S \subseteq \mathbb{N}$ and $S \neq \varnothing$. Then, $\exists x \in S$ such that $x \leq y, \forall y in S$.
An important tool that arises from it is called 'Induction'. We can state it as:

Proof by induction

Let $P(n)$ be a statement depending on $n \in \mathbb{N}$. Assume:

1. Base case: $P(1)$ is true
2. Inductive case: If $P(m)$ is true, so is $P(m + 1)$.

From it, we conclude $P(n)$ is true for all $n \in \mathbb{N}$.

Example Prove that

$$1 + c + c^2 + ... + c^n = \frac{1 - c^{n+1}}{1 - c}, \forall c \neq 1, \forall n \in \mathbb{N} \tag{2.5}$$

using induction.

Proof. Following the algorithm presented before:

1. Base case.
$$1 + c = \frac{1 - c^2}{1 - c} = \frac{(1 - c)(1 + c)}{1 - c} = 1 + c \tag{2.6}$$
As expected.

2. Inductive case. Assume
$$1 + c + c^2 + ... + c^m = \frac{1 - c^{m+1}}{1 - c} \tag{2.7}$$
is true. Now, for $m + 1$:
$$\begin{aligned} 1 + c + c^2 + ... + c^{m+1} &= (1 + c + c^2 + ... + c^m) + c^{m+1} \\ &= \frac{1 - c^{m+1}}{1 - c} + c^{m+1} \\ &= \frac{1 - c^{m+1} + c^{m+1} + c^{m+2}}{1 - c} \\ &= \frac{1 - c^{m+2}}{1 - c} \end{aligned} \tag{2.8}$$
Hence, the relation still holds.

□

2.3 Defining ℝ

2.3.1 The incompleteness of ℚ

Now, let's revisit our initial problem, namely $\sqrt{2} \notin \mathbb{Q}$. First, we start with a theorem:

Theorem 5. *The set $E = \{x \in \mathbb{Q} : 0 < x < \sqrt{2}\}$ is bounded above and does not have a supremum in $\mathbb{Q}$.*

Proof. First, consider $q \in \mathbb{Q}$ then $q^2 < 2 < 4 \Rightarrow q^2 - 4 < 0 \Rightarrow (q - 2)(q + 2) < 0$. Since $q > 0$ we have $q - 2 < 0 \Rightarrow a < 2$. Hence, 2 is an upper bound for E.
Next, to show that $\nexists \sup E \in \mathbb{Q}$ we begin by assuming $x = \sup E \in Q$.
Assume, for contradiction, $x^2 < 2$. Define

$$h = \min \left\{ \frac{1}{2}, \frac{2 - x^2}{2(2x + 1)} \right\} < 1$$

Then, $h > 0$. Now we prove $h + x \in E$. Computing $(x+h)^2 = x^2 + 2xh + h^2 < x^2 + 2xh + h$ since $h < 1$. So

$$\begin{aligned}(x+h)^2 < x^2 + (2x+1)h &= x^2 + (2x+1)\frac{2-x^2}{2(2x+1)} \\ &= x^2 + 2 - x^2 \\ &= 2\end{aligned}$$

Therefore $(x+h)^2 < 2$ which implies $x+h \in E$ and $x+h > x$ so $x \neq \sup E$ which is a contradiction. Therefore, $x^2 \geq 2$.
Now, assume for contradiction $x > 2$. Then, define

$$h = \frac{x^2 - 2}{2x}$$

Note that $x^2 > 2 \Rightarrow h > 0 \Rightarrow x - h < x$. Now we prove $x - h$ is an upper bound for E. Compute $(x-h)^2 = x^2 - 2xh + h^2 = x^2 - (x^2 - 2) + h^2 = 2 + h^2 > 2$. Let $q \in E$, *i.e.* $0 < q < \sqrt{2}$. Then, $q^2 < 2 < (x-h)^2 \Rightarrow 0 < (x-h)^2 - q^2 \Rightarrow 0 < (x-h+q)(x-h-q)$ and

