Empirical Finance: Methods & Applications

Linear Asset Pricing Tests

Pasquale Della Corte
Imperial College London

Week 5

Introduction

A central question in finance:

Why different financial assets earn different returns on average?

Finance theory suggests that

Assets with riskier payoffs should earn higher average returns.

But what risk matters?

Only systematic (non-diversifiable) risk command a premium,

diversifiable (or idiosyncratic) risk should not priced.

US Stocks: 1926-2017

Average return on small stocks → small value stocks 17.7% per annum

→ small growth stocks 11.6% per annum

Average return on big stocks → Big value stocks 14.4% per annum

→ Big growth stocks 10.9% per annum

US Stocks: 1926–2017

	Small Growth (1)	(2)	Small Value (3)	Big Growth (4)	(5)	Big Value (6)	MKT	SMB	HML
Full Sample: 1926M7 – 2017M1									
mean (%)	11.60	15.17	17.68	10.87	11.61	14.37	7.84	2.54	4.79
sdev (%)	26.12	24.24	28.31	18.38	19.68	24.81	18.59	11.12	12.14
SR	0.44	0.63	0.62	0.59	0.59	0.58	0.42	0.23	0.39
Pre-Crisis: 1926M7 – 2007M12									
mean (%)	11.85	15.57	18.36	11.03	12.05	15.08	7.77	2.54	5.28
sdev (%)	26.57	24.62	28.85	18.75	19.98	25.12	18.85	11.41	12.35
SR	0.45	0.63	0.64	0.59	0.60	0.60	0.41	0.22	0.43
Post-Crisis: 2008M1 – 2017M1									
mean (%)	9.35	11.58	11.58	9.48	7.67	7.98	8.45	2.46	0.37
sdev (%)	21.74	20.57	22.91	14.66	16.75	21.89	16.19	8.10	9.94
SR	0.43	0.56	0.51	0.65	0.46	0.36	0.52	0.30	0.04

Source: Kenneth French's website (6 portfolios sorted by size and book-to-market)

US Stocks: 1926–2017

MKT $\rightarrow$ excess return on the stock market index

$$MKT = E(R_M - R_f) \longrightarrow \text{Market Risk Premium}$$

SMB $\rightarrow$ excess return of small-cap over large-cap stocks

$$SMB = \frac{1+2+3}{3} - \frac{4+5+6}{3} \longrightarrow \text{Size Factor}$$

HML $\rightarrow$ excess return of value (high book-to-market) over growth stocks (low book-to-market)

$$HML = \frac{6+3}{2} - \frac{4+1}{2} \longrightarrow \text{Value Factor}$$

Traded Factors

Traded Factor

What is a traded factor?

- A traded factor is a portfolio of assets whose return is observable and investable.

Why a portfolio?

- To diversify away idiosyncratic risk, which is asset-specific,
- To isolate a systematic source of risk compensation, common across assets.

Example of traded factors

- Market excess return, size, value, or momentum portfolios,
- Carry, short-term reversal, or long-term reversal.

How to Construct a Traded Factor?

Select N assets, each denoted by i

- Construct the simple excess return $R^e_{i,t}$ from $t-1$ to t,
- Construct the signal $s_{i,t-1}$ observed at time $t-1$,
- Ensure no selection bias and no look-ahead bias

Sort assets into deciles (quintiles or terciles) based on their signal

- Deciles $\longrightarrow$ maximize signal separation, but increase noise,
- Quintiles $\longrightarrow$ balance signal strength and diversification,
- Terciles $\longrightarrow$ reduce estimation error, but weaken the signal,
- Quintiles or deciles are most commonly used, depending on sample size.

How to Construct a Traded Factor?

