Empirical Finance: Methods & Applications

Volatility Trading

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Week 6

Introduction

The trade-off between risk and return is a key relationship in finance

- We want to quantify the amount of risk an investor faces over a given investment period,
- The return standard deviation (or simply volatility), is commonly as a measure of risk,
- Volatility is a key input for portfolio allocation, derivatives pricing, and risk management.

No matter how we model it, volatility is a backward-looking measure of risk

- Do we have a forward-looking measure of risk?
- Can market participants trade volatility as an asset class?
- Why would they trade volatility?

Volatility Trading

As a speculator you can bet on the future path of volatility

- If you expect high volatility after a forthcoming election, you can take a position in order to profit if actual volatility increases.

As an hedger you can offset your portfolio against market uncertainty:

- During a financial turmoil volatility tends to increase and you may need to rebalance your portfolio quite often incurring in large transaction costs,
- You can hedge against such an event by trading a volatility derivative.

Volatility trading took off

- After the meltdown of LTCM, when volatility reached unprecedented levels (Gatheral, 2006),
- The first volatility contract was traded in 1993 by UBS (Carr and Lee, 2009).

European Options

	Call Option	**Put Option**
Buyer	The right to buy one unit of the underlying asset for a given price on a given date	The right to sell one unit of the underlying asset for a given price on a given date
Seller	The conditional obligation to sell one unit of the underlying asset for a given price on a given date	The conditional obligation to buy one unit of the underlying asset for a given price on a given date

Pricing European Options

Consider the BlackScholesMerton model

Value at time of an option with
strike price K and maturity T-t
on one unit of the underlyng asset

$$\boxed{V_t\left(K,T\right)} = \phi B_{t,T}\left[F_{t,T}\mathcal{N}\left(\phi d_+\right) - K\mathcal{N}\left(\phi d_-\right)\right]$$

$\phi = +1$ for a call option

$\phi = -1$ for a put option

Pricing European Options

$\mathcal{N}(\cdot)$ denotes the CDF of a standard normal with arguments defined as

Forward rate at time t with delivery date T on one unit of the underlying asset

Strike price at time t with delivery date T on one unit of the underlying asset

$$d_{\pm} = \frac{\ln\left(F_{t,T}/K\right)}{\sigma_{t,T}\sqrt{T-t}} \pm \frac{1}{2}\sigma_{t,T}\sqrt{T-t}$$

Implied volatility between times t and T (or maturity T-t) per annum

Option's maturity as $T\text{-}t = \frac{\text{calendar days}}{365}$

Pricing European Options

The forward price with maturity $T - t$ is

Continuously compounded risk-less between t and T per annum

$$F_{t,T} = P_t e^{(r_{t,T} - q_{t,T})(T-t)}$$

Underlying asset price at time t for the underlying

Continuously compounded dividend yield between t and T per annum

Pricing European Options

The present value of expected dividend yield between times t and T is

Expected dividends between t and T

$$DY_{t,T} = \frac{\sum_{\tau \in (t,T)} \mathbb{E}_t[D_\tau] \, e^{-r_{t,\tau}(\tau - t)}}{P_t}$$

Discount rates based on riskless rates from t to τ

The continuously compounded dividend yield between times t and T is

$$q_{t,T} = -\frac{1}{T-t} \ln(1 - DY_{t,T})$$

To make $q_{t,T}$ in annual terms

The option is on the final price and dividends are 'stolen' before maturity.

Pricing European Options

The bond price with maturity $T - t$ is

Continuously compounded risk-less between t and T per annum

$$B_{t,T} = e^{-r_{t,T}(T-t)}$$

Pricing European Options

By setting $\phi = 1$, we have a call option

$$V_t(K,T) = C_t(K,T)$$

Call at time t with strike K and maturity T-t

By setting $\phi = -1$, we have a put option

$$V_t(K,T) = P_t(K,T)$$

Put at time t with strike K and maturity T-t

Volatility Trading

Can we use plain-vanilla call (or put) to trade volatility? No

$$C_{t,T}(K) = f(P_t, \sigma_{t,T}, t) \longrightarrow$$ depends on the underlying asset price, volatility, and time to maturity

Using a second-order Taylor approximation, we can show that

Change in Option Value $\longleftarrow$ $$\boxed{dC_{t,T}} \approx \underbrace{\frac{\partial C_{t,T}}{\partial P_t}}_{\text{Directional risk (or Delta } \Delta)} dP_t + \underbrace{\frac{\partial C_{t,T}}{\partial \sigma_{t,T}}}_{\text{Volatility risk (or Vega } \nu)} d\sigma_{t,T} + \underbrace{\frac{\partial C_{t,T}}{\partial t}}_{\text{Time decay (or Theta } \Theta)} dt + \frac{1}{2}\underbrace{\frac{\partial^2 C_{t,T}}{\partial P_t^2}}_{\text{Convexity risk (or Gamma } \Gamma)} (dP_t)^2 + \ldots$$

A vanilla option is a joint bet on (P, σ, t), not a pure vol instrument

Volatility Trading

How does the option respond to price changes?

Option sensitivity to the underlying asset price ← $\boxed{\Delta_{t,T}} = \frac{\partial C_{t,T}}{\partial P_t} > 0$ → As P increases, the probability of exercise rises, making the call more valuable (opposite for put).

How does the option respond to volatility changes?

Option sensitivity to implied volatility ← $\boxed{\nu_{t,T}} = \frac{\partial C_{t,T}}{\partial \sigma_{t,T}} > 0$ → Higher σ raises the option value by increasing the chance of a "big win" (same for put)

Volatility Trading

What is the cost of time?

Option sensitivity to time decay $\leftarrow$ $\boxed{\Theta_{t,T}} = \dfrac{\partial C_{t,T}}{\partial t} < 0$ $\rightarrow$ As time passes (T-t falls), the option loses value each day, all else equal (same for puts).

How does Δ change with price?

Delta's sensitivity to price changes $\leftarrow$ $\boxed{\Gamma_{t,T}} = \dfrac{\partial^2 C_{t,T}}{\partial P_t^2} > 0$ $\rightarrow$ Measures how fast Δ changes as P moves. High Γ means the option price can accelerate quickly (same for put).