$$0 < \left(\frac{x^2+2}{2x} + q\right)(x - h - q).$$

Since $q > 0$ and $(x^2+2)/(2x) > 0$ then $0 < x-h-q \Rightarrow q < x-h$. Thus, $\forall q \in E, q < x-h \Rightarrow x-h$ is an upper bound for E. Since $x = \sup E \Rightarrow x \leq x + h \Rightarrow h \leq 0$, which is a contradiction.
Thus, $x^2 = 2$ and $x > 1$.
For contradiction, assume $\exists m, n \in \mathbb{N}$ such that $m > n, x = m/n$. Then, $\exists n \in \mathbb{N}$ such that $nx \in \mathbb{N}$. Let $S = \{k \in \mathbb{N} : kx \in \mathbb{N}\}$, note that $n \in S \Rightarrow S \neq \varnothing$. By the well-ordering of $\mathbb{N}$, S has the least element $k_0 \in S$. Define $k_1 = k_0 x - k_0 \in \mathbb{Z}$. Since $x > 1, k_1 = k_0(x-1) > 0 \Rightarrow k_1 \in \mathbb{N}$. Since $x^2 = 2 \Rightarrow 4 - x^2 > 0 \Rightarrow (2-x)(2+x) > 0 \Rightarrow 2 - x > 0 \Rightarrow x < 2$. Then $k_1 = k_0(x-1) < k_0(2-x) = k_0$. Thus, $k_1 \in \mathbb{N}$ and $k_1 < k_0$. Computing $xk_1 = x(xk_0 - k_0) = x^2 k_0 - xk_0 = 2k_0 - xk_0 = k_0 + (k_0 - xk_0) = k_0 - k_1 \in \mathbb{N}$. Thus, $k_1 \in S$ and $k_1 < k_0$ which means k_0 is not the least element in S and $\sup E$ does not exist in $\mathbb{Q}$. $\square$

2.3.2 The definition of $\mathbb{R}$

First, in order to define $\mathbb{R}$ the previous result about the lack of an upper bound for $E = \{q \in \mathbb{Q} : 0 < q < \sqrt{2}\}$ allows us to introduce a definition.

Definition 20 (Least upper bound property)**.** *An ordered set S has the least upper bound property if every nonempty and bounded above subset $E \subseteq S$ has a supremum in S.*

The previous definition could be stated about a 'Greatest upper bound property'. Clearly $\mathbb{Q}$ does not have the Least upper bound property as the previous subsection has shown.

Now, for the real numbers,

Theorem 6 (Existence of $\mathbb{R}$). *There exists a unique ordered field which contains $\mathbb{Q}$ and has the least upper bound property. This field is denoted by $\mathbb{R}$.*

Theorem 7. *There exists a unique $r \in \mathbb{R}$ such that $r > 0$ and $r^2 = 2$.*

Proof. First, let $\tilde{E} = \{x \in \mathbb{R} : 0 < x < \sqrt{2}\}$. Then, $\tilde{E}$ is bounded above. Take $r = \sup \tilde{E}$. The same proof as before show $r > 1$ and $r^2 = 2$. We now prove r is unique. Suppose $\tilde{r} \in \mathbb{R}, \tilde{r} > 0$ and $\tilde{r}^2 = 2$. Then, $0 = \tilde{r}^2 - r^2 = (\tilde{r} - r)(\tilde{r} + r) \Rightarrow 0 = \tilde{r} - r \Rightarrow r = \tilde{r}$. □

2.3.3 The density of $\mathbb{Q}$ in $\mathbb{R}$

The set $\mathbb{Q}$ contains $\mathbb{N}$, and $\mathbb{R}$ contains $\mathbb{Q}$. The following theorems shows how $\mathbb{N}$ and $\mathbb{Q}$ sit inside $\mathbb{R}$:

Theorem 8 (Archimedean property). *If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx < y$.*

Proof. Suppose $x, y \in \mathbb{R}$ and $x > 0$. We need to show $\exists n \in \mathbb{N}$ such that $nx < y$, *i.e.* $x < y/n$. Assume for contradiction $\forall n \in \mathbb{N} : n \leq y/x$. Then $\mathbb{N} \subseteq \mathbb{R}$ is bounded above, hence it has a supremum, by the least upper bound property of $\mathbb{R}$ with value $a \in \mathbb{R}$. Since a is the supremum of $\mathbb{N}$ then $a - 1$ is not an upper bound for $\mathbb{N}$. Therefore, $\exists m \in \mathbb{N}$ such that $a - 1 < m \Rightarrow a < m + 1$ which implies a is not an upper bound for $\mathbb{N}$ contradiction our initial claim. □

