Investments and Portfolio Management

Tutorial 2

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Introduction

- My office hours: 18:30-19:30 on Monday on Zoom.
- Link to the Zoom meeting: Click
- Passcode to enter the Zoom session: Fin25!
- Outside office hours, feel free to send all your questions to a.izadyar23@imperial.ac.uk

Notation Guide

- r_i : Return on asset i
- r_m : Return on market portfolio.
- r_f : Risk-free rate
- R_i : Excess return on asset i, defined as

$$R_i = r_i - r_f$$

- R_M : Excess return on the market portfolio

$$R_M = r_M - r_f$$

CAPM Problem 1

The market price of a security is \$50. Its expected rate of return is 14%. The risk- free rate is 6%, and the market risk premium is 8.5%.

What will be the market price of the security if its correlation coefficient with the market portfolio doubles (and all other variables remain unchanged)? Assume that the stock is expected to pay a constant dividend in perpetuity.

CAPM Problem 1

We have a stock that pays a dividend in perpetuity:

$$P_0 = D\left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots\right) = \frac{D}{r}$$

If the rate of return is 14%, we can deduce that the dividend amount is 14% of \$50, or \$7.

We can find the beta of the stock from the CAPM – knowing the risk free rate and the market risk premium.

$$E(r_P) = r_f + \beta(E(r_M) - r_f)$$

$$\beta = \frac{E(r_P) - r_f}{E(r_M) - r_f}$$

$$\beta = \frac{0.14 - 0.06}{0.085} = 0.94$$

CAPM Problem 1

If the correlation coefficient between the stock and the market doubles, the beta necessarily doubles.

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_M)}{\mathrm{Var}(r_M)} = \frac{\rho_{iM}\,\sigma_i\,\sigma_M}{\sigma_M^2} = \rho_{iM}\,\frac{\sigma_i}{\sigma_M}.$$

The new rate of return for the stock is therefore:

$$E(r_S) = r_f + \beta^*(E(r_M) - r_f)$$

$$\Rightarrow E(r_S) = 0.06 + 1.88(0.085)$$

$$\Rightarrow E(r_S) = 0.2198 \sim 22\%$$

The dividend amount of \$7 is fixed, so the price of the stock must adjust to ensure this rate of return.

CAPM Problem 1

The price for which \$7 is a 22% rate of return is:

$$P_s = \frac{7}{0.22} \sim \$31.82$$

$$\$50 \to \$31.82$$

Price declines by 36.4%.

CAPM Problem 2

Karen Kay, a portfolio manager at Collins Asset Management, is using the capital asset pricing model for making recommendations to her clients. Her research department has developed the information below:

	Forecast return	Standard deviation	Beta
Stock X	14%	23%	0.6
Stock Y	16%	14%	1.3
Market index	18%	13%	1.0
Risk-free rate	7.0%		

What are the alphas for stocks X and Y?

CAPM Problem 2

The CAPM model predicts the expected return on a stock i, based on the principle that the risk-to-reward ratio of the stock should be the same as the market portfolio. The prediction is expressed as

$$E(r_i) = r_f + \beta_i(r_M - r_f)$$

Alpha is the difference between the SML-implied rate of return and the actual rate of return of an asset – can be interpreted as performance against the market (adjusting for factor exposure).

CAPM Problem 2

Applying this formula to the stock X:

$$E(r_X) = r_f + \beta_X(r_M - r_f)$$

$$\Rightarrow E(r_X) = 0.07 + 0.6(0.18 - 0.07)$$

$$\Rightarrow E(r_X) = 0.136 \text{ or } 13.6\%$$

Given that the forecast return is 14%,

$$\alpha_X = 14\% - 13.6\% = 0.4\%$$

X delivers a positive alpha.

CAPM Problem 2

Applying this formula to the stock Y:

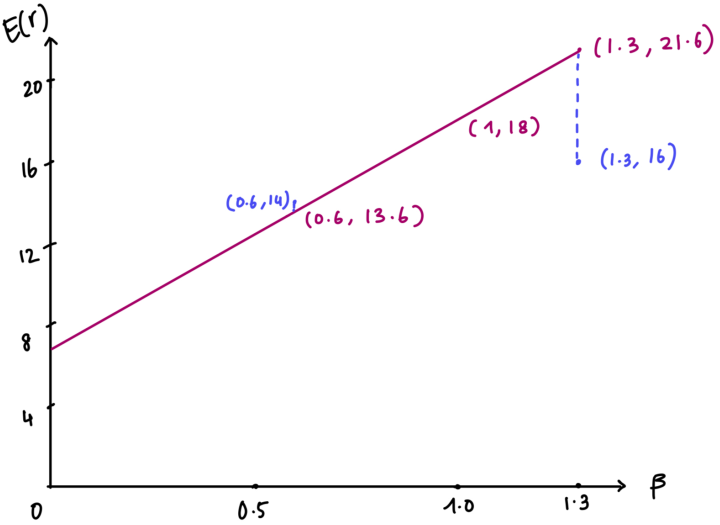
$$E(r_Y) = r_f + \beta_Y(r_M - r_f)$$
$$\Rightarrow E(r_Y) = 21.3\%$$