Volatility Trading

Can we use a delta-hedged strategy to trade volatility? No

$$\Pi_{t,T} = C_{t,T} - \Delta \times P_t$$

Buy a call + sell Δ units of the underlying asset

We neutralize Δ at the inception but ...

$$\underbrace{\boxed{d\Pi_{t,T}}}_{\text{Change in Portfolio Value}} \approx \underbrace{\frac{\partial \Pi_{t,T}}{\partial \sigma_{t,T}}}_{\nu} d\sigma_{t,T} + \underbrace{\frac{\partial \Pi_{t,T}}{\partial t}}_{\Theta} dt + \frac{1}{2} \underbrace{\frac{\partial^2 \Pi_{t,T}}{\partial P_t^2}}_{\Gamma} (dP_t)^2 + \ldots$$

Change in Portfolio Value

ν
Profit if implied volatility rises

Θ
The daily cost of holding the trade

Γ
The benefit of large price swings

Volatility Trading

A delta-hedged portfolio must be frequently adjusted to be riskless

- Γ risk implies that $\Delta_{t,T}$ changes as prices move, requiring continuous rebalancing.
- Θ risk implies that even with perfect hedging, the portfolio loses value as time passes.

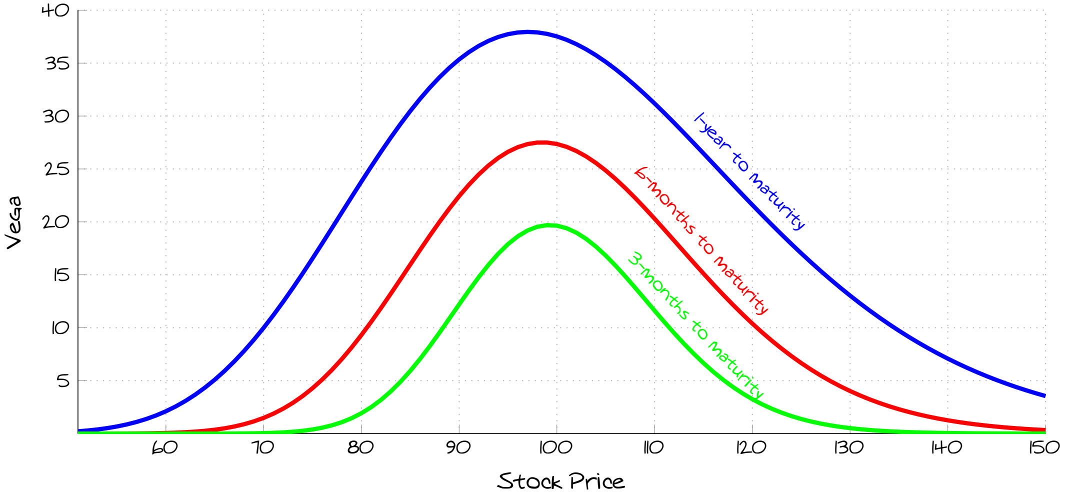
Also, the volatility of delta-hedged portfolio is not constant over time

$$\nu_{t,T} = \frac{\partial C_{t,T}(K)}{\partial \sigma_{t,T}} = P_t \sqrt{T-t} N'(d_+) e^{-q_{t,T}(T-t)} \longrightarrow$$

Maximum value for ATM options and declines for ITM and OTM options.

A delta-hedged portfolio provides an imperfect bet on volatility.

Option's Vega



Data source: Author's calculation using $K = 100$, $r = 5\%$ and $\sigma = 20\%$.

Volatility Trading

Can we use a straddle to trade volatility? Yes, but imperfectly

$$\Pi_{t,T} = C_{t,T}(K) + P_{t,T}(K) \longrightarrow$$

Buy a call and a put with the same K, T

We neutralize Δ at the inception but ...

Change in Portfolio Value $\longleftarrow$

$$\boxed{d\Pi_{t,T}} \approx \underbrace{\boxed{\frac{\partial \Pi_{t,T}}{\partial \sigma_{t,T}}}}_{\nu} d\sigma_{t,T} + \underbrace{\boxed{\frac{\partial \Pi_{t,T}}{\partial t}}}_{\Theta} dt + \frac{1}{2} \underbrace{\boxed{\frac{\partial^2 \Pi_{t,T}}{\partial P_t^2}}}_{\Gamma} (dP_t)^2 + \dots$$

ν: Profit if implied volatility rises

Θ: The daily cost of holding the trade

Γ: The benefit of large price swings

A straddle provides a more direct, but still imperfect, bet on volatility

Other hidden terms in the "..." for straddles

What is the cost volatilitydirection interaction?

Delta's sensitivity to implied volatility $\longleftarrow$

$$\boxed{\text{Vanna}_{t,T}} = \frac{\partial^2 C_{t,T}}{\partial P_t \partial \sigma_{t,T}} = \frac{\partial \Delta_{t,T}}{\partial \sigma_{t,T}}$$

$\longrightarrow$ Large volatility moves can cause call and put deltas to adjust at different speeds, "leaking" directional exposure into a straddle.

What is the cost of convexity in volatility?

Vega's sensitivity to implied volatility $\longleftarrow$

$$\boxed{\text{Volga}_{t,T}} = \frac{\partial^2 C_{t,T}}{\partial \sigma_{t,T}^2} = \frac{\partial \nu_{t,T}}{\partial \sigma_{t,T}}$$

$\longrightarrow$ A straddle is long two options, so Volga is high. If σ rises, Vega increases and you are "more long" volatility. If σ falls, Vega shrinks, making premium recovery much harder.

Volatility Trading

The volatility swap is the cleanest and simplest way to trade volatility

- A forward contract on the future realized volatility of a given asset,
- It requires a dynamic replication strategy.

The variance swap is not a pure bet but is related volatility

- A forward contract on the future realized variance of a given asset,
- It requires a static replication strategy.

Players talk about volatility but variance swaps are easier to replicate than volatility swaps

- Variance swaps are more popular in equity (traded on exchanges),
- Volatility swaps are more popular in currency (traded over-the-counter).

