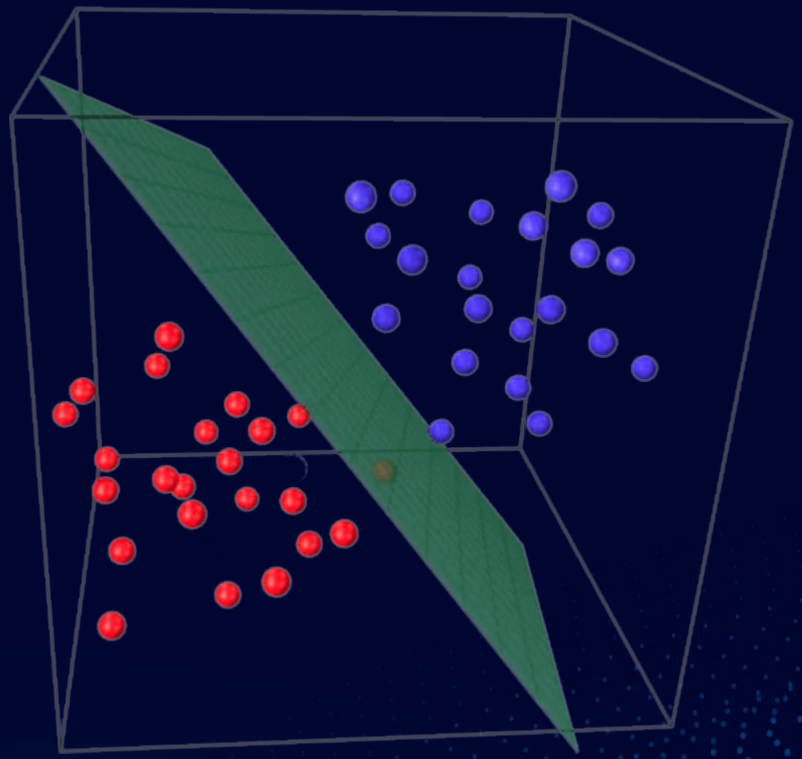


Linear Classifiers

Most of the following slides are adapted from ***Fei-Fei Li*** *and* ***Andrej Karpathy***

Last modified on 2022/05/07
Send typos and comments to f.maire@qut.edu.au

Outline

- Introductory example: image classification
- Nearest Neighbour classifier (review)
- L1 distance
- Linear classification
  - Losses (SVM loss, Cross-entropy loss)
  - Error landscapes
  - Training algorithms

Image Classification: a core task in Computer Vision

X

(assume given set of discrete labels)
{dog, cat, truck, plane, ...}

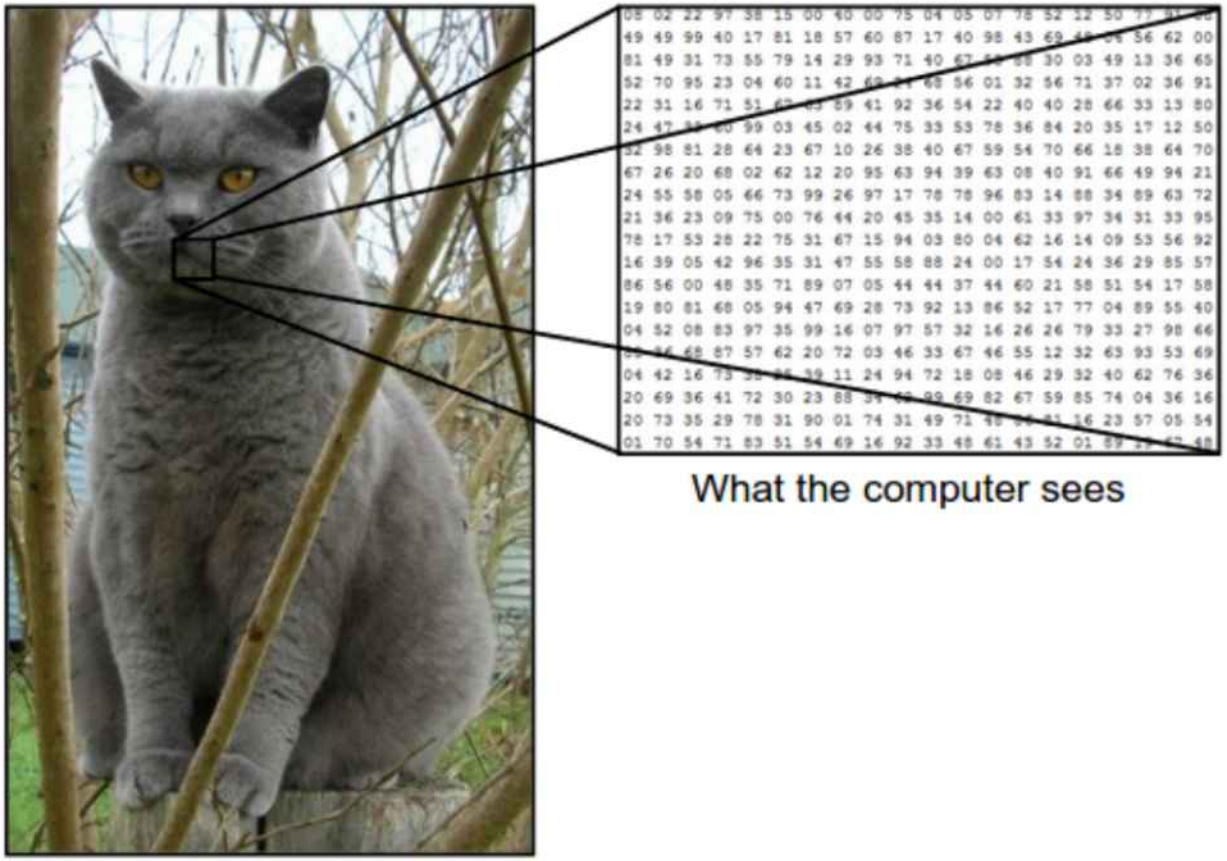
→ cat

y

The problem:
semantic gap

Images are represented as R^d arrays of numbers

- E.g. R^3 with integers between [0, 255], where d=3 represents 3 color channels (RGB)



What the computer sees

The complex process of image processing: leading from raw images, through subsequent stages to the semantic interpretation of the image.

An image classifier

```
def predict(image):
    # ????
    return class_label
```

Unlike e.g. sorting a list of numbers,

no obvious way to hard-code the algorithm for recognizing a cat, or other classes.

Data-driven approach:

1. Collect a dataset of images and label them
2. Use Machine Learning to train an image classifier
3. Evaluate the classifier on a withheld set of test images

```
def train(train_images, train_labels):
    # build a model of images -> labels

def predict(image):
    # evaluate the model on the image
    return class_label
```