Given that the forecast return is 16%,

$$\alpha_Y = 16\% - 21.3\% = -5.3\%$$

Y delivers a negative alpha.

CAPM Problem 2



CAPM Problem 3-a

	Forecast return	Standard deviation	Beta
Stock X	14%	23%	0.6
Stock Y	16%	14%	1.3
Market index	18%	13%	1.0
Risk-free rate	7.0%		

Identify and justify which stock would be more appropriate for an investor who wants to add his stock to a well diversified portfolio.

CAPM Problem 3-a

Stock X is the better choice to add to a well-diversified portfolio because it has a **positive alpha**. Additional considerations:

- A low beta has a good impact on overall portfolio risk.
- The high individual risk of the stock can be offset by diversification.

CAPM Problem 3-b

	Forecast return	Standard deviation	Beta
Stock X	14%	23%	0.6
Stock Y	16%	14%	1.3
Market index	18%	13%	1.0
Risk-free rate	7.0%		

Identify and justify which stock would be more appropriate for an investor who wants to hold this stock as a single stock portfolio.

CAPM Problem 3-b

If the investor wants to hold a single-stock portfolio, beta as a risk measure is irrelevant.

Instead, when a stock is held as a single stock portfolio, standard deviation is the relevant risk measure.

	Forecast return	Standard deviation	Beta
Stock X	14%	23%	0.6
Stock Y	16%	14%	1.3

Stock Y is better. It offers a higher return for a lower level of risk.

APT Problem 1-a

Assume that stock market returns have the market index as a common factor, and that all stocks in the economy have a beta of 1 on the market index. Firm-specific returns all have a standard deviation of 30%.

Suppose that an analyst studies 20 stocks and finds that one-half of them have an alpha of +2%, and the other half have an alpha of -2%. Suppose the analyst invests $1 million in an equally weighted portfolio of the positive alpha stocks, and shorts $1 million of an equally weighted portfolio of the negative alpha stocks.

What is the expected profit (in dollars) and standard deviation of the analyst's profit?

APT Problem 1-a

$$E[R_i] = \alpha_i + \beta_{iM} R_M$$

If the stocks each have a beta of 1 on the market index, their returns can be expressed in the following way

$$E[R_i] = \alpha_i + R_M$$

For the portfolios of positive and negative alpha stocks respectively:

$$E(R_+) = 0.02 + R_M$$

$$E(R_-) = -0.02 + R_M$$

APT Problem 1-a

We are shorting \$1 million of a portfolio of 10 equally-weighted stocks with the negative alpha and investing the proceeds in a portfolio of 10 equally-weighted stocks with positive alpha. This eliminates market risk exposure.

Expected excess return is

$$1000000[E(R_+)] - 1000000[E(R_-)]$$

$$\Rightarrow 1000000[0.02 + R_M] - 1000000[-0.02 + R_M]$$

$$\Rightarrow 1000000[0.04] = \$40000$$

APT Problem 1-a

The long-short portfolio has market risk exposure, therefore systematic (market) risk cancels out, but firm-specific risks remain. Note that the firm-specific (idiosyncratic) standard deviation of returns is 0.3.

For \$100,000 invested in one stock, the variance of the profit/excess return on that stock is

$$(100,000)^2 V(R) = (100,000)^2 (0.3)^2 = 900,000,000$$

APT Problem 1-a

The total idiosyncratic variance of the portfolio is simply the sum of each component's idiosyncratic variance.

$$Variance = 20 * 900,000,000 = 18,000,000,000$$

Standard deviation is $\sqrt{18,000,000,000} = \$134,164$

APT Problem 1-b

How does the answer change if the analyst examines 50 stocks instead of 20 stocks? 100 stocks?

APT Problem 1-b

The expected excess return is not going to change, because we're still making two portfolios with positive and negative alpha respectively, and we still eliminate market risk exposure.