Variance Swap

Variance Swap

What is a variance swap? A forward contract on future realized variance

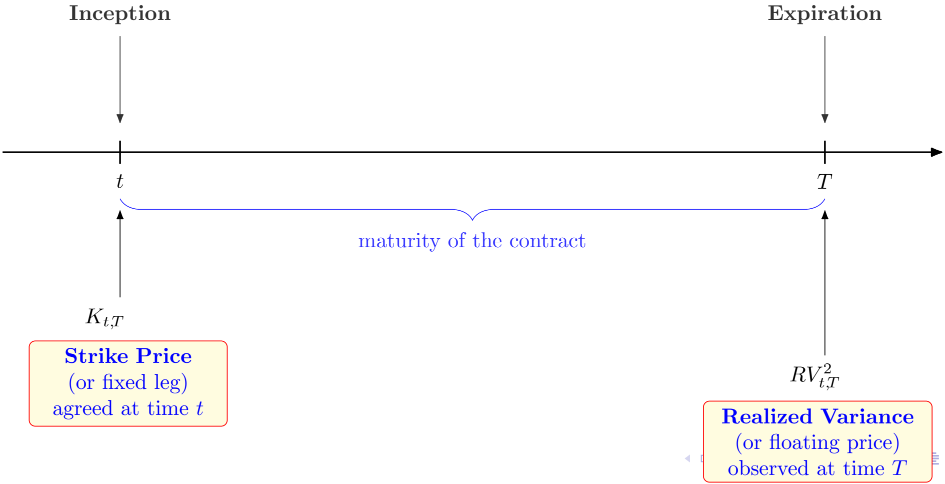
- A derivative instrument that offers a direct exposure to the return variance of an asset,
- You enter a contract on day t but the payoff is calculate on the maturity day T.

What is the payoff of a variance swap at maturity?

Buyer's payoff $\longleftarrow$ $$X_{t,T} = \left(RV^2_{t,T} - K_{t,T}\right) \times N_{var}$$ $\longrightarrow$ She pays the strike and receives the realized variance

- $RV^2_{t,T}$ is the annualized realized variance known ex-post at time T
- $K_{t,T}$ is the annualized variance strike known ex-ante at time t
- N_{var} is the variance notional amount.

Variance Swap



Variance Swap

Take the S&P500 index as an example

- The market expects a 20% volatility over the next year,
- You expect volatility to be higher and wish to gain exposure,
- You decide to buy a variance swap with variance notional $N_{var} = \$2,500$.

At maturity, the realized volatility is 15% and your payoff is

$$X_{t,T} = (15^2 - 20^2) \times \$2,500 = -\$437,500$$

The buyer will make a payment to the seller

Variance Swap

The notional of a variance swap can be expressed either as variance notional or vega notional

- The variance notional is the true notional for the calculation of the payoff,
- But variance is not intuitive and market participants think in terms of volatility,
- It is thus common to express the notional amount in terms of vega notional.

The vega notional is expressed as

$$N_{vega} = N_{var} \times 2K_{t,T}$$

Variance Swap

Take the S&P500 index as an example

- The market expects a 20% volatility over the next year,
- You expect volatility to be higher and wish to gain exposure,
- You decide to buy a variance swap with vega notional $N_{vega} = \$100,000$.

At maturity, the realized volatility is 15% and your payoff is

$$X_{t,T} = (15^2 - 20^2) \times \frac{\$100,000}{2 \times 20} = -\$437,500$$

The variance notional is implied from the vega notional

Variance Swap

The risk-neutral valuation (present value of the future expected payoff) is

$$B_{t,T} E_t^Q (X_{t,T}) = B_{t,T} E_t^Q \left(RV_{t,T}^2 - K_{t,T} \right) = 0$$

We thus have the fair price of a variance swap

$$K_{t,T} = E_t^Q \left[RV_{t,T}^2 \right]$$

The strike price of variance swap is the risk-neutral expectation of its future realized variance

Realized Variance

The annualized realized variance measured from day t to T

$$RV_{t,T}^2 = \frac{365}{T-t} \sum_{t=1}^{T} r_t^2$$

- where r_t is the calendar asset return ($r_t = 0$ for holidays/weekends),
- it is also common to use business days with 252 to annualize.

Let x be a random variable

$$Var\,(x) = E(x^2) - [E(x)]^2 \simeq E(x^2)$$

as $E(x) \approx 0$ for daily observations.

Pricing Variance Swap

Pricing Variance Swaps

We can price variance swaps using Britten-Jones and Neuberger (2000)

- They derive the so-called model-free implied variance,
- Volatility can be either deterministic or stochastic,

Assumptions

- The stock price has a continuous sample path (no jumps) and positive at all times,
- There is no assumption regarding the underlying volatility process.

Pricing Variance Swaps

The dynamics of the stock price under risk neutrality follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = (r - q)\,dt + \sigma_t dW_t$$

where σ_t is the volatility which is

$$\sigma_t \longrightarrow \quad \text{an arbitrary function of other parameters}$$

- $S_t \longrightarrow$ stock price at time t,
- $dW_t \longrightarrow$ standard Brownian increments,
- $r \longrightarrow$ riskless rate rate,
- $q \longrightarrow$ dividend yield.

Pricing Variance Swaps

By taking the Ito's lemma, we obtain the standard Brownian motion for log price

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t.$$

Take the difference between the geometric and standard Brownian motion

$$\frac{dS_t}{S_t} - d\ln S_t = \frac{1}{2}\sigma_t^2 dt.$$

and rearrange as

$$\sigma_t^2 dt = 2\left[\frac{dS_t}{S_t} - d\ln S_t\right]. \tag{1}$$

Pricing Variance Swaps

The risk-neutral integrated variance between t to T is defined

$$V_{t,T}^2 = \frac{1}{T-t}\int_t^T \sigma_t^2 dt \tag{2}$$

Combine Equations (1) and (2), and obtain

$$V_{t,T}^2 = \frac{2}{T-t}\left[\int_t^T \frac{dS_\tau}{S_\tau} - \boxed{\int_t^T d\ln S_\tau}\right]$$

This is the sum of daily log returns

$\ln(S_T/S_t)$

Pricing Variance Swaps

This identity dictates the replication strategy

$$V_{t,T}^2 = \frac{2}{T-t}\left[\int_t^T \frac{dS_\tau}{S_\tau} - \ln\frac{S_T}{S_t}\right]$$

- The first term denotes gains/losses on a continuously rebalancing strategy where you hold $1/S_\tau$ units of the underlying asset at time τ and your position is always worth $\$1$,
- The second term is a static position into a contract that pays the log of the stock price at T normalised by its current price at time t. This is the *log contract* of Neuberger (1994).

