Graded Problem Set: Investment and Portfolio Management

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```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.dates as mdates
```

Let's begin the analysis by looking at the structure of the dataset.

```
stock_prices = pd.read_excel('Problem set data.xls', sheet_name='Stock Prices')
```

```
stock_prices.info()
```

```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 216 entries, 0 to 215
Data columns (total 6 columns):
 #   Column         Non-Null Count  Dtype
---  ------         --------------  -----
 0   Date           216 non-null    object
 1   Ticker         216 non-null    object
 2   Company Name   216 non-null    object
 3   Close Price    216 non-null    float64
 4   High Price     216 non-null    float64
 5   Low Price      216 non-null    float64
dtypes: float64(3), object(3)
memory usage: 10.3+ KB
```

```
stock_prices['Date'] = pd.to_datetime(
    stock_prices['Date'], format="%Y-%m", errors='coerce')
```

```
# Pivot the DataFrame to have dates as index and tickers as columns

prices_wide = stock_prices.pivot_table(
    index='Date', columns='Ticker', values='Close Price').sort_index()

prices_wide.head()
```

Table 1: Pivoted stock prices dataframe

Ticker Date	AAPL	AMZN	GE
2019-01-01	166.44	1718.73	10.16
2019-02-01	173.15	1639.83	10.39

Ticker Date	AAPL	AMZN	GE
2019-03-01	189.95	1780.75	9.99
2019-04-01	200.67	1926.52	10.17
2019-05-01	175.07	1775.07	9.44

```
plt.figure(figsize=(12, 6))
ax = plt.gca()
for ticker in prices_wide.columns:
    ax.plot(prices_wide.index, prices_wide[ticker], label=ticker)
ax.xaxis.set_major_locator(mdates.MonthLocator(interval=2))
ax.xaxis.set_major_formatter(mdates.DateFormatter('%Y-%m'))

plt.xlabel('Date')
plt.ylabel('Close Price')
plt.title('Close Price over time')
plt.legend(title='Ticker', bbox_to_anchor=(1.02, 1), loc='upper left')
plt.grid(alpha=0.3)
plt.xticks(rotation=45)
plt.tight_layout()
plt.show()
```

Figure 1: Close Price over time for different tickers

We clearly see that there is some issue on the close price, likely due to stock splits. Let's try to clean our data so that we escape the stock split issue. First, let's identify the dates where stock splits happened for each ticker.

```
stock_prices = stock_prices.sort_values(by=['Ticker', 'Date']).reset_index(drop=True)
stock_prices['Return'] = stock_prices.groupby('Ticker')['Close Price'].pct_change()
stock_prices = stock_prices.dropna(subset=['Return'])

plt.figure(figsize=(12, 6))
ax = plt.gca()
for ticker in stock_prices['Ticker'].unique():
    ticker_data = stock_prices[stock_prices['Ticker'] == ticker]
    ax.plot(ticker_data['Date'], ticker_data['Return'], label=ticker)
ax.xaxis.set_major_locator(mdates.MonthLocator(interval=2))
ax.xaxis.set_major_formatter(mdates.DateFormatter('%Y-%m'))

plt.ylabel('Return')
plt.legend(title='Ticker', bbox_to_anchor=(1.02, 1), loc='upper left')
plt.title('Stock Returns over time')
plt.grid(alpha=0.3)
plt.xticks(rotation=45)
plt.tight_layout()
plt.show()
```

Figure 2: Stock Returns over time for different tickers