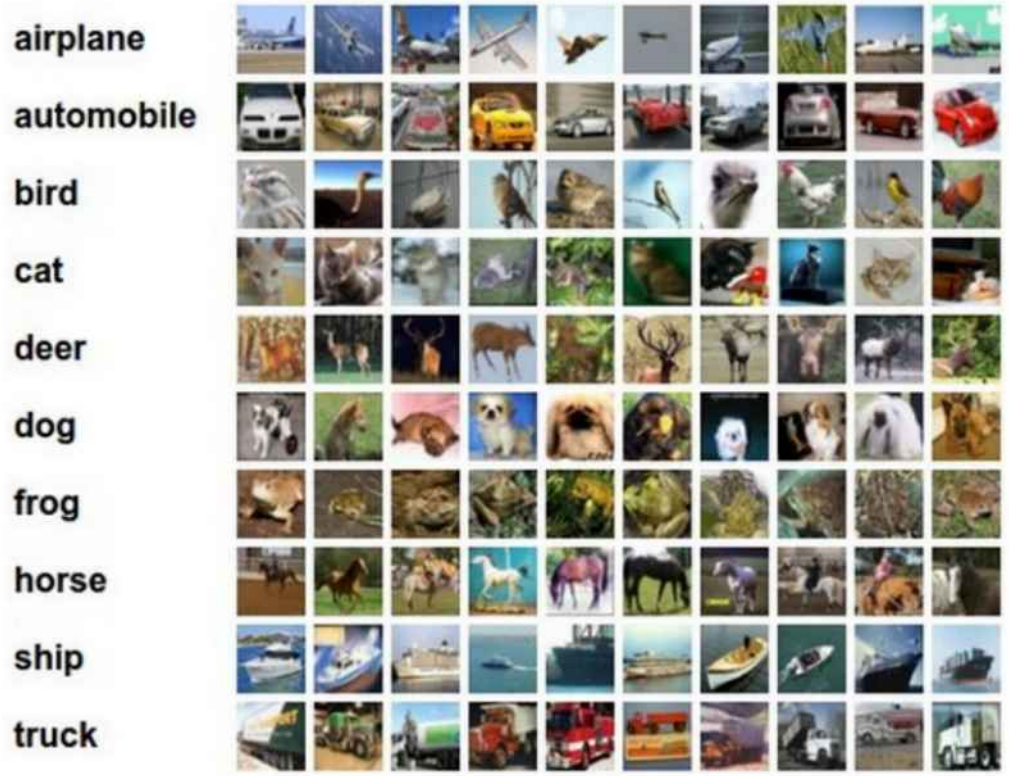
First classifier: **Nearest Neighbor Classifier**

```
def train(train_images, train_labels):
    # build a model of images -> labels

def predict(image):
    # evaluate the model on the image
    return class_label
```

Remember all training images and their labels

Predict the label of the most similar training image

Example dataset: **CIFAR-10**
10 labels
50,000 training images
10,000 test images.



For every test image (first column), examples of nearest neighbors in rows

How do we compare the images? What is the **distance metric**?

L1 distance: $$d_1(I_1, I_2) = \sum_p |I_1^p - I_2^p|$$

Where I_1 denotes image 1, and p denotes each pixel

test image

56	32	10	18
90	23	128	133
24	26	178	200
2	0	255	220

\-

training image

10	20	24	17
8	10	89	100
12	16	178	170
4	32	233	112

=

pixel-wise absolute value differences

46	12	14	1
82	13	39	33
12	10	0	30
2	32	22	108

→ 456

Linear Classification

1. define a **score function**

(assume CIFAR-10 example so
32 x 32 x 3 images, 10 classes)

data (image)
[3072 x 1]

$$f(x_i, W, b) = Wx_i + b$$

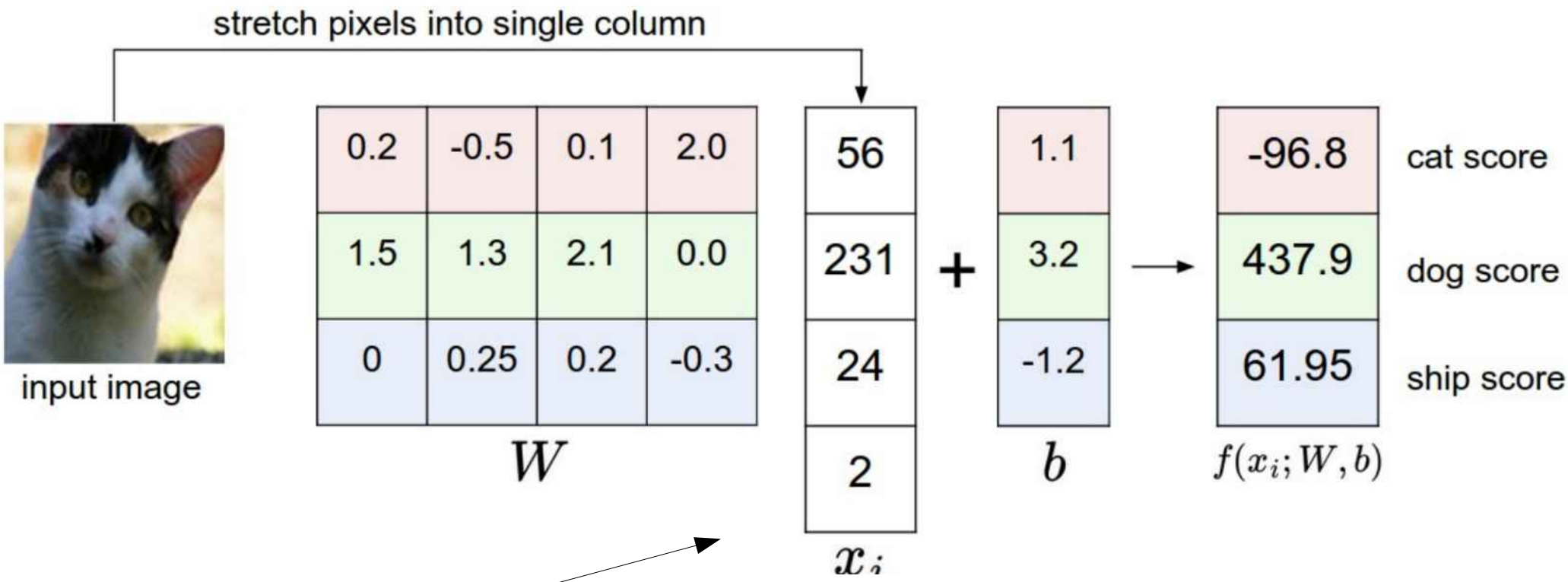
class scores
[10 x 1]

weights
[10 x 3072]

bias vector
[10 x 1]

Linear Classification

$$f(x_i, W, b) = Wx_i + b$$

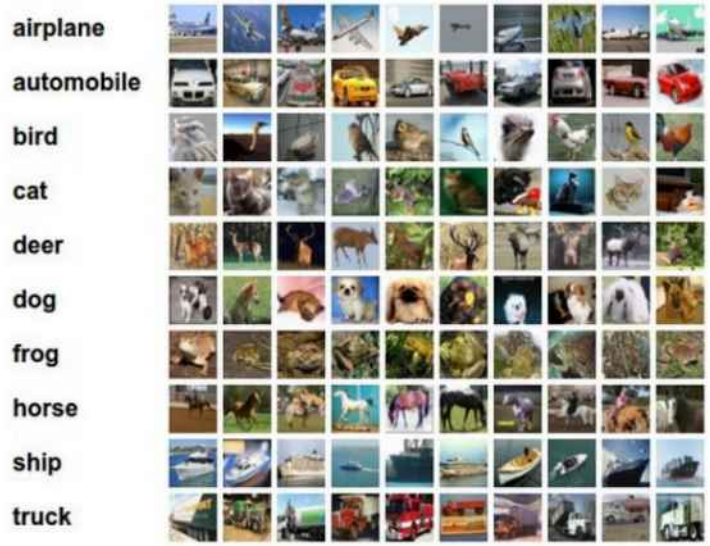


You have to imagine that the image has only 4 pixels!