Pricing Variance Swaps

The swap rate K_t is the risk-neutral expectation of the integrated variance

$$K_t = E_t^Q\left[V_{t,T}^2\right]$$

$$= \frac{2}{T-t} E_t^Q\left[\int_t^T \frac{dS_\tau}{S_\tau} - \ln\frac{S_T}{S_t}\right]$$

$$= \frac{2}{T-t}\left\{E_t^Q\left[\int_t^T \frac{dS_\tau}{S_\tau}\right] - E_t^Q\left[\ln\frac{S_T}{S_t}\right]\right\}$$

Pricing Variance Swaps

The first risk-neutral expectation is

$$E_t^Q\left[\int_t^T \frac{dS_\tau}{S_\tau}\right] = E_t^Q\left[\int_t^T (r-q)\,d\tau + \int_t^T \sigma_\tau^2 dW_\tau\right]$$

$$= (r-q)\,(T-t)$$

the expected terminal value of the dynamically rebalanced strategy.

Pricing Variance Swaps

The second risk-neutral expectation is

$$E_t^Q \left[\ln \frac{S_T}{S_t} \right] = \ ?$$

the expected payoff of a log contract (Neuberger, 1994).

How to relicate such a payoff?

- Bakshi and Madan (2000) show that a payoff function with bounded expectations can be spanned by a continuum of out-of-the-money call and put options.
- We will use this result and replicate the payoff of a log contract using a static position in plain-vanilla call & put options.

Bakshi and Madan (2000)

Bakshi and Madan (2000)

Consider a payoff function with bounded expectations

$$g\left(S_T\right) \longrightarrow$$ This payoff depends on the terminal value S_T

and define the following quantities

$$g'(\cdot) = \frac{\partial g(\cdot)}{\partial S_T} \longrightarrow$$ first-order derivative

$$g''(\cdot) = \frac{\partial^2 g(\cdot)}{\partial S_T^2} \longrightarrow$$ second-order derivative

Bakshi and Madan (2000)

The payoff function $g\left(S_T\right)$ can be spanned as follows

$$
\begin{aligned}
g\left(S_T\right) &= g(\overline{S}) + g'(\overline{S})(S_T - \overline{S}) \longrightarrow \text{tangent approximation} \\
&+ \int_0^{\overline{S}} g''\left(K\right)\left(K - S_T\right)^{+} dK \longrightarrow \text{tangent correction} \\
&+ \int_{\overline{S}}^{\infty} g''\left(K\right)\left(S_T - K\right)^{+} dK \longrightarrow \text{tangent correction}
\end{aligned}
$$

where

$$\overline{S} \geq 0 \longrightarrow \text{scalar for evaluation}$$

Bakshi and Madan (2000)

Apply the risk-neutral expectation and obtain the arbitrage-free price as

$$
\begin{aligned}
B_{t,T}E_t^Q\left[g\left(S_T\right)\right] = {} & B_{t,T}E_t^Q\left[g(\overline{S})\right] - B_{t,T}E_t^Q\left[g'(\overline{S})\overline{S}\right] \\
& + g'(\overline{S})B_{t,T}E_t^Q\left[S_T\right] \\
& + \int_0^{\overline{S}} g''(K)B_{t,T}E_t^Q\left[(K-S_T)^+\right]dK \\
& + \int_{\overline{S}}^{\infty} g''(K)B_{t,T}E_t^Q\left[(S_T-K)^+\right]dK
\end{aligned}
$$

Bakshi and Madan (2000)

Apply the risk-neutral expectation and obtain the arbitrage-free price as

$$\underbrace{B_{t,T}}_{\text{Bond Price } e^{-r(T-t)}} E_t^Q\left[g\left(S_T\right)\right] = B_{t,T}\underbrace{E_t^Q\left[g(\bar{S})\right]}_{\text{constant terms}} - B_{t,T}\underbrace{E_t^Q\left[g'(\bar{S})\bar{S}\right]}_{\text{constant terms}}$$

$$+ g'(\bar{S})\underbrace{B_{t,T}E_t^Q\left[S_T\right]}_{\text{Current Asset Price } S_t}$$

$$+ \int_0^{\bar{S}} g''(K)\underbrace{B_{t,T}E_t^Q\left[(K-S_T)^+\right]}_{\text{Put Option } P_{t,T}(K)}dK$$

$$+ \int_{\bar{S}}^{\infty} g''(K)\underbrace{B_{t,T}E_t^Q\left[(S_T-K)^+\right]}_{\text{Call Option } C_{t,T}(K)}dK$$

Bakshi and Madan (2000)

Apply the risk-neutral expectation and obtain the arbitrage-free price as

$$B_{t,T}E_t^Q\left[g\left(S_T\right)\right] = B_{t,T}g(\overline{S}) - B_{t,T}g'(\overline{S})\overline{S}$$

$$+ g'(\overline{S})S_t$$

$$+ \int_0^{\overline{S}} g''\left(K\right)P_{t,T}(K)dK$$

$$+ \int_{\overline{S}}^{\infty} g''\left(K\right)C_{t,T}(K)dK$$

Bakshi and Madan (2000)

Similar to Taylor expansion, any payoff can be replicated by trading

$$B_{t,T}E_t^Q\left[g\left(S_T\right)\right] = B_{t,T}\left[g(\overline{S}) - g'(\overline{S})\overline{S}\right] \longrightarrow$$

Long position in a $\$1$ zero coupon bond with weight $g(\overline{S}) - g'(\overline{S})\overline{S}$

$$+\ g'(\overline{S})S_t \longrightarrow$$

Long position in the asset S_t with weight $g'(\overline{S})$

$$+\int_0^{\overline{S}} g''\left(K\right) P_{t,T}(K)dK \longrightarrow$$

Long position in a portfolio of put options with weights $g''\left(K\right)$

$$+\int_{\overline{S}}^{\infty} g''\left(K\right) C_{t,T}(K)dK \longrightarrow$$

Long position in a portfolio of call options with weights $g''\left(K\right)$

Replication of a Log Contract

Replication of a Log Contract

Take the forward price $F_{t,T}$ and rewrite

$$\ln \frac{S_T}{S_t} = \ln \frac{F_{t,T}}{S_t} + \ln \frac{S_T}{F_{t,T}}$$

independent of the final price S_T
and requires no replication

depends on the final price S_T
and must be replicated

and use Bakshi and Madan (2000) to replicate the second term.