```
stock_prices
```

	Date	Ticker	Company Name	Close Price	High Price	Low Price	Return
1	2019-02-01	AAPL	APPLE INC	173.15	175.87	165.9300	0.040315
2	2019-03-01	AAPL	APPLE INC	189.95	197.69	169.5000	0.097026
3	2019-04-01	AAPL	APPLE INC	200.67	208.48	188.3800	0.056436
4	2019-05-01	AAPL	APPLE INC	175.07	215.31	174.9900	-0.127573
5	2019-06-01	AAPL	APPLE INC	197.92	201.57	170.2700	0.130519
...	...	...	...	...	...	...	...
211	2024-08-01	GE	GE AEROSPACE	174.62	175.97	150.2001	0.025969
212	2024-09-01	GE	GE AEROSPACE	188.58	190.88	160.5900	0.079945
213	2024-10-01	GE	GE AEROSPACE	171.78	194.80	170.4300	-0.089087
214	2024-11-01	GE	GE AEROSPACE	182.16	187.47	171.4500	0.060426
215	2024-12-01	GE	GE AEROSPACE	166.79	182.90	159.6000	-0.084376

Here we get the stock split or reverse stock split events for each ticker (outlier).

```
df_split_events = stock_prices[np.abs(
    stock_prices['Return']) >= 0.50].copy()
df_split_events
```

Table 3: Identified stock split or reverse stock split events

	Date	Ticker	Company Name	Close Price	High Price	Low Price	Return
19	2020-08-01	AAPL	APPLE INC	129.04	131.00	107.8925	-0.696405
113	2022-06-01	AMZN	AMAZON.COM INC	106.21	128.99	101.4300	-0.955823
175	2021-08-01	GE	GE AEROSPACE	105.41	107.23	98.1100	7.139768

After some research on the internet, I've found the following stock split events: - AAPL: 2020-08 (4-for-1) - GE: 2021-08 (1-for-8 reverse split) - AMZN: 2022-06 (20-for-1)

So now we can adjust the closing prices accordingly.

```
split_events = {
    ('AAPL','2020-08-01'): 4,
    ('AMZN','2022-06-01'): 20,
    ('GE','2021-08-01'): 1/8,
}

for (ticker, date_str), split_factor in split_events.items():
    date = pd.to_datetime(date_str)
    index_t = stock_prices[(stock_prices['Ticker'] == ticker) &
                           (stock_prices['Date'] == date)].index

    index_t = index_t[0]
    index_t_minus_1 = index_t - 1
    P_t = stock_prices.loc[index_t, 'Close Price']
    P_t_minus_1 = stock_prices.loc[index_t_minus_1, 'Close Price']

                # La formule utilise le facteur de split
    adjusted_return = ((P_t * split_factor) / P_t_minus_1) - 1
    stock_prices.loc[index_t, 'Return'] = adjusted_return
```

```
df_final_returns_adj = stock_prices[['Date', 'Ticker', 'Company Name',
                                     'Return']].dropna(subset=['Return']).copy()

plt.figure(figsize=(12, 6))
ax = plt.gca()
for ticker in df_final_returns_adj['Ticker'].unique():
    ticker_data = df_final_returns_adj[df_final_returns_adj['Ticker'] == ticker]
    ax.plot(ticker_data['Date'], ticker_data['Return'], label=ticker)
ax.xaxis.set_major_locator(mdates.MonthLocator(interval=2))
ax.xaxis.set_major_formatter(mdates.DateFormatter('%Y-%m'))