Form a dollar-neutral portfolio

- Long in the highest-signal basket and short in the lowest-signal basket as

High-minus-Low portfolio
(or viceversa)

$$f_t = \underbrace{\sum_{i \in P_H}}_{\text{Assets in the top basket}} \underbrace{w_{i,t-1}}_{\text{portfolio weights}} R^e_{i,t} - \underbrace{\sum_{i \in P_L}}_{\text{Assets in the bottom basket}} w_{i,t-1} R^e_{i,t}$$

Common weights

- Equally-weighted, value-weighted, or rank-weighted long and short baskets.

How to Construct a Traded Factor?

Direction of the trade

- `High-minus-Low (HML)` factor $\longrightarrow$ buy high and sell low signal assets (e.g., momentum),
- `Low-minus-High (LMH)` factor $\longrightarrow$ buy low and sell high signal assets (e.g., reversal).

Harvesting a positive excess return

- If a high $s_{i,t-1}$ identifies assets that perform better in good times relative to bad times, the appropriate traded factor is `HML`,
- If a high $s_{i,t-1}$ identifies assets that perform better in bad times relative to good times, the appropriate traded factor is `LMH`.

Time-Series Regressions

Suppose you have N test assets

- A test asset can be an individual asset or a portfolio of assets whose excess return is used to evaluate an asset pricing model,
- In practice, test assets are often portfolios of assets, constructed to reduce idiosyncratic noise and form a balanced panel (e.g., 25 Fama-French portfolios).

Asset Pricing Tests

- We want to test whether the test assets are priced by a set of tradable factors,
- When the risk factors are tradable, we only need time-series regressions.

Time-Series Regressions

We run time series regression for each test asset i

Test asset
(excess returns)

$$R^e_{i,t} = \alpha_i + \beta_{i,1} f_{1,t} + \ldots + \beta_{i,k} f_{k,t} + \varepsilon_{i,t}$$

unexplained average
excess return

quantity of risk

risk factor
(excess return)

quantity of risk

risk factor
(excess return)

idiosyncratic component

Time-Series Regressions

In compact form, we have that

$$\underbrace{R^e_{i,t}}_{\text{Test asset (excess returns)}} = \underbrace{\alpha_i}_{\text{unexplained average excess return}} + \underbrace{\beta_i'}_{1 \times k \text{ vector of quantities of risk}} \underbrace{F_t}_{k \times 1 \text{ vector of risk factors (excess returns)}} + \underbrace{\varepsilon_{i,t}}_{\text{idiosyncratic component}}$$

Examples of Linear Factor Models

CAPM model

$$R^e_{i,t} = \alpha_i + \beta_i MTK_t + \varepsilon_{i,t} \longrightarrow$$ MKT is the single risk factor

ICAPM model

$$R^e_{i,t} = \alpha_i + \beta_{i,1} MTK_t + \beta_{i,2} H_t + \beta_{i,3} L_t + \cdots + \varepsilon_{i,t} \longrightarrow$$ + other factors to hegde changes in the future investment opportunity set

Three-factor Fama-French model

$$R^e_{i,t} = \alpha_i + \beta_{i,1} MTK_t + \beta_{i,2} SMB_t + \beta_{i,3} HML_t + \varepsilon_{i,t} \longrightarrow$$ An example of ICAPM

Asset Pricing Tests

What is the objective of an asset pricing model?

- Explain the cross-sectional variation in expected returns across assets,
- Such variation should be driven by exposure to systematic risk factors.

What is the asset pricing restriction?

Average excess return ← $\mathbb{E}[R^e_{i,t}] = \beta_i{}'\lambda$

Quantity of risk
(measures how much
asset i co-moves with
risk factors)

Price of risk
(compensation investors
require for bearing
that risk)

This is the beta-representation model

Asset Pricing Tests

In practice, the beta-representation model is written as

$$\underbrace{\alpha_i}_{\text{Pricing error}} = \underbrace{\mathbb{E}[R^e_{i,t}]}_{\text{Average excess return}} - \underbrace{\beta_i'\lambda}_{\text{risk premium on asset } i}$$

What is the interpretation?