Replication of a Log Contract

The payoff function is defined as

$$g(S_T) = \ln \frac{S_T}{F_{t,T}}$$

The derivatives are given by

$$g'(S_T) = \frac{1}{S_T}$$

$$g''(S_T) = -\frac{1}{S_T^2}$$

Replication of a Log Contract

Use the replicating strategy of Bakshi and Madan (2000) while setting $\overline{S} = F_{t,T}$ as

$$g(S_T) = g(F_{t,T}) + g'(F_{t,T})(S_T - F_{t,T})$$

$$+ \int_0^{F_{t,T}} g''(K)(K - S_T)^+ dK + \int_{F_{t,T}}^{\infty} g''(K)(S_T - K)^+ dK$$

Replication of a Log Contract

Use the replicating strategy of Bakshi and Madan (2000) while setting $\overline{S} = F_{t,T}$ as

$$\ln \frac{S_T}{F_{t,T}} \longleftarrow \underbrace{g(S_T)} = \underbrace{g(F_{t,T})}_{\ln \frac{F_{t,T}}{F_{t,T}}} + \underbrace{g'(F_{t,T})}_{\frac{1}{F_{t,T}}}(S_T - F_{t,T})$$

$$+ \int_0^{F_{t,T}} \underbrace{g''(K)}_{-\frac{1}{K^2}}(K - S_T)^+ dK + \int_{F_{t,T}}^{\infty} \underbrace{g''(K)}_{-\frac{1}{K^2}}(S_T - K)^+ dK$$

Replication of a Log Contract

Rearrange and obtain

$$\ln\frac{S_T}{F_{t,T}} = \frac{S_T - F_{t,T}}{F_{t,T}} - \left(\int_0^{F_{t,T}} \frac{1}{K^2}(K - S_T)^+ dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK\right)$$

Take the risk-neutral valuation and obtain

$$\begin{aligned} B_{t,T}E_t^Q\left[\ln\frac{S_T}{F_{t,T}}\right] &= B_{t,T}E_t^Q\left[\frac{S_T - F_{t,T}}{F_{t,T}}\right] \\ &- \left(\int_0^{F_{t,T}} \frac{1}{K^2} B_{t,T}E_t^Q\left[(K - S_T)^+\right] dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2} B_{t,T}E_t^Q\left[(S_T - K)^+\right] dK\right) \end{aligned}$$

Replication of a Log Contract

Recall

$$P_{t,T}(K) = B_{t,T} E_t^Q \left[(K - S_T)^+ \right]$$

$$C_{t,T}(K) = B_{t,T} E_t^Q \left[(S_T - K)^+ \right]$$

$$F_{t,T}(K) = E_t^Q \left[S_T \right]$$

and simplify as

$$E_t^Q \left[\ln \frac{S_T}{F_{t,T}} \right] = -\frac{1}{B_{t,T}} \left(\int_0^{F_{t,T}} \frac{1}{K^2} P_{t,T}(K) dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2} C_{t,T}(K) dK \right)$$

Pricing Variance Swap

Pricing Variance Swaps

We now have the ingredients to solve the risk-neutral expectation

$$K_t = E_t^Q\left[V_{t,T}^2\right]$$

$$= \frac{2}{T-t}E_t^Q\left[\int_t^T \frac{dS_\tau}{S_\tau} - \ln\frac{S_T}{S_t}\right]$$

$$= \frac{2}{T-t}\left\{E_t^Q\left[\int_t^T \frac{dS_\tau}{S_\tau}\right] - E_t^Q\left[\ln\frac{S_T}{S_t}\right]\right\}$$

Pricing Variance Swaps

We now have the ingredients to solve the risk-neutral expectation

$$K_t = \frac{2}{T-t}\left\{(r-q)(T-t) - \ln\frac{F_{t,T}}{S_t} - E_t^Q\left[\ln\frac{S_T}{F_{t,T}}\right]\right\}$$

$$= \frac{2}{T-t}\left\{(r-q)(T-t) - \ln\frac{F_{t,T}}{S_t} + \frac{1}{B_{t,T}}\left(\int_0^{F_{t,T}}\frac{1}{K^2}P_{t,T}(K)dK + \int_{F_{t,T}}^{\infty}\frac{1}{K^2}C_{t,T}(K)dK\right)\right\}$$

Pricing Variance Swaps

Recall that

$$F_{t,T} = S_t e^{(r-q)(T-t)}$$

thus having

$$K_t = \frac{2}{T-t}\left\{(r-q)(T-t) - \ln\frac{S_t e^{(r-q)(T-t)}}{S_t} + \frac{1}{B_{t,T}}\left(\int_0^{F_{t,T}} \frac{1}{K^2}P_{t,T}(K)dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2}C_{t,T}(K)dK\right)\right\}$$

$$= \frac{2}{T-t}\left\{(r-q)(T-t) - (r-q)(T-t) + \frac{1}{B_{t,T}}\left(\int_0^{F_{t,T}} \frac{1}{K^2}P_{t,T}(K)dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2}C_{t,T}(K)dK\right)\right\}$$