plt.ylabel('Return')
plt.legend(title='Ticker', bbox_to_anchor=(1.02, 1), loc='upper left')
plt.title('Stock Returns over time')
plt.grid(alpha=0.3)
plt.xticks(rotation=45)
plt.tight_layout()
plt.show()
```

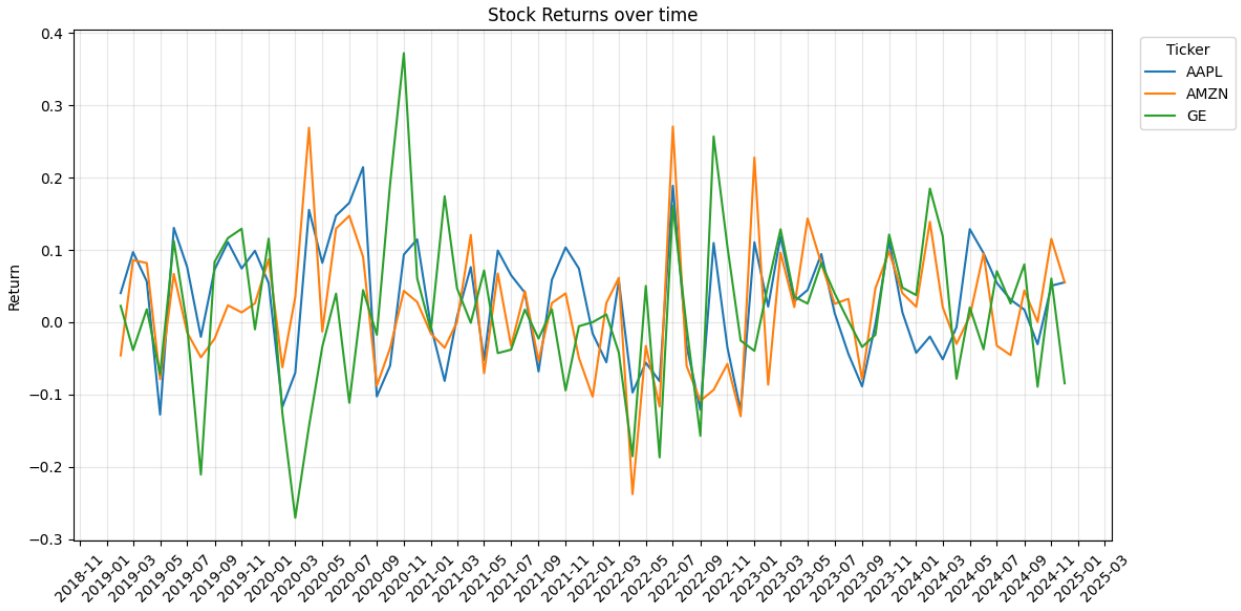

Figure 3: Adjusted Stock Returns over time for different tickers

```
mkt_index = pd.read_excel('Problem set data.xls', sheet_name='Market Index')
```

```
mkt_index["Market return"] = mkt_index["Level of the S&P 500 Index"]\
    .pct_change().dropna()
mkt_index['Date'] = pd.to_datetime(
    mkt_index['Date'], format="%Y-%m", errors='coerce')
```

```
rf_rate = pd.read_excel('Problem set data.xls', sheet_name='Risk Free Rate')
rf_rate['Risk Free Rate (Proportion)'] = rf_rate['Return on the T bill (in %)']\
    .apply(lambda x: ((1+(x/100))**(1/12)) - 1)

rf_rate['Date'] = pd.to_datetime(
    rf_rate['Date'], format="%Y-%m", errors='coerce')
```

```
df_all = pd.merge(df_final_returns_adj, mkt_index[['Date', 'Market return']],\
                  on='Date', how='left')
df_all = pd.merge(df_all, rf_rate[['Date', 'Risk Free Rate (Proportion)']],\
                  on='Date', how='left')
df_all['Stock excess return'] = df_all['Return'] \
    - df_all['Risk Free Rate (Proportion)']
df_all['Market excess return'] = df_all['Market return'] \
    - df_all['Risk Free Rate (Proportion)']
```

A. Covariance Matrix over the three stocks

First, let's rework the stock prices data to compute the monthly returns for each stock.

```
returns_pivot = df_final_returns_adj.pivot(
    index='Date',
    columns='Ticker',
    values='Return'
).dropna()

returns_pivot.head()
```

Table 4: Pivoted adjusted stock returns dataframe

Ticker Date	AAPL	AMZN	GE
2019-02-01	0.040315	-0.045906	0.022638
2019-03-01	0.097026	0.085936	-0.038499
2019-04-01	0.056436	0.081859	0.018018
2019-05-01	-0.127573	-0.078613	-0.071780
2019-06-01	0.130519	0.066792	0.112288

Now that we do know the returns, we can compute the covariance matrix of the three stocks over the entire period. Covariance between two stocks i and j is computed as:

$$Cov(R_i, R_j) = \frac{1}{N-1} \sum_{t=1}^{N} (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)$$

Where:

- $R_{i,t}$ and $R_{j,t}$ are the return of stock i and j at the time t
- $\bar{R}_i$ and $\bar{R}_j$ are the mean of each column (stock)
- N is the number of observations (periods).