Theorem 9 (Densit of $\mathbb{Q}$ in $\mathbb{R}$). *If $x, y \in \mathbb{R}$ and $x < y$ then $\exists q \in \mathbb{Q}$ such that $x < r < y$.*

Proof. Let $x, y \in \mathbb{R}$ and $x < y$, then:

1. If $x < 0 < y$ we have $r = 0 \in \mathbb{Q}$.
2. If $0 \leq x < y$ then by the Archimedean property, $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ and $\exists l \in \mathbb{N}$ such that $l > nx$. Thus, $S = \{k \in \mathbb{N} : k > nx\} \neq \varnothing$. By the well ordering property of $\mathbb{N}$, S has the least element m.
   Since $m \in S \Rightarrow nx < m$. Since m is the least element of S, $m - 1 \notin S \Rightarrow m - 1 \leq nx \Rightarrow m \leq nx + 1$. Thus, $nx < m < nx + 1 \Rightarrow x < m/n < y$. So, $r = m/n \in \mathbb{Q}$ is the solution.
3. If $x < y \leq 0$, then $0 \leq -y < x$. Then by the previous result $\exists \tilde{r} \in \mathbb{Q}$ such that $-y < \tilde{r} < -x$, or equivalently $x < -\tilde{r} < y$ and $r = -\tilde{r}$ is the solution.

□

2.4 Cardinality

Now, we turn our attention to cardinality, which is an approach to compare the size of sets.

Definition 21 (Cardinality). *Two sets, A and B, have the same cardinality if there exists a bijective function $f : A \to B$.*

Notation

- If A and B have the same cardinality we write $|A| = |B|$
- If $|A| = |\{1, 2, 3, ..., n\}|$ we write $|A| = n$
- If there exists an injective function $f : A \to B$ we write $|A| \leq |B|$
- If $|A| \leq |B|$ and $|A| \neq |B|$ then $|A| < |B|$

Theorem 10 (Cantor-Schorer-Bernstein). *If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$*

2.4.1 Countable, uncountable and countably infinite sets

Definition 22 (Countably infinite). *If $|A| = |\mathbb{N}|$ then A is countably infinite.*

Definition 23 (Countable sets). *If A is countably infinite or finite, then A is countable.*

Definition 24 (Uncountable set). *If A is neither countably infinite nor finite, then A is uncountable.*

Since both cardinality and countability have been introduced it is time to appreciate some very interesting results.

> There are twice as many numbers as numbers
>
> *Richard Feynman*

Theorem 11. *The set of positive even numbers is countable,* i.e. $|\{2 \times n : n \in \mathbb{N}\}| = |\mathbb{N}|$. *And so are the odd numbers,* $|\{2 \times n - 1 : n \in \mathbb{N}\}|$.

Proof. Let $f : \mathbb{N} \to \{2 \times n : n \in \mathbb{N}\}$. So, $f(n) = 2n, \forall n \in \mathbb{N}$.
First, if $f(n_1) = f(n_2)$ then $2n_1 = 2n_2$, hence $n_1 = n_1$ and f is injective.
Second, let $m \in \{2 \times k : k \in \mathbb{N}\}$. Then, $\exists n \in \mathbb{N}$ such that $m = 2n$, the function $f(n) = 2n = m \Rightarrow n = m/2$. Therefore, f is also surjective, and by result there exists a bijective function $f : \mathbb{N} \to \{2 \times n : n \in \mathbb{N}\}$, and we conclude $|N| = |\{2 \times n : n \in \mathbb{N}\}|$.
The proof for the odd numbers is similar. □

Theorem 12 (Countability of $\mathbb{Z}$). *The set of integers is countable,* i.e. $|\mathbb{N}| = |\mathbb{Z}|$.

Proof. Define $f : \mathbb{N} \to \mathbb{Z}$ as

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{-n}{2} & \text{if } n \text{ is even} \end{cases}$$

$f(n)$ is both injective and surjective. Hence, $|\mathbb{Z}| = |\mathbb{N}|$. □

Theorem 13 (Countability of $\mathbb{Q}$)**.** $|\mathbb{Q}| = |\mathbb{N}|$

Proof. Set $A_1 = \{0\}$, for $n \geq 2$ define $A_n = \{\pm p/q : p, q \in \mathbb{N}$ and are in the lowest terms with $p + q = n\}$. For example, $A_1 = \{0\}$, $A_2 = \{1/1, -1/1\}$, $A_3 = \{1/2, -1/2, 2/1, -2/1\}$, and so on. Since each rational number appear in only one A_n and every rational number can be represented by the relation above, $|\mathbb{Q}| = |\mathbb{N}|$. □

Theorem 14. *If $A \subseteq B$ and B is countable, then A is either countable or finite.*

Theorem 15.