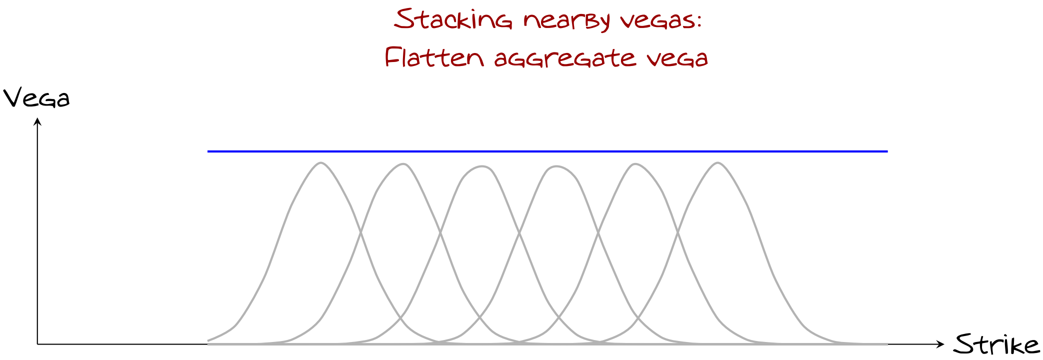
Pricing Variance Swaps

Rearrange as

$$K_t = \frac{2B_{t,T}^{-1}}{T-t}\left(\int_0^{F_{t,T}} \frac{1}{K^2} P_{t,T}(K)dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2} C_{t,T}(K)dK\right)$$

The strike price of a variance swap can be replicated using a portfolio of out-of-the money call and put options, each weighted by the inverse of their squared strike price.

Pricing Variance Swaps



A continuum of nearby strikes delivers approximately constant volatility exposure.

Volatility Swap

Volatility Swap

What is a volatility swap? A forward contract on future realized volatility

- A derivative instrument that offers a direct exposure to the return volatility of an asset,
- You enter a contract on day t but the payoff is calculate on the maturity day T.

What is the payoff of a volatility swap at maturity?

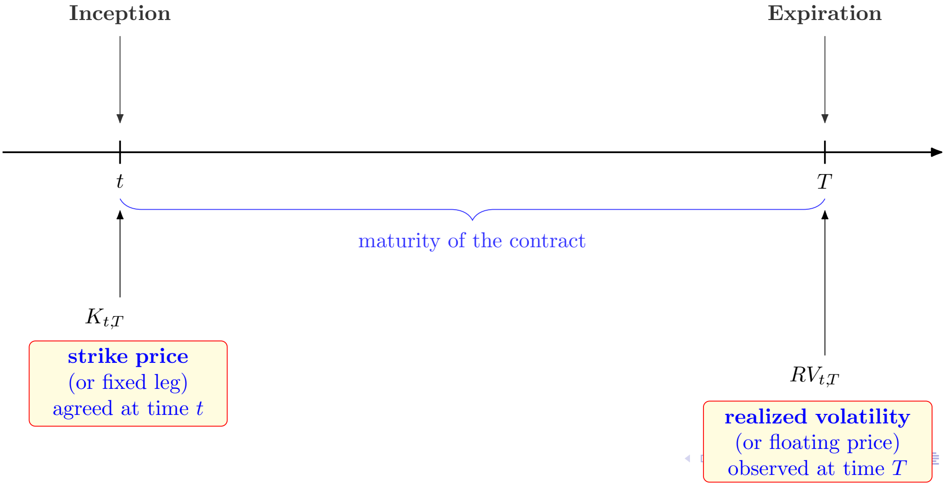
$$\underbrace{X_{t,T}}_{\text{Buyer's payoff}} = (RV_{t,T} - K_{t,T}) \times N_{vol}$$

Buyer's payoff

She pays the strike and receives the realized volatility

- $RV_{t,T}$ is the annualized realized volatility known ex-post at time T
- $K_{t,T}$ is the annualized volatility strike known ex-ante at time t
- N_{vol} is the volatility (or vega) notional amount.

Volatility Swap



Volatility Swap

Take the EURUSD exchange rate as an example

- The market expects a 10% volatility over the next year,
- You expect volatility to be higher and wish to gain exposure,
- You decide to buy a volatility swap with $N_{vol} = \$100,000$.

At maturity, the realized volatility is 15% and your payoff is

$$X_{t,T} = (15 - 10) \times \$100,000 = \$500,000$$

The buyer will receive a payment from the seller

Volatility Swap

The risk-neutral valuation (present value of the future expected payoff) is

$$B_{t,T} E_t^{\mathbb{Q}}(X_{t,T}) = B_{t,T} E_t^{\mathbb{Q}}\left(RV_{t,T} - K_{t,T}\right) = 0$$

We thus have the fair price of a volatility swap

$$K_{t,T} = E_t^{\mathbb{Q}}\left[RV_{t,T}\right]$$

The strike price of volatility swap is the risk-neutral expectation of its future realized volatility

Realized Volatility

The annualized realized variance measured from day t to T

$$RV_{t,T}^2 = \frac{365}{T-t} \sum_{t=1}^{T} r_t^2$$

- where r_t is the calendar asset return ($r_t = 0$ for holidays/weekends),
- it is also common to use business days with 252 to annualize.

The annualized realized volatility is simply

$$RV_{t,T} = \sqrt{RV_{t,T}^2}$$

Variance Swap vs. Volatility Swap

Variance Swap vs. Volatility Swap

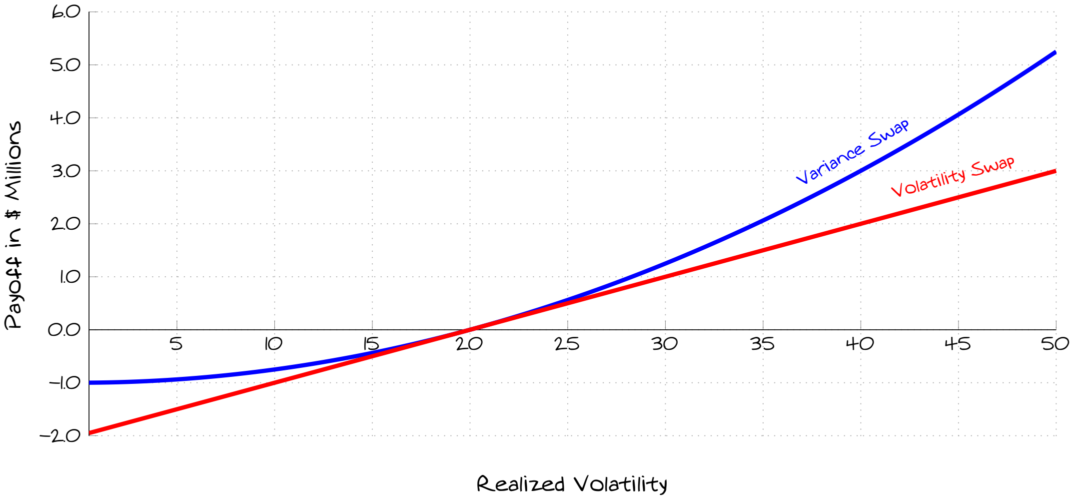
The payoff of a variance swap is linear with variance but convex with volatility

- A long variance swap always gains more from an increase in volatility than it loses lose for a corresponding decrease in volatility,
- This difference between the magnitude of gains and losses increases with the change in volatility, which is known as the convexity of the variance swap.