```
demeaned_returns = returns_pivot - returns_pivot.mean()
n = demeaned_returns.shape[0]
cov_matrix_manual = (demeaned_returns.T @ demeaned_returns) / (n - 1)
```

```
cov_matrix_manual
```

Table 5: Covariance matrix between AAPL, AMZN and GE.

Ticker Ticker	AAPL	AMZN	GE
AAPL	0.006766	0.004986	0.002481
AMZN	0.004986	0.008038	0.001691
GE	0.002481	0.001691	0.011339

Other method to compute the covariance matrix

Method	Description	Pros	Cons
1. CPAM	Uses only one factor: market return. Covariance is estimated via the β of each asset in relation to the market.	**Simplicity**: Requires only one data series (the market). Covariance matrix always defined positive.	Maybe **too simplistic**: ignores other factors that may affect returns.
2. Multi-Factor Models (i.e. Fama-French)	Uses several factors (e.g. market, company size - SMB, value - HML) to model returns.	**More comprehensive**: Captures various sources of risk.	**Complexity**: Requires more data and sophisticated modeling techniques.
3. Linear Shrinkage	Combine la matrice de covariance historique (Σ) avec une matrice cible (Σ_{cible}),souvent une matrice à facteur unique ou constante, en utilisant un poids $\delta \in [0, 1]$.	**Forecast improved** Reduces noise and estimation errors. Ensures that the matrix is defined positive	Performance depends on the relevance of the chosen target matrix and the smoothing factor
4. Historical Method (chosen)	Relies solely on historical return data to estimate the covariance matrix.	**Simplicity**: Easy to implement and understand.	**Data sensitivity**: Highly dependent on the chosen historical period.

B. Historical Sharpe Ratios for three different portfolios

Quick recap of the different portfolios:

Portfolio	Weights in Apple	Weights in Amazon	Weights in GE
Portfolio 1	33.33%	33.33%	33.33%
Portfolio 2	40%	40%	20%
Portfolio 3	25%	25%	50%

1. General Methodology & function redaction

Sharpe Ratio Calculation

Sharpe Ratio is computed as:

$$SR = \frac{E[R_p] - R_f}{\sigma_p}$$

Where:

- $E[R_p]$ is the expected return of the portfolio
- R_f is the risk-free rate
- σ_p is the standard deviation of the portfolio returns

Define the function to compute the portfolio expected returns

```
def portfolio_expected_excess_return(dic_weight: dict, df_all: pd.DataFrame) \
    -> pd.Series:
    keys = list(dic_weight.keys())
    df_tmp = df_all.pivot(
        index='Date',
        columns='Ticker',
        values='Stock excess return'
    ).dropna()

    excess_return = df_tmp[keys[0]]*dic_weight[keys[0]] \
                  + df_tmp[keys[1]]*dic_weight[keys[1]] \
                  + df_tmp[keys[2]]*dic_weight[keys[2]]

    return excess_return
```

2. Portfolio A

Each stock has equal weights of 33.33%.

```
dic_weight = {'AAPL': 1/3, 'AMZN': 1/3, 'GE': 1/3}
portfolio_expected_excess_return_A = portfolio_expected_excess_return(
    dic_weight,
    df_all
    )

sharpe_ratio_A = \
    portfolio_expected_excess_return_A.mean()\
  / portfolio_expected_excess_return_A.std()

print(
    f"The sharpe ratio for Portfolio A is: {sharpe_ratio_A}")
```

```
The sharpe ratio for Portfolio A is: 0.265069079018295
```

3. Portfolio B

Apple and Amazon have weights of 40% each, while GE has a weight of 20%.