- $\alpha_i > 0 \longrightarrow$ asset i earns too much relative to its risk exposure,
- $\alpha_i = 0 \longrightarrow$ asset i is correctly priced by the model,
- $\alpha_i < 0 \longrightarrow$ asset i earns too little relative to its risk exposure.

Pricing error = average abnormal return unexplained by systematic risk

Asset Pricing Tests

How to estimate λ?

With traded factors, we observe factor risk premia from the data $\longleftarrow$ $\lambda = \mathbb{E}[F_t]$

Does risk exposure explain all average asset returns?

Risk factors have to price the entire cross-section of asset excess returns $\longleftarrow$ $H_0: \alpha_1 = \alpha_2 = \ldots = \alpha_N = 0$

What is the recipe for asset pricing tests with traded factors?

- Estimate β_i via time-series regressions and λ as the mean of F_t,
- Test whether pricing errors are jointly zero.

Asset Pricing Tests

We can conveniently stack N equations into a single system

$$\underbrace{R_t^e}_{N\times 1 \text{ vector of excess returns}} = \underbrace{\alpha}_{N\times 1 \text{ vector of pricing errors}} + \underbrace{\beta}_{N\times K \text{ matrix of systematic exposures}} \underbrace{F_t}_{K\times 1 \text{ vector of Traded factors}} + \underbrace{\varepsilon_t}_{N\times 1 \text{ vector of residual errors}}$$

Asset Pricing Tests

With $N = 5$ and $K = 3$, the system of equations looks like

$$\begin{bmatrix} R^e_{1,t} \\ R^e_{2,t} \\ R^e_{3,t} \\ R^e_{4,t} \\ R^e_{5,t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} + \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \\ \beta_{4,1} & \beta_{4,2} & \beta_{4,3} \\ \beta_{5,1} & \beta_{5,2} & \beta_{5,3} \end{bmatrix} \begin{bmatrix} MKT_t \\ SMB_t \\ HML_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \\ \varepsilon_{4,t} \\ \varepsilon_{5,t} \end{bmatrix}$$

Asset Pricing Tests

Calculate the sample vector of average excess returns

$$\text{N} \times \text{1 vector} \longleftarrow \mu_R = \mathbb{E}[R_t^e] = \frac{1}{T}\sum_{t=1}^{T} R_t^e$$

Compute the sample vector of average factor risk premia

$$\text{K} \times \text{1 vector} \longleftarrow \mu_F = \mathbb{E}[F_t] = \frac{1}{T}\sum_{t=1}^{T} F_t$$

Asset Pricing Tests

Calculate the sample return-factor covariance matrix

$$\text{N}\times\text{K matrix} \longleftarrow \Sigma_{RF} = \frac{1}{T}\mathbb{C}\text{ov}(R_t^e, F_t) = \sum_{t=1}^{T}(R_t^e - \mu_R)(F_t - \mu_F)'$$

Calculate the sample factor covariance matrix

$$\text{K}\times\text{K matrix} \longleftarrow \Sigma_F = \mathbb{V}\text{ar}(F_t) = \frac{1}{T}\sum_{t=1}^{T}(F_t - \mu_F)(F_t - \mu_F)'$$

Asset Pricing Tests

Calculate the matrix of risk exposures

$$\underbrace{\beta}_{N\times K \text{ matrix}} = \Sigma_{RF}\Sigma_F^{-1}$$

Calculate the vector of pricing errors

$$\underbrace{\alpha}_{N\times 1 \text{ vector}} = \mu_R - \beta\mu_F$$

Testing Pricing Errors

We can now test the null that all pricing errors are zero

$$H_0 : \alpha = 0$$

If R_t^e and F_t are jointly normal and *iid*, we can use

$$J_\alpha = \left(\frac{T-N-K}{N}\right)\left(1+\mu_F'\Sigma_F^{-1}\mu_F\right)^{-1}\alpha'\Sigma_\varepsilon^{-1}\alpha$$

where

N×N covariance matrix of regression errors

$$\Sigma_\varepsilon = \mathbb{V}\mathrm{ar}(\varepsilon_t) = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\varepsilon_t'$$

Testing Pricing Errors

Gibbons, Ross and Shanken (1989) show that in finite sample

This is the 'GRS' test statistic $\longleftarrow$ $\boxed{J_\alpha} \sim F_{N,T-N-K}$

How do we decide in practice?