1. *If $A_1, A_2, ..., A_m$ are each countable sets, then $\bigcup_{n=1}^{m} A_n$ is countable*
2. *If A_n is a countable $\forall n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable*

2.4.2 Cantor's theorem

Cantor's diagonalization method

Theorem 16. *The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.*

Proof. For contradiction, assume there exists $f : \mathbb{N} \to (0, 1)$ bijective. Next, for each $m \in \mathbb{N}, \exists f(m) \in (0, 1)$ we write the decimal representation $f(m) = 0.a_{m,1}a_{m,2}....$ Next, define $b \in (0, 1)$ such that $b = 0.b_1b_2...$ with each digit following

$$b_n = \begin{cases} 2 \text{ if } a_{n,n} \neq 2 \\ 3 \text{ if } a_{n,n} = 2 \end{cases} \tag{2.9}$$

Since $b_n \neq a_{n,n} \forall n \in \mathbb{N}$ there exists $b \in (0, 1)$ such that $\nexists n \in \mathbb{N} : f(n) = b$ and $f(m)$ is not surjective. □

Power sets and Cantor's theorem

Given a set A, the power set $\mathcal{P}(A)$ refers to the collection of all subsets of A.

Theorem 17 (Cantor's theorem)**.** *Given a set A, there is no function $f : A \to \mathcal{P}(A)$ which is onto.*

Proof. Assume $f : A \to \mathcal{P}(A)$ is bijective. We prove f cannot be surjective by finding a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$. Take $B = \{a \in A : a \notin f(a)\}$. If f is surjective, then $B = f(a')$ for some $a' \in A$. However:

1. If $a' \in B$ then $a' \notin f(a')$. However, since $B = f(a')$ this implies $a' \notin B$
2. Else, if $a' \notin B$ then $a' \in f(a') = B$, which is again a contradiction. So, there is no function $f : A \to P(A)$ which is onto.

$\square$

Theorem 18. $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < ...$

Informally, there exists an infinite number of infinitudes.

Theorem 19. *If* $|A| = n$ *then* $\mathcal{P}(A) = 2^n$.

2.5 Epilogue

Cardinality allows us to create an equivalence relation between sets. In this sense, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are grouped together and are called countable sets. On the other hand, $\mathbb{R}, (a, b), P(\mathbb{N})$ are uncountable, and belong to a separate group.

Because of the importance of the countable sets, it is usual to denote $\aleph_0 = |\mathbb{N}|$. In terms of cardinal numbers, if $|X| < \aleph_0$ then X is finite. In this way, $\aleph_0$ is the smallest infinite cardinal number. The cardinality of $\mathbb{R}$ also deserves its special designation $\boldsymbol{c} = |\mathbb{R}| = |(0, 1)|$. Hence, $\aleph_0 < \boldsymbol{c}$.

From this point, one possible question to ask is: "is there a set $A \subseteq \mathbb{R} : \aleph_0 < |A| < \boldsymbol{c}$"?

Cantor believed there was no such set, leading to the "continuum hypothesis" i.e. $\nexists A \subseteq \mathbb{R} : \aleph_o < |A| < \boldsymbol{c}$. In 1940, Kurt Gödel showed there was no way to disprove this hypothesis from the axioms of set theory. Latter, in 1964 Paul Cohen showed it was also impossible to prove this conjecture. Hence, the problem of the continuum hypothesis is undecidable.

Chapter 3

Sequences and series

3.1 The starting problem

Basically a series is a sum of infinite terms. On the following example, some problems will appear as we try to manipulate the series as standard mathematical entities. Consider, for instance:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... \tag{3.1}$$

We can consider the partial sum, s_n, *i.e.* the sum of the n first terms of the series. In this case we would obtain: $s_1 = 1$, $s_2 = 1/2$, $s_3 = 5/6$,... and so on. Interestingly, the odd sums decrease ($s_1 > s_3 > s_5 > ...$), while the even sums increase ($s_2 < s_4 < s_6 < ...$). It gives the idea that $\{s_n\}$ converges to some number S. And we may feel tempted to write:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

However, the use of standard mathematical notation $(+, -, =)$ for series can be misleading. Take the previous equation, multiply it for 1/2 and add it to itself. We would get:

$$\frac{3}{2}S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

Which seems to be a contradiction to our initial claim. In a certain sense, addition in this infinite setting is not commutative.