```
dic_weight = {'AAPL': 0.4, 'AMZN': 0.4, 'GE': 0.2}
portfolio_excess_return_B = portfolio_expected_excess_return(
    dic_weight,
    df_all
    )

sharpe_ratio_B = \
    portfolio_excess_return_B.mean()\
  / portfolio_excess_return_B.std()

print(f"The sharpe ratio for Portfolio B is: {sharpe_ratio_B}")
```

```
The sharpe ratio for Portfolio B is: 0.2747369701297241
```

4. Portfolio C

GE has a weight of 50%, while Apple and Amazon have weights of 25% each.

```
dic_weight = {'AAPL': 0.25, 'AMZN': 0.25, 'GE': 0.5}

portfolio_excess_return_C = portfolio_expected_excess_return(
    dic_weight,
    df_all)

sharpe_ratio_C = \
    portfolio_excess_return_C.mean()\
  / portfolio_excess_return_C.std()

print(f"The sharpe ratio for Portfolio C is: {sharpe_ratio_C}")
```

```
The sharpe ratio for Portfolio C is: 0.2365065375896759
```

```
df_sharpe = pd.DataFrame({
    "Portfolio": ["A", "B", "C"],
    "Sharpe": [sharpe_ratio_A.round(4),
               sharpe_ratio_B.round(4),
               sharpe_ratio_C.round(4)]
})
df_sharpe
```

Table 8: Summary of sharpe ratios for Portfolios A B and C.

	Portfolio	Sharpe
0	A	0.2651
1	B	0.2747
2	C	0.2365

TODO: ADD conclusion on the sharpe ratios

C. Risky portfolio weights

1. Context and methodology

Here, we get to find the weight that each of the investors would invest in the risky portfolio (composed of the three stocks) versus the risk-free asset, based on their risk aversion.

The Expected return of the complete portfolio is computed as:

$$E[R_c] = \omega_r E[R_p] + (1 - \omega_r)R_f = R_f + \omega_r(E[R_p] - R_f)$$

Where: - ω_r is the weight in the risky portfolio - p is the risky portfolio - R_f is the risk-free rate - c is the complete portfolio

Utility function is defined as:

$$U = E[R_c] - \frac{1}{2}A\sigma_c^2$$

Where: - A is the risk aversion coefficient

To maximize the utility function, we need to first substitute the expected return and variance of the complete portfolio into the utility function, then derive regarding ω_r and set it to zero to find the optimal weight ω_r^*. We start by substituting $E[R_c]$ and σ_c^2 into the utility function:

$$U = R_f + \omega_r(E[R_p] - R_f) - \frac{1}{2}A\omega_r^2\sigma_p^2$$

$$\Rightarrow \frac{\partial U}{\partial \omega_r} = (E[R_p] - R_f) - A\omega_r\sigma_p^2 = 0$$

$$\Leftrightarrow \omega_r^* = \frac{E[R_p] - R_f}{A\sigma_p^2}$$

Now, we only need to find the risk aversion for each investor to compute their optimal weight in the risky portfolio.

```python
def get_risk_aversion(a: int) -> float:
    """
    Here we compute the risk aversion from the utility function.
    Args:
        a (int): The coefficient of risk aversion.

    Returns:
        float: The risk aversion coefficient.
    """
    return 2*a

def compute_optimal_weight(risk_aversion: float,
                           excess_return: float,
                           portfolio_variance: float) -> float:

    return excess_return / (risk_aversion * portfolio_variance)
```

2. Computation of risky portfolio weights for each investor

Here are the utility functions for each investor:

Investor	Utility Function
X	$U = E[R_c] - \sigma_c^2$
Y	$U = E[R_c] - 2\sigma_c^2$
Z	$U = E[R_c] - 0.1\sigma_c^2$