- Compute the p-value as

$$p = 1 - F_{cdf}(\text{GRS};\, N,\ T - N - K).$$

- If $p > 5\%$, we fail to reject the null hypothesis.
- If $p \leq 5\%$, we reject the null hypothesis.

Testing Pricing Errors

What is the interpretation?

$$J_\alpha \propto SR_R^2 - SR_F^2$$

- $SR_R \rightarrow$ Sharpe ratio of the tangency portfolio spanned by the N assets,
- $SR_F \rightarrow$ Sharpe ratio of the tangency portfolio spanned by the K factors.

If factors correctly price assets, the maximum Sharpe ratio achievable with factors should equal that from all assets.

Non-Traded Factors

Non-Traded Factor

What is a non-traded factor?

- A non-traded factor is not a return portfolio and cannot be directly invested in.
- We cannot use the factor mean to infer its price of risk.

Recall the asset pricing restriction

Average excess return $\leftarrow$ $\mathbb{E}[R^e_{i,t}] = \beta_i{}'\lambda$

Quantity of risk (measures how much asset i co-moves with risk factors)

Price of risk (compensation investors require for bearing that risk)

How do we estimate λ with non-traded factors?

We need cross-sectional regressions

Two-Stage Regressions

First-stage: Run a time-series regression (TSR) to obtain estimates of betas

$$R_t^e = a + \beta F_t + \varepsilon_t$$

Second-stage: Run a cross-sectional regression (CSR) to obtain estimates of lambdas

N×1 vector of average excess returns

$$\mu_R = \beta\lambda + \alpha$$

K×1 vector of pricing of risk (regression slope)

N×1 vector of pricing errors (regression errors)

Cross-Sectional R-squared

How well does the model explain average returns?

variance of pricing errors

Cross-sectional R-squared

$$R^2_{cs} = 1 - \frac{\alpha'\alpha}{(\mu_R - \overline{\mu}_R)'(\mu_R - \overline{\mu}_R)}$$

variance of average excess returns

1×1 mean of μ_R

What is the interpretation?

- $R^2_{cs} = 1 \longrightarrow$ perfect pricing.
- $R^2_{cs} \leq 0 \longrightarrow$ factor exposures does not help.

Fraction of cross-sectional dispersion in average returns explained

Two-Stage OLS Regressions

The cross-sectional parameters can be estimated via OLS

$$\lambda = (\beta'\beta)^{-1}\beta'\mu_R$$

$$\alpha = \mu_R - \beta\lambda$$

If we assume that factors and errors are *iid*

$$\mathbb{V}\mathrm{ar}(\lambda) = \frac{1}{T}\left[(\beta'\beta)^{-1}(\beta'\Sigma_\varepsilon\beta)(\beta'\beta)^{-1} + \Sigma_F\right]$$

$$\mathbb{C}\mathrm{ov}(\alpha) = \frac{1}{T}[I_N - \beta(\beta'\beta)^{-1}\beta]\Sigma_\varepsilon[I_N - \beta(\beta'\beta)^{-1}\beta]'$$

Two-Stage OLS Regressions

We can test the null of zero pricing errors with the Wald statistic

$$W = \alpha' \, \mathbb{C}\mathrm{ov}(\alpha)^{-1}\alpha \sim \chi^2_{N-K}$$

How do we decide in practice?