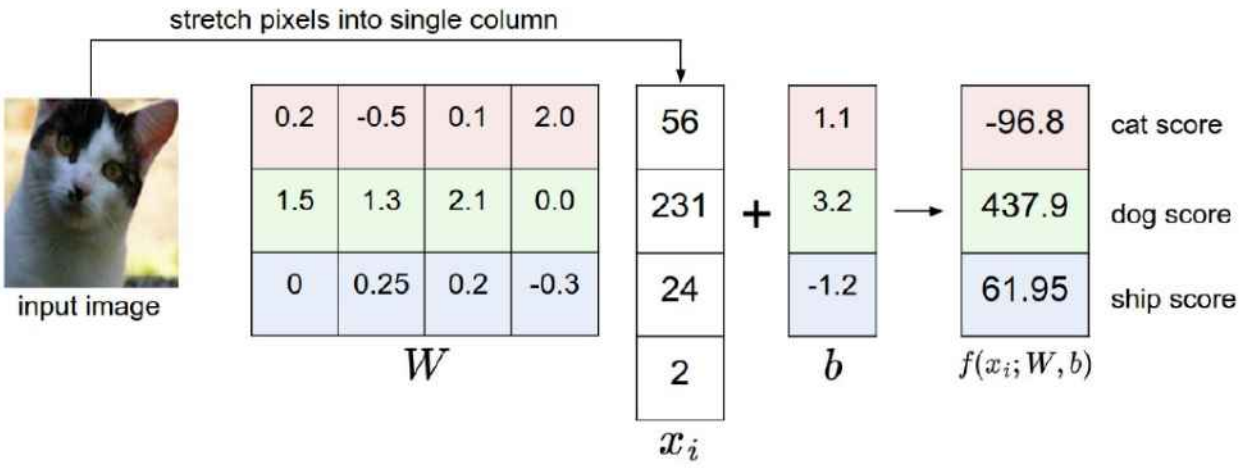
Interpreting a Linear Classifier

Question:

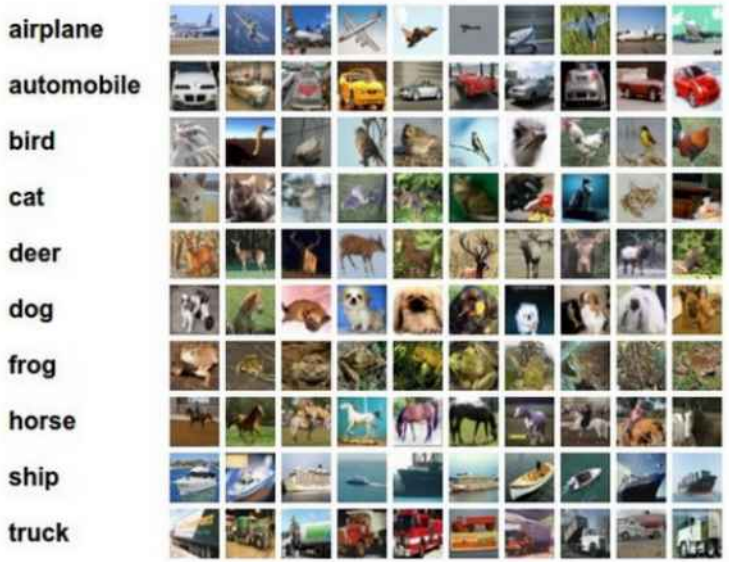
what can a linear classifier do?



$$f(x_i, W, b) = Wx_i + b$$



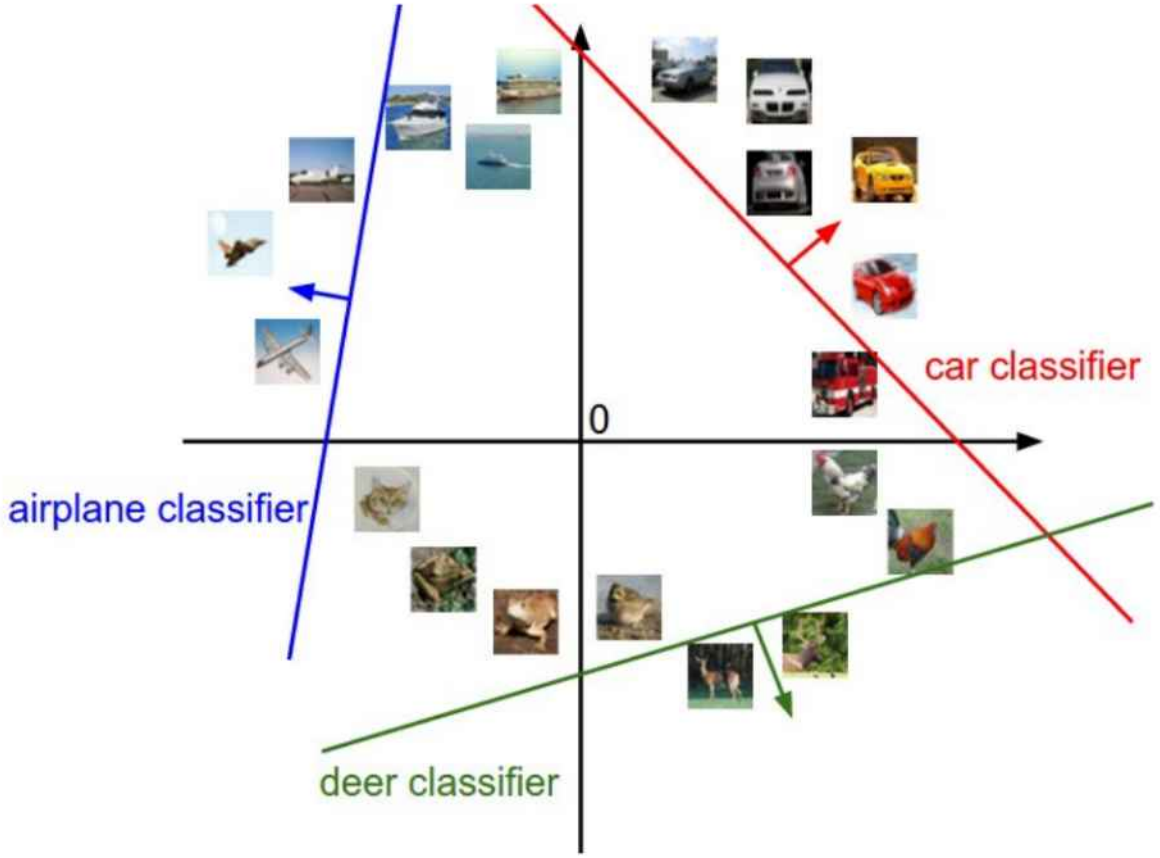
Interpreting a Linear Classifier



$$f(x_i, W, b) = Wx_i + b$$

Example training classifiers on CIFAR-10:

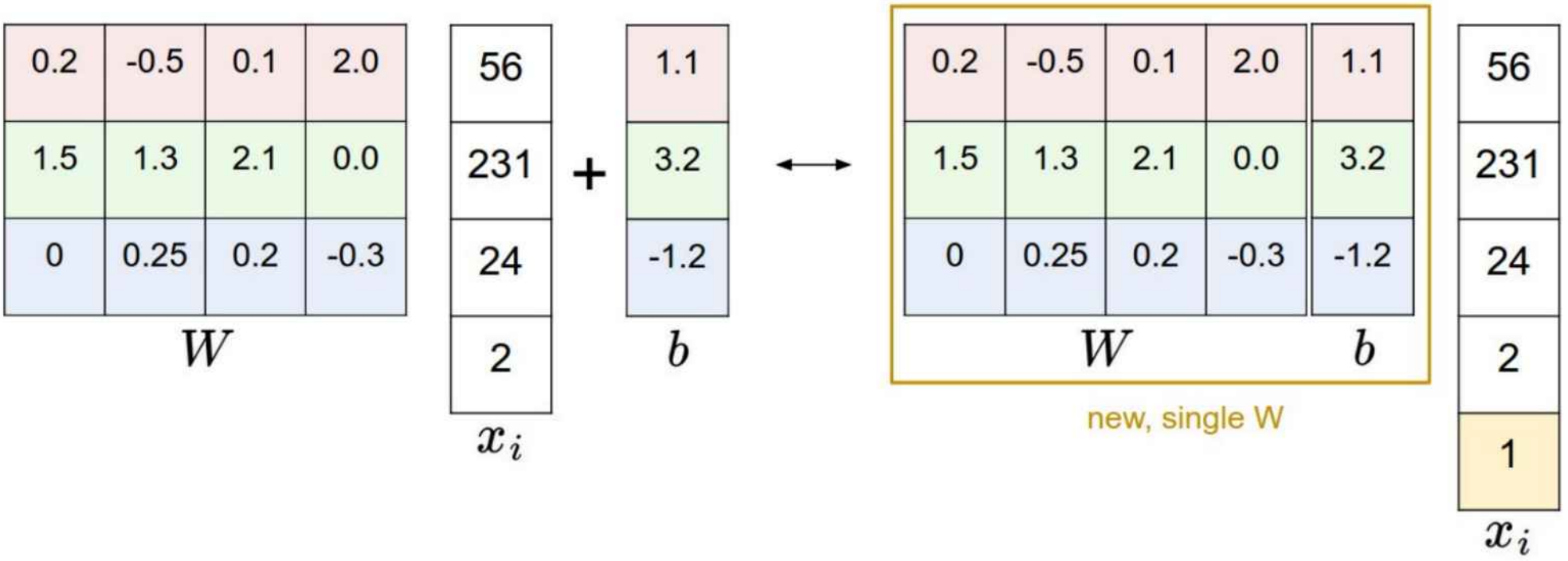
Interpreting a Linear Classifier



$$f(x_i, W, b) = Wx_i + b$$

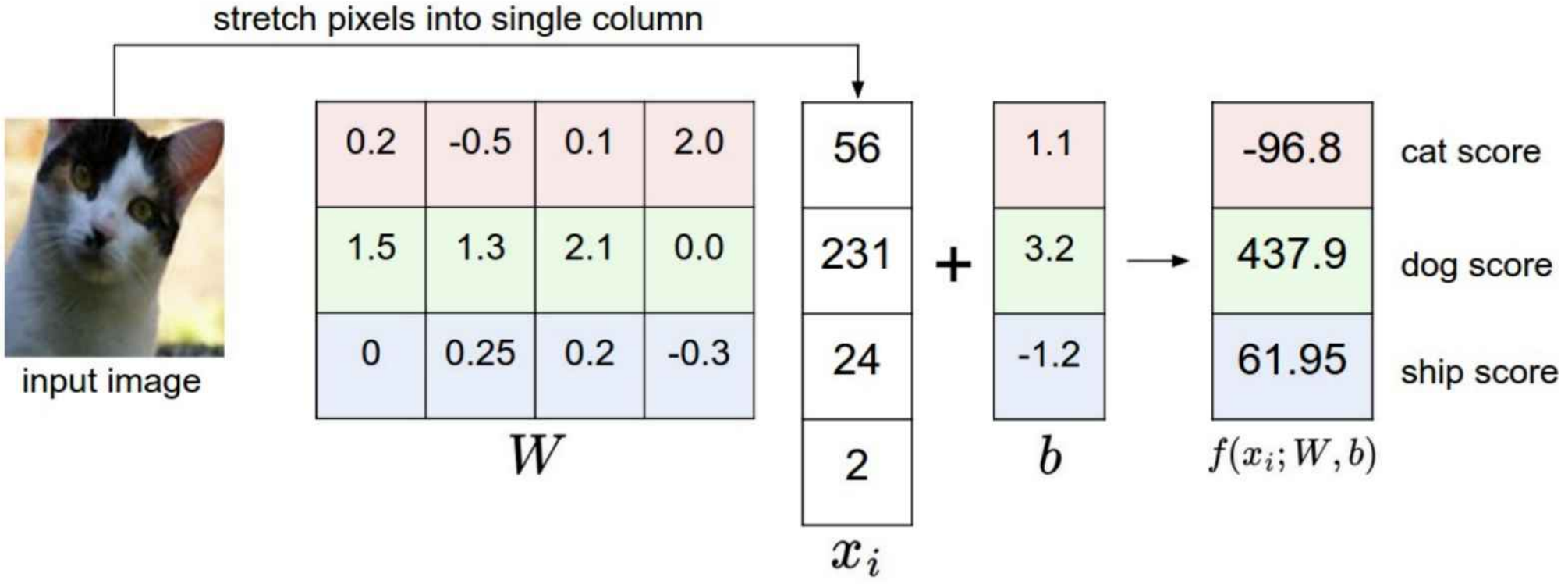
Bias trick

$$f(x_i, W, b) = Wx_i + b \longrightarrow f(x_i, W) = Wx_i$$



So far:

We defined a (linear) **score function**: $f(x_i, W, b) = Wx_i + b$



Define a **loss function** (or cost function, or objective)

- scores, label ⟶ *loss*.

$f(x_i, W)$ y_i L_i

Example:

$$f(x_i, W) = [13, -7, 11]$$

$$y_i = 0$$

Question: if you were to assign a single number to how "unhappy" you are with these scores, what would you do?

Define a **loss function** (or cost function, or objective)
One (of many ways) to do it: **Multiclass SVM Loss**

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

loss due to example i

sum over all incorrect labels

difference between the correct class score and incorrect class score

Example: $$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

e.g. 10

$$f(x_i, W) = [13, -7, 11]$$

$$y_i = 0$$

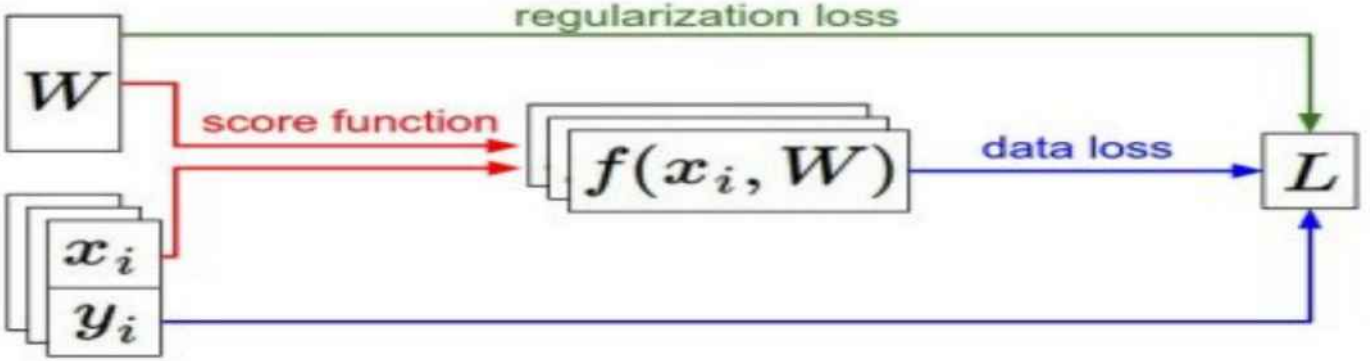
$$L_i = \max(0, -7 - 13 + 10) + \max(0, 11 - 13 + 10)$$

L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

$$L = \frac{1}{N} \sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta) \right] + \lambda R(W)$$