Another example is the series:

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + ... \tag{3.2}$$

Depending on how we group the terms we would find different results:

$$(-1 + 1) + (-1 + 1) + (-1 + 1) + ... = 0$$

On the other hand,

$$-1 + (1 - 1) + (1 - 1) + (1 - 1) + ... = 1$$

In order to deal with the tricks hidden in infinite series, we begin by discussing sequences.

3.2 Sequences

3.2.1 Convergent sequences

Definition 25 (Sequence)**.** *A sequence is a function, f, whose domain is $\mathbb{N}$. In this way, $f : \mathbb{N} to \mathbb{R}$. Hence, $f(n)$ is the n-th term of the sequence. Notation: usually, a sequence is presented in the form $\{x_n\}$, or $\{x_n\}_{n=1}^{\infty}$, or $x_1, x_2, x_3, \ldots$.*

Definition 26 (Convergence of a sequence)**.** *A sequence $\{x_n\}$ converges to x if $\forall \varepsilon > 0, \exists N \in N$ such that $|x_n - x| < \varepsilon, \forall n \geq N$. There are a few different ways to denote convergence, such as $\{x_n\} \to x$, $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.*

The negation of the convergence of a sequence would be:

Definition 27. *A sequence $\{x_n\}$ does not converge to x if $\exists \varepsilon_0 > 0$ such that $\exists m \in \mathbb{N}$ such that $|x_n - x| \geq \varepsilon, \forall n \geq m$.*

Example

$$\lim_{n\to\infty} \frac{1}{n^2 + 30n + 1} = 0$$

Proof. We need to find $N \in \mathbb{N}$ such that

$$\frac{1}{n^2 + 30n + 1} < \varepsilon, \forall n \geq N$$

But

$$\frac{1}{n^2 + 30n + 1} \leq \frac{1}{n^2 + 30n} \leq \frac{1}{30n} \leq \frac{1}{n}$$

Hence, if $1/n < \varepsilon$ then we would obtain the initial inequality.
Let $\varepsilon > 0$, set $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then, for all $n \geq N$:

$$\left|\frac{1}{n^2 + 30n + 1} - 0\right| = \frac{1}{n^2 + 30n + 1} \leq \frac{1}{30n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

□

Definition 28 (Bounded sequences)**.** *A sequence $\{x_n\}$ is bounded if there exists a number $M > 0$ such that $|x_n| < M, \forall n \in \mathbb{N}$.*

Theorem 20. *If $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded.*

Proof. Suppose $\{x_n\} \to x$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N, |x_n - x| < \varepsilon, \forall \varepsilon > 0$. Regardless if x is positive or negative, we can write $|x_n| < |x| + \varepsilon$. Define $M = \max\{||x_1|, |x_2|, \ldots, |x_{N-1}|, |x| + \varepsilon\}$. Then, $|x_n| \leq M, \forall n \in \mathbb{N}$. □

Definition 29. *A sequence $\{x_n\}$ is:*

1. *Monotone increasing, if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$,*
2. *Monotone decreasing, if $x_n \geq x_{n+1}, \forall n \in \mathbb{N}$,*
3. *If it is either monotone increasing or decreasing, then it is called monotone.*

Theorem 21. *A monotonic sequence is convergent if, and only if, it is bounded.*

Proof. Suppose $\{x_n\}$ is a monotonic increasing sequence. Then,

1. $(\Rightarrow)$ follows from the previous theorem.
2. $(\Leftarrow)$. Suppose $\{x_n\}$ is bounded. Then, $\{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is a bounded set. Let $x = \sup\{x_n : n \in \mathbb{R}\}$. We claim
$$x = \lim_{n\to\infty} x_n$$
Let $\varepsilon > 0$. Since $x - \varepsilon$ is not an upper bound for $\{x_n : n \in \mathbb{N}\}$, $\exists M_0 \in \mathbb{N}$ such that $x_n - \varepsilon < x_{M_0} < x$. Choose $M = M_0$, then $\forall n \geq M, x - \varepsilon < x_{M_0} < x_n \leq x + \varepsilon$, or $x - \varepsilon < x_M < x + \varepsilon$.