However, if the investor is investing in 50 stocks, their position in each stock is smaller - \$40,000. Therefore, the variance is:

$$50 * (40,000)^2 * V(R) = 50 * (40,000)^2 (0.3)^2 = 7,200,000,000$$

Standard deviation is $\sqrt{7,200,000,000} = \$84,853$

APT Problem 1-b

For 100 stocks, the investor's position in every stock is \$20,000.

$100 * (20,000)^2 * V(R) = 100 * (20,000)^2 (0.3)^2 = 3,600,000,000$

Standard deviation is $\sqrt{3,600,000,000} = \$60,000$

When we increase the number of stocks in each portfolio, we are mitigating the effect of the idiosyncratic risk e_i, without affecting the expected excess return.

APT Problem 2-a

Assume that security returns are generated by the single index model, where R_i is the excess return for security i and R_M is the market's excess return. The risk-free rate is 2%. Suppose also that there are three securities, A, B, and C:

	β_i	$E(r_i)$	$\sigma(e_i)$
A	0.8	10%	25%
B	1.0	12%	10%
C	1.2	14%	20%

If $\sigma_M = 20\%$, calculate the variance of returns of securities A, B, and C.

APT Problem 2-a

As per the standard single index model, we have

$$R_i = \alpha_i + \beta_i R_M + e_i$$

Derivation

$$\begin{aligned}
V(R_i) &= V(\alpha_i + \beta_i R_M + e_i) \\
&= V(\beta_i R_M + e_i) \qquad (\text{since } V(\alpha_i) = 0) \\
&= \beta_i^2 V(R_M) \ + \ V(e_i) \ + \ 2\beta_i \operatorname{Cov}(R_M, e_i) \\
&= \beta_i^2 \sigma_M^2 \ + \ \sigma^2(e_i) \qquad \big(\text{note } \operatorname{Cov}(R_M, e_i) = 0\big).
\end{aligned}$$

Therefore, we have

$$V(R_i) = (\beta_i)^2 \sigma_M^2 + \sigma^2(e_i)$$

APT Problem 2-a

	β_i	$E(r_i)$	$\sigma(e_i)$
A	0.8	10%	25%
B	1.0	12%	10%
C	1.2	14%	20%

We plug in the details from above, and the fact that $\sigma_M = 20\%$ into this:

$$V(R_i) = (\beta_i)^2 \sigma_M^2 + \sigma^2(e_i)$$

and obtain

$$V(R_1) = (0.8)^2(20)^2 + (25)^2 = 881$$
$$V(R_2) = (1.0)^2(20)^2 + (10)^2 = 500$$
$$V(R_3) = (1.2)^2(20)^2 + (20)^2 = 976$$

APT Problem 2-b

Now assume that there are an infinite number of assets with return characteristics identical to those of A, B, and C, respectively. If one forms a well-diversified portfolio of type A securities, what will be the mean and variance of the portfolio's excess returns? What about portfolios composed only of type B or C stocks?

APT Problem 2-b

If there are an infinite number of assets with identical characteristics, then a **well-diversified portfolio** of each type will have only **systematic risk** since the nonsystematic (idiosyncratic) risk approaches zero as the number of assets n becomes very large.

Every portfolio's variance therefore is:

$$V(R_i) = (\beta_i)^2 \sigma_M^2$$

This gives us

$$V(R_1) = (0.8)^2(20)^2 = 256$$
$$V(R_2) = (1.0)^2(20)^2 = 400$$
$$V(R_3) = (1.2)^2(20)^2 = 576$$

Expected portfolio returns are the same as individual stocks.

APT Problem 2-c

Is there an arbitrage opportunity in this market? What is it? Analyze the opportunity graphically.

APT Problem 2-c

	β_i	$E(r_i)$	$\sigma(e_i)$
A	0.8	10%	25%
B	1.0	12%	10%
C	1.2	14%	20%

Check against the Security Market Line (SML)

SML: $E(r_i) = r_f + \beta_i\big(E(r_M) - r_f\big)$.

Compute the implied market risk premium using any row:

$$\frac{E(r_i) - r_f}{\beta_i} = \begin{cases} (10\% - 2\%)/0.8 = 10\% \\ (12\% - 2\%)/1.0 = 10\% \\ (14\% - 2\%)/1.2 = 10\% \end{cases} \quad \Rightarrow \; E(r_M) - r_f = 10\% \,.$$