- Compute the p-value as $1 - \chi^2_{cdf}(W, N-K)$
- If $p > 5\%$, we fail to reject the null hypothesis,
- If $p \leq 5\%$, we reject the null hypothesis.

Two-Stage GLS Regressions

The pricing errors are cross-sectionally correlated and GLS estimates are more efficient

$$\lambda = (\beta' \Sigma_\varepsilon^{-1} \beta)^{-1} \beta' \Sigma_\varepsilon^{-1} \mu_R$$

$$\alpha = \mu_R - \beta \lambda$$

If we assume that factors and errors are *iid*

$$\mathbb{V}\mathrm{ar}(\lambda) = \frac{1}{T} \left[(\beta' \Sigma_\varepsilon^{-1} \beta)^{-1} + \Sigma_F \right]$$

$$\mathbb{C}\mathrm{ov}(\alpha) = \frac{1}{T} [\Sigma_\varepsilon - \beta (\beta' \Sigma_\varepsilon^{-1} \beta)^{-1} \beta']$$

Two-Stage GLS Regressions

We can test the null of zero pricing errors with the Wald statistic

$$T\alpha'\Sigma_{\varepsilon}^{-1}\alpha \sim \chi^2_{N-K}$$

How do we decide in practice?

- Compute the p-value as $1 - \chi^2_{cdf}(W, N-K)$
- If $p > 5\%$, we fail to reject the null hypothesis,
- If $p \leq 5\%$, we reject the null hypothesis.

Shanken's Correction

The β estimates may suffer from errors-in-variable and Shanken (1992) proposes a correction.

For OLS regressions

$$\mathbb{V}\mathrm{ar}(\lambda) = \frac{1}{T}\left[(\beta'\beta)^{-1}(\beta'\Sigma_\varepsilon\beta)(\beta'\beta)^{-1}\left(1+\lambda'\Sigma_F^{-1}\lambda\right)+\Sigma_F\right]$$

$$\mathbb{C}\mathrm{ov}(\alpha) = \frac{1}{T}[I_N-\beta(\beta'\beta)^{-1}\beta]\Sigma_\varepsilon[I_N-\beta(\beta'\beta)^{-1}\beta]'\times\left(1+\lambda'\Sigma_F^{-1}\lambda\right)$$

For GLS regressions

$$\mathbb{V}\mathrm{ar}(\lambda) = \frac{1}{T}[(\beta'\Sigma_\varepsilon^{-1}\beta)^{-1}\left(1+\lambda'\Sigma_F^{-1}\lambda\right)+\Sigma_F]$$

$$\mathbb{C}\mathrm{ov}(\alpha) = \frac{1}{T}[\Sigma_\varepsilon-\beta(\beta'\Sigma_\varepsilon^{-1}\beta)^{-1}\beta']\left(1+\lambda'\Sigma_F^{-1}\lambda\right)$$

Fama-MacBeth Regressions

First Stage: run TSR and obtain estimates of betas as

$$R_t^e = a + \beta f_t + \varepsilon_t$$

Second Stage: run CSR at each time period t as

$$R_t^e = \beta \lambda_t + \alpha_t$$

Third Stage: Construct the factor risk premia as

$$\lambda = \frac{1}{T} \sum_{t=1}^{T} \lambda_t$$

$$var(\lambda) = \frac{1}{T^2} \sum_{t=1}^{T} (\lambda_t - \lambda)(\lambda_t - \lambda)'$$

Fama-MacBeth Regressions

Regarding the pricing error, we have that

$$\alpha = \frac{1}{T} \sum_{t=1}^{T} \alpha_t$$

$$var(\alpha) = \frac{1}{T^2} \sum_{t=1}^{T} (\alpha_t - \alpha)(\alpha_t - \alpha)'.$$

Then, use the following test

$$\alpha \, \mathrm{Cov}(\alpha)^{-1} \alpha \sim \chi^2_{N-K}$$

What if I want to construct HAC standard errors?