□

Theorem 22 (Algebraic limit theorem)**.** *Let $\{a_n\} \to a$ and $\{b_n\} tob$. Then,*

1. $\{ca_n\} \to ca, \forall c \in \mathbb{R}$
2. $\{a_n + b_n\} \to a + b$
3. $\{a_n b_n\} \to ab$
4. $\{a_n/b_n\} \to a/b$, *given* $b \neq 0$

Proof. Let's take each item individually:

1. First, note $|ca_n - ca| = |c||a_n - a|$. Hence, for $\varepsilon > 0$ we have $|ca_n - ca| < \varepsilon \Leftrightarrow |a_n - a| < \varepsilon/|c|$. Since $\{a_n\} \to a$ then $\exists N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/|c|$, so $|ca_n - ca| = |c||a_n - a| < |c|\varepsilon/|c|, \forall n \geq N$.
2. From the triangle inequality, $|(a_n + b_n) - (a - b)| \leq |a_n - a| + |b_n - b|$. Set $N_1 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2, \forall n \geq N_1$ with $\varepsilon > 0$. And set $N_2 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/2, \forall n \geq N_2$. Then, for $N = \max\{N_1, N_2\}$ we obtain: $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
3. First, $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n b_n - ab_n| + |ab_n - ab| = |b_n||a_n - a| + |a||b_n - b|$. Take $N_1 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/(2|a|), \forall n \geq N_1$ with $\varepsilon > 0$. Since every convergent sequence is bounded, take $M > 0$ so that $|b_n| < M, \forall n \in \mathbb{N}$. Then, set $N_2 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/(2M), \forall n \geq N_2$. Finally, for $N = \max\{N_1, N_2\}$ we obtain $|a_n b_n - ab| \leq |b_n||a_n - a| + |a||b_n - b| < M\varepsilon/(2M) + |a|\varepsilon/(2|a|) = \varepsilon$.

4. $\{a_n/b_n\} \to a/b$ follows from the previous result by noting $\{(1/b_n)\} \to 1/b$, provided $b \neq 0$. So,
$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|}$$
$|b_n - b|$ can be made arbitrarily small. On the other hand, considering $\varepsilon_0 = |b|/2$, define $N_1 \in \mathbb{N}$ such that $|b_n - b| < |b|/2, \forall n \geq N_1$, hence $|b_n| > |b|/2, \forall n \geq N_1$. Now, set $N_2 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon|b|^2/2$. Taking $N = \max\{N_1, N_2\}$ leads to
$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = |b - b_n|\frac{1}{|b||b_n|} < \frac{\varepsilon|b|^2}{2}\frac{1}{|b|\frac{|b|}{2}} = \varepsilon, \forall n \geq N$$

□

Theorem 23 (Order limit theorem). *Assume $\{a_n\} \to a$ and $\{b_n\} \to b$, then:*

1. *If $a_n \geq 0, \forall n \in \mathbb{N} \Rightarrow a \geq 0$,*
2. *If $a_n \leq b_n, \forall n \in \mathbb{N} \Rightarrow a \leq b$,*
3. *If there exists $c \in \mathbb{R}$ such that $c \leq b_n, \forall n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c, \forall n \in \mathbb{N}$ then $a \leq c$.*

Proof. Proof for each statement:

1. Assume $a < 0$. Consider $\varepsilon = |a|$, since $\{a_n\} \to a$ we have $|a_n - a| < |a|, \forall n \geq N$. In particular, $|a_N - a| < |a|$ hence $a_N < 0$ which is a contradiction. Therefore, $a \geq 0$.
2. From the algebraic theorem, $\{b_n - a_n\} \to a - b$. Since $b_n - a_n \geq 0$ from the previous result, we get $b - a \geq 0$.
3. Take $a_n = c, \forall n \in \mathbb{N}$. From the previous theorem, if $c \leq b_n$ then $b - c \geq 0$. Hence, $b \geq c$.

□

3.2.2 Operations involving convergent sequences

In order to find out if a sequence converges, and to what value, there are a few tools at our disposal. We begin with the very popular Squeeze theorem, sometimes refered here as ST.