Therefore, for each asset

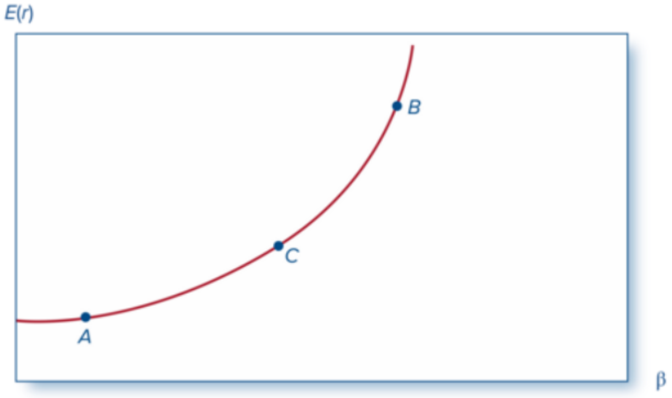
$$E(r_i) \stackrel{!}{=} 2\% + \beta_i \times 10\% \quad \text{holds exactly.}$$

There is no opportunity for arbitrage, as all the assets are on the SML. All the assets are fairly priced.

APT Problem 3-a

The SML relationship states that the expected risk premium on a security in a one-factor model must be *directly* proportional to the security's beta. Suppose that this were not the case. For example, suppose that expected return rises more than proportionately with beta as in the figure on the next slide.

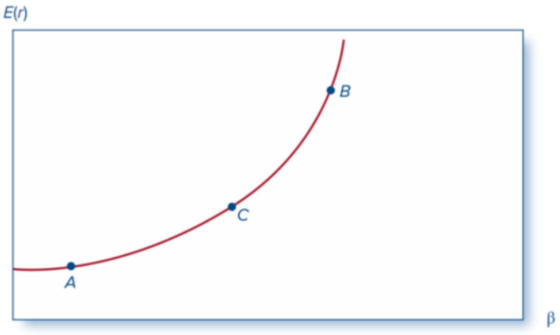
APT Problem 3-a



APT Problem 3-a

How could you construct an arbitrage portfolio?

APT Problem 3-a



Say we have a portfolio P such that it contains portfolio A weighted by w_A and portfolio B by w_B.

APT Problem 3-a

The expected return of this portfolio P is

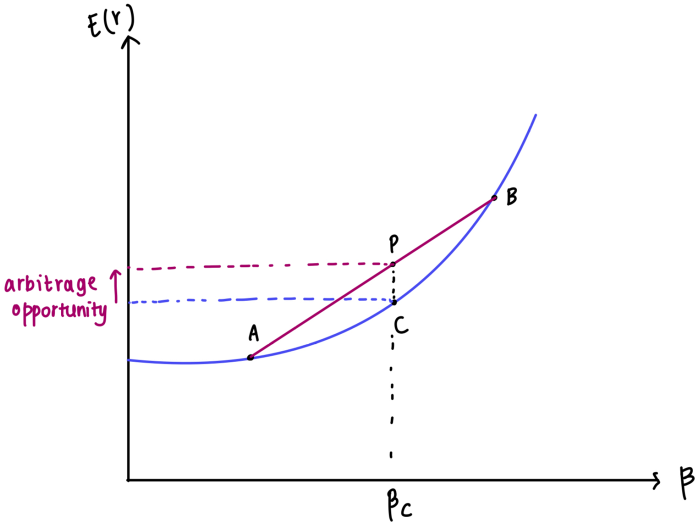
$$E(r_P) = w_A E(r_A) + w_B E(r_B)$$

The beta of this portfolio P is given by:

$$\beta_P = w_A \beta_A + w_B \beta_B$$

- Suppose we choose weights such that $\beta_P = \beta_C$.
- Since the curve is convex, $E(r_P) > E(r_C)$.
- Arbitrage opportunity: Go long portfolio P (combination of A and B) and short portfolio C in equal amounts.

Problem 3-a



Problem 3-b

Some researchers have examined the relationship between average returns on diversified portfolios and the β and β^2 of those portfolios. What should they have discovered about the effect of β^2 on portfolio return?

Problem 3-b

Researchers test whether the expected return depends nonlinearly on beta by estimating:

$$E(R_i) = \gamma_0 + \gamma_1 \beta_i + \gamma_2 \beta_i^2.$$

Under CAPM / APT theory:

$$E(R_i) = E(R_M)\beta_i.$$

This is a *linear* relation between $E(R_i)$ and β_i.

Interpretation:

- A nonzero γ_2 (i.e., curvature in the SML) would mean that expected returns rise more or less than proportionally with β.
- This curvature implies that some portfolios would lie *above* and others *below* the straight-line SML.
- Investors could then form a combination of portfolios on either side of the curve to create an **arbitrage portfolio**:

Conclusion

Because any curvature would allow arbitrage, the theoretical SML must be perfectly linear. Therefore $\gamma_2 = 0$.