Cross-Sectional Regressions

This is also known as the two-pass regression

- in the first-pass, regress each return on F_t and obtain estimates of β_i,
- in the second-pass, regress the sample average of each return on β_i to get a λ estimate.

Should we include a constant in the second pass?

- Include a constant if theory allows a return unrelated to factor risk (a zero-beta rate),
- Impose a zero intercept if theory predicts all average returns are fully explained by risk,
- In practice, estimate the constant and check whether it is close to zero.

Stochastic Discount Factor

Stochastic Discount Factor

Security prices are determined by discounting future payoffs because of

1. Time value of money (cash/consumption worth less in the future),
2. Risk (unknown states of nature in the future).

Risk-neutral investors only cares about the time value of money

$$P_{i,t-1} = \frac{1}{R_{f,t-1}} \mathbb{E}^{Q}_{t-1}[X_{i,t}]$$

Risk-averse investors cares about both the time value of money and risk

$$P_{i,t-1} = \mathbb{E}_{t-1}[M_t X_{i,t}]$$

- $E^{Q}(\cdot) \to$ risk-neutral expectation,
- $E(\cdot) \to$ real-world expectation,
- $M_t \to$ stochastic discount factor (or SDF).

Stochastic Discount Factor

The 'Euler condition' can be rearranged as

$$\mathbb{E}_{t-1}[M_t R_{i,t}] = 1$$

where the gross asset return is given by

$$R_{i,t} = \frac{X_{i,t}}{P_{i,t-1}}$$

Stochastic Discount Factor

The Euler condition applied to a risk-free bond

$$\mathbb{E}_{t-1}[M_t R_{f,t-1}] = 1 \Longrightarrow \mathbb{E}_{t-1}[M_t] = \frac{1}{R_{f,t-1}}$$

The Euler condition applied to the excess return

$$\begin{aligned}
\mathbb{E}_{t-1}[M_t R_{i,t}] = 1 &\Longrightarrow \mathbb{E}_{t-1}[M_t R_{i,t} - 1] = 0 \\
&\Longrightarrow \mathbb{E}_{t-1}[M_t R_{i,t} - M_t R_{f,t-1}] = 0 \\
&\Longrightarrow \mathbb{E}_{t-1}[M_t (R_{i,t} - R_{f,t-1})] = 0 \\
&\Longrightarrow \mathbb{E}_{t-1}[M_t R^e_{i,t}] = 0
\end{aligned}$$

Stochastic Discount Factor

The 'Euler condition' applies to each asset (or strategy) i

$$\mathbb{E}_{t-1}\left[M_t R^e_{i,t}\right] = 0$$

- $R^e_{i,t} \rightarrow$ excess return on the asset i at time t,
- $M_t \rightarrow$ SDF that prices at time $t-1$ a payoff that occur at time t,
- $\mathbb{E}_{t-1}\left[\cdot\right] \rightarrow$ real-world expectation operator based on information available at date $t-1$.

Most asset pricing models, including CAPM and ICAPM, can be recast into an SDF framework

Stochastic Discount Factor

Conditional Euler equation

$$\mathbb{E}_{t-1}\left[M_t R^e_{i,t}\right] = 0$$

- Conditional expectations depend on investors' information and beliefs,
- We need to specify how investors form expectations and their information set,
- Modeling this information set is difficult and leads to joint inference.

(Unconditional) Euler equation

$$\mathbb{E}\left[M_t\, R^e_{i,t}\right] = 0$$

- It is convenient to examine the unconditional expectations,
- This removes the need to model investors' information sets.