What is effect of convexity?

- A loss from a long variance swap is limited,
- But the loss from a short variance swap is not unlimited!

Variance Swap vs. Volatility Swap



Data source: Author's calculation using $N_{vega} = 100,000$, and $K_vol = 20\%$.

Pricing Volatility Swap

Pricing Volatility Swaps

Can we infer the model-free implied volatility from the model-free implied variance?

What we need $\longleftarrow \boxed{E_t^Q\left(\sqrt{RV_{t,T}^2}\right)} \leq \boxed{\sqrt{E_t^Q\left(RV_{t,T}^2\right)}} \longrightarrow$ What we have from Britte-Jones and Neuberger (2000)

We take a shortcut and calculate

$$E_t^Q\left(RV_{t,T}\right) \simeq \sqrt{E_t^Q\left(RV_{t,T}^2\right)}$$

the model-free implied volatility as the square root of the model-free implied variance.

Pricing Volatility Swaps

Convexity bias arise from the pricing approximation

$$\text{Convexity} = \underbrace{\sqrt{E_t^Q\left(RV_{t,T}^2\right)}}_{\sqrt{K_{var}}} - \underbrace{E_t^Q\left(RV_{t,T}\right)}_{K_{vol}}$$

Brockhaus and Long (2000) derive the convexity using a second order Taylor expansion as

$$K_{vol} = \sqrt{K_{var}^*} - \frac{Var[RV_{t,T}^2]}{8\sqrt{(E[RV_{t,T}^2])^3}}.$$

Pricing Volatility Swaps

Recall the second-order Taylor expansion around x_0 of random variable x

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

By applying this expansion to $\sqrt{x}$, we have

$$\sqrt{x} \approx \sqrt{x_0} + \frac{(x - x_0)}{2\sqrt{x_0}} - \frac{1}{8} \frac{(x - x_0)^2}{\sqrt{x_0^3}}$$

$$\approx \frac{(x + x_0)}{2\sqrt{x_0}} - \frac{(x - x_0)^2}{8\sqrt{x_0^3}}$$

Pricing Volatility Swaps

By taking the expectation on both sides and rearranging, we obtain

$$\begin{aligned} E\left[\sqrt{x}\right] &\approx \frac{E\left[x\right]+x_0}{2\sqrt{x_0}} - \frac{E\left[x^2\right]-2x_0E\left[x\right]+x_0^2}{8\sqrt{x_0^3}} \\ &\approx \frac{E\left[x\right]+x_0}{2\sqrt{x_0}} - \frac{E\left[x^2\right]-E\left[x\right]^2+E\left[x\right]^2-2x_0E\left[x\right]+x_0^2}{8\sqrt{x_0^3}} \\ &\approx \frac{E\left[x\right]+x_0}{2\sqrt{x_0}} - \frac{V\left[x\right]+\left(E\left[x\right]-x_0\right)^2}{8\sqrt{x_0^3}}. \end{aligned}$$

Pricing Volatility Swaps

If we set

$$x_0 = E\left[x\right]$$

We can further simplify the expansion as follows

$$E\left[\sqrt{x}\right] \approx \sqrt{E\left[x\right]} - \frac{V\left[x\right]}{8\sqrt{\left(E\left[x\right]\right)^3}}$$

where the approximate value of the convexity bias is

$$\frac{V\left[x\right]}{8\sqrt{E\left[x\right]^3}}$$

An Application

Volatility Risk Premium

The **volatility risk premium** is the ex-ante payoff to a volatility swap

$$VRP_t = E_t(RV_{t,T}) - K_{t,T}$$

- E_t denotes the real-world expectation at time t,
- $E_t(RV_{t,T}) = RV_{t-T,t}$ as in Bollerslev, Tauchen and Zhou (2009).

The **empirical evidence** suggests

- The volatility risk premium is generally negative,
- A volatility swap viewed as an insurance contract to hedge unexpected changes in volatility,
- The protection buyer pays a premium to the insurance seller.

Currency Strategy Based on Volatility Risk Premia

At the end of each month t

- Sort currencies into 5 baskets using the 1-year volatility risk premia,
- First Portfolio (P_C): Currencies with cheap volatility insurance (or high VRP),
- Last Portfolio (P_E): Currencies with expensive volatility insurance (or low VRP).

At the end of next month $t + 1$

- Currencies are equally-weighted within each portfolios,
- Long the first portfolio and short the last portfolio, i.e., $P_C - P_E$
- This is the VRP strategy.

Constructing Other Currency Portfolios

Carry Portfolios

- CAR is a portfolio which is long high interest rate and short low interest rate currencies

Momentum Portfolios

- MOM is a portfolio which is long winner and short loser currencies (1-month FX return).

Value Portfolios

- VAL is a portfolio which is long undervalued $(RER < 1)$ and short overvalued $(RER > 1)$ currencies according to PPP.

Risk Reversal Portfolios

- RR is a portfolio which is long low risk reversal (negative skew) and short high risk reversal (positive skew) currencies.

Dataset

The 'Developed & Emerging Countries' sample (20):

- Australia, Brazil, Canada, Czech Republic, Denmark, Euro Area, Hungary, Japan, Mexico, New Zealand, Norway, Poland, Singapore, South Africa, South Korea, Sweden, Switzerland, Taiwan, Turkey, United Kingdom.

The 'Developed Countries' sample (10):

- highlighted in blue. I will only present this evidence.