Regularization strength



Softmax Classifier

$$f(x_i, W) = Wx_i$$

score function is the same

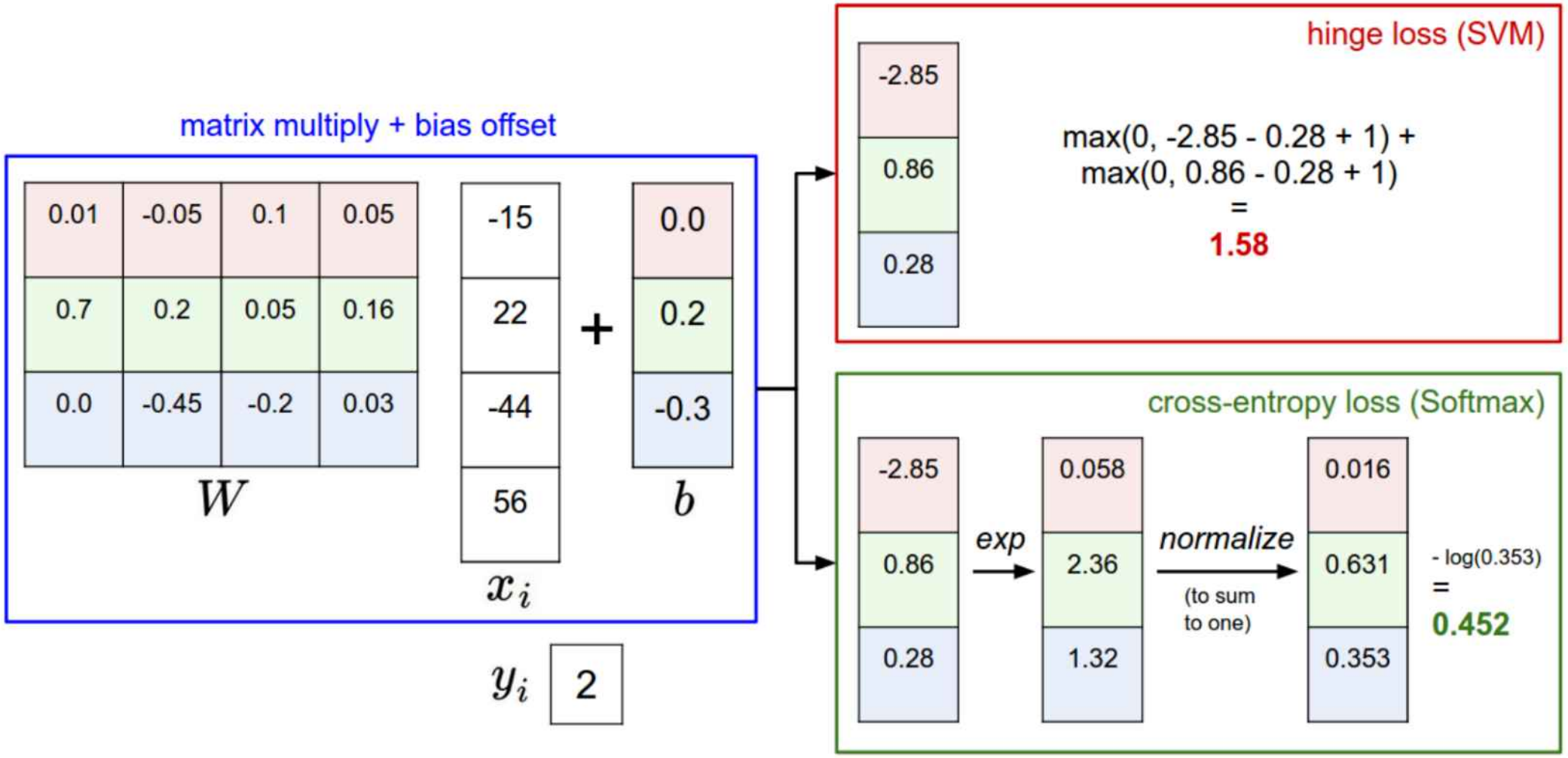
$$L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$$

softmax function

$$P(y_i \mid x_i; W)$$

i.e. we're minimizing the negative log likelihood.

$$[1, -2, 0] \rightarrow [e^1, e^{-2}, e^0] = [2.71, 0.14, 1] \rightarrow [0.7, 0.04, 0.26]$$


matrix multiply + bias offset
0.01
-0.05
0.1
0.05
0.7
0.2
0.05
0.16
0.0
-0.45
-0.2
0.03
W
-15
22
-44
56
x_i
+
0.0
0.2
-0.3
b
y_i
2
hinge loss (SVM)
-2.85
0.86
0.28
max(0, -2.85 - 0.28 + 1) +
max(0, 0.86 - 0.28 + 1)
=
1.58
cross-entropy loss (Softmax)
-2.85
0.86
0.28
exp
0.058
2.36
1.32
normalize
(to sum
to one)
0.016
0.631
0.353
- log(0.353)
=
0.452

Softmax vs. SVM

- Interpreting the probabilities from the Softmax

$$[1, -2, 0] \rightarrow [e^1, e^{-2}, e^0] = [2.71, 0.14, 1] \rightarrow [0.7, 0.04, 0.26]$$

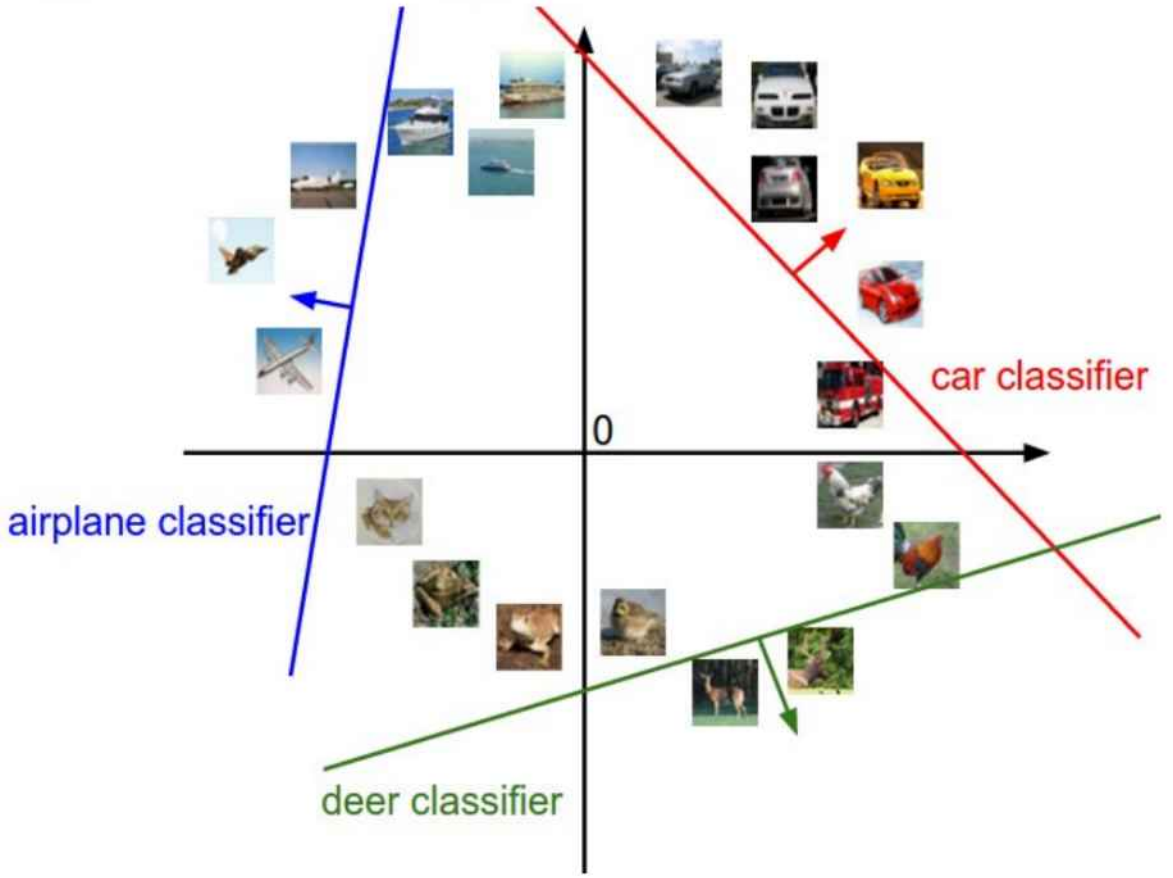
suppose the weights W were only half as large:

$$[0.5, -1, 0] \rightarrow [e^{0.5}, e^{-1}, e^0] = [1.65, 0.37, 1] \rightarrow [0.55, 0.12, 0.33]$$