Stochastic Discount Factor

Recall that

$$\mathbb{C}\mathrm{ov}(y,x) = \mathbb{E}\left(yx\right) - \mathbb{E}(y)E(x)$$

If the Euler condition holds

$$\mathbb{E}\left[M_t R^e_{i,t}\right] = \mathbb{C}\mathrm{ov}\left(R^e_{i,t}, M_t\right) + \mathbb{E}\left[R^e_{i,t}\right]\mathbb{E}\left[M_t\right] = 0$$

which implies

$$\mathbb{E}\left[R^e_{i,t}\right] = -\frac{\mathbb{C}\mathrm{ov}(R^e_{i,t}, M_t)}{\mathbb{E}\left[M_t\right]} \longrightarrow$$

Higher expected return for assets that have a lower $\mathbb{C}\mathrm{ov}\left(R_{i,t}, M_t\right)$

Stochastic Discount Factor

The beta representation can be then derived as follows

$$\mathbb{E}\left[R^e_{i,t}\right] = -\frac{\mathbb{C}\text{ov}(R^e_{i,t}, M_t)}{\mathbb{E}\left[M_t\right]} = \underbrace{\frac{\mathbb{C}\text{ov}\left(R^e_{i,t}, M_t\right)}{\mathbb{V}\text{ar}\left(M_t\right)}}_{\beta_i} \times \underbrace{\left(-\frac{\mathbb{V}\text{ar}\left(M_t\right)}{\mathbb{E}\left[M_t\right]}\right)}_{\lambda}$$

- $\beta_i \rightarrow$ quantity of risk obtained from TSR,
- $\lambda \rightarrow$ price of risk obtained from CRS.

Stochastic Discount Factor

The beta pricing model suggests that

- λ is the same across all assets,
- β_i is different across assets.

Each asset expected return is proportional to its quantity of risk.

- λ is the risk premium for an asset with $\beta_i = 1$.
- $\beta_i \lambda$ is the risk premium for an asset with a given β_i.

Stochastic Discount Factor

The C-CAPM, for example, specifies the following SDF

$$M_t = \beta \frac{u'(c_t)}{u'(c_{t-1})}$$

- $u(c_t) \rightarrow$ utility of consumption
- $u'(c_t) \rightarrow$ marginal utility of consumption,
- $\beta \rightarrow$ subjective discount factor.

The utility $u(c_t)$ captures the desire for more consumption, i.e., $u(c_t) \uparrow$ when $c_t \uparrow$

- The marginal utility is decreasing in consumption, i.e., $u'(c_t) \downarrow$ when $c_t \uparrow$,
- Why? The last bite is never as satisfying as the first one.

Stochastic Discount Factor

If an asset pays well when consumption is high

$$\text{Cov}(R^e_{i,t}, M_t) < 0 \longrightarrow$$

Negative covariance between $R^e_{i,t}$ and $u'(c_t)$

This asset makes consumption more volatile

$$E\left[R^e_{i,t}\right] = -\frac{\text{Cov}(R^e_{i,t}, M_t)}{E\left[M_t\right]} > 0 \longrightarrow$$

Must pay higher expected returns to induce investor to hold it.

Stochastic Discount Factor

If an asset pays well when consumption is low

$$\mathbb{C}\text{ov}(R^e_{i,t}, M_t) > 0 \longrightarrow$$

Positive covariance between $R^e_{i,t}$ and $u'(c_t)$

This asset makes consumption less volatile

$$\mathbb{E}\left[R^e_{i,t}\right] = -\frac{\mathbb{C}\text{ov}(R^e_{i,t}, M_t)}{\mathbb{E}\left[M_t\right]} < 0 \longrightarrow$$

Must pay lower expected returns as it acts as an insurance.

Stochastic Discount Factor

The C-CAPM, despite being intuitive and elegant, does not work well in practice.

Which variable is a good proxy for the marginal utility growth?

$$M_t = \beta \frac{u'(c_t)}{u'(c_{t-1})} \approx a + b' F_t$$

The literature has found a number of variables that proxy for 'bad states' of nature. Empirically, we often need to discriminate among competing factors (or factor zoo).

Stochastic Discount Factor

The Euler equation does not allow to separately identify a and b for any given λ.