Over-the-Counter Currency Option Data

- 1-year implied volatility data from JP Morgan from Jan98 to Dec13,
- five deltas: ATM, 10Δ Calls & Puts, 25Δ Calls & Puts.

Currency Strategies: Summary Statistics

Panel A: Excess Returns					
	CAR	*MOM*	*VAL*	*RR*	*VRP*
Mean	4.10	0.92	3.66	5.10*	4.95**
Sdev	10.73	9.81	8.97	11.49	8.15
Skew	−0.71	0.26	−0.16	−0.47	−0.03
Kurt	5.25	3.75	3.71	5.41	3.97
SR	0.38	0.09	0.41	0.44	0.61
MDD	−37.8	−22.8	−15.1	−35.1	−17.0
$Freq_L$	0.10	0.48	0.09	0.08	0.29
$Freq_S$	0.07	0.44	0.07	0.22	0.35
Panel B: FX Returns					
Mean	−0.81	0.94	2.15	1.84	5.45***
Sdev	10.76	9.87	9.02	11.58	8.12
Skew	−0.72	0.33	−0.24	−0.50	−0.03
Kurt	5.43	3.94	3.76	5.63	4.04
SR	−0.08	0.09	0.24	0.16	0.67
MDD	−43.3	−23.2	−22.5	−40.3	−14.5
$Freq_L$	0.10	0.48	0.09	0.08	0.29
$Freq_S$	0.07	0.44	0.07	0.22	0.35

Excess Return Decomposition

Excess Return

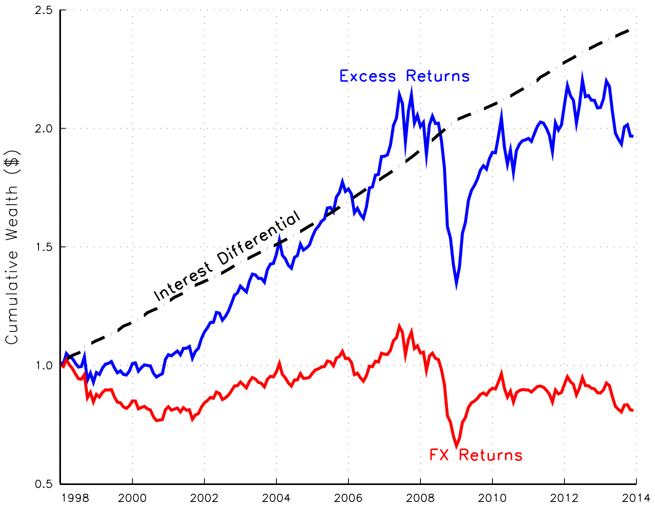
- Buy the foreign currency at time t in forward market and offset the position at time $t+1$ in the spot market

$$RX_{t+1} = \frac{S_{t+1} - F_t}{S_t}$$

- Perform the simple decomposition

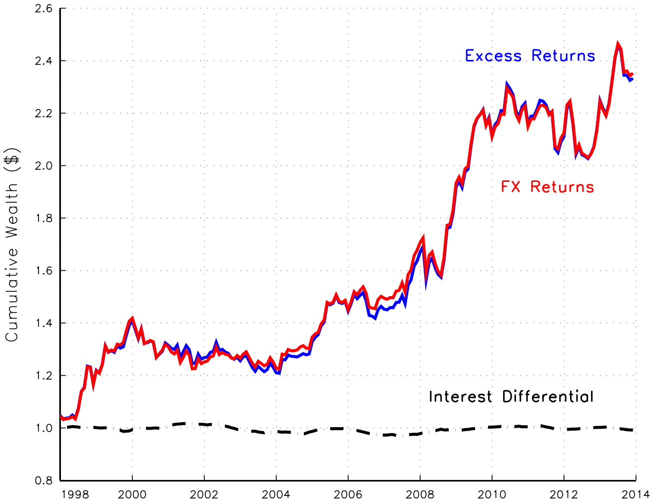
$$\begin{aligned} RX_{t+1} &= \frac{S_{t+1} - S_t}{S_t} - \frac{F_t - S_t}{S_t} \\ &\simeq \frac{S_{t+1} - S_t}{S_t} + i_t^* - i_t \end{aligned}$$

Carry Strategy



Carry returns depend mostly on the interest rate differential

VRP Strategy

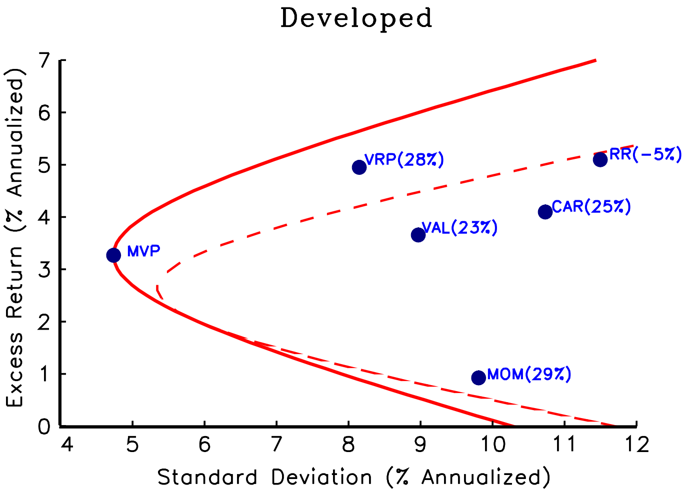


VRP returns depend mostly on FX returns.

Currency Strategy: Correlations

	CAR	*MOM*	*VAL*	*RR*
CAR	1.00			
MOM	−0.20	1.00		
VAL	0.30	−0.20	1.00	
RR	0.76	−0.23	0.46	1.00
VRP	−0.08	0.11	0.19	0.10

Global Minimum Variance Portfolio



VRP adds diversification benefits to a global currency strategy

What is the Story?

VRP has strong predictive power for the cross-section of currency returns

- Economically valuable and statistically significant,
- VRP information relates to future spot exchange rate returns, not to interest differentials.

VRP returns essentially uncorrelated with traditional currency strategies

- Potential diversification gains from adding VRP strategy to those normally followed by currency managers,
- Evidence consistent with VRP returns driven by speculator-hedger interactions in the presence of time-varying capital constraints.