Theorem 24 (Squeeze theorem). *Let $\{a_n\}, \{b_n\}, \{x_n\}$ be sequences such that $\forall n \in \mathbb{N}$, $a_n \leq x_n \leq b_n$. Suppose $\{a_n\}$ and $\{b_n\}$ both converge and*
$$\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$$
So, $\{x_n\} \to x$.

Proof. Let $\varepsilon > 0$. Since $\{a_n\} \to x$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0, |a_n - x| < \varepsilon \Rightarrow x - \varepsilon < a_n$.
Since $\{b_n\} \to x$, then $\exists M_1 \in \mathbb{N}$ such that $\forall n \geq M_1, |b_n - x| < \varepsilon \Rightarrow b_n < x + \varepsilon$.
Choose $M = \max\{M_0, M_1\}$. Then, if $n \geq M, x - \varepsilon < a_n \leq x_n \leq b_n < x - \varepsilon \Rightarrow |x_n - x| > \varepsilon$. □

The limit of a function can be also expressed in another form, which can be pretty useful.

Theorem 25. *Another way to check if $\{x_n\} \to x$ would be*

$$\lim_{n\to\infty} x_n = x \Longleftrightarrow \lim_{n\to\infty} |x_n - x| = 0$$

Example Show that

$$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1$$

Proof. We have

$$\left|\frac{n^2}{n^2+n+1} - 1\right| = \left|\frac{-n-1}{n^2+n+1}\right| = \frac{n+1}{n^2+n+1} \leq \frac{n+1}{n^2+n} = \frac{1}{n}$$

Thus,

$$0 \leq \left|\frac{n^2}{n^2+n+1} - 1\right| \leq \frac{1}{n} \Longrightarrow \lim_{n\to\infty} \left|\frac{n^2}{n^2+n+1} - 1\right| = 0$$

by the Squeeze theorem. □

3.2.3 Some special sequences

Theorem 26. *If $\{x_n\}$ is a convergent sequence such that $\forall n \in \mathbb{N}, x_n \geq 0$, then $\{\sqrt{x_n}\}$ is convergent and*

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n} \tag{3.3}$$

Proof. Let $x = \lim_{n\to infty} x_n$, then:

1. If $x = 0$. Let $\varepsilon > 0$ then, since $\{x_n\} \to 0$, $\exists N_0 \in \mathbb{N}$ such that $x_n = |x_n - 0|\varepsilon^2, \forall n \geq N_0$. Choose $N = N_0$, then $\sqrt{x_n} - \sqrt{0} = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon$

2. If $x > 0$. Then,

$$|\sqrt{x_n} - \sqrt{x}| = \left|\frac{\sqrt{x_n} - \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}(\sqrt{x_n} + \sqrt{x})\right| \tag{3.4}$$

$$= \frac{1}{\sqrt{x_n} - \sqrt{x}}|x_n - x| \tag{3.5}$$

$$\leq \frac{1}{\sqrt{x}}|x_n - x|, \forall n \in \mathbb{N} \tag{3.6}$$

And,

$$0 \leq |\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}}|x_n - x|, \forall n \in \mathbb{N}$$

So, by the squeeze theorem, $\lim_{n\to\infty} |\sqrt{x_n} - \sqrt{x}| = 0$

□

Theorem 27. *If $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = x$, then $\{|x_n|\}$ is convergent and $\lim_{n\to\infty} |x_n| = |x|$.*

Proof. Note that $|x| = \sqrt{x^2}, \forall x \in \mathbb{R}$. Then,

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} \sqrt{x_n^2} = \sqrt{x^2} = x$$

□

Theorem 28. *If $x \in (0,1)$ then $\lim_{n\to\infty} c^n = 0$. If $c > 1$ then $\{c^n\}$ is unbounded.*

Proof. For each case:

1. If $0 < c < 1$. Note that $0 < c^{n+1} < c^n < 1, \forall n \in \mathbb{N}$. This can be shown by induction:
   - Base case, consider $0 < c^2 < c < 1$, since $0 < c < 1$.
   - Inductive case, consider $0 < c^{m+1} < c^m < 1$ to be true. Multiplying the former inequality by c we obtain $0 < c^{m+2} < c^{m+1}$.