It is common to choose the following normalization of the SDF

$$M_t = 1 - b'(F_t - \mu_F)$$

such that

$$\mathbb{E}\left[M_t\right] = 1.$$

In this case, the model is estimated by exploiting the moment restrictions

$$\mathbb{E}[M_t R^e_{i,t}] = 0$$

$$\mathbb{E}[F_t] = \mu_F$$

Stochastic Discount Factor

The beta pricing model can be rewritten as follows

$$\mathbb{E}[R^e_{i,t}] = -\frac{\mathbb{C}\mathrm{ov}[R^e_{i,t}, 1 - b'(F_t - \mu_F)]}{\mathbb{E}[1 - b'(F_t - \mu_F)]} = b'\,\mathbb{C}\mathrm{ov}\left(R^e_{i,t}, F_t\right)$$

We can further rearranged it as

$$\mathbb{E}[R^e_{i,t}] = \underbrace{\mathbb{C}\mathrm{ov}(R^e_{i,t}, F_t)\,\mathbb{V}\mathrm{ar}\,(F_t)^{-1}}_{\beta_i} \times \underbrace{b'\,\mathbb{V}\mathrm{ar}\,(F_t)}_{\lambda}$$

What is the economic interpretation?

Stochastic Discount Factor

Suppose we have a single factor

$$F_t \longrightarrow f_t$$

f_t is traded and yields positive (negative) excess returns in good (bad) times

- For example: book-to-market and carry factors,
- We should expect $b > 0$ and, in turn, $\lambda > 0$,
- A risky asset performs well in good times and vice-versa ($\beta_i > 0$),
- An insurance asset performs well in bad times and vice versa ($\beta_i < 0$).

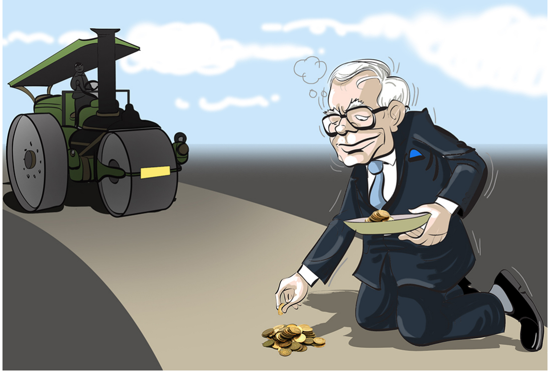
Stochastic Discount Factor

f_t is non-traded is high (low) in bad (good) states of nature

- For example: volatility shocks or liquidity shocks,
- We should expect $b < 0$ and, in turn, $\lambda < 0$,
- A risky asset performs well in good times and vice-versa ($\beta_i < 0$),
- An insurance asset performs well in bad times and vice versa ($\beta_i > 0$).

Asset Pricing Tests: Carry Trade

Carry Trade



"Such a strategy has its dangers. It has been likened to "picking up nickels in front of steamrollers"You have a long run of small gains but eventually get squashed."

The Economist, 22 February 2007

Carry Trade

How to rationalize the carry profits?

- The academic literature has proposed a number of different interpretations.

A risk-based explanation

- The difference between forward rates and expected future spot rates may reflect an equilibrium risk premium
- Several papers find evidence that carry returns are exposed to priced risk factors (e.g., Lustig et al., 2011; Menkhoff et al., 2012; Mancini et al., 2013; Lettau al., 2014).

Currency Portfolios

At time t,

- group currencies into 6 portfolios using their forward discounts fd_t
- P_1: low-yielding currencies (funding currencies)
- P_6: high-yielding currencies (investment currencies)

At time $t+1$,

- each portfolio's excess return is computed as the average individual excess return.

Lustig, Roussanov & Verdelhan (2011)

- The first principal component implies an equally weighted strategy across all portfolios, and defines the DOL factor,
- The second principal component is equivalent to a long-short strategy between P_6 and P_1, and is denoted as CAR factor.