Softmax vs. SVM

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + 1)$$

$$L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$$

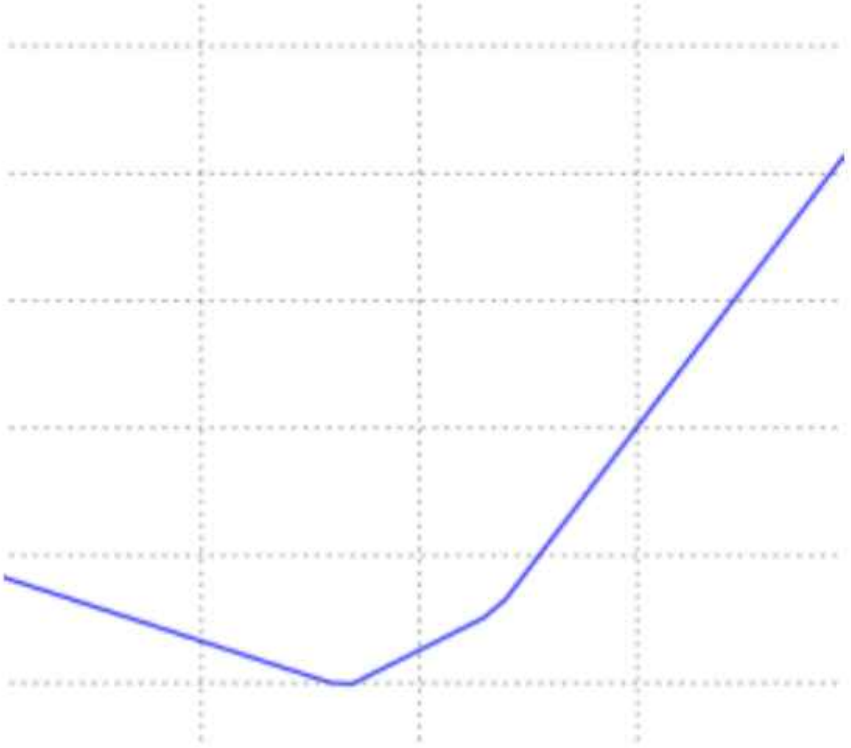


How do we find *W* and *b*?

Visualizing the (SVM) loss function

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

$$L_i(W + aW_1)$$

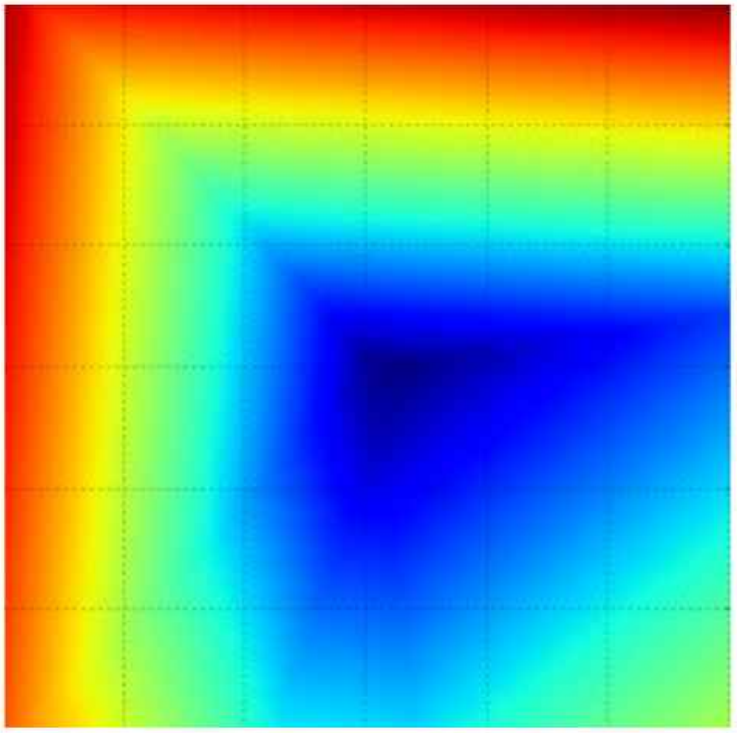


Plot of L_i versus a

Visualizing the (SVM) loss function

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

$$L_i(W + aW_1 + bW_2)$$



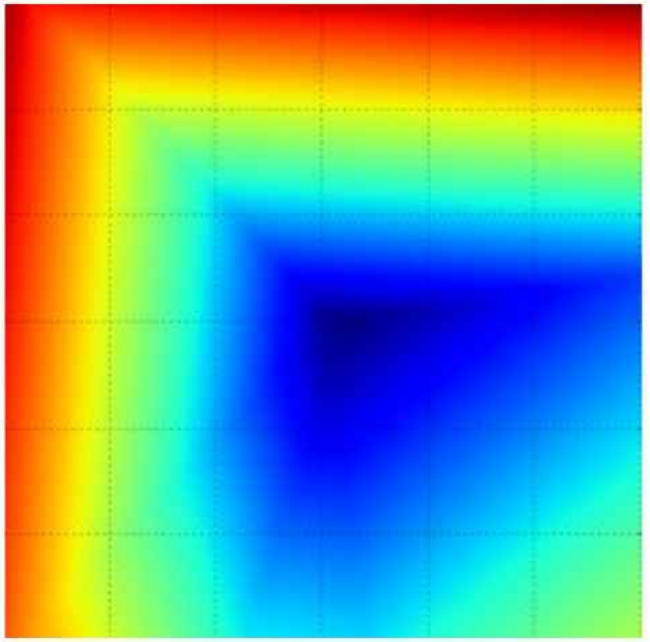
Plot of L_i versus *a* and *b*

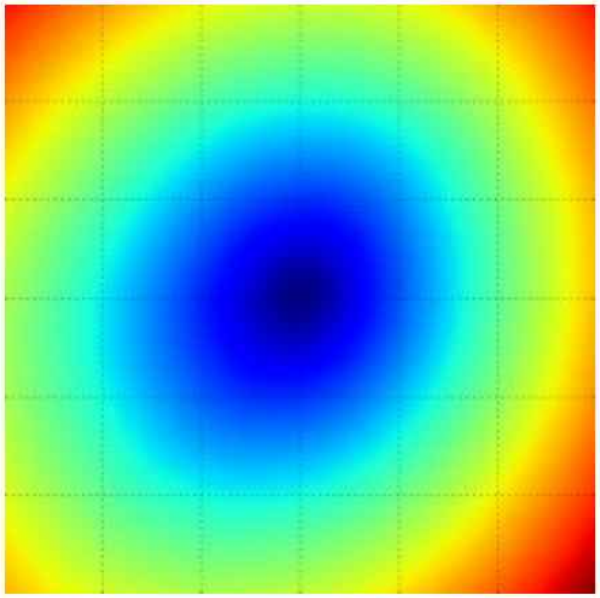
Visualizing the (SVM) loss function

$$L_i = \sum_{j \neq y_i} \max(0, f(x_i, W)_j - f(x_i, W)_{y_i} + \Delta)$$

the full data loss:

$L_i(W + aW_1 + bW_2)$



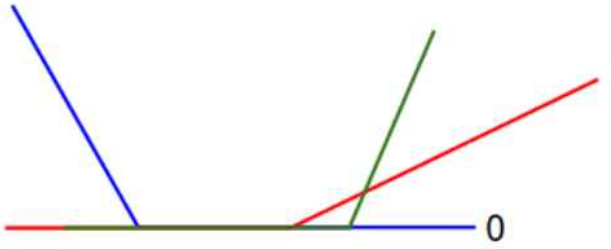


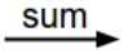
Visualizing the (SVM) loss function

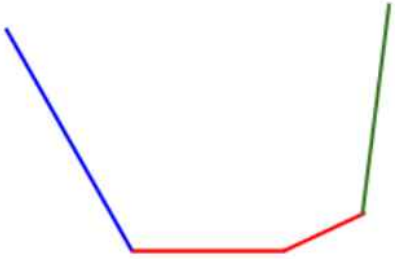
$$L_i = \sum_{j \neq y_i} \left[\max(0, w_j^T x_i - w_{y_i}^T x_i + 1) \right]$$

Suppose there are 3 examples with 3 classes (class 0, 1, 2 in sequence), then this becomes:

$$L_0 = \max\ (0, w_1^T x_0 - w_0^T x_0 + 1) + \max(0, w_2^T x_0 - w_0^T x_0 + 1)$$
$$L_1 = \max\ (0, w_0^T x_1 - w_1^T x_1 + 1) + \max(0, w_2^T x_1 - w_1^T x_1 + 1)$$
$$L_2 = \max\ (0, w_0^T x_2 - w_2^T x_2 + 1) + \max(0, w_1^T x_2 - w_2^T x_2 + 1)$$
$$L = (L_0 + L_1 + L_2)/3$$