   Thus, $\{c^n\}$ is monotone decreasing sequence and is bounded below, which implies $\{c^n\}$ is convergent. Set $L = \lim_{n\to\infty} c^n$. Take $\varepsilon > 0$, then $\exists N \in \mathbb{N}$ such that

$$\begin{aligned}(1-c)|L| = |L - cL| &= |L - c^{M+1} + c^{M+1} - cL| \\ &\leq |L - c^{M+1}| + c|c^M - L| \\ &< (1-c)\frac{\varepsilon}{2} + c(1-c)\frac{\varepsilon}{2} \\ &< (1-c)\varepsilon, \forall n \geq N\end{aligned}$$

   Hence, $|L| < \varepsilon, \forall \varepsilon > 0 \longrightarrow L = 0$.

2. For $c > 1$. Note that $\forall B \geq 0, \exists n \in \mathbb{N}$ such that $c^n > B$. For $n \in \mathbb{N}$ such that $n > B/(c-1)$ then $c^n = (1 + (1-c))^n \geq 1 + n(c-1) \geq n(c-1) > B$. Hence, $\{c^n\}$ is unbounded $\forall c > 1$ and the sequence does not converge.

□

Theorem 29. *If $p > 0$ then $\lim_{n\to\infty} n^{-p} = 0$*

Proof. Let $\varepsilon > 0$. Take $N > (1/\varepsilon)^{1/p}$, then

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{|n^p|} \leq \frac{1}{N^p} < \varepsilon$$

$\square$

Theorem 30. *If $p > 0$ then $\lim_{n\to\infty} p^{1/n} = 1$.*

3.2.4 Subsequences and Bolzano–Weierstrass theorem

Definition 30 (Subsequence)**.** *Let $\{x_n\}$ be a sequence of real numbers, and $\{n_k\}$ be a strictly increasing sequence of natural numbers. Then, $\{x_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{x_n\}$.*

Theorem 31. *If $\{x_n\}$ converges to x then any subsequence of $\{x_n\}$ will converge to x.*

Proof. Suppose $\{x_n\} \to x$. Let $\varepsilon > 0$, then $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0, |x_n - x| < \varepsilon$. Choose, $M = M_0$. If $k \geq M$, then $n_k \geq k \geq M = M_0$, hence for $\varepsilon > 0, \exists M \in \mathbb{N}$ such that $|x_{n_k} - x| < \varepsilon, \forall n_k \geq M$. $\square$

From the decision of subsequence we may ask: "does a bounded sequence have a convergent subsequence?". The answer is yes, before we show it, we need to define some specific limits.

Definition 31 (Limsup/liminf)**.** *Let $\{x_n\}$ be a bounded sequence. If the limit exists, we can define:*

- *Limit superior:* $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup\{x_k : k \geq n\}\right)$
- *Limit inferior:* $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf\{x_k : k \geq n\}\right)$

Now we proceed to show an interesting result: these limits always exist.

Theorem 32. *Let $\{x_n\}$ be a bounded sequence, and*

- $a_n = \sup\{x_k : k \geq n\}$
- $b_n = \inf\{x_k : k \geq n\}$

Then, the following statements are true:

1. *$\{a_n\}$ is monotone decreasing and bounded,*
2. *$\{b_n\}$ is monotone increasing and bounded,*

3. $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

Proof. Proving each of the results:

1. First, $\{x_k : k \geq n+1\} \subseteq \{x_k : k \geq n\}, \forall n \in N$, so $a_{n+1} = \sup\{x_k : k \geq n+1\} \leq \sup\{x_k : k \geq n\} = a_n$.
2. Similarly, $b_{n+1} \geq b_n, \forall n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, $\exists M \geq 0$ such that $-B \leq x_n \leq B, \forall n \in \mathbb{N}$. So, $-B \leq b_n \leq a_n \leq B$.
3. By the previous result, $b_n \leq a_n, \forall n \in \mathbb{N} \implies \liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$

□

Example Consider the series $\{x_n\}$ with $x_n = (-1)^n$. Calculate the limit superior and limit inferior.

Proof. First, notice that $\{(-1)^k : k \geq n\} = \{-1, 1\}, \forall n \in \mathbb{N}$. Hence, the supremum is always 1 and the infimum is always -1. So,

$$\limsup_{n\to\infty} = 1$$

$$\liminf_{n\to\infty} = -1$$

□