```python
risk_aversion_X = get_risk_aversion(1)
risk_aversion_Y = get_risk_aversion(2)
risk_aversion_Z = get_risk_aversion(0.1)

excess_return = 0.1
variance_portfolio = 0.3

optimal_weight_X = compute_optimal_weight(
    risk_aversion_X,
    excess_return,
    variance_portfolio
)

optimal_weight_Y = compute_optimal_weight(
    risk_aversion_Y,
    excess_return,
```

```
    variance_portfolio
)

optimal_weight_Z = compute_optimal_weight(
    risk_aversion_Z,
    excess_return,
    variance_portfolio
)
print(f"Optimal weight for investor X: {optimal_weight_X:.4f}\
 for a risk aversion of {risk_aversion_X}")
print(f"Optimal weight for investor Y: {optimal_weight_Y:.4f}\
 for a risk aversion of {risk_aversion_Y}")
print(f"Optimal weight for investor Z: {optimal_weight_Z:.4f}\
 for a risk aversion of {risk_aversion_Z}")
```

```
Optimal weight for investor X: 0.1667 for a risk aversion of 2
Optimal weight for investor Y: 0.0833 for a risk aversion of 4
Optimal weight for investor Z: 1.6667 for a risk aversion of 0.2
```

At first sight it is quite surprising to have such a weight for investor Z. However, looking at his risk aversion coefficient of 0.2 (very low), it makes sense that he would invest a lot in the risky portfolio since he is not very averse to risk. We can then interprete the result as he would borrow money at the risk-free rate to invest even more than what he has in the risky portfolio.

D. Return on stocks assuming APT model

1. Methodology

The single factor chosen is the Market excess return r_M. The objective is to calculate the expected return ($E(R_i)$) for each asset for the next period.

the relationship between a stock's excess return r_i and the factor's excess return (here, the market r_M) is modeled by a simple linear regression:

$$r_i = \alpha_i + \beta_i r_{Mkt} + \epsilon_i$$

Where:

- r_i is the stock i's excess return
- r_{Mkt} is the market's excess return
- α_i is the regression intercept
- β_i is the sensitivity of stock i to the market excess return
- ϵ_i is the stock specific error term

To estimate the Expected Total Return $E(R_i)$ for the future period, we use the expected form of the APT equation, which, in this single-factor case, is identical to the CAPM. We assume the alpha and the specific error term are zero in the forecast:

$$E(R_i) = R_{f,nextmonth} + \beta_i E(r_{Mkt,nextmonth})$$

1. Compute each stock's β using linear regression

To compute the β of each stock, we will use the formula:

$$\beta_i = \frac{Cov(R_i, R_{Mkt})}{Var(R_{Mkt})}$$

```
dic_results = {'beta':{}, 'return':{}}
for tic in df_all['Ticker'].unique():
    df_tmp = df_all[df_all['Ticker'] == tic].copy()
    dic_results['beta'][tic] = round(np.cov(df_tmp['Return'],
            df_tmp['Market return'])[0, 1]/df_tmp['Market return'].var(), 4)
```

2. Estimate the market excess return for the next month

Estimated using the historical average of the Market Excess Return from the dataset.

```
df_tmp = df_all[df_all['Ticker'] == 'AAPL'] # To get only values once (otherwise we get

historical_market_excess_return = df_tmp['Market excess return'].mean()
```

3. Find risk free rate for the next month

Estimated using the last available risk-free rate in the dataset.

```
rf_next = df_tmp['Risk Free Rate (Proportion)'].iloc[-1]
```

4. Compute the expected return for each stock using the APT Formula

$$E(R_i) = R_{f,nextmonth} + \beta_i E(r_{Mkt,nextmonth})$$

```
for tic in dic_results['beta'].keys():
    dic_results["return"][tic] = rf_next + dic_results['beta'][tic] * historical_market_

pd.DataFrame(dic_results)
```

Table 10: Estimated betas and expected returns for each stock using the APT formula

	beta	return
AAPL	1.2591	0.016587
AMZN	1.1408	0.015374
GE	1.2355	0.016345