One plot for each L_i

Plot of *L*

Strategy #1: A first very bad idea solution: Random search

```
# assume X_train is the data where each column is an example (e.g. 3073 x 50,000)
# assume Y_train are the labels (e.g. 1D array of 50,000)
# assume the function L evaluates the loss function

bestloss = float("inf") # Python assigns the highest possible float value
for num in xrange(1000):
  W = np.random.randn(10, 3073) * 0.0001 # generate random parameters
  loss = L(X_train, Y_train, W) # get the loss over the entire training set
  if loss < bestloss: # keep track of the best solution
    bestloss = loss
    bestW = W
  print 'in attempt %d the loss was %f, best %f' % (num, loss, bestloss)

# prints:
# in attempt 0 the loss was 9.401632, best 9.401632
# in attempt 1 the loss was 8.959668, best 8.959668
# in attempt 2 the loss was 9.044034, best 8.959668
# in attempt 3 the loss was 9.278948, best 8.959668
# in attempt 4 the loss was 8.857370, best 8.857370
# in attempt 5 the loss was 8.943151, best 8.857370
# in attempt 6 the loss was 8.605604, best 8.605604
# ... (trunctated: continues for 1000 lines)
```

Lets see how well this works on the test set...

```
# Assume X_test is [3073 x 10000], Y_test [10000 x 1]
scores = Wbest.dot(Xte_cols) # 10 x 10000, the class scores for all test examples
# find the index with max score in each column (the predicted class)
Yte_predict = np.argmax(scores, axis = 0)
# and calculate accuracy (fraction of predictions that are correct)
np.mean(Yte_predict == Yte)
# returns 0.1555
```

Strategy #2: A better but still very bad idea solution: **Random local search**

```
W = np.random.randn(10, 3073) * 0.001 # generate random starting W
bestloss = float("inf")
for i in xrange(1000):
  step_size = 0.0001
  Wtry = W + np.random.randn(10, 3073) * step_size
  loss = L(Xtr_cols, Ytr, Wtry)
  if loss < bestloss:
    W = Wtry
    bestloss = loss
  print 'iter %d loss is %f' % (i, bestloss)
```

gives 21.4%!

Strategy #3: **Following the gradient**

In 1-dimension, the derivative of a function:

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In multiple dimension, the **gradient** is the vector of (partial derivatives).

Evaluation the gradient numerically

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

in practice:

$$[f(x+h) - f(x-h)]/2h$$

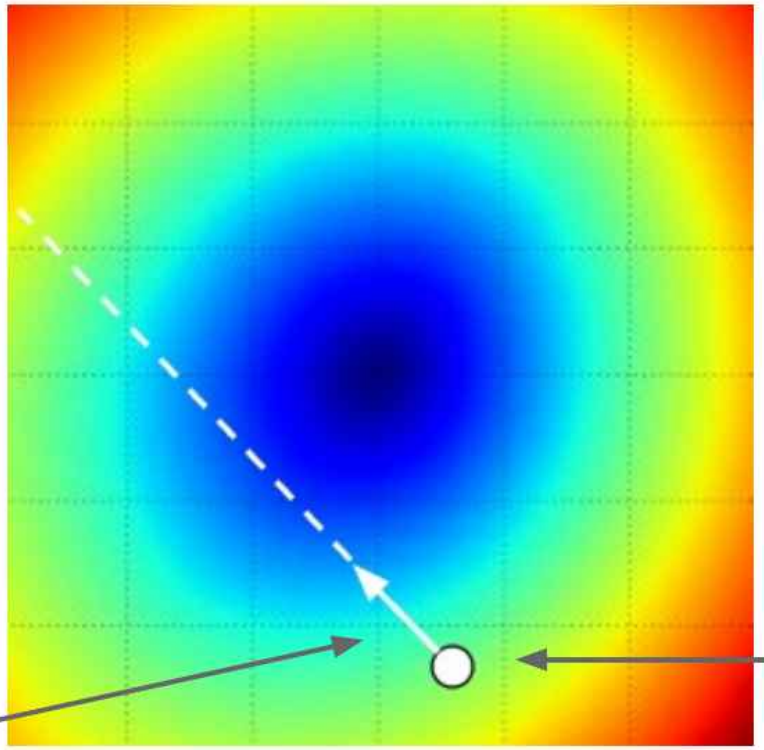
"centered difference formula"

```python
def eval_numerical_gradient(f, x):
  """
  a naive implementation of numerical gradient of f at x
  - f should be a function that takes a single argument
  - x is the point (numpy array) to evaluate the gradient at
  """

  fx = f(x) # evaluate function value at original point
  grad = np.zeros(x.shape)
  h = 0.00001

  # iterate over all indexes in x
  it = np.nditer(x, flags=['multi_index'], op_flags=['readwrite'])
  while not it.finished:

    # evaluate function at x+h
    ix = it.multi_index
    old_value = x[ix]
    x[ix] = old_value + h # increment by h
    fxh = f(x) # evalute f(x + h)
    x[ix] = old_value # restore to previous value (very important!)

    # compute the partial derivative
    grad[ix] = (fxh - fx) / h # the slope
    it.iternext() # step to next dimension

  return grad
```

```python
# to use the generic code above we want a function that takes a single argument
# (the weights in our case) so we close over X_train and Y_train
def CIFAR10_loss_fun(W):
  return L(X_train, Y_train, W)

W = np.random.rand(10, 3073) * 0.001 # random weight vector
df = eval_numerical_gradient(CIFAR10_loss_fun, W) # get the gradient
```

performing a parameter update

```
loss_original = CIFAR10_loss_fun(W) # the original loss
print 'original loss: %f' % (loss_original, )

# lets see the effect of multiple step sizes
for step_size_log in [-5,-4,-3,-2,-1,0,1,2]:
  step_size = 10 ** step_size_log
  W_new = W - step_size * df # new position in the weight space
  loss_new = CIFAR10_loss_fun(W_new)
  print 'for step size %f new loss: %f' % (step_size, loss_new)

# prints:
# original loss: 2.200718
# for step size 1.000000e-10 new loss: 2.200652
# for step size 1.000000e-09 new loss: 2.200057
# for step size 1.000000e-08 new loss: 2.194116
# for step size 1.000000e-07 new loss: 2.135493
# for step size 1.000000e-06 new loss: 1.647802
# for step size 1.000000e-05 new loss: 2.844355
# for step size 1.000000e-04 new loss: 25.558142
# for step size 1.000000e-03 new loss: 254.086573
# for step size 1.000000e-02 new loss: 2539.370888
# for step size 1.000000e-01 new loss: 25392.214036
```



original W

negative gradient direction

The problems with numerical gradient:

- approximate
- very slow to evaluate

```python
# to use the generic code above we want a function that takes a single argument
# (the weights in our case) so we close over X_train and Y_train
def CIFAR10_loss_fun(W):
  return L(X_train, Y_train, W)

W = np.random.rand(10, 3073) * 0.001 # random weight vector
df = eval_numerical_gradient(CIFAR10_loss_fun, W) # get the gradient
```

```python
def eval_numerical_gradient(f, x):
  """
  a naive implementation of numerical gradient of f at x
  - f should be a function that takes a single argument
  - x is the point (numpy array) to evaluate the gradient at
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  fx = f(x) # evaluate function value at original point
  grad = np.zeros(x.shape)
  h = 0.00001

  # iterate over all indexes in x
  it = np.nditer(x, flags=['multi_index'], op_flags=['readwrite'])
  while not it.finished:

    # evaluate function at x+h
    ix = it.multi_index
    x[ix] += h # increment by h
    fxh = f(x) # evalute f(x + h)
    x[ix] -= h # restore to previous value (very important!)

    # compute the partial derivative
    grad[ix] = (fxh - fx) / h # the slope
    it.iternext() # step to next dimension

  return grad
```

We need something better...

$$L_i = \sum_{j \neq y_i} \left[\max(0, w_j^T x_i - w_{y_i}^T x_i + 1) \right]$$

$$\begin{aligned} L_i = & \max\ (0, w_1^T x_i - w_0^T x_i + 1) \\ & + \max\ (0, w_2^T x_i - w_0^T x_i + 1) \\ & + \max\ (0, w_3^T x_i - w_0^T x_i + 1) \end{aligned}$$

$$\nabla_{w_j} L_i = 1(w_j^T x_i - w_{y_i}^T x_i + \Delta > 0) x_i$$

$$\nabla_{w_{y_i}} L_i = - \left(\sum_{j \neq y_i} 1(w_j^T x_i - w_{y_i}^T x_i + \Delta > 0) \right) x_i$$

- Numerical gradient: approximate, slow, easy to write

- Analytic gradient: exact, fast, error-prone

=>

In practice: Always use analytic gradient, but check implementation with numerical gradient. This is called a **gradient check.**

Gradient Descent

```
# Vanilla Gradient Descent

while True:
  weights_grad = evaluate_gradient(loss_fun, data, weights)
  weights += - step_size * weights_grad # perform parameter update
```

Mini-batch Gradient Descent

- only use a small portion of the training set to compute the gradient.

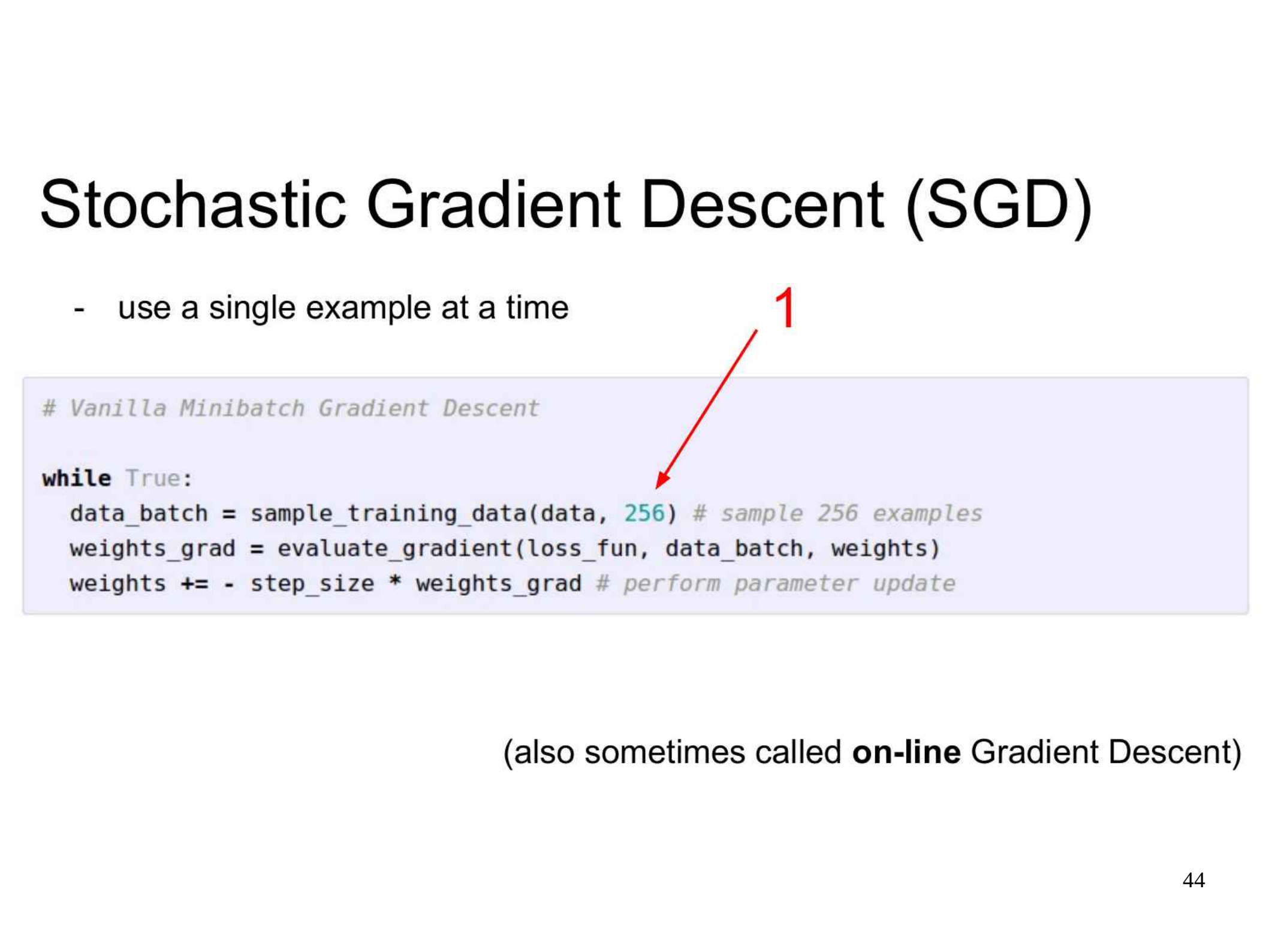
```
# Vanilla Minibatch Gradient Descent

while True:
  data_batch = sample_training_data(data, 256) # sample 256 examples
  weights_grad = evaluate_gradient(loss_fun, data_batch, weights)
  weights += - step_size * weights_grad # perform parameter update
```

Common mini-batch sizes are ~100 examples.
e.g. Krizhevsky ILSVRC ConvNet used 256 examples

Stochastic Gradient Descent (SGD)

- use a single example at a time




```
# Vanilla Minibatch Gradient Descent

while True:
  data_batch = sample_training_data(data, 256) # sample 256 examples
  weights_grad = evaluate_gradient(loss_fun, data_batch, weights)
  weights += - step_size * weights_grad # perform parameter update
```

(also sometimes called **on-line** Gradient Descent)

- Always use mini-batch gradient descent

- Incorrectly refer to it as “doing SGD” as everyone else

(or call it batch gradient descent)

- The mini-batch size is a hyperparameter, but it is not very common to cross-validate over it (usually based on practical concerns, e.g. space/time efficiency)

The dynamics of Gradient Descent

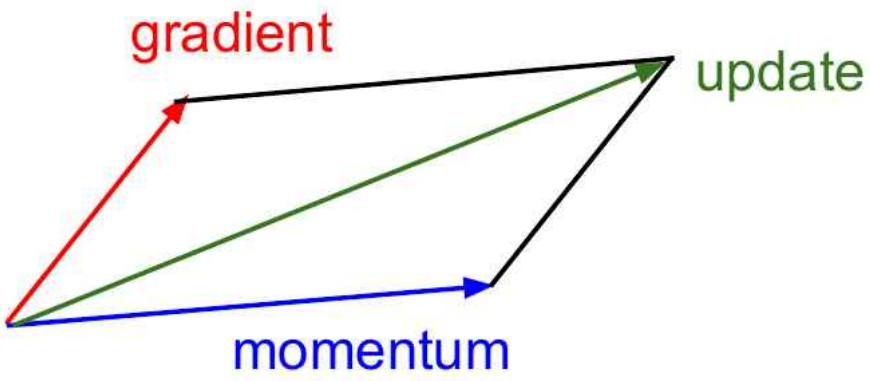
pull some weights up and some down

$$L = \frac{1}{N}\sum_i \sum_{j \neq y_i} \left[\max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + \Delta)\right] + \lambda \sum_k \sum_l W_{k,l}^2$$

$$L = \frac{1}{N}\sum_i -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right) + \lambda \sum_k \sum_l W_{k,l}^2$$

always pull the weights down

Momentum Update




```
weights_grad = evaluate_gradient(loss_fun, data, weights)
vel = vel * 0.9 - step_size * weights_grad
weights